Flow around a $2D$ cylinder

1 Laplace's equation in polars

Consider Laplace's equation in polars

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0. \tag{1}$$

Looking for separable solutions of the form $\Phi(r,\theta) = R(r)H(\theta)$ we find

$$\frac{r^2}{R}\left(R'' + \frac{1}{r}R'\right) = -\frac{H''}{H} = \lambda^2. \tag{2}$$

Choosing the separation constant negative anticipates solutions for $H(\theta)$ that need to be periodic. Solving $H'' + \lambda^2 H = 0$ gives

$$H(\theta) = A\cos\lambda\theta + B\sin\lambda\theta. \tag{3}$$

When $\lambda \neq 0$ solving $R'' + \frac{1}{r}R' - \frac{\lambda^2}{r^2}R = 0$ gives

$$R(r) = a\, r^{\lambda} + b\, r^{-\lambda} \tag{4}$$

If we require $\Phi(r,\theta)$ to be continuous[1] in θ; that is, $\Phi(r,\theta) = \Phi(r,\theta + 2n\pi)$, then $\lambda = n$ (an integer). The general 2π-periodic solution of (1) is

$$\Phi(r,\theta) = \sum_{n=1}^{\infty} \left(a_n\, r^n + b_n\, r^{-n}\right)\left(A_n \cos n\theta + B_n \sin n\theta\right). \tag{5}$$

2 Flow around a cylinder

Consider an incompressible irrotational 2D fluid with velocity vector $\boldsymbol{u}$. Incompressibility implies that $\operatorname{div}\boldsymbol{u} = 0$ and irrotationality (no vorticity) implies that $\operatorname{curl}\boldsymbol{u} = 0$.

(i) The $\operatorname{div}\boldsymbol{u} = 0$ condition means that a stream function $\psi(x,y)$ exists such that

$$\boldsymbol{u} = (\psi_y, -\psi_x) = \boldsymbol{i}\psi_y - \boldsymbol{j}\psi_x.$$

Thus $\operatorname{curl}\boldsymbol{u} = 0$ means that

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial_x & \partial_y & \partial_z \\ \psi_y & -\psi_x & 0 \end{vmatrix} = 0$$

Thus we have Laplace's equation for the stream function

$$\psi_{xx} + \psi_{yy} = 0 \qquad \Rightarrow \qquad \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0. \tag{6}$$

(ii) The alternative way, using the potential, starts from $\operatorname{curl}\boldsymbol{u} = 0$. This means that a potential function ϕ exists such that $\boldsymbol{u} = \nabla\phi = \boldsymbol{i}\phi_x + \boldsymbol{j}\phi_y$. From $\operatorname{div}\boldsymbol{u} = 0$, we have Laplace's equation $\nabla^2\phi = \phi_{xx} + \phi_{yy} = 0$ which is also (1) in polar co-ordinates.

[1] The case with $\lambda = 0$ where $H(\theta) = \tilde{A}\theta + \tilde{B}$ and $R(r) = \tilde{a}\,\ln r + \tilde{b}$ is not 2π-periodic in θ.

Thus we want to solve (6) under the circumstance where the fluid, of constant horizontal speed U at infinity, flows past a solid cylinder of radius a centred at the origin. The fact that no fluid can cross the surface of the cylinder translates into the boundary condition

$$\left.\frac{\partial\psi}{\partial\theta}\right|_{r=a} = 0. \tag{7}$$

Since the flow at $r = \infty$ is horizontal we have $\boldsymbol{u} = U\boldsymbol{i} + 0\boldsymbol{j}$ there, which means that

$$\psi = Uy = Ur\sin\theta \qquad \text{at} \qquad r = \infty. \tag{8}$$

We want to solve Laplace's equation (6) in the infinite domain around the cylinder of radius a with prescribed BCs (7) and (8). Separating the $n = 1$ term from the rest of the infinite sum in (5) we have

$$\begin{aligned} \psi(r,\theta) &= \left(a_1\, r + b_1\, r^{-1}\right)(A_1\cos\theta + B_1\sin\theta) \\ &+ \sum_{n=2}^{\infty}\left(a_n\, r^n + b_n\, r^{-n}\right)(A_n\cos n\theta + B_n\sin n\theta) \end{aligned} \tag{9}$$

Applying the BC in (8) we find that

$$a_1 B_1 = U \qquad\qquad A_1 = 0 \tag{10}$$

and all coefficients $a_n = b_n = 0$ for $n \geq 2$. This leaves us with

$$\psi = U\left(r + \frac{b_1}{a_1}\frac{1}{r}\right)\sin\theta. \tag{11}$$

Finally applying the BC (7) at $r = a$ we find $b_1/a_1 = -a^2$ giving the stream function as

$$\psi = U\left(r - \frac{a^2}{r}\right)\sin\theta. \tag{12}$$