THE CHAIN RULE IN PARTIAL DIFFERENTIATION

1 Simple chain rule

If $u = u(x, y)$ and the two independent variables x and y are each a function of just *one* other variable t so that $x = x(t)$ and $y = y(t)$, then to find du/dt we write down the differential of u

$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \dots. \tag{1}$$

Then taking limits $\delta x \to 0$, $\delta y \to 0$ and $\delta t \to 0$ in the usual way we have

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}. \tag{2}$$

Note we only need straight 'd's' in dx/dt and dy/dt because x and y are function of one variable t whereas u is a function of both x and y.

2 Chain rule for two sets of independent variables

If $u = u(x, y)$ and the two independent variables x, y are each a function of *two* new independent variables s, t then we want relations between their partial derivatives.

1. When $u = u(x, y)$, for guidance in working out the chain rule, write down the differential

$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \dots \tag{3}$$

then when $x = x(s, t)$ and $y = y(s, t)$ (which are known functions of s and t), the chain rule for u_s and u_t in terms of u_x and u_y is

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} \tag{4}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}. \tag{5}$$

2. Conversely, when $u = u(s, t)$, for guidance in working out the chain rule write down the differential

$$\delta u = \frac{\partial u}{\partial s}\delta s + \frac{\partial u}{\partial t}\delta t + \dots \tag{6}$$

then when $s = s(x, y)$ and $t = t(x, y)$ (which are known functions of x and y) the chain rule for u_x and u_y in terms of u_s and u_t is

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} \tag{7}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}. \tag{8}$$

3. **It is important to note that:** $\frac{\partial s}{\partial x} \neq \left(\frac{\partial x}{\partial s}\right)^{-1}$ etc. Why? Because $\frac{\partial s}{\partial x}$ means differentiating s w.r.t x holding y constant whereas $\frac{\partial x}{\partial s}$ means differentiating x w.r.t s holding t constant. This is the most commonly made mistake.

3 Polar co-ordinates

We want to transform from Cartesian co-ordinates in the two independent variables (x, y) to two new independent variables (r, θ) which are polar co-ordinates. The pair (r, θ) therefore play the role of (s, t) in (4), (5), (7) and (8). The relation between these two sets of variables with x and y expressed in terms of r and θ is

$$x = r\cos\theta, \qquad y = r\sin\theta \tag{9}$$

whereas the other way round we have

$$r^2 = x^2 + y^2, \qquad \theta = \tan^{-1}\frac{y}{x}. \tag{10}$$

From (9) we have

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial y}{\partial r} = \sin\theta, \quad \frac{\partial x}{\partial \theta} = -r\sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta. \tag{11}$$

From (10) we have

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta, \qquad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta, \tag{12}$$

and[1]

$$\frac{\partial \theta}{\partial x} = \frac{-y}{x^2+y^2} = -\frac{\sin\theta}{r}, \qquad \frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r}. \tag{13}$$

Now we are ready to use the chain rule as in (3) and (4):

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \tag{14}$$

and

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{\partial u}{\partial x}(r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta). \tag{15}$$

Conversely

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\left(\frac{\sin\theta}{r}\right) \tag{16}$$

and

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\left(\frac{\cos\theta}{r}\right). \tag{17}$$

Exercise: From (16) and (17) we can write the derivative operations $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ as

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \left(\frac{\sin\theta}{r}\right)\frac{\partial}{\partial \theta} \qquad \frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \left(\frac{\cos\theta}{r}\right)\frac{\partial}{\partial \theta}. \tag{18}$$

Use the expression for $\frac{\partial}{\partial x}$ on $\frac{\partial u}{\partial x}$ in (16) to find u_{xx} in terms of u_{rr}, $u_{r\theta}$, $u_{\theta\theta}$ and u_r and u_θ. Do the same to find u_{yy}. Then show

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}. \tag{19}$$

[1] Note that $\frac{\partial r}{\partial x} = \cos\theta$ whereas $\frac{\partial x}{\partial r} = \cos\theta$, illustrating Item 3 at the bottom of the previous page.

4 Laplace's equation: changing from Cartesian to polar co-ordinates

Laplace's equation (a *partial differential equation* or PDE) in Cartesian co-ordinates is

$$u_{xx} + u_{yy} = 0. \tag{20}$$

We would like to transform to polar co-ordinates. In the handout on the chain rule (side 2) we found that the x and y-derivatives of u transform into polar co-ordinates in the following way:

$$u_x = (\cos\theta)\, u_r - \left(\frac{\sin\theta}{r}\right) u_\theta \qquad\qquad u_y = (\sin\theta)\, u_r + \left(\frac{\cos\theta}{r}\right) u_\theta. \tag{21}$$

Likewise the operation $\frac{\partial}{\partial x}$ becomes

$$\frac{\partial}{\partial x} = (\cos\theta)\,\frac{\partial}{\partial r} - \left(\frac{\sin\theta}{r}\right)\frac{\partial}{\partial\theta} \tag{22}$$

and the operation $\frac{\partial}{\partial y}$ becomes

$$\frac{\partial}{\partial y} = (\sin\theta)\,\frac{\partial}{\partial r} + \left(\frac{\cos\theta}{r}\right)\frac{\partial}{\partial\theta}. \tag{23}$$

Hence

$$u_{xx} = \frac{\partial u_x}{\partial x} = \underbrace{\left[(\cos\theta)\,\frac{\partial}{\partial r} - \left(\frac{\sin\theta}{r}\right)\frac{\partial}{\partial\theta}\right]}_{\frac{\partial}{\partial x}\text{ from (22)}}\underbrace{\left[(\cos\theta)\, u_r - \frac{\sin\theta}{r} u_\theta\right]}_{u_x\text{ from (21)}}. \tag{24}$$

Now we work this out using the product rule. Remember that u_r and u_θ are functions of both r and θ. We get

$$u_{xx} = (\cos^2\theta) u_{rr} + \left(\frac{\sin^2\theta}{r}\right) u_r + 2\left(\frac{\cos\theta\sin\theta}{r^2}\right) u_\theta - 2\left(\frac{\cos\theta\sin\theta}{r}\right) u_{r\theta} + \left(\frac{\sin^2\theta}{r^2}\right) u_{\theta\theta}. \tag{25}$$

Now we do the same for u_{yy} to get

$$u_{yy} = \frac{\partial u_y}{\partial y} = \underbrace{\left[(\sin\theta)\,\frac{\partial}{\partial r} + \left(\frac{\cos\theta}{r}\right)\frac{\partial}{\partial\theta}\right]}_{\frac{\partial}{\partial y}\text{ from (23)}}\underbrace{\left[(\sin\theta)\; u_r + \frac{\cos\theta}{r} u_\theta\right]}_{u_y\text{ from (21)}} \tag{26}$$

and therefore

$$u_{yy} = (\sin^2\theta) u_{rr} + \left(\frac{\cos^2\theta}{r}\right) u_r - 2\left(\frac{\cos\theta\sin\theta}{r^2}\right) u_\theta + 2\left(\frac{\cos\theta\sin\theta}{r}\right) u_{r\theta} + \left(\frac{\cos^2\theta}{r^2}\right) u_{\theta\theta}. \tag{27}$$

Summing (25) and (27) and remembering that $\cos^2\theta + \sin^2\theta = 1$, we find that

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \tag{28}$$

and so Laplace's equation converts to

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0. \tag{29}$$