Ae2 Mathematics: 1st and 2nd order PDEs

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These notes are not identical word-for-word with my lectures which will be given on a WB. Some of these notes may contain more examples than the corresponding lecture while in other cases the lecture may contain more detailed working. I will not be handing out copies of these notes – **you are therefore advised to attend lectures and take your own**.

Contents

[1] Do not confuse me with Dr J. Gibbons who is also in the Mathematics Dept.

1 1st order PDEs & the method of characteristics

1.1 The derivation of the auxiliary equations

Consider the semi-linear 1st order partial differential equation[2] (PDE)

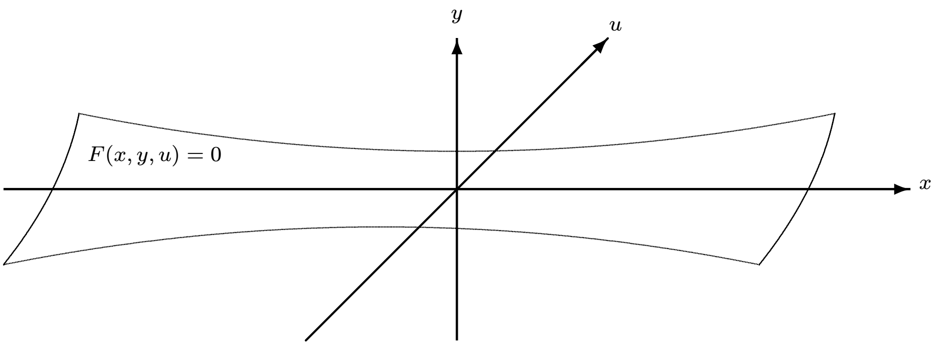
$$P(x,y)u_x + Q(x,y)u_y = R(x,y,u) \tag{1.1}$$

where P and Q are continuous functions and R is not necessarily linear[3] in u.

Consider solutions represented as a family of surfaces (which one depends on our boundary conditions). Below is a picture of one of these surfaces which we'll call

$$F(x,y,u) = 0 \qquad\qquad u = u(x,y) \tag{1.2}$$

in (x,y,u)-space.



Because $F = 0$ in (1.2), it must be true that $dF = 0$ and so the chain rule gives

$$0 = dF = F_x dx + F_y dy + F_u du \tag{1.3}$$

$$du = u_x dx + u_y dy \tag{1.4}$$

Combining these two gives

$$0 = F_x dx + F_y dy + \big(u_x dx + u_y dy\big) F_u\,. \tag{1.5}$$

Re-arranging terms we have

$$u_x\big[F_u dx\big] + u_y\big[F_u dy\big] = -\big[F_x dx + F_y dy\big]\,. \tag{1.6}$$

Now compare this with our PDE in (1.1) : a comparison of coefficients gives

$$F_u dx = P\,; \qquad F_u dy = Q\,; \qquad -\big[F_x dx + F_y dy\big] = R\,. \tag{1.7}$$

Now, because $-\big[F_x dx + F_y dy\big] = F_u du$ we can represent (1.7) as a series of ratios which are called the **auxiliary equations**

[2]The subscript notation $u_x = \partial u/\partial x$ and $u_{xy} = \partial^2 u/\partial x \partial y$ etc is used throughout.
[3]This PDE is also said to be quasi-linear if P and Q are dependent on u.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{du}{R} \qquad R \neq 0 \tag{1.8}$$

$$\frac{dx}{P} = \frac{dy}{Q} = 0 \qquad \text{and} \qquad du = 0 \qquad (R = 0) \tag{1.9}$$

1. The first pair in the auxiliary equations can be re-written as a differential equation in x, y without reference to u
$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}\,. \tag{1.10}$$
In principle, this can be solved to give
$$\lambda(x,y) = c_1 \tag{1.11}$$
where c_1 is a constant of integration. These curves or lines are called the **characteristics** or **characteristic curves** of the PDE[4]. They form a family of curves because of the arbitrariness of the constant c_1.

2. If $R = 0$ we have $du = 0$ as in the second line of (1.8), in which case $u = \text{const} = c_2$ on characteristics.

   If $R \neq 0$ as in the first line of (1.8) then one of the other pair of differential equations must be solved to get $u = g(x, y, c_2)$ on characteristics $\lambda(x,y) = c_1$, where c_2 is another constant of integration.

3. The two arbitrary constants c_1 and c_2 can be thought of as being related by an arbitrary function $c_2 = f\big(c_1\big)$.

1.2 Seven examples

Example 1 : Consider the simple PDE

$$u_x + u_y = 0\,. \tag{1.12}$$

Solution : Obviously $P = 1\,,\; Q = 1$ and $R = 0$. Therefore the auxiliary equations (1.8) are

$$\frac{dx}{1} = \frac{dy}{1} \qquad \text{and} \qquad du = 0\,. \tag{1.13}$$

Clearly the characteristics are the family of curves $y = x + c_1$ on which $u = \text{const} = c_2$. The arbitrary constants c_1 and c_2 are related by $c_2 = f\big(c_1\big)$ in which case $u = f\big(x - y\big)$ for an

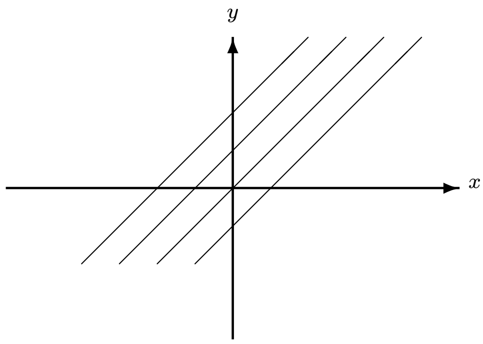
[4] They are also sometimes referred to as Riemann invariants,

arbitrary differentiable function f : **this is the general solution.** It can easily be checked that this is indeed a solution of (1.13) by writing $X = x - y$ and $u = f(X)$. Then

$$u_x = X_x f'(X) \qquad\qquad u_y = X_y f'(X) \tag{1.14}$$

However, $X_x = 1$ and $X_y = -1$ and so $u_x + u_y = 0$.

For Example 1, the characteristics are the family of straight lines $y = x + c_1$.



Example 2 : Consider the simple PDE

$$xu_x - yu_y = 0\,. \tag{1.15}$$

subject to the boundary conditions $u = x^4$ on the line $y = x$.

Solution : Obviously $P = x$, $Q = -y$ and $R = 0$. Therefore the auxiliary equations (1.8) are

$$\frac{dx}{x} = -\frac{dy}{y} \qquad \text{and} \qquad du = 0\,. \tag{1.16}$$

Clearly the characteristics come from

$$\int \frac{dx}{x} + \int \frac{dy}{y} = \text{const} \tag{1.17}$$

from which we discover that $\ln(xy) = $ const. **Thus the characteristics are the family of hyperbolae $xy = c_1$.** On these characteristics $u = $ const $= c_2$ in which case

$$u = f\big(xy\big) \tag{1.18}$$

for an arbitrary differentiable function f : this is the general solution. It can easily be checked that this is indeed a solution of (1.15) by writing $X = xy$ and $u = f(X)$. Then $u_x = X_x f'(X)$ and $u_y = X_y f'(X)$ with $X_x = y$ and $X_y = x$ and so $xu_x - yu_y = 0$.

Application of the BCs $u = x^4$ on the line $y = x$ now determines f because on $y = x$

$$x^4 = f\big(x^2\big) \tag{1.19}$$

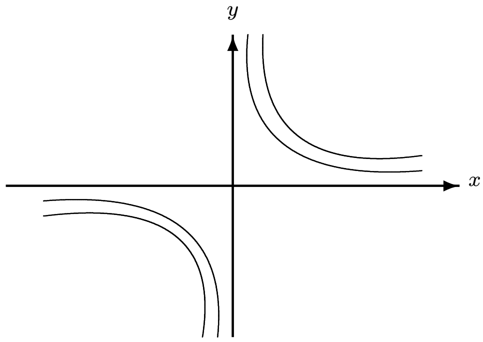
and so $f(t) = t^2$: however, $f(t)$ is only defined[5] for $t \geq 0$. Thus our solution is

$$u(x, y) = x^2 y^2 \qquad\qquad xy \geq 0\,, \tag{1.20}$$

which means that it is only valid in the 1st and 3rd quadrants of the characteristic plane.

[5] The variable t is simply the argument of the function $f(t)$: the fact that it is called t has no meaning – we could designate it by any symbol we wish.

For Example 2, the characteristics are the family of hyperbolae $xy = c_1 \geq 0$.



Example 3 : Consider the PDE
$$xu_x + yu_y = u\,. \tag{1.21}$$
subject to the boundary conditions $u = y^2$ on the line $x = 1$.

Solution : Clearly $P = x\,,\ Q = y$ and $R = u$. Therefore the auxiliary equations (1.8) are
$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{u}\,. \tag{1.22}$$
Clearly the characteristics come from
$$\int \frac{dy}{y} - \int \frac{dx}{x} = \text{const} \tag{1.23}$$
from which we discover that $\ln\big(y/x\big) = $ const. **Thus the characteristics are the family of lines $y = x\,c_1$: these are a fan of straight lines all passing through the origin.** Now integrate one of the other pair (either will do) : $\ln\big(u/x\big) = $ const which means that $u = x\,c_2$. Therefore, on characteristics
$$u = xf\big(y/x\big) \tag{1.24}$$
for an arbitrary differentiable function f : this is the general solution. Now applying the BCs : $u = y^2$ on $x = 1$ we obtain $f(y) = y^2$. Therefore, with these BCs, the solution is
$$u = x\big(y/x\big)^2 = y^2/x\,. \tag{1.25}$$

Example 4 : Consider the PDE
$$yu_x + xu_y = x^2 + y^2\,, \tag{1.26}$$
subject to the boundary conditions
$$u = \begin{cases} 1 + x^2 & \text{on} \quad y = 0 \\ 1 + y^2 & \text{on} \quad x = 0 \end{cases} \tag{1.27}$$
Solution : $P = y\,,\ Q = x$ and $R = x^2 + y^2$. Therefore the auxiliary equations (1.8) are
$$\frac{dx}{y} = \frac{dy}{x} = \frac{du}{x^2 + y^2}\,. \tag{1.28}$$

Characteristics come from the integral

$$\int (xdx - ydy) = \text{const} \tag{1.29}$$

which gives $x^2 - y^2 = c_1$ and

$$\begin{aligned} du &= y^{-1}\big(x^2 + y^2\big)dx \\ &= ydx + x^2y^{-1}dx \\ &= ydx + xdy \qquad \text{on characteristics} \\ &= d(xy) \end{aligned} \tag{1.30}$$

which integrates to

$$u = xy + c_2\,. \tag{1.31}$$

Therefore, as the general solution, we have

$$u = xy + f\big(x^2 - y^2\big)\,. \tag{1.32}$$

Applying the BCs:

$$\begin{aligned} 1 + x^2 &= f(x^2) \qquad \Rightarrow f(t) = 1 + t\,, \qquad t \geq 0\,, \\ 1 + y^2 &= f(-y^2) \qquad \Rightarrow f(t) = 1 - t\,, \qquad t \leq 0\,. \end{aligned} \tag{1.33}$$

Thus we end up with

$$f(t) = 1 + |t| \tag{1.34}$$

so

$$u = xy + 1 + |x^2 - y^2|\,. \tag{1.35}$$

Example 5: (Exam 2001) Show that the PDE

$$yu_x - 3x^2yu_y = 3x^2u\,, \tag{1.36}$$

has a general solution of the form

$$yu(x,y) = f\big(x^3 + y\big) \tag{1.37}$$

where f is an arbitrary function.

(i) If you are given that

$$u(0,y) = y^{-1}\tanh y \tag{1.38}$$

on the line $x = 0$, show that

$$yu(x,y) = \tanh\big(x^3 + y\big)\,. \tag{1.39}$$

(ii) If you given that $u(x,1) = x^6$ on $y = 1$ show that

$$yu(x,y) = \big(x^3 + y - 1\big)^2\,. \tag{1.40}$$

Solution : $P = y$, $Q = -3x^2y$ and $R = 3x^2u$. Thus the auxiliary equations are

$$\frac{dx}{y} = -\frac{dy}{3x^2y} = \frac{du}{3x^2u}\,, \tag{1.41}$$

which gives characteristics as solutions of $dy/dx = -3x^2$. These are the family of curves $y + x^3 = c_1$. Then we also have

$$\frac{du}{u} = -\frac{dy}{y}\,, \tag{1.42}$$

from which we discover that $\ln uy = \text{const}$ or $uy = c_2$. Therefore the general solution is

$$yu(x, y) = f(y + x^3)\,. \tag{1.43}$$

Then, on $x = 0$,

$$\frac{f(y)}{y} = \frac{\tanh y}{y} \tag{1.44}$$

in which case $f(y) = \tanh y$ and so

$$yu(x, y) = \tanh\,(y + x^3)\,. \tag{1.45}$$

However, for the other BC $u(x, 1) = x^6$, we have $x^6 = f(1 + x^3)$ from which we find $f(t) = (t - 1)^2$ where $t = 1 + x^3$. With these BCs, the solution is

$$yu(x, y) = \big(y + x^3 - 1\big)^2\,. \tag{1.46}$$

Example 6 : (Exam 2002) Show that the PDE

$$yu_x + xu_y = 4xy^3\,, \tag{1.47}$$

has a general solution of the form

$$u(x, y) = y^4 + f\big(y^2 - x^2\big) \tag{1.48}$$

where f is an arbitrary function. If you are given that $u(0, y) = 0$ and $u(x, 0) = -x^4$, show that the solution is

$$u(x, y) = 2x^2y^2 - x^4\,. \tag{1.49}$$

Solution : The auxiliary equations are

$$\frac{dx}{y} = \frac{dy}{x} = \frac{du}{4xy^3}\,. \tag{1.50}$$

Characteristics come from the integration of $xdx = ydy$ thereby giving and so $y^2 - x^2 = c_1$. We also have $du = 4y^3dy$ resulting in $u = y^4 + c_2$, thereby giving the general solution

$$u(x, y) = y^4 + f\big(y^2 - x^2\big)\,. \tag{1.51}$$

Applying the two boundary conditions gives

$$\begin{aligned} 0 &= y^4 + f\big(y^2\big) \qquad \Rightarrow \qquad f(t) = -t^2 \quad t \geq 0 \\ -x^4 &= f\big(-x^2\big) \qquad \Rightarrow \qquad f(t) = -t^2 \quad t \leq 0 \end{aligned} \tag{1.52}$$

Thus we have

$$u(x,y) = y^4 - \big(y^2 - x^2\big)^2 = 2x^2y^2 - x^4\,. \tag{1.53}$$

One can check directly that this is indeed a solution.

Example 7 : (Exam 2003) Show that the PDE

$$y^2 u_x + x^2 u_y = 2xy^2\,, \tag{1.54}$$

has a general solution of the form

$$u(x,y) = x^2 + f\big(y^3 - x^3\big) \tag{1.55}$$

where f is an arbitrary function.

(i) If $u(0,y) = -y^6$ and $u(x,0) = x^2 - x^6$ show that

$$u(x,y) = x^2 - x^6 + 2x^3y^3 - y^6 \tag{1.56}$$

and

(ii) If $u(0,y) = \exp(y^3)$ and $u(x,0) = x^2 + \exp(-x^3)$ show that

$$u(x,y) = x^2 + \exp(y^3 - x^3)\,. \tag{1.57}$$

Solution : The auxiliary equations are

$$\frac{dx}{y^2} = \frac{dy}{x^2} = \frac{du}{2xy^2}\,. \tag{1.58}$$

Characteristics come from the integration of $x^2dx = y^2dy$ thereby giving the family of curves $y^3 - x^3 = c_1$. We also have $du = 2xdx$ giving $u = x^2 + c_2$. Thus the general solution is

$$u(x,y) = x^2 + f\big(y^3 - x^3\big)\,. \tag{1.59}$$

Applying the two boundary conditions gives :

(i) For $u(0,y) = -y^6$ and $u(x,0) = x^2 - x^6$

$$\begin{aligned} -y^6 &= f(y^3) \qquad \Rightarrow \qquad f(t) = -t^2 \\ x^2 - x^6 &= x^2 + f\big(-x^3\big) \qquad \Rightarrow \qquad f(t) = -t^2 \end{aligned} \tag{1.60}$$

Therefore

$$u(x,y) = x^2 - \big(y^3 - x^3\big)^2 = x^2 - x^6 + 2x^3y^3 - y^6\,. \tag{1.61}$$

(ii) For $u(0,y)=\exp(y^3)$ and $u(x,0)=x^2+\exp(-x^3)$

$$\begin{aligned}\exp(y^3) &= f(y^3) \qquad \Rightarrow \qquad f(t)=\exp t\\ x^2+\exp(-x^3) &= x^2+f(-x^3) \qquad \Rightarrow \qquad f(t)=\exp t\end{aligned} \tag{1.62}$$

Therefore, with $f(t)=\exp t$ the solution with these BCs is

$$u(x,y)=x^2+\exp\left(y^3-x^3\right). \tag{1.63}$$

2 Characteristics and 2nd order PDEs

2.1 Derivation of two sets of characteristics

Consider the class of 2nd order PDEs

$$Ru_{xx}+2Su_{xy}+Tu_{yy}=f \tag{2.1}$$

where $u_{xx},\ u_{yy}$ & u_{xy} are 2nd derivatives & $R,\ S,\ T$ and f are functions of $x,\ y,\ u,\ u_x$ & u_y. For motivational purposes let us return to the class of 1st order semi-linear equations

$$Pu_x+Qu_y=R\,. \tag{2.2}$$

Together with $u_xdx+u_ydy=du$, these can be written as

$$\begin{pmatrix} P & Q\\ dx & dy\end{pmatrix}\begin{pmatrix} u_x\\ u_y\end{pmatrix}=\begin{pmatrix} R\\ du\end{pmatrix}. \tag{2.3}$$

However, from the auxiliary equations for (2.2)

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{du}{R} \tag{2.4}$$

which, can be re-expressed as

$$\det\begin{pmatrix} P & Q\\ dx & dy\end{pmatrix}=0\,, \tag{2.5}$$

the 2×2 matrix on the LHS in (2.3) has zero determinant. This means that solutions for u_x and u_y are not unique: characteristics are a *family* of curves, so u_x and u_y may differ on each curve within the family.

Keeping this property in mind for the 2nd order class in (2.1) we use the chain rule to find dF for a function

$$dF=F_xdx+F_ydy \tag{2.6}$$

and then take $F=u_x$ and $F=u_y$ in turn.

$$d(u_x) = u_{xx}dx+u_{xy}dy \tag{2.7}$$

$$d(u_y) = u_{xy}dx+u_{yy}dy\,. \tag{2.8}$$

Together with (2.1) we now have a 3×3 system :

$$\begin{pmatrix} R & 2S & T \\ dx & dy & 0 \\ 0 & dx & dy \end{pmatrix} \begin{pmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{pmatrix} = \begin{pmatrix} f \\ d(u_x) \\ d(u_y) \end{pmatrix} . \tag{2.9}$$

Zero determinant of the 3×3 on the LHS side of (2.9) gives[6]

$$R(dy)^2 - 2S\,dxdy + T(dx)^2 = 0 \tag{2.10}$$

which leads to the following formal classification :

Classification :

$$R\left(\frac{dy}{dx}\right)^2 - 2S\left(\frac{dy}{dx}\right) + T = 0\,, \tag{2.11}$$

which has two roots

$$\frac{dy}{dx} = \frac{S \pm \sqrt{S^2 - RT}}{R}\,. \tag{2.12}$$

In principle, this provides us with two ODEs to solve : call these solutions $\xi(x, y) = c_1$ and $\eta(x, y) = c_2$; **these are our two sets of characteristic curves.**

1. When $S^2 > RT$ the two roots are real : the PDE is classed as **HYPERBOLIC** ;
2. When $S^2 < RT$ the roots form a complex conjugate pair : the PDE is classed as **ELLIPTIC** ;
3. When $S^2 = RT$ the double root is real : the PDE is classed as **PARABOLIC** .

A transformation of the PDE from derivatives in x, y into one in ξ, η produces the *canonical form* of the PDE :

1. In the hyperbolic case we use $\xi(x, y)$ and $\eta(x, y)$ as the new co-ordinates in place of x , y : these arise from integration of the two real solutions of (2.12).
2. The new co-ordinates $\xi(x, y)$ and $\eta(x, y)$ arise from the real and imaginary parts of the complex conjugate pair of solutions of (2.12).
3. In the parabolic there is only one real (double) root $\xi(x, y)$ of (2.12) : the other $\eta(x, y)$ may be chosen at will, usually for convenience ; for instance, if $\xi = x + y$ then it might be convenient to choose $\eta = x + y$ for simplicity.

[6] Note the negative sign on the central term $-2S\,dxdy$ in contrast to the positive sign in the PDE (2.1).

2.2 Six Examples

Example 1 : The standard form of the wave equation is $u_{xx} - c^{-2}u_{tt} = 0$ but under the transformation $y = ct$ we obatin $u_{xx} - u_{yy} = 0$.

Solution : $R = 1$, $S = 0$ and $T = -1$. Thus $R^2 - ST = 1$ and we have a **hyperbolic PDE**. (2.11) is

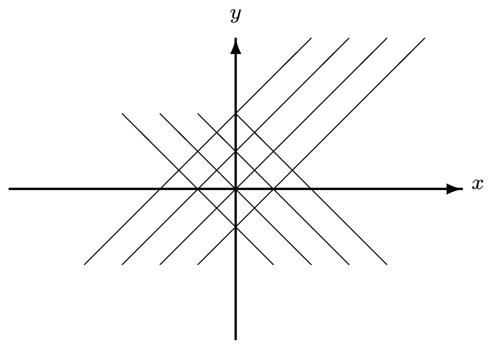
$$\left(\frac{dy}{dx}\right)^2 - 1 = 0 \tag{2.13}$$

which has two real roots $dy/dx = \pm 1$. Thus our two sets of characteristics are

$$\xi = x + y = c_1 \qquad \eta = x - y = c_2 \,. \tag{2.14}$$

Clearly, therefore, the characteristics are two families of straight lines, the first of gradient $+1$ and the second -1.

For both Examples 1 & 2, the characteristics are the 2 families of straight lines $x - y = c_2$ and $x + y = c_1$.



Now transform into the new co-ordinates $\xi = x + y\,,\ \ \eta = x - y$. The chain rule gives

$$\frac{\partial}{\partial x} = \xi_x \frac{\partial}{\partial \xi} + \eta_x \frac{\partial}{\partial \eta}\,; \qquad \frac{\partial}{\partial y} = \xi_y \frac{\partial}{\partial \xi} + \eta_y \frac{\partial}{\partial \eta} \tag{2.15}$$

into which the definitions of ξ, η allow us to write $\xi_x = 1,\ \eta_x = 1,\ \xi_y = 1$ and $\eta_y = -1$.

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\,; \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \tag{2.16}$$

Thus we have $u_x = u_\xi + u_\eta$ and $u_y = u_\xi - u_\eta$. Moreover,

$$u_{xx} = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)(u_\xi + u_\eta)\,; \qquad u_{yy} = \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right)(u_\xi - u_\eta) \tag{2.17}$$

Thus we have

$$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}\,; \qquad u_{yy} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta} \tag{2.18}$$

and so our PDE transforms to

$$0 = u_{xx} - u_{yy} = 4u_{\xi\eta}\,. \tag{2.19}$$

The **canonical form** is $u_{\xi\eta} = 0$. This can be integrated wrt ξ directly to give

$$u_\eta = F(\eta)\,, \tag{2.20}$$

where F is an arbitrary function of η, and then again wrt η

$$\begin{aligned} u(\xi,\,\eta) &= \int F(\eta)\,d\eta + g(\xi) \\ &= = f(\eta) + g(\xi)\,. \end{aligned} \tag{2.21}$$

Both f and g are arbitrary functions. Thus we have the general solution

$$u(x,y) = f(x-y) + g(x+y)\,. \tag{2.22}$$

Example 2 : Consider the PDE $u_{xx} + 2u_{xy} + u_{yy} = 0$: in this case $R = 1$, $S = 1$ and $T = 1$ so $R^2 - ST = 0$. Thus the PDE is **parabolic** : (2.11) is

$$\left(\frac{dy}{dx} - 1\right)^2 = 0 \tag{2.23}$$

which has a double real root $dy/dx = 1$. Thus one characteristic curve is

$$\eta = x - y \tag{2.24}$$

and we have a free choice with the other : for convenience we choose this as $\xi = x + y$, which makes (ξ, η) the same as Example 1. Then we have

$$\begin{aligned} u_{xx} &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \\ u_{yy} &= u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta} \\ u_{xy} &= u_{\xi\xi} - u_{\eta\eta} \end{aligned} \tag{2.25}$$

and so our PDE transforms to

$$0 = u_{xx} + 2u_{xy} + u_{yy} = 4u_{\xi\xi}\,. \tag{2.26}$$

Integration wrt ξ gives

$$u_\xi = f(\eta) \tag{2.27}$$

for arbitrary f, and again

$$u(\xi,\eta) = \xi f(\eta) + g(\eta) \tag{2.28}$$

for arbitrary g. In terms of x, y this becomes

$$u(x,y) = (x+y)f(x-y) + g(x-y)\,. \tag{2.29}$$

One can check by direct differentiation – provided f, g have continuous second derivatives – that (2.29) is a solution.

Example 3 : Consider the PDE $u_{xx} + x^2 u_{yy} = 0$: in this case $R = 1$, $S = 0$ and $T = x^2$ so $R^2 - ST = -x^2 < 0$. Thus the PDE is **elliptic**. (2.11) is

$$\left(\frac{dy}{dx}\right)^2 + x^2 = 0\,. \tag{2.30}$$

Is there a natural canonical form? The formal solution of (2.30) is the complex function

$$y \pm \tfrac{1}{2} i x^2 = c_{1,2}\,. \tag{2.31}$$

We could choose ξ and η as the real and imaginary parts respectively (or v-v). Take $\xi = \frac{1}{2}x^2$ and $\eta = y$, then

$$\frac{\partial}{\partial x} = x\frac{\partial}{\partial \xi} \qquad\qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial \eta} \tag{2.32}$$

Thus $u_x = x u_\xi$ and $u_y = u_\eta$. Differentiating again is tricky because we have mixed old/new derivatives on the RHS of $u_x = x u_\xi$. To find u_{xx} we use the product rule, differentiating wrt x first and then using the chain rule

$$\begin{aligned} u_{xx} &= u_\xi + x\frac{\partial}{\partial x}u_\xi = u_\xi + x^2 u_{\xi\xi}\,, \\ u_{yy} &= u_{\eta\eta}\,. \end{aligned} \tag{2.33}$$

Thus the PDE is

$$0 = u_{xx} + x^2 u_{yy} = x^2\left(u_{\xi\xi} + u_{\eta\eta}\right) + u_\xi \tag{2.34}$$

and so the canonical form is

$$u_{\xi\xi} + u_{\eta\eta} + \frac{1}{2\xi}u_\xi = 0\,. \tag{2.35}$$

Example 4 (exam 05) : Consider the PDE $8u_{xx} - 6u_{xy} + u_{yy} + 4 = 0$. Show that this is hyperbolic and that the characteristics are $\xi = x + 2y$ and $\eta = x + 4y$. Hence show the canonical form is $u_{\xi\eta} = 1$. If $u = \cosh x$ & $u_y = 2\sinh x$ on $y = 0$, show that the solution is

$$u = \xi\eta - \tfrac{1}{2}(\xi^2 + \eta^2) + \cosh\xi\,. \tag{2.36}$$

Solution : In this case $R = 8$, $S = -3$ and $T = 1$ so $S^2 - RT = 1$. Thus the PDE is **hyperbolic**. (2.11) is

$$8\left(\frac{dy}{dx}\right)^2 + 6\frac{dy}{dx} + 1 = 0\,, \tag{2.37}$$

which factorizes to

$$\left(4\frac{dy}{dx} + 1\right)\left(2\frac{dy}{dx} + 1\right) = 0\,, \tag{2.38}$$

so $\xi = x + 2y$ and $\eta = x + 4y$ as required. Now we transform to canonical variables

$$u_x = u_\xi + u_\eta \qquad\qquad u_y = 2u_\xi + 4u_\eta \tag{2.39}$$

and

$$\begin{aligned} u_{xx} &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}\,, \\ u_{xy} &= 2u_{\xi\xi} + 6u_{\xi\eta} + 4u_{\eta\eta}\,, \\ u_{yy} &= 4u_{\xi\xi} + 16u_{\xi\eta} + 16u_{\eta\eta}\,. \end{aligned} \tag{2.40}$$

Therefore

$$\begin{aligned} 0 &= 8u_{xx} - 6u_{xy} + u_{yy} + 4 \\ &= u_{\xi\xi}(8-12+4) + u_{\xi\eta}(16-36+16) + u_{\eta\eta}(8-24+16) + 4 \\ &= 4 - 4u_{\xi\eta}\,. \end{aligned} \tag{2.41}$$

Thus we have the **canonical form** $u_{\xi\eta} = 1$ which integrates to

$$u = \xi\eta + F(\eta) + G(\xi)\,. \tag{2.42}$$

Applying the BCs: on $y = 0$ we have $\xi = \eta = x$: with $u = \cosh x$

$$\cosh x = F(x) + G(x) + x^2\,. \tag{2.43}$$

and with $u_y = 2\sinh x$

$$\begin{aligned} 2\sinh x &= \left\{2\left(\frac{\partial}{\partial\xi} + 2\frac{\partial}{\partial\eta}\right)[F(\eta) + G(\xi) + \xi\eta]\right\}_{y=0} \\ &= 2\left\{G'(x) + 3x + 2F'(x)\right\} \end{aligned} \tag{2.44}$$

Integrating this gives

$$G(x) + 2F(x) = \cosh x - \frac{3}{2}x^2 + c \tag{2.45}$$

Solving for $F(x)$ and $G(x)$ between (2.45) and (2.43) gives

$$F(x) = c - \tfrac{1}{2}x^2 \qquad\qquad G(x) = \cosh x - \tfrac{1}{2}x^2 - c \tag{2.46}$$

in which case (2.42) becomes

$$u(\xi,\,\eta) = \cosh\xi - \tfrac{1}{2}\left(\xi^2 + \eta^2\right) + \xi\eta\,. \tag{2.47}$$

Expressing this in x, y-coordinates it is found that

$$u(x, y) = \cosh\,(x + 2y) - 2y^2\,. \tag{2.48}$$

Example 5 (exam 2003): Consider the 2nd order PDE

$$y^2\frac{\partial^2 u}{\partial x^2} = x^2\frac{\partial^2 u}{\partial y^2}\,. \tag{2.49}$$

Show firstly that is is hyperbolic in nature. Secondly show that it has characteristics

$$\xi = y^2 + x^2 = \text{const}\,, \qquad\qquad \eta = y^2 - x^2 = \text{const}\,. \tag{2.50}$$

Thirdly, show that its canonical form in characteristic variables is given by

$$\frac{\partial^2 u}{\partial\xi\partial\eta} = \frac{1}{2(\xi^2-\eta^2)}\left(\eta\frac{\partial u}{\partial\xi} - \xi\frac{\partial u}{\partial\eta}\right)\,. \tag{2.51}$$

Solution: (i) $R = y^2$, $S = 0$ and $T = -x^2$. Thus

$$y^2\left(\frac{dy}{dx}\right)^2 = x^2 \tag{2.52}$$

so we have a hyperbolic PDE with two roots: $\xi = y^2 + x^2 =$ const and $\eta = y^2 - x^2 =$ const.

(ii) Using the chain rule we have

$$u_x = \xi_x u_\xi + \eta_x u_\eta = 2x(u_\xi - u_\eta) \qquad\qquad u_y = \xi_y u_\xi + \eta_y u_\eta = 2y(u_\xi + u_\eta) \tag{2.53}$$

Using the product rule, and the fact that

$$\frac{\partial}{\partial x} = 2x\left(\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\eta}\right) \qquad\qquad \frac{\partial}{\partial y} = 2y\left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta}\right) \tag{2.54}$$

we have

$$\begin{aligned} u_{xx} &= 2(u_\xi - u_\eta) + 4x^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) \\ u_{yy} &= 2(u_\xi + u_\eta) + 4y^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) \end{aligned} \tag{2.55}$$

Substituting this into $y^2u_{xx} - x^2u_{yy} = 0$, we get the answer, using the fact that $y^2 = \frac{1}{2}(\xi+\eta)$ and $x^2 = \frac{1}{2}(\xi-\eta)$ so $4x^2y^2 = \xi^2 - \eta^2$.

Example 6 (exam 2004): Consider the 2nd order PDE

$$y^2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0\,. \tag{2.56}$$

Show firstly that is is hyperbolic in nature. Secondly show that it has characteristics

$$\xi = \tfrac{1}{2}y^2 + x = \text{const} \qquad\qquad \eta = \tfrac{1}{2}y^2 - x = \text{const}\,. \tag{2.57}$$

Thirdly, show that its canonical form in characteristic variables is given by

$$\frac{\partial^2 u}{\partial\xi\partial\eta} + \frac{1}{4(\xi+\eta)}\left(\frac{\partial u}{\partial\xi} + \frac{\partial u}{\partial\eta}\right) = 0\,. \tag{2.58}$$

Solution: (i) $R = y^2$, $S = 0$ and $T = -1$. Thus $S^2 - RT = y^2 > 0$ so we have a hyperbolic PDE with

$$y^2\left(\frac{dy}{dx}\right)^2 = 1\,. \tag{2.59}$$

Integration gives two roots: $\xi = \frac{1}{2}y^2 + x =$ const and $\eta = \frac{1}{2}y^2 - x =$ const.

(ii) Using the chain rule we have

$$u_x = \xi_x u_\xi + \eta_x u_\eta = u_\xi - u_\eta \qquad\qquad u_y = \xi_y u_\xi + \eta_y u_\eta = y(u_\xi + u_\eta) \tag{2.60}$$

Using the product rule, and the fact that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \qquad\qquad \frac{\partial}{\partial y} = y\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) \tag{2.61}$$

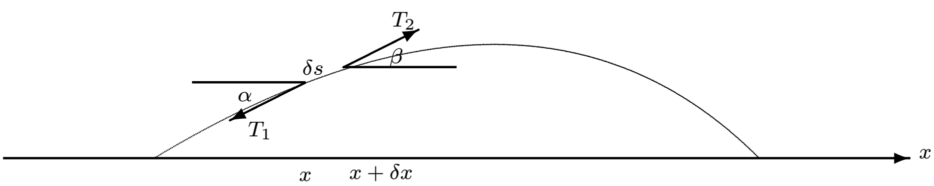
we have

$$u_{xx} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta} \qquad\qquad u_{yy} = (u_\xi + u_\eta) + y^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) \tag{2.62}$$

Substituting this into $y^2 u_{xx} - u_{yy} = 0$, we get the answer, using the fact that $y^2 = \xi + \eta$ and $x = \frac{1}{2}(\xi - \eta)$.

3 The wave equation – a hyperbolic PDE

3.1 Physical derivation



In the figure consider a string in motion whose vertical displacement is $u(x,t)$ at the point x is taken as a snapshot at time t: it is assumed that (i) the vertical displacement is very small so that the angles $|\alpha|$ and $|\beta|$ are small; (ii) stretching of the string is sufficiently negligible that there is no horizontal motion. Thus, resolving horizontally, $T_1 \cos\alpha = T_2 \cos\beta \approx T$ (the tension). Now consider the small arc-length of string δs between the co-ordinate points x and $x + \delta x$. Because the angles are small $\delta s \simeq \delta x$. If ρ is the string mass/unit density then the vertical equation of motion for our small element of string of mass $\rho\delta x$ is

$$\rho\delta x \frac{\partial^2 u}{\partial t^2} = T_2 \sin\beta - T_1 \sin\alpha \tag{3.1}$$

The smallness of $|\alpha|$ and $|\beta|$ allow us to write $\sin\alpha \approx \tan\alpha$ and $\sin\beta \approx \tan\beta$ to convert (3.1) to

$$\rho\delta x \frac{\partial^2 u}{\partial t^2} = T(\tan\beta - \tan\alpha) \tag{3.2}$$

However

$$\tan\alpha = \left(\frac{\partial u}{\partial x}\right)_x \qquad\qquad \tan\beta = \left(\frac{\partial u}{\partial x}\right)_{x+\delta x} \tag{3.3}$$

Thus (3.2) can be written as

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\left(\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x}\right) \tag{3.4}$$

Therefore, in the limit $\delta x \to 0$ (3.4) becomes

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2} \tag{3.5}$$

$T\rho^{-1}$ has the dimensions of a squared velocity, denoted as c^2, which is constant for a chosen string with a fixed tension T. With

$$c^2 = \frac{T}{\rho} \tag{3.6}$$

(3.5) becomes the wave equation

$$\boxed{\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0\,.} \tag{3.7}$$

3.2 d'Alembert's solution of the wave equation

We now wish to solve the wave equation (3.7) subject to initial conditions on the initial shape $u(x,0)$ and the initial velocity $\partial u(x,0)/\partial t$

$$u(x,0) = h(x) \qquad\qquad \frac{\partial}{\partial t}u(x,0) = \frac{\partial}{\partial t}u(x,t)\Big|_{t=0} = v(x) \tag{3.8}$$

where $h(x)$ and $v(x)$ are given functions. In example 1 in §2.2 we found the general solution of $u_{xx} - u_{yy} = 0$ in (2.22). With $y = ct$ this is

$$u(x,t) = f(x-ct) + g(x+ct) \tag{3.9}$$

where, so far, f and g are arbitrary functions. Applying (3.8)

$$f(x) + g(x) = h(x) \qquad\qquad g'(x) - f'(x) = \frac{1}{c}v(x)\,. \tag{3.10}$$

Integrating the latter equation from an arbitrary point $x = a$ to x and then adding and subtracting, it is found that

$$\begin{aligned} f(x) &= \tfrac{1}{2}h(x) - \frac{1}{2c}\int_a^x v(\xi)d\xi - \frac{1}{2c}\left[g(a) - f(a)\right] \\ g(x) &= \tfrac{1}{2}h(x) + \frac{1}{2c}\int_a^x v(\xi)d\xi + \frac{1}{2c}\left[g(a) - f(a)\right] \end{aligned} \tag{3.11}$$

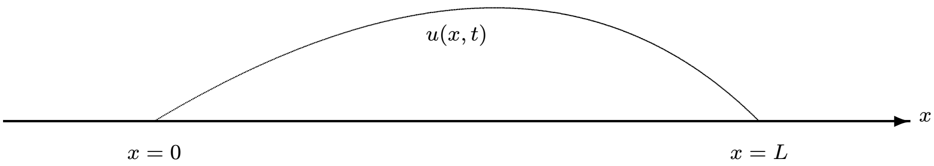
Now substitute this into (3.9) with $x \to x - ct$ in $f(x)$ and $x \to x + ct$ in $g(x)$ to get

$$u(x,t) = \tfrac{1}{2}\left\{h(x-ct) + h(x+ct)\right\} + \frac{1}{2c}\int_{x-ct}^{x+ct} v(\xi)\, d\xi\,. \tag{3.12}$$

This is the **d'Alembert's solution** which is valid on an infinite domain : note that the pair of terms that contain the point $x = a$ cancel leaving no trace.

3.3 Waves on a guitar string : Separation of variables

The same initial conditions as above in (3.2) are now used but now with boundary conditions that fix the ends of a **finite** string down at $x = 0$ and $x = L$.



Now try a solution in the form

$$u(x,t) = X(x)T(t) \tag{3.13}$$

which is substituted into the wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \tag{3.14}$$

to get

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}\,. \tag{3.15}$$

Note that the LHS is a function of x but not t while the RHS is a function of t but not x. Thus we can write

$$\frac{X''}{X} = -\lambda^2 \qquad\qquad \frac{T''}{T} = -\lambda^2 c^2\,, \tag{3.16}$$

where $-\lambda^2$ is an arbitrary constant[7]. The ODE for X is $X'' + \lambda^2 X = 0$ which has a solution

$$X(x) = A\cos\lambda x + B\sin\lambda x\,. \tag{3.17}$$

Applying the BC that $u(x,0) = 0$ for all values of t means that $X(0) = 0$ from which it is deduced that $A = 0$: likewise from $X(L) = 0$ it is deduced that

$$B\sin\lambda L = 0\,. \tag{3.18}$$

[7] The choice of a negative constant is explained lower down.

$B = 0$ is the trivial solution: $\sin \lambda L = 0$ gives an infinite number of solutions for λ, namely

$$\lambda_n = \frac{n\pi}{L} \qquad n = 0,\, \pm 1,\, \pm 2 \ldots \tag{3.19}$$

giving an infinite set of solutions

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right) . \tag{3.20}$$

Here is the reason for a negative choice of the constant in (3.16): a positive choice of constant $+\lambda^2$ would have made $\sin(\lambda L)$ into $\sinh(\lambda L)$. This has only one root at $\lambda = 0$ which corresponds to the trivial solution.
The time part in (3.16) can now be easily solved

$$T_n = C_n \sin(\omega_n t) + D_n \cos(\omega_n t) . \tag{3.21}$$

where the infinite set of frequencies[8] ω_n are defined by $\omega_n = \frac{n\pi c}{L}$. This means that there is an infinite set of solutions $u_n = X_n T_n$ which can be summed to form the general solution. In so doing the products of arbitrary constants $B_n C_n$ etc are re-labelled

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left[c_n \sin(\omega_n t) + d_n \cos(\omega_n t)\right] . \tag{3.22}$$

Now apply the initial conditions from (3.8)

$$u(x,0) = h(x)\,; \qquad\qquad \frac{\partial}{\partial t} u(x,0) = v(x)\,. \tag{3.23}$$

The first says that

$$h(x) = \sum_{n=1}^{\infty} d_n \sin\left(\frac{n\pi x}{L}\right) \tag{3.24}$$

This is the half-range Fourier series of $h(x)$ on $[0,\, L]$ which was discussed regarding "periodic extension"; this means that the series can be inverted to find d_n

$$d_n = \frac{2}{L}\int_0^L h(x) \sin\left(\frac{n\pi x}{L}\right)\, dx\,. \tag{3.25}$$

Applying the second initial condition gives

$$v(x) = \sum_{n=1}^{\infty} \tilde{c}_n \sin\left(\frac{n\pi x}{L}\right) \tag{3.26}$$

where $\tilde{c}_n = c_n n\pi c/L$. We have

$$\tilde{c}_n = \frac{2}{L}\int_0^L v(x) \sin\left(\frac{n\pi x}{L}\right)\, dx\,. \tag{3.27}$$

[8] ω_1 is the fundamental frequency; ω_2 is the 1st harmonic etc. Note that all harmonics are summed in the solution. It is the balance of these that gives a musical instrument its quality.

Question : Is this consistent with d'Alembert's solution? For simplicity, take $v = 0$ so the string is released from rest. The solution in (3.22) is

$$u(x,t) = \sum_{n=1}^{\infty} d_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right). \tag{3.28}$$

Now use a standard trig formula to write this as

$$u(x,t) = \sum_{n=1}^{\infty} \tfrac{1}{2} d_n \left\{\sin\left(\frac{n\pi x + ct)}{L}\right) + \sin\left(\frac{n\pi(x - ct)}{L}\right)\right\} \tag{3.29}$$

which is in the D'Alembert form.

Example Take the string from rest ($v = 0$) and $h(x)$ as a "tent function" of height d at the mid-point $x = \frac{1}{2}L$.

$$h(x) = \begin{cases} \frac{2d}{L}x & 0 \le x \le \frac{1}{2}L \\ 2d\left(1 - \frac{x}{L}\right) & \frac{1}{2}L \le x \le L\,. \end{cases}$$

The Fourier series for this – with no working – contains only odd sine-terms

$$u(x,t) = \frac{8d}{\pi^2}\sum_{r=0}^{\infty}\frac{(-1)^r}{(2r+1)^2}\sin\left(\frac{(2r+1)\pi x}{L}\right)\cos\left(\frac{(2r+1)\pi ct}{L}\right). \tag{3.30}$$

Note that the coefficients of the higher harmonics die off as n^{-2}.

4 Laplace's equation – an elliptic PDE

The simplest elliptic PDE is Laplace's equation in cartesian co-ordinates where $R = T = 1$ and $S = 0$

$$u_{xx} + u_{yy} = 0 \qquad\qquad S^2 - RT = -1 < 0\,. \tag{4.1}$$

In two-dimensions, the method of separation of variables is useful but needs to be considered in the context of the BCs. Solutions in terms of polar co-ordinates will be our concern of the subsection §4.2 concerning flow around a cylinder. First we look at a simpler problem.

4.1 An infinite strip

Physically Laplace's equation often occurs in situations where the diffusive flow of heat or some other scalar in a two-dimensional piece of material is governed by the *diffusion* or *heat equation* $u_t = \alpha\nabla^2 u$ where ∇^2 is the Laplacian $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. When the system has reached a steady state – so $u_t = 0$ – we are left with the problem of solving Laplace's equation (4.1). The strip below is an example of how to solve this with a set of given boundary conditions (BCs).

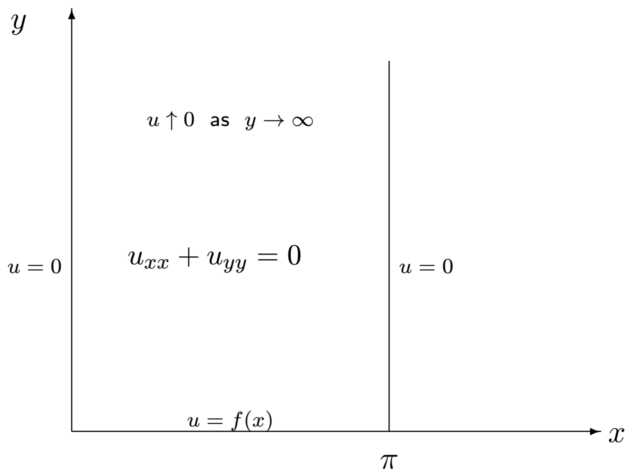


Figure : *The region is a strip bounded between $x = 0$ (y-axis) and $x = \pi$ on which $u = 0$ while $u = f(x)$ on $y = 0$.*

The infinite strip, as in the figure above, has $u = 0$ on the sides and $u = f(x)$, a given function, on the bottom edge. To remain physical it is also necessary to insist that $u \to 0$ as $y \to \infty$. Inside the strip u satisfies Laplace's equation (4.1) which we attempt to solve by the method of separation of variables

$$u(x, y) = X(x)Y(y) \tag{4.2}$$

and thus (4.1) becomes $X''Y + XY'' = 0$. Therefore

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2 \tag{4.3}$$

the choice of $\pm$ on the far RHS is dependent on the BCs. Clearly we have the two ODEs

$$X'' + \lambda^2 X = 0 \qquad\qquad Y'' - \lambda^2 Y = 0 \tag{4.4}$$

whose solution is

$$X = A\cos\lambda x + B\sin\lambda x\,, \qquad Y = Ce^{\lambda y} + De^{-\lambda y}\,. \tag{4.5}$$

The BC at $x = 0$ insists that $A = 0$ and at $x = \pi$ that $\sin\lambda\pi = 0$. Thus $\lambda_n = n$ where n is an integer. For $n > 0$ we must also choose $C = 0$ to be sure that there is no exponential growth as $y \to \infty$. We are left with a summed infinite set of solutions

$$u(x, y) = \sum_{n=1}^{\infty} b_n e^{-ny} \sin nx \tag{4.6}$$

To find the b_n requires the use of the last BC $u = f(x)$ on $y = 0$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx\,. \tag{4.7}$$

This is the Fourier sine-series expansion of $f(x)$ on $[0,\, \pi]$ which inverts to

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx\,. \tag{4.8}$$

For example, if $f(x) = 1$ – that is, a uniform value – then

$$b_n = \begin{cases} 0 & n\ even \\ \frac{4}{n\pi} & n\ odd \end{cases} \tag{4.9}$$

With $n = 2r + 1$, our solution is

$$u(x,y) = \frac{4}{\pi}\sum_{r=1}^{\infty} e^{-(2r+1)y}\left(\frac{\sin(2r+1)x}{2r+1}\right)\,. \tag{4.10}$$

Note that this solution correctly decays exponentially as $y \to \infty$ and is zero at $x = 0$ and $x = \pi$.

4.2 Fluid flow around a cylinder

4.2.1 Laplace's equation in polar co-ordinates

Consider Laplace's equation in polar co-ordinates (see handout on The Chain Rule)

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = 0. \tag{4.11}$$

Looking for separable solutions of the form $\Phi(r,\theta) = R(r)H(\theta)$ we find

$$\frac{r^2}{R}\left(R'' + \frac{1}{r}R'\right) = -\frac{H''}{H} = \lambda^2. \tag{4.12}$$

Choosing the separation constant negative anticipates solutions for $H(\theta)$ that need to be periodic. Solving $H'' + \lambda^2 H = 0$ gives

$$H(\theta) = A\cos\lambda\theta + B\sin\lambda\theta. \tag{4.13}$$

When $\lambda \neq 0$ solving $R'' + \frac{1}{r}R' - \frac{\lambda^2}{r^2}R = 0$ gives

$$R(r) = a\,r^{\lambda} + b\,r^{-\lambda}\,. \tag{4.14}$$

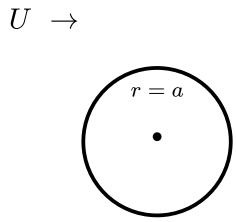
If we require $\Phi(r,\theta)$ to be continuous[9] in θ; that is, $\Phi(r,\theta) = \Phi(r,\theta + 2n\pi)$, then $\lambda = n$ (an integer). The general 2π-periodic solution of (4.11) is

$$\Phi(r,\theta) = \sum_{n=1}^{\infty}\left(a_n\,r^n + b_n\,r^{-n}\right)\left(A_n\cos n\theta + B_n\sin n\theta\right). \tag{4.15}$$

[9]The case with $\lambda = 0$ where $H(\theta) = \tilde{A}\theta + \tilde{B}$ and $R(r) = \tilde{a}\,\ln r + \tilde{b}$ is not 2π-periodic in θ.

4.2.2 Calculating the flow around the cylinder

Consider an incompressible irrotational 2D fluid with velocity vector $\boldsymbol{u}$ flowing past a cylinder of radius a, as in the figure: the centre of the cylinder can be considered to be at $r=0$. At $r=\pm\infty$ the flow is laminar: that is, $\boldsymbol{u}=(0,\,U)$ where U is a constant.



(i) The divergence-free condition div $\boldsymbol{u}=0$ means that a stream function $\psi(x,y)$ exists

$$\boldsymbol{u}=(\psi_y,-\psi_x)=\hat{\boldsymbol{i}}\psi_y-\hat{\boldsymbol{j}}\psi_x.$$

Irrotational flow (curl $\boldsymbol{u}=0$) means that

$$\begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \partial_x & \partial_y & \partial_z \\ \psi_y & -\psi_x & 0 \end{vmatrix}=0$$

Thus we have Laplace's equation for the stream function

$$\psi_{xx}+\psi_{yy}=0 \qquad \Rightarrow \qquad \frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}=0. \tag{4.16}$$

(ii) The alternative way, using the potential, starts from curl $\boldsymbol{u}=0$. This means that a potential function ϕ exists such that $\boldsymbol{u}=\nabla\phi=\hat{\boldsymbol{i}}\phi_x+\hat{\boldsymbol{j}}\phi_y$. From div $\boldsymbol{u}=0$, we have Laplace's equation $\nabla^2\phi=\phi_{xx}+\phi_{yy}=0$ which is also (4.11) in polar co-ordinates.

Thus we want to solve (4.16) under the circumstance where the fluid, of constant horizontal speed U at infinity, flows past a solid cylinder of radius a centred at the origin. The fact that no fluid can cross the surface of the cylinder translates into the boundary condition

$$\left.\frac{\partial\psi}{\partial\theta}\right|_{r=a}=0\,. \tag{4.17}$$

Since the flow at $r=\pm\infty$ is horizontal we have $\boldsymbol{u}=U\hat{\boldsymbol{i}}+0\hat{\boldsymbol{j}}$ there, which means that

$$\psi=Uy=Ur\sin\theta \qquad \text{at} \qquad r=\infty. \tag{4.18}$$

We want to solve Laplace's equation (4.16) in the infinite domain around the cylinder of radius a with prescribed BCs (4.17) and (4.18). Separating the $n=1$ term from the rest of the infinite sum in (4.15) we have

$$\begin{aligned}\psi(r,\theta) &= \left(a_1\, r + b_1\, r^{-1}\right)\left(A_1 \cos\theta + B_1 \sin\theta\right) \\ &+ \sum_{n=2}^{\infty}\left(a_n\, r^n + b_n\, r^{-n}\right)\left(A_n \cos n\theta + B_n \sin n\theta\right) . \end{aligned} \tag{4.19}$$

Applying the BC in (4.18) we find that

$$a_1 B_1 = U \qquad\qquad A_1 = 0 \tag{4.20}$$

and all coefficients $A_n = B_n = 0$ for $n \geq 2$. This leaves us with

$$\psi = U\left(r + \frac{b_1}{a_1}\frac{1}{r}\right)\sin\theta. \tag{4.21}$$

Finally applying the BC (4.17) at $r = a$ we find $b_1/a_1 = -a^2$ giving the stream function as

$$\psi = U\left(r - \frac{a^2}{r}\right)\sin\theta. \tag{4.22}$$

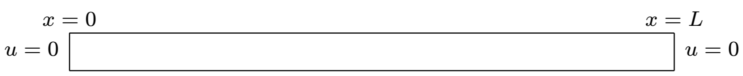
5 The diffusion equation – a parabolic PDE

Consider a very thin metal bar on the x-axis on $[0,\, L]$, as in the figure below, with temperature $u = 0$ at both ends. For standard materials, the equation that normally governs heat flow is the **diffusion equation**[10]

$$u_t = \kappa u_{xx} \tag{5.1}$$

where κ is a material constant (thermal conductivity) which has the dimensions (length)^2/time. In this section we solve two problems: on a finite one-dimensional domain $[0,\, L]$ and similarity solutions on an infinite domain.

5.1 Separation of variables on a finite domain



$u(x, 0) = f(x)$ at $t = 0$

The BCS are $u = 0$ on both $x = 0$ and $x = L$ with[11] an initial distribution of temperature $u(x, 0) = f(x)$. Separation of variables

$$u(x,t) = X(x)T(t) \tag{5.2}$$

[10] In 2 dimensions the equivalent is $u_t = \kappa\nabla^2 u$ where ∇^2 is the Laplacian $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

[11] If the end conditions are different, say $u = 0$ at $x = 0$ and $u = u_0$ at $x = L$, then the following trick is useful: define $u(x,t) = u_0 x/L + v(x,t)$ with $v = 0$ on $x = 0$ and $x = L$ with v satisfying $v_t = \kappa v_{xx}$, then the problem reduces to the one solved above with $u = 0$ at both ends.

gives

$$\frac{X''}{X} = \frac{1}{\kappa}\frac{T'}{T} = -\lambda^2 \tag{5.3}$$

for which we write

$$X'' + \lambda^2 X = 0 \qquad \text{with} \qquad X(0) = X(L) = 0\,. \tag{5.4}$$

This we have solved before: (5.4) gives $X = A\cos\lambda x + B\sin\lambda x$ in which $A = 0$ because $X(0) = 0$, whereas

$$\sin\lambda L = 0 \qquad \Rightarrow \qquad \lambda_n = \frac{n\pi}{L} \qquad \text{with} \qquad X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right)\,. \tag{5.5}$$

The time part $T' = -\lambda^2\kappa T$ solves to become

$$T_n(t) = T_{n,0}\,\exp\left(-\frac{n^2\pi^2\kappa t}{L^2}\right) \tag{5.6}$$

Thus the general solution is a linear sum of all the solutions for each n

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp\left(-\frac{n^2\pi^2\kappa t}{L^2}\right)\sin\left(\frac{n\pi x}{L}\right)\,, \tag{5.7}$$

where the constants $B_n T_{n,0} = b_n$. Applying the ICs gives

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\,, \tag{5.8}$$

and, as before, this Fourier half-range series can be inverted to give the b_n

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx\,. \tag{5.9}$$

5.2 Similarity solutions on an infinite domain

The diffusion equation in one-dimension is $u_t = \kappa u_{xx}$ has been solved above on a domain of *finite length*. What if $L = \infty$? Clearly, the method of separation of variables no longer works and we need a different approach. The key lies in κ, the diffusion coefficient, which has dimension L^2T^{-1}. If we are looking for solutions on an infinite domain $-\infty \le x \le \infty$ where there is no natural length scale, then we can use the **dimensionless** variable

$$\eta = \frac{x}{\sqrt{\kappa t}} \tag{5.10}$$

and look for solutions in the form

$$u(x,t) = t^p g(\eta) \tag{5.11}$$

where the number p and the function $g(\eta)$ are to be determined. Substituting (5.11) into $u_t = \kappa u_{xx}$ we find that

$$t^{p-1}\left(pg - \frac{\eta}{2}g' - g''\right) = 0 \tag{5.12}$$

and so

$$g'' + \frac{\eta}{2}g' = pg. \tag{5.13}$$

This is difficult to solve for arbitrary values of p but for special values we can do something.

1. Take $p = 0$ and (5.13) is easily solved to give

$$g'(\eta) = A\,e^{-\eta^2/4} \tag{5.14}$$

where A is a constant. Integrating again we have

$$g(\eta) = A\int_{-\infty}^{\eta} e^{-\eta'^2/4}\,d\eta'. \tag{5.15}$$

This gives a full solution for $u(x,t)$

$$u(x,t) = A\int_{-\infty}^{\frac{x}{\sqrt{\kappa t}}} e^{-\eta'^2/4}\,d\eta' = 2A\sqrt{\pi}\,\mathsf{erf}\left(\frac{x}{2\sqrt{\kappa t}}\right) \tag{5.16}$$

where the *error function* $\mathsf{erf}\,(\xi)$ is defined as $\mathsf{erf}\,(\xi) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\xi} e^{-y^2}\,dy$. This has the property that $\mathsf{erf}\,(\infty) = 1$.

2. Now define $G = g\,e^{\eta^2/4}$ and we observe that 5.13) can be transformed into

$$G'' - \frac{\eta}{2}G' = (p + 1/2)G. \tag{5.17}$$

This has the trivial solution $G = b = \text{const}$ provided $p = -1/2$. Hence

$$g(\eta) = b\,e^{-\eta^2/4}. \tag{5.18}$$

This gives a full solution for $u(x,t)$ in the form

$$u(x,t) = b\,t^{-1/2}e^{-\frac{x^2}{4\kappa t}}. \tag{5.19}$$