

Research statement

Jeffrey D. Carlson

The main line of my work lies in the algebraic topology of Lie group actions; in lay terms, I study aspects of smooth symmetry. I began using a tool called *Borel cohomology* to situations of classical interest to geometers, such as the *isotropy actions* on (generalized) homogeneous spaces, the *biquotients* of two-sided actions, and *cohomogeneity-one actions*. An at times more precise but also more demanding approach to studying these actions involves *equivariant K-theory* (Section 2). A fair deal of my work has centered on a family of important algebraic simplicity conditions on the Borel cohomology and equivariant K-theory of an action, called *equivariant formality* (Section 3). Trying to answer one such question, involving circular symmetry, led me to accidentally reprove a little-known classical result on the rational cohomology of homogeneous spaces and then (this time intentionally) write a textbook on the subject currently under revision for Springer (Section 1).

Equivariant topology in general depends heavily on fixed point sets of group actions, and one can ask what kind of “local models” of potential actions near fixed point sets actually arise. Such questions led me into the realm of *equivariant cobordism*, which in a way attempts to understand not symmetries of individual spaces, but the totality of *all* spaces admitting certain kinds of symmetries. My collaborators and I have been particularly successful with a class of well-behaved actions called *GKM actions*, which have been of perennial interest to differential and symplectic geometers. My existing work and plans in this area are described in Section 4.

I proved in the case of isotropy actions the notion of equivariant formality above is linked with a more general notion of *formality* connecting *rational homotopy theory* with *Galois cohomology*, two other fields I am pursuing projects in; see Section 6. To recover the same results with coefficient rings other than \mathbb{Q} , a less rigid notion of formality is called for, leading to the study of *A_∞ -algebras* and other up-to-homotopy algebra structures. My projects in this field are summarized in Section 5.

I have also done work in low-dimensional topology and dynamics [AkC12, C10].

1. Rational cohomology of homogeneous spaces

A *homogeneous space* is the orbit of a single point under a Lie group action. The geometry of such a space is highly symmetric, being identical at every point, and homogeneous spaces have long been studied by differential geometers. The most famous algebraic result about homogeneous spaces may be H. Cartan’s result that if G and a subgroup K are compact, connected Lie groups, then the real cohomology ring of the homogeneous space G/K is given by

$$H^*(G/K; \mathbb{R}) \cong \mathrm{Tor}_{H^*(BG; \mathbb{R})}^*(\mathbb{R}, H^*(BK; \mathbb{R})).$$

In order to characterize equivariant formality of isotropic circle actions (see Section 3.3) in my thesis, I derived a consequence of Cartan’s theorem which states that unless a circle subgroup S^1 of a connected Lie group G is nullhomotopic in G , the rational cohomology ring of G/S^1 is isomorphic to that of a product $S^2 \times \prod S^{2n_\ell+1}$ of spheres [C19b, Appendix A].

1.1. My book

My later discovery that this result is not actually original, having been announced without proof by Leray and Koszul in the late 1940s, solidified what had been growing into a general discontent with the secondary literature in this area. Convinced that this material needed to be better publicized, I resolved to write a text on the rational non-equivariant cohomology of homogeneous spaces [C20], which I have

since submitted for publication. I am now preparing a revision with more background material on Lie groups at the request of Springer’s editors.

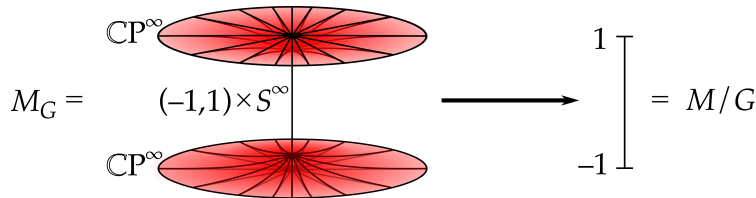
The manuscript uses a touch of rational homotopy theory to streamline the approach to the “Cartan” cochain model for $H^*(G/K; \mathbb{Q})$ which Borel developed in his thesis, and is meant in part to be a gentle introduction to spectral sequences suitable for a second year graduate student. The necessary algebra is developed along the way and the resulting exposition is substantially faster than previously published approaches. Several aspects of my approach do not seem to appear elsewhere.

2. Borel equivariant cohomology and equivariant K-theory

It is a well-known disappointment that the orbit space M/G of the action of a Lie group G on a topological space M does not distinguish between orbit types; for example, when one passes to the quotient $S^2/S^1 \approx [-1, 1]$ of a standard globe S^2 under the action of the circle S^1 by rotation, both poles $*$ and latitudes S^1 become simply points. One wants to have one’s cake and eat it too by taking the quotient in a way that somehow retains the distinction between orbit types, and does this via *Borel equivariant cohomology*, a central tool since its inception around 1960 [BorelSem]. One forms the *homotopy quotient* or *Borel construction*

$$M_G := EG \times M / (eg, m) \sim (e, gm),$$

where EG is the total space of the universal principal G -bundle, a contractible space with free G -action. Homotopically speaking, $EG \times M$ is no different than M , but the diagonal action on $EG \times M$ is free, so orbit types now remain distinct and we may regard M_G as a homotopically-correct replacement for M/G . The *Borel cohomology* $H_G^*(M)$ of the action is the singular cohomology $H^*(M_G)$ of this new construction. For example, the homotopy quotient $(S^2)_{S^1}$ of the rotation action on the 2-sphere can be visualized as in the following cartoon.



Here forgetting the EG coordinate induces a projection to the naive quotient, whose fiber over any point of the open interval $(-1, 1)$ is the (contractible) infinite-dimensional sphere $ES^1 = S^\infty$, and whose fibers over ± 1 are infinite complex projective spaces $BS^1 = \mathbb{C}P^\infty = S^\infty/S^1$. Thus M_G is homotopy equivalent to the wedge sum $\mathbb{C}P^\infty \vee \mathbb{C}P^\infty$. Its cohomology $\mathbb{Z}[x, y]/(xy)$ encodes much of the structure of the action; for example, the two fixed points show up in the fact that the ring is free of rank two over the coefficient ring $H_{S^1}^*(*) \cong \mathbb{Z}[x + y]$. In general, the orbit types can be read off of the ideal structure of $H_G^*(X)$ [Hsiang, Ch. IV], so Borel cohomology makes orbit structure legible in ring theory.

Another approach to analyzing an action studies bundles over the space. Given a G -space M , one can consider the notion of a *G-equivariant vector bundle* $V \rightarrow M$ whose total space admits a G -action such that the projection preserves the group action. These can be directly summed and tensored just as ordinary vector bundles can, and formally inverting the direct sum yields the *equivariant K-theory* ring $K_G^*(M)$. As in the nonequivariant case, equivariant K-theory is inherently less computable than Borel cohomology but often better-behaved algebraically.

In the rest of this section we describe some of my computations.

2.1. ... of real Grassmannians

The real Grassmannians $G_k(\mathbb{R}^n)$ of oriented k -planes in n -dimensional Euclidean space are important parametrizing objects, well-studied as manifolds in their own right. Accordingly, their rational singular cohomology rings have long been known [Ler49, Tak62][Cartan51, p. 71][Bor53, p. 192]. Chen He [He16, Thms. 5.2.2, 6.3.1, Cor. 5.2.1] applied his extension of GKM-theory to odd-dimensional and nonorientable manifolds to compute the rational Borel cohomology rings of the *isotropy actions* on these spaces, defined as the left multiplication action of K (here $\mathrm{SO}(k) \times \mathrm{SO}(n-k)$) on the right quotient homogeneous space $G/K \approx \mathrm{SO}(n)/(\mathrm{SO}(k) \times \mathrm{SO}(n-k))$. I showed [C19a] that one can compute these rings much more simply using existing models and a structure result, Theorem 3.1 below.

2.2. ... of cohomogeneity-one actions

The next simplest actions after homogeneous ones are the *cohomogeneity-one actions*, those with one-dimensional orbit space, which are the subject of a vast geometric literature and classified in low dimensions [GGZ18]. Topologically, they all are mapping tori of G -equivariant self-homeomorphisms of homogeneous spaces or double mapping cylinders of certain pairs of G -equivariant maps $G/K^- \leftarrow G/H \rightarrow G/K^+$, but they furnish many interesting examples of positively-curved manifolds with large isometry groups.

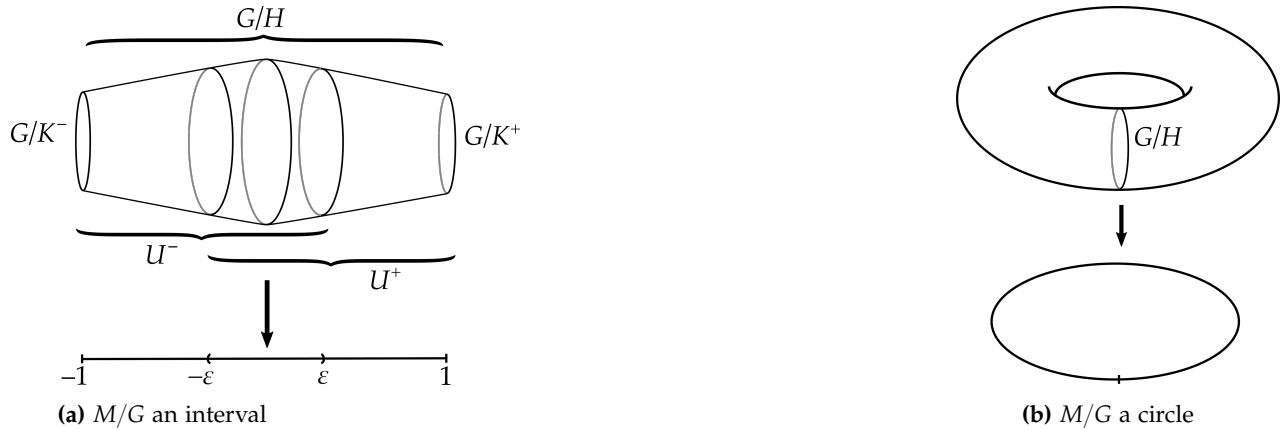


Figure 2.1: Schematic of the orbit projection $M \rightarrow M/G$ of a cohomogeneity-one action

It is natural to explore the algebraic invariants of this class, and Oliver Goertsches, Chen He, Liviu Mare, and I computed their rational Borel cohomology [CGHM19, Theorem 1.2]. I independently computed the equivariant K-theory of a cohomogeneity-one action a few months later [C18a]; the result looks similar but the proof requires substantially different techniques from Lie and representation theory. As in the cohomological case, though, the proof uses an additional structure on the Mayer–Vietoris sequence induced by the standard cover of $[-1, 1]$, which turns out to obtain in great generality.

Theorem 2.2 ([C18a, Proposition A.11]). *For any \mathbb{Z} -graded multiplicative G -equivariant cohomology theory E^* , the connecting map in the Mayer–Vietoris sequence of a triad $(X; U, V)$ of G -CW complexes with $X = U \cup V$ preserves a natural E^*X -module structure.*

This fact does not seem to appear in the literature and is needed to obtain the ring structure. Similarly [C18a, Lemma A.3], I prove a result computing E^* of a mapping torus with suitable coefficients, relying on an equivariant Atiyah–Hirzebruch spectral sequence; this works even when there is no transfer map because the E_2 page is Bredon cohomology [Matu73, §4], where there is always such a map.

3. Formality and equivariant formality

There is a natural map from a G -space M to its homotopy quotient M_G , given by $M \times \{e_0\} \hookrightarrow M \times EG \rightarrow (M \times EG)/\sim$ for any point $e_0 \in EG$, which is the fiber inclusion of an M -bundle $M_G \rightarrow G \backslash EG = BG$. This fiber inclusion induces a pullback map $H_G^*(M) \rightarrow H^*(M)$ which in the algebraic best-case scenario is *surjective*. For example, in the example of S^1 rotating S^2 above, the map

$$\mathbb{Z}[x, y]/(xy) \cong H_{S^1}^*(S^2) \rightarrow H^*(S^2) \cong \mathbb{Z}[t]/(t^2)$$

is given by $x \mapsto t$ and $y \mapsto -t$. In this instance, the action of G on M is called *equivariantly formal*, and a preimage $\tilde{c} \in H_G^*(M)$ of $c \in H^*(M)$ is called an *equivariant extension* of c . While Borel already made use of this condition in his seminar, it was given its present name by Goresky, Kottwitz, and MacPherson [GKM98] in the paper that began what is now called *GKM theory*. This theory allows the equivariant cohomology $H_T^*(M)$ of a *GKM manifold*, a certain kind of well-behaved manifold with equivariantly formal action and finitely many fixed points, to be computed in terms of the combinatorics of the orbits of 0- and 1-dimensional orbits using a lemma of Chang and Skjelbred. This is the simplifying condition figuring in Theorem 4.1. Equivariant formality guarantees all classes in $H^*(X)$ admit equivariant extensions in $H_T^*(X)$, to which the Atiyah–Bott–Berline–Vergne localization theorem applies, yielding the restrictions on isotropy data mentioned in Section 4.

3.1. Equivariant cohomology and K-theory of isotropy actions

Equivariant formality simplifies the computation of equivariant cohomology. I showed the following around the time of my thesis, generalizing classical results that come to the same conclusion when $\text{rk } G = \text{rk } H$.

Theorem 3.1 ([C20, Theorem 10.1.1][C18b, Theorem C]). *Let G be a compact, connected Lie group, and H a closed, connected subgroup such that the action of H on G/H is equivariantly formal.¹ Then there is a ring isomorphism*

$$H_H^*(G/H; \mathbb{Q}) \cong H^*(BH; \mathbb{Q}) \otimes_{H^*(BG; \mathbb{Q})} H^*(BH; \mathbb{Q}) \otimes_{\mathbb{Q}} \text{im}(H^*(G/H; \mathbb{Q}) \rightarrow H^*(G; \mathbb{Q})),$$

where the $H^*(BG; \mathbb{Q})$ -algebra structure on $H^*(BH; \mathbb{Q})$ is induced from the inclusion $H \hookrightarrow G$.

Example 3.2. The group of orientation-preserving isometries stabilizing the three-plane $\mathbb{R}^3 \times \{0\}^3$ in \mathbb{R}^6 is $\text{SO}(3) \times \text{SO}(3)$. The associated $\text{SO}(3)^2$ -equivariant cohomology of the Grassmannian of oriented 3-planes in \mathbb{R}^6 is

$$\frac{\mathbb{Q}[p_1, p'_1, \pi_1, \pi'_1]}{(p_1 + p'_1 - \pi_1 - \pi'_1, p_1 p'_1 - \pi_1 \pi'_1)} \otimes \Lambda[\eta], \quad |p_1| = |p'_1| = |\pi_1| = |\pi'_1| = 4, \quad |\eta| = 5.$$

This result implies the classical computation of $H^*(G/H; \mathbb{Q})$ in these cases. Our proof relies on a Sullivan model for biquotients due to Vitali Kapovitch [Kapo4, Prop. 1]² which also applies to homotopy biquotients [C19a]. The model can be viewed as a compression of the Serre spectral sequence of the fibration $G \rightarrow G_{H \times H} \rightarrow BH \times BH$. Although there is no cochain-level model of equivariant K-theory, I conjectured and was eventually able to prove a related result under more stringent hypotheses [C18b, Theorem K], which still apply up to taking a finite cover, in all cases where equivariant formality of an isotropy action is known, except those I determined in the case H is a circle. As with the cohomological case, this result generalizes the classical computations of $K^*(G/H)$ in these cases [Min75]. The K-theoretic and cohomological results are connected by a map of spectral sequences from the Künneth spectral sequence in equivariant K-theory [Hodgkin] to that in Borel cohomology, constructed by showing

¹ We will discuss when this hypothesis is satisfied in Section 3.3.

² and independently, much later, the present author

one “geometric resolution” will work for both theories and applying the equivariant Chern character, which [CF18, Thm. 5.3] identifies $H_G^*(X; \mathbb{Q})$ with the completion of $K_G^*(X; \mathbb{Q})$ with respect to IG (discussed in Section 3.2). In our case of interest, $X = Y = G/H$, and the target sequence collapses, essentially because its E_2 term is the cohomology of the Kapovitch model, which then forces the collapse of the K-theoretic sequence.

The strong commutative-algebraic hypotheses come from an unexpected source, the fact that one still does not know in general when a surjection from one finitely generated polynomial ring over \mathbb{Z} to another has kernel generated by a regular sequence. Particularly, in algebro-geometric terms, we still do not know the answer to the longstanding *Abhyankar–Sathaye conjecture* addressing when a regular embedding of the affine plane \mathbb{A}^k in \mathbb{A}^n of affine planes can be taken by an algebraic automorphism of \mathbb{A}^n to the standard embedding as $\mathbb{A}^k \times \{0\}^{n-k}$.

3.2. Weak K-theoretic equivariant formality

As it turns out [Fok19][CF18, Thm. 5.6], equivariant formality is equivalent rationally to surjectivity of the *forgetful map* $f: K_G^*(X) \rightarrow K^*(X)$ induced by discarding the G -structure on an equivariant bundle [MatsM86]. An equivariant bundle over a point is just a representation, so $K_G^*(*)$ is the representation ring RG . The trivial G -map $X \rightarrow *$ induces a map $RG \rightarrow K_G^*(X)$, and the composition with f sends a representation to its dimension, annihilating the virtual representations IG of dimension 0. Thus f annihilates the ideal $(IG \cdot 1)$ of K_G^*X and factors as

$$K_G^*X \twoheadrightarrow K_G^*X \otimes_{RG} \mathbb{Z} \xrightarrow{\bar{f}} K^*X.$$

Harada–Landweber [HaLo7, Prop. 4.2] observe that f is surjective if and only if \bar{f} is, and say that the action is *weakly equivariantly formal* if \bar{f} is an isomorphism. By definition, weak equivariant formality implies equivariant formality in our sense, and Fok also showed that *rationally*, weak equivariant formality is *equivalent* to equivariant formality [Fok19][CF18, Thm. 5.6]. I was able to improve this to an integral result.

Theorem 3.3 ([C18b, Theorem W]). *If a compact, connected Lie group G such that $\pi_1 G$ is free abelian acts on a compact Hausdorff space X in such a way that K_G^*X is finitely generated over RG and the forgetful map $f: K_G^*X \rightarrow K^*X$ is surjective, then the action is weakly equivariantly formal.*

The proof involves the map from the Atiyah–Hirzebruch spectral sequence of BG to the Atiyah–Hirzebruch–Leray–Serre spectral sequence of $X \rightarrow X_G \rightarrow BG$, which induces a tensor decomposition of the E_2 page of the former which which can be shown to persist to E_∞ .

3.3. When is an isotropy action equivariantly formal?

We’ve now computed the Borel cohomology and K-theory of an equivariantly formal isotropy action, so it seems only fair to say *when* an isotropy action is equivariantly formal.

Question 3.4. Let G be a compact Lie group and K a closed subgroup. When is the isotropy action of K on G/K equivariantly formal?

At the beginning of 2014, only three classes of examples were known: generalized flag manifolds, those for which $H^*(G; \mathbb{Q}) \rightarrow H^*(K; \mathbb{Q})$ is surjective, and *generalized symmetric spaces* [GN16]. In collaboration with Fok, the author was able to extend this to a complete characterization [CF18, Thm. 1.4, Prop. 3.13] which particularly shows that if the action is equivariantly formal, then G/K is formal in the sense of rational homotopy theory. The tools involved include Kapovitch’s model, a result of Shiga–Takahashi [Shi96, Thm. A, Prop. 4.1][ShTa95, Thm. 2.2], and classical invariant theory in the form of the Chevalley–Shepherd–Todd theorem [Kane94, p. 82].

This work follows on a string of reductions established in my dissertation [C19b], essentially reducing the situation to the case where G is simply-connected and K is a torus. I applied these reductions to exhaustively analyze the case when K is a circle, $SO(3)$, or $SU(2)$, providing an explicit algorithm. Particularly, the action is *always* equivariantly formal if K is $SO(3)$ or $SU(2)$. It is natural to ask if a similar classification is possible for tori S of codimension one in a maximal torus of G . The result of joint work with Chen He is the following:

Theorem 3.5. *Let (G, S) be an pair of compact, connected Lie groups such that G is semisimple and S is a torus of codimension one in a maximal torus T of G . If S^\perp is the circle orthogonal to S at $1 \in T$ under the Killing form and H is the subgroup generated by S and the commutator subgroup of the centralizer $Z_G(S^\perp)$, then (G, S) is isotropy-formal if either (a) G/H is a rational cohomology sphere or (b) G/H is a rational cohomology $S^n \times S^m$, with n even and m odd, and the number of components of the normalizer $N_G(S)$ is greater than that of $N_H(S)$.*

Characterizing when this occurs leads to a classification result expected to be complete by the beginning of December 2019.

4. Equivariant complex cobordism and fixed points

One can study smooth symmetry in terms of individual manifolds or the *totality* of manifolds. *Equivariant complex cobordism* is one such approach; one attempts to understand when two *stably complex* G -manifolds, meaning roughly manifolds locally modeled by \mathbb{C}^n or $\mathbb{C}^n \times \mathbb{R}$ and equipped with the action of a Lie group G , together bound another stably complex G -manifold, and views them as equivalent in this case. This equivalence relation makes of all stably complex G -manifolds a ring Ω_*^G which has been studied since the 1960s but is to this day only completely understood when G is an abelian p -group.

A related question attempts to characterize an action of a torus T on a stably complex manifold in terms of the normal T -equivariant bundle to the fixed-point set, (in the event of an isolated fixed point, this is just a T -representation). These *isotropy data* in fact determine the manifold up to equivariant cobordism and are not arbitrary, but highly interdependent by the integral localization theorem of Atiyah–Bott–Berline–Vergne (ABBV) [BeV82, AtB84], which expresses this dependency as a web of identities in the fraction field of the cohomology ring H^*BT of the classifying space. These constraints are so restrictive that one might well wonder if any family of putative normal/representation data so constrained must necessarily arise from a T -action.

Realization question (Viktor L. Ginzburg, Yael Karshon, and Susan Tolman, late 1990s). *Can any abstract isotropy data satisfying all the ABBV relations be realized as the isotropy data of some torus action on a compact, oriented, equivariantly stably complex manifold?*

Elisheva Adina Gamse, Karshon, and I settled the question in the affirmative for an important class of well-behaved examples, the GKM manifolds already mentioned at the beginning of Section 3.

Theorem 4.1 ([CGK18]). *Let T be a torus. Given GKM abstract isotropy data $(X_p, \sigma_p)_{p \in P}$ satisfying the ABBV relations, there exists a compact, oriented, stably complex GKM T -manifold M with this isotropy data.*

In independent work analyzing the realization question in the so-called *semifree* case when S^1 is a circle whose orbits are all either free or fixed points, I found the following [C19c].

Theorem 4.2. *Any semifree abstract isotropy data $(V_p, \sigma_p)_{p \in P}$ satisfying the ABBV identities is the isotropy data of a compact, oriented, stably complex, semifree S^1 -manifold M^{2n} with isolated fixed points.*

Unexpectedly, this enables one to more constructively recover a 2004 result of Dev Sinha characterizing a case of semifree bordism.

Theorem 4.3 (Sinha [Sino5]). *Every compact, oriented, stably complex, semifree S^1 -manifold with isolated fixed points is bordant to a disjoint union of direct powers of S^2 with the standard rotation action of S^1 . That is, the bordism ring of such manifolds is isomorphic to the polynomial ring $\mathbb{Z}[S^2]$ on one generator.*

In the general case, even the precise statement of the realization question requires some work to nail down. In this generality, a positive answer to the realization question is less likely to lead to a concrete generators-and-relators-level presentation of the cobordism ring, but already seems to be within reach in some special cases.

5. In progress: A_∞ -algebraic methods

Many of the theorems I have proven regarding (equivariant) cohomology of homogeneous spaces depend in an essential way on using rational or real coefficients, which is necessary because there are no functorial commutative differential graded algebras computing cohomology with \mathbb{Z} or \mathbb{F}_p coefficients. Thus the comparatively simple techniques proving Cartan's ring isomorphism $H^*(G/K; \mathbb{R}) \cong \mathrm{Tor}_{H^*(BG; \mathbb{R})}^*(\mathbb{R}, H^*(BK; \mathbb{R}))$, the rational analogue of which is discussed in the monograph mentioned in Section 1.1, do not go through when we try to replace \mathbb{R} with an arbitrary principal ideal domain k .

Recent work of Matthias Franz [Fra19] has finally removed this dependence by analyzing the singular cochain algebra in terms of certain up-to-coherent-homotopy generalization of commutativity. Briefly, one already knows $H^*(G/K; k) \cong \mathrm{Tor}_{C^*(BG; k)}^*(k, C^*(BK; k))$, where the Tor can be computed via a functorial bar construction. This bar construction can be shown to carry the structure of a *homotopy Gerstenhaber algebra*, a differential graded algebra equipped with two infinite families of operations $\mu_n, F_{j, \ell}$ making it an A_∞ -algebra with some extra structure. A map $C^*(BK; k) \rightarrow H^*(BK; k)$ preserving sufficiently much structure will induce a quasi-isomorphism of homotopy Gerstenhaber algebras between the bar constructions. Franz shows that the multiplication in a homotopy Gerstenhaber algebra satisfies a condition called *strong homotopy commutativity* that enables one to show a certain known map can be modified to induce a *ring isomorphism* in cohomology, whereas proofs from the 1970s could only obtain a k -module isomorphism.

5.1. Isotropy actions and biquotients

Once one has done this for homogeneous spaces G/K , of course, one is tempted to see how far one can take it. My structure theorem 3.1 came essentially from the fact that if the action is equivariantly formal, then $H_H^*(G/H; \mathbb{Q})$ can be computed as $\mathrm{Tor}_{H^*(BG; \mathbb{Q})}^*(H^*(BH; \mathbb{Q}), H^*(BH; \mathbb{Q}))$, the cohomology of a *two-sided bar construction*. It is expected that arguments similar to those of Franz's papers will enable one to compute $H_K^*(G/H; k)$ for $K, H \leq G$ under analogous hypotheses, assuming only that one can define a multiplication making the two-sided bar construction a differential graded algebra. Perhaps surprisingly, this obvious purely algebraic question has not been addressed in the literature, and it is in fact not obvious how to generalize from the product on the one-sided bar construction, so it is a current project.

As a special case, when it is done, we will be able to compute the cohomology of biquotients $K \backslash G / H$, the special case of $H_K^*(G/H; k)$ where the two-sided action of $K \times H$ on G is free, in far more generality than was possible previously.

5.2. Generalized homogeneous spaces

The algebraic hypotheses permitting the k -module isomorphisms $H^*(G/K; k) \cong \mathrm{Tor}_{H^*(BG; k)}^*(k, H^*(BK; k))$ understood from the 1970s turn out to apply much more generally. J. Peter May and Frank Neumann noted that the same hypotheses apply to what are called *finite loop spaces* so long as they admit a suitable analogue of a maximal torus [MayNo2]. Using the new homotopy-Gerstenhaber-algebraic methods, a further extension of these results to show a ring isomorphism in these cases is expected to be routine.

6. Selected other work in progress

A number of other projects do not directly involve the objects discussed so far, but are generally clustered around the theme of formality.

6.1. Galois cohomology and the Bloch–Kato conjecture

The Bloch–Kato conjecture states that for a field k containing a primitive p^{th} root of unity, a certain homomorphism from the quotient $K_*^M(k)/(p)$ of the Milnor K-theory of k to the cohomology $H^*(\text{Gal}(k^{\text{sep}}/k); \mathbb{F}_p)$ of the absolute Galois group of k is an isomorphism. The conjecture’s eventual proof due to Voevodsky relied on techniques from \mathbb{A}^1 -homotopy theory not available at the time of its formulation and on a higher level of abstraction than one expects from the statement. A more constructive proof might enable one to extract more of the structure of $G_k = \text{Gal}(k^{\text{sep}}/k)$ itself from the proof than is visible from the isomorphism alone.

1. For example, the isomorphism shows the cohomology groups are generated by elements of H^1 , but is not explicit how to identify elements of H^n as polynomials in the elements of H^1 .
2. A complete understanding of this might enable us to recover a presentation for the maximal pro- p quotient $G_k(p)$ of G_k , which has been known since work of J. Labute in the 1960s to be a so-called *Demushkin group* in the case k is a local field, but is not well-understood even in the global case.
3. A presentation could be used to resolve a question of Positselski: the cohomology of $G_k(p)$ is known to be a quadratic algebra, but it is not known to be a Koszul algebra.
4. The *Mináč–Tân conjecture* that all n -fold Massey products of elements of $H^1(G_k(p); \mathbb{F}_p)$ vanish for $n \geq 3$ is known in several important cases, in particular for number fields due to recent work of Harpaz–Wittenberg, but open in general. It holds at least whenever the cochain algebra $C^*(G_k(p); \mathbb{F}_p)$ is formal. This is known not to always be the case due to counterexamples of Positselski, but all existing counterexamples arise in cases when primitive $(p^n)^{\text{th}}$ -roots do not exist in k for all n , causing certain cohomology operations to be nonzero, so there remains the possibility that if this additional hypothesis were assumed, the cochain algebra would be formal and the Mináč–Tân conjecture would also hold in these cases.

Formality properties, Koszulity properties and n -Massey vanishing properties of Galois cohomology are tightly connected, but not all connections and precise implications are clear. Part of the proposed project is to clearly delineate these connections. Joint work in progress with Ján Mináč, Federico Pasini, and Xin Fu is expected to use techniques analogous to those effective in the computation of cohomology of homogeneous spaces to provide a more nuts-and-bolts proof of the Bloch–Kato isomorphism and resolve several of the problems above. For context as to the developments in Galois cohomology leading to this proposal, we refer the reader to Harpaz–Wittenberg and Mináč–Tân [HaW19, MT17].

6.2. The toral rank conjecture for nilmanifolds

A *nilmanifold* N is³ a manifold which can be represented as an iterated principal torus bundle over a torus: i.e., N can be written as the total space of a principal torus bundle $T \rightarrow N \rightarrow B$ where B is again a nilmanifold. One can thus ask if it satisfies the following conjecture.

Conjecture 6.1. Let N be a space of finite topological dimension admitting an action of a torus T with finite stabilizers. Then $\dim_{\mathbb{Q}} H^*(N; \mathbb{Q}) \geq \dim_{\mathbb{Q}} H^*(T; \mathbb{Q})$.

Sullivan models are a common method of attack for this conjecture, which has been settled in certain special cases but remains open in general. The differentials in the Serre spectral sequence of $N \rightarrow B \rightarrow$

³ among other things

BT converging to $H^*(B; \mathbb{Q}) = H_T^*(N; \mathbb{Q})$ are determined by certain higher cohomology operations on $H^*(N; \mathbb{Q})$ [GKM98, §13], and Steven Amelotte and I are in the process of using these operations to establish bounds on the dimension of $H^*(N; \mathbb{Q})$ and hence verify the conjecture in this case.

6.3. The Halperin conjecture for biquotients

Recall that equivariant formality is the surjectivity of the fiber restriction i^* of the bundle $M \xrightarrow{i} M_G \rightarrow BG$. One can ask the same question about other fiber bundles, and one of the principal developers and historical protagonists of rational homotopy theory, Stephen Halperin, made the following conjecture.

Conjecture 6.2. Let F be a simply-connected CW complex such that $\dim_{\mathbb{Q}}(\pi_* F \otimes \mathbb{Q})$ is finite and the Euler characteristic of F is positive. Then for any fiber bundle $F \rightarrow E \rightarrow B$, the fiber restriction $H^*(E; \mathbb{Q}) \rightarrow H^*(F; \mathbb{Q})$ is surjective.

The conjecture was verified by Shiga and Tezuka [ShTe87] in the case M is a *complete flag manifold*, a homogeneous space which can be written as G/H where G and H are connected compact Lie groups and H contains a maximal torus of G . Their proof involved a careful analysis that, among other things, invoked Cartan’s theorem about G/H described in Section 1. I believe that the case of a biquotient $K \backslash G/H$ with $\text{rk } K + \text{rk } H = \text{rk } G$ will yield to a similar analysis using the Kapovitch model discussed in Section 3.1.

6.4. Representations up to homotopy as K-theory classes

Given an action of a Lie group G on a space X , the Atiyah–Segal completion theorem [AtS69], which identifies the map $K_G^*(X) \rightarrow K^*(X_G)$ with the completion of $K_G^*(X)$ with respect to the augmentation ideal IG of RG discussed in Section 3.2, can be formulated as a statement about the transformation groupoid $G \times X$. By viewing the action maps on total space V and base X as topological groupoids, then taking geometric realizations of topological nerves to get a vector bundle $V_G = B(G \times V) \rightarrow B(G \times X) = X_G$, one obtains a natural map

$$\text{Rep}(G \times X) \rightarrow K^*(B(G \times X))$$

that can be identified with the Atiyah–Segal map. Phrased this way, the construction generalizes to an arbitrary topological groupoid $\mathcal{G} \rightrightarrows M$, yielding an analogous correspondence⁴

$$\begin{aligned} \text{Rep}(\mathcal{G}) &\rightarrow K^*(B\mathcal{G}), \\ (V \rightarrow M) &\mapsto [B(\mathcal{G} \times_{\pi} V)]. \end{aligned}$$

One can ask if the same can be done with *representations up to homotopy*, generalizations necessary in the Lie groupoid world to, *inter alia*, recapture the notion of an adjoint representation [AbC13]. These are conjecturally equivalent to vector bundles $E \rightarrow M$ equipped with a map $(g, e) \mapsto \lambda_g e: \mathcal{G} \times_{\pi} E \rightarrow E$ such that each λ_g is linear and the differences $\lambda_g \circ \lambda_h - \lambda_{gh}$, etc., are ruled by a set of coherent homotopies. The geometric realization of the topological nerve no longer applies because the last face map $d_n: (\vec{g}, g_n, e) \mapsto (\vec{g}, \lambda_{g_n} e)$ only satisfies the simplicial identities up to coherent homotopy, but there is an analogous one-sided bar construction $B(*, X, Y)$ for an A_{∞} -action $X \times Y \rightarrow Y$ [HoeLS16, Def. 2.8]. Applied to a representation up to homotopy, this bar construction yields a vector bundle $B(*, \mathcal{G}, E) \rightarrow B(*, \mathcal{G}, M) \simeq B\mathcal{G}$. The resulting map

$$\text{Rep}^{\infty}(\mathcal{G}) \rightarrow K^*(B\mathcal{G})$$

has yet to be studied.

⁴ Cantarero [Can12] considers a related question, obtaining a completion theorem for finite \mathcal{G} -CW complexes and domain, instead, a \mathcal{G} -equivariant K-theory $K_{\mathcal{G}}^*(-)$ comprising classes of bundles over a \mathcal{G} -space X which arise as summands of pullbacks, under $X \rightarrow M$ of bundles, over M .

6.5. The Weyl integral formula via ABBV

Let G be a compact, connected Lie group, T its maximal torus, W the Weyl group, and $\Phi \subseteq \text{Hom}(T, S^1)$ the set of global roots of G . If $f: G \rightarrow \mathbb{R}$ is a continuous, conjugation-invariant function on G and dg and dt are Haar measures on G and T respectively, the Weyl integral formula states that

$$\int_G f(g) dg = \frac{1}{|W|} \int_T f(t) \prod_{\alpha \in \Phi} (1 - \alpha(t)^{-1}) dt.$$

On the other hand, an application of Atiyah–Bott–Berline–Vergne localization to the conjugation action of T on G yields a superficially similar equation

$$\int_G f(g) dg = \int_T \frac{[\widetilde{f \text{ vol}}]_T}{\prod_{\alpha \in \Phi^+} c_1(S_\alpha)},$$

where S_α is the line bundle $ET \times \mathbb{C} / (et, z) \sim (e, \alpha(t)z)$ over BT and the equivariant form $[\widetilde{f \text{ vol}}]$ in $\Omega(G)^T[u_1, \dots, u_{\text{rk}G}]$ represents a closed T -equivariant extension of the top form $[f \text{ vol}]$ in $H^*(G)$.

One might expect the former to follow from the latter. The Weyl and Kirillov character formulas are known to be essentially equivalent in the case G is compact, the former being implied by the Weyl integral formula and the latter by the Atiyah–Bott/Berline–Vergne equivariant localization formula, so a proof may lie in analyzing this equivalence.

References

- [AbC13] Camilo Arias Abad and Marius Crainic. Representations up to homotopy and Bott’s spectral sequence for Lie groupoids. *Adv. Math.*, 248:416–452, 2013. [arXiv:0911.2859](#), [doi:10.1016/j.aim.2012.12.022](#).
- [AkC12] Ethan Akin and Jeffrey D. Carlson. Conceptions of topological transitivity. *Topology Appl.*, 159(12):2815–2830, 2012. [arXiv:1108.4710](#), [doi:10.1016/j.topol.2012.04.016](#).
- [AtB84] Michael F. Atiyah and Raoul Bott. The moment map and equivariant cohomology. *Topology*, 23(1):1–28, 1984. [doi:10.1016/0040-9383\(84\)90021-1](#).
- [AtS69] Michael F. Atiyah and Graeme B. Segal. Equivariant K -theory and completion. *J. Diff. Geom.*, 3(1-18):9, 1969. URL: <http://www.maths.ed.ac.uk/~aar/papers/atyahsegal1.pdf>, [doi:10.4310/jdg/1214428815](#).
- [BeV82] Nicole Berline and Michèle Vergne. Classes caractéristiques équivariantes. Formule de localisation en cohomologie équivariante. *C. R. Acad. Sci. Paris*, 295(2):539–541, Nov 1982. URL: <http://gallica.bnf.fr/ark:/12148/bpt6k62356694/f77>.
- [Bor53] Armand Borel. Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts. *Ann. of Math. (2)*, 57(1):115–207, 1953. URL: <http://web.math.rochester.edu/people/faculty/doug/otherpapers/Borel-Sur.pdf>, [doi:10.2307/1969728](#).
- [BorelSem] Armand Borel, Glen Bredon, Edwin E. Floyd, Deane Montgomery, and Richard Palais. *Seminar on transformation groups*. Number 46 in Ann. of Math. Stud. Princeton Univ. Press, 1960. URL: http://indiana.edu/~jfdavis/seminar/Borel_Seminar_on_Transformation_Groups.pdf.
- [Can12] José Cantarero. Equivariant K -theory, groupoids and proper actions. *Journal of K-Theory*, 9(3):475–501, 2012. [arXiv:0803.3244](#), [doi:10.1017/is011011005jkt173](#). URL: <http://eudml.org/doc/112227>.
- [C10] Jeffrey D. Carlson. Commensurability of two-multitwist pseudo-Anosovs. 2010. [arXiv:1011.0247](#).
- [C18a] Jeffrey D. Carlson. The equivariant K -theory of a cohomogeneity-one action. 2018. [arXiv:1805.00502](#).
- [C18b] Jeffrey D. Carlson. The equivariant K -theory of isotropy actions. 2018. [arXiv:1810.09685](#).
- [C19a] Jeffrey D. Carlson. The Borel equivariant cohomology of real Grassmannians. *Proc. Amer. Math. Soc.*, Nov 2019. [arXiv:1611.01175](#).
- [C19b] Jeffrey D. Carlson. Equivariant formality of isotropic torus actions. *J. Homotopy Relat. Struct.*, 14(1):199–234, 2019. [arXiv:1410.5740](#), [doi:10.1007/s40062-018-0207-5](#).
- [C19c] Jeffrey D. Carlson. Fixed points and semifree bordism. 2019. [arXiv:1908.06906](#).
- [C20] Jeffrey D. Carlson. On the equivariant cohomology of homogeneous spaces. Manuscript monograph, 2019. URL: <http://dropbox.com/s/bczvs caxuixibac/book.pdf>.

- [CF18] Jeffrey D. Carlson and Chi-Kwong Fok. Equivariant formality of isotropy actions. *J. Lond. Math. Soc.*, Mar 2018. [arXiv:1511.06228](#), [doi:10.1112/jlms.12116](#).
- [CGK18] Jeffrey D. Carlson, Elisheva Adina Gamse, and Yael Karshon. Realization of abstract GKM isotropy data. 2018. URL: <http://www.math.toronto.edu/jcarlson/realization.pdf>.
- [CGHM19] Jeffrey D. Carlson, Oliver Goertsches, Chen He, and Augustin-Liviu Mare. The equivariant cohomology ring of a cohomogeneity-one action. *Geometriae Dedicata*, 203(1):205–223, Dec 2019. [arXiv:1802.02304](#), [doi:10.1007/s10711-019-00434-4](#).
- [Cartan51] Henri Cartan. La transgression dans un groupe de Lie et dans un espace fibré principal. In *Colloque de topologie (espace fibrés), Bruxelles 1950*, pages 57–71, Liège/Paris, 1951. Centre belge de recherches mathématiques, Georges Thone/Masson et compagnie.
- [Fok19] Chi-Kwong Fok. Equivariant formality in K -theory. *New York J. Math.*, 25:315–327, Mar 2019. URL: <http://nyjm.albany.edu/j/2019/25-15.html>, [arXiv:1704.04796](#).
- [Fra19] Matthias Franz. The cohomology rings of homogeneous spaces. 2019. [arXiv:1907.04777](#).
- [GGZ18] Fernando Galaz-García and Masoumeh Zarei. Cohomogeneity one topological manifolds revisited. *Math. Z.*, 288(3–4):829–853, 2018. URL: <http://link.springer.com/article/10.1007/s00209-017-1915-y>, [arXiv:1503.09068](#), [doi:10.1007/s00209-017-1915-y](#).
- [GKM98] Mark Goresky, Robert Kottwitz, and Robert MacPherson. Equivariant cohomology, Koszul duality, and the localization theorem. *Invent. Math.*, 131(1):25–83, 1998. URL: <http://math.ias.edu/~goresky/pdf/equivariant.jour.pdf>, [doi:10.1007/s002220050197](#).
- [GN16] Oliver Goertsches and Sam Haghshenas Noshari. Equivariant formality of isotropy actions on homogeneous spaces defined by lie group automorphisms. *J. Pure Appl. Algebra*, 220(5):2017–2028, 2016. [arXiv:1405.2655](#), [doi:10.1016/j.jpaa.2015.10.013](#).
- [HaLo7] Megumi Harada and Gregory Landweber. Surjectivity for Hamiltonian G -spaces in K -theory. *Trans. Amer. Math. Soc.*, 359(12):6001–6025, 2007. [arXiv:0503609](#), [doi:10.1090/S0002-9947-07-04164-5](#).
- [HaW19] Yonatan Harpaz and Olivier Wittenberg. The Massey vanishing conjecture for number fields. 2019. [arXiv:1904.06512](#).
- [Hodgkin] Luke Hodgkin. The equivariant Künneth theorem in K -theory. In *Topics in K-theory*, pages 1–101. Springer, 1975. [doi:10.1007/BFb0082285](#).
- [HoeLS16] Eduardo Hoefel, Muriel Livernet, and Jim Stasheff. A_∞ -actions and recognition of relative loop spaces. *Topology Appl.*, 206:126–147, 2016. [arXiv:1312.7155](#), [doi:10.1016/j.topol.2016.03.023](#).
- [He16] Chen He. GKM theory, characteristic classes and the equivariant cohomology ring of the real Grassmannian. September 2016. [arXiv:1609.06243](#).
- [Hsiang] Wu-Yi Hsiang. *Cohomology theory of topological transformation groups*. Springer, 1975.
- [Kane94] Richard Kane. Poincaré duality and the ring of coinvariants. *Canad. Math. Bull.*, 37(1):82–88, 1994. URL: <http://books.google.ca/books?id=aZqyMg8v1TEC&pg=PA82>.
- [Kap04] Vitali Kapovitch. A note on rational homotopy of biquotients. 2004. URL: <http://math.toronto.edu/vtk/biquotient.pdf>.
- [Ler49] Jean Leray. Détermination, dans les cas non exceptionnels, de l’anneau de cohomologie de l’espace homogène quotient d’un groupe de Lie compact par un sous-groupe de même rang. *C. R. Acad. Sci. Paris*, 228(25):1902–1904, 1949. URL: <http://gallica.bnf.fr/ark:/12148/bpt6k31801/f1902>.
- [MatsM86] Hiromichi Matsunaga and Haruo Minami. Forgetful homomorphisms in equivariant K -theory. *Publ. Res. Inst. Math. Sci.*, 22(1):143–150, 1986. [doi:10.2977/prims/1195178377](#).
- [Matu73] Takao Matumoto. Equivariant cohomology theories on G -CW complexes. *Osaka J. Math.*, 10(1):51–68, 1973. URL: http://ir.library.osaka-u.ac.jp/repo/ouka/all/11621/ojm10_01_07.pdf.
- [MayNo2] J. Peter May and Frank Neumann. On the cohomology of generalized homogeneous spaces. *Proc. Amer. Math. Soc.*, 130(1):267–270, 2002.
- [MT17] Jan Mináč and Nguyễn Duy Tân. Triple Massey products and Galois theory. *J. Eur. Math. Soc.*, 19:255–284, 2017. [arXiv:1307.6624](#), [doi:10.4171/JEMS/665](#).
- [Min75] Haruo Minami. K -groups of symmetric spaces I. *Osaka J. Math.*, 12:623–634, 1975. URL: <http://projecteuclid.org/euclid.ojm/1200758166>.
- [Shi96] Hiroo Shiga. Equivariant de Rham cohomology of homogeneous spaces. *J. Pure Appl. Algebra*, 106(2):173–183, 1996. [doi:10.1016/0022-4049\(95\)00018-6](#).
- [ShTa95] Hiroo Shiga and Hideo Takahashi. Remarks on equivariant cohomology of homogeneous spaces. Technical report 17, Tech. Univ. Nagaoka, May 1995. URL: dl.ndl.go.jp/info:ndljp/pid/8760355.
- [ShTe87] H. Shiga and M. Tezuka. Rational fibrations, homogeneous spaces with positive Euler characteristics and Jacobians. *Ann. Inst. Fourier (Grenoble)*, 37(1):81–106, 1987. [doi:10.5802/aif.1078](#).

- [Sino5] Dev P. Sinha. Bordism of semi-free S^1 -actions. *Math. Z.*, 249(2):439–454, 2005. [arXiv:math/0303100](https://arxiv.org/abs/math/0303100), [doi:10.1007/s00209-004-0707-3](https://doi.org/10.1007/s00209-004-0707-3).
- [Tak62] Masaru Takeuchi. On Pontrjagin classes of compact symmetric spaces. *J. Fac. Sci. Univ. Tokyo Sect. I*, 9:313–328, 1962. URL: http://dropbox.com/s/858u6qp89gmp88e/Pont_cpt_sym_sp%28Takeuchi%2C1962%29.pdf.

THE FIELDS INSTITUTE AND WESTERN UNIVERSITY
jcarlson@math.toronto.edu