Realization of fixed-point data for GKM actions

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Abstract

[This version is still being edited and should be considered preliminary.]

Given a compact manifold with an action of a torus T with isolated fixed points and with a T equivariant stable complex structure, the isotropy weights at the fixed points satisfy certain identities that are obtained by applying the ABBV localization formulae to the fixed point sets of closed subgroups of T with prescribed isotropy representation on the normal bundle. We formulate the "realization challenge" – the question whether every list of abstract fixed point data that satisfies these identities can be obtained in this way. When the dimension of the manifold is 2 or 4 and the torus action is locally standard, we obtain a positive answer from an explicit construction. When the dimension of the manifold is arbitrary and the torus action is locally standard, or, more generally, GKM, we obtain a positive answer.

1. Introduction: the "realization challenge"

Much of the information about a smooth action on a manifold can be extracted from the isotropy representations on the tangent spaces to the fixed points of the action. The representations that occur are not independent; the topology of the manifold dictates relations between them. In particular, the Atiyah–Bott–Berline–Vergne fixed point theorem, applied to characteristic classes of the tangent bundle and normal bundles of isotropy strata, imposes a list of conditions that the weights of the representations must satisfy. We ask conversely whether, given a finite list of representations that satisfy these conditions, it necessarily arises as fixed-point data for some torus action on a manifold. Schematically, we observe there is a map

$$\text{torus actions on manifolds} \longrightarrow \text{isotropy data satisfying certain conditions}$$

and ask whether we can construct a section.

To describe our notion of isotropy data precisely, we will need a well-defined notion of *weights*, and for this we will equip the manifolds we consider with an *equivariant stable complex structure* (we lay out this and other necessary background in Section 2). To make this one level more precise, let T be a torus and consider a family $(X_p, \sigma_p)_{p \in P}$ indexed by a finite set P, where each X_p is a multiset consisting of $(\dim T)$ elements of $\mathrm{Hom}(T, S^1) \backslash \{1\}$ and each σ_p is 1 or -1.

Conjecture 1.1. Given abstract isotropy data $(X_p, \sigma_p)_{p \in P}$ satisfying all relations obtained from ABBV, there exists a compact oriented stably complex T-manifold M whose fixed point set M^T can be identified with P in such a way that the weights of the isotropy representation on T_pM are X_p and the orientation on T_pM inherited from M agrees with the orientation on T_pM induced by the stable complex structure precisely when $\sigma_p = 1$.

A more precise statement involves spelling out these relations, and can be found in Problem A.1. The amount of data to be dealt with in general is substantial, and here we deal only with the case of a GKM action, where we show the answer to be in the affirmative.

Theorem A. *Given GKM abstract isotropy data $(X_p, \sigma_p)_{p \in P}$ satisfying (2.5), there exists a compact, oriented, stably complex GKM T-manifold M whose abstract isotropy data is $(X_p, \sigma_p)_{p \in P}$.*

When M is 2-dimensional this result is encapsulated in known examples, as we discuss in Section 3, and when M is 4-dimensional, we provide an explicit construction in Section 4. The proof of Theorem A is in Section 5, and in Appendix A we discuss what the conjecture should look like in the general case.

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2. Definitions and set-up

In this section we establish notation, definitions, and a few lemmas. As there are differing conventions for some terms, we go into more detail than we might otherwise.

Notation 2.1. In all that follows we denote by T a compact torus of real dimension k, i.e., a Lie group isomorphic to a finite product $\mathrm{U}(1)^k$ of circle groups, and by M an orientable smooth manifold of real dimension $2n$ equipped with a smooth T-action.

Definition 2.2. An ***orientation*** of a real vector bundle $V \to B$ is a smoothly varying choice of orientation of each fibre. Explicitly, if $\mathrm{Fr}(V) \longrightarrow B$ denotes the frame bundle of V and $\mathrm{GL}^+(V)$ the identity component of the group $\mathrm{GL}(V)$ of bundle automorphisms of V over id_B, then an orientation of V, if one exists, is a global section of the bundle $\mathrm{Fr}(V)/\mathrm{GL}^+(V) \to B$. A (*fibrewise*) ***complex vector bundle*** $V \to B$ is a real vector bundle equipped with a (*fibrewise*) ***complex structure***, meaning an automorphism J of bundles over B such that $J^2 v = -v$ for each $v \in V$. It follows each fibre of a complex vector bundle is a complex vector space under $i \cdot v := Jv$. A ***almost complex manifold*** is a smooth manifold M together with a complex structure on its tangent bundle $TM \to M$.

Any vector space of real dimension 2ℓ equipped with a complex structure J admits a basis with respect to which J is a block-diagonal matrix with blocks $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$, which is to say $\mathrm{GL}(2\ell, \mathbb{R})$ acts transitively by conjugation on the space of complex structures, and one checks the stabilizer is the subgroup $\mathrm{GL}(\ell, \mathbb{C})$ under the standard embedding. It follows a complex structure on a real vector bundle $V \to B$ of rank 2ℓ can be identified with a section of the $\mathrm{GL}(2\ell, \mathbb{R})/\mathrm{GL}(\ell, \mathbb{C})$-bundle $\mathrm{Fr}(V)/\mathrm{GL}(\ell, \mathbb{C}) \longrightarrow B$ associated to the frame bundle.

An ***isomorphism*** of complex vector bundles is an isomorphism of real bundles intertwining the complex structures. A T-***equivariant complex vector bundle*** $\pi \colon V \to B$ is a complex vector bundle equipped with T-actions on V and B such that π is T-equivariant and the map $V \longrightarrow V$ induced by each element of T is an automorphism of complex vector bundles. It follows that a T-equivariant complex structure can be identified with a T-equivariant global section of the bundle $\mathrm{Fr}(V)/\mathrm{GL}(n, \mathbb{C}) \longrightarrow B$. Given a complex vector bundle $V \to B$ a local section $(v_1, \ldots, v_n)$ of the complex frame bundle $\mathrm{Fr}(V) \to B$ determines the local section $(v_1, Jv_1, \ldots, v_n, Jv_n)$ of the real frame bundle $\mathrm{Fr}(V_{\mathbb{R}}) \to B$ and hence an ordered basis at each fiber, inducing what is called the ***complex orientation*** of V. Specializing to the case of the tangent $TM \to M$ to a smooth manifold, this means an almost complex structure on M determines an orientation of M, again called the ***complex orientation***.

We denote by $\mathbb{C}_\alpha$ the one-dimensional complex representation determined by a homomorphism $\alpha \in \mathrm{Hom}(T, S^1)$. Reciprocally, we call such an α the ***weight*** of a one-dimensional representation isomorphic to $\mathbb{C}_\alpha$. We always write the operation in $\mathrm{Hom}(T, S^1)$ ***additively***, so that, particularly, $-\alpha$ is the composition of α and complex conjugation $S^1 \longrightarrow S^1$.

Given a manifold M with a T-action, a ***T-equivariant stable complex structure*** on M is the equivalence class of a T-equivariant complex structure on some stabilization $TM \oplus \underline{\mathbb{R}}^r$ of its tangent bundle, where T acts on TM by the tangent action and trivially on $\underline{\mathbb{R}}^r$. Two such complex structures J_1 on $TM \oplus \underline{\mathbb{R}}^{r_1}$ and J_2 on $TM \oplus \underline{\mathbb{R}}^{r_2}$ are defined to be equivalent if there exist non-negative integers s_1, s_2 and a smooth T-equivariant isomorphism

$$(TM \oplus \underline{\mathbb{R}}^{r_1}, J_1) \oplus \underline{\mathbb{C}}^{s_1} \xrightarrow{\sim} (TM \oplus \underline{\mathbb{R}}^{r_2}, J_2) \oplus \underline{\mathbb{C}}^{s_2}$$

of complex vector bundles over M, where the stabilizing trivial bundles $\underline{\mathbb{C}}^{s_1}$ and $\underline{\mathbb{C}}^{s_2}$ each carry the standard complex structure.The manifold M, equipped with such structures, is called a ***stably complex T-manifold***. Note that the choice of a *stable* complex structure on M does *not* determine an orientation of M.

Remark 2.3. As real T-representations $\mathbb{C}_\alpha$ and $\mathbb{C}_{-\alpha}$ are isomorphic, but as complex representations they are not. This distinction is important, as it means the weights of irreducible factors are defined for isotropy representations on a stably complex T-manifold.

Stable complex structures behave well under restriction to fixed-point submanifolds.

Proposition 2.4. *Let H be a closed subgroup of T and N a connected component of the set M^H of H-fixed points of M. An orientation of M induces an orientation on N. A T-equivariant stable complex structure on M induces a unique T-equivariant stable complex structure on N and a unique T-invariant complex structure on the normal bundle $\nu_M N$ to N in M. An orientation on M thus uniquely determines an orientation on N. Moreover, if H is contained in another closed subgroup K of H and P is a connected component of N^K, then the T-equivariant stable complex structure on P induced from N agrees with that induced directly from M, the T-equivariant complex structures on normal bundles are compatible in the sense that*

$$0 \to \nu_N P \longrightarrow \nu_M P \longrightarrow (\nu_M N)|_P \to 0 \tag{2.1}$$

is a short exact sequence of complex vector bundles over P, and the orientation on P inherited from N agrees with that inherited directly from M.

Proof. Fix a representative $(TM \oplus \underline{\mathbb{R}}^r, J)$ of an equivariant stable complex structure on M. Restricting, we obtain a T-equivariant complex vector bundle $(TM \oplus \underline{\mathbb{R}}^r)|_N$ over N whose H–fixed point set is the subbundle $TN \oplus (\underline{\mathbb{R}}^r|_N)$. This is in fact a *complex* subbundle, as J commutes with the tangent action of H on TM, and so we also obtain a complex structure on the quotient bundle. The inclusion $TM|_N \hookrightarrow (TM \oplus \underline{\mathbb{R}}^r)|_N$ induces a T-equivariant bundle isomorphism

$$\nu_M N = \frac{TM|_N}{TN} \xrightarrow{\sim} \frac{(TM \oplus \underline{\mathbb{R}}^r)|_N}{TN \oplus (\underline{\mathbb{R}}^r|_N)}$$

and hence a T-equivariant complex structure on the normal bundle $\nu_M N$. It can be checked that complex structures on stabilizations of TM representing the same equivariant stable complex structure on M give rise to the same equivariant stable complex structure on N and the same invariant complex structure on $\nu_M N$. The given fibrewise orientation on TM and the complex orientation on $\nu_M N$ uniquely induce a fibrewise orientation on the kernel TN of the bundle map $TM|_N \twoheadrightarrow \nu_M N$.

The statement about restriction of equivariant stable complex structures follows again from the fact J commutes with the tangent action of each element of T. The exact sequence (2.1) of normal bundles follows from the third isomorphism theorem:

$$(\nu_M N)|_P \cong \left.\frac{(TM \oplus \underline{\mathbb{R}}^r)|_N}{TN \oplus \underline{\mathbb{R}}^r}\right|_P \cong \frac{TM|_P \oplus \underline{\mathbb{R}}^r \big/ TP \oplus \underline{\mathbb{R}}^r}{TN|_P \oplus \underline{\mathbb{R}}^r \big/ TP \oplus \underline{\mathbb{R}}^r} \cong \frac{\nu_M P}{\nu_N P}.$$

The transitivity of induced orientations follows from this sequence. □

We filter a T-manifold not only by its stabilizers but by its isotropy representations.

Definition 2.5. Given a complex representation $\rho\colon H \longrightarrow \operatorname{Aut} V_\rho$ of a subgroup H of the torus T with trivial invariant subspace $(V_\rho)^H$, we define the corresponding ***isotypic submanifold*** M^ρ to be the closure in M of the set of points p whose stabilizer is equal to H and such that the isotropy representation of H on $T_pM/(T_pM)^H$ is isomorphic to ρ.

Remark 2.6. Note that given a equivariant stable complex structure on M, because every connected component N of M^ρ is a connected component of M^H, such N are T-submanifolds of M, which by Proposition 2.4 inherit T-equivariant stable complex structures, and whose normal bundles in M are T-equivariant complex bundles. Particularly, if p is a fixed point, then the tangent space $T_pM = \nu_M\{p\}$ inherits an equivariant almost complex structure and hence a complex orientation. By Proposition 2.4, this inherited orientation does not depend whether we view $\{p\}$ as a submanifold of M or of some isotypic submanifold M^ρ.

Definition 2.7. Given a fixed point p of a stably complex T-manifold, the orientation on T_pM induced by the stable complex structure on M is called the ***complex orientation***. If M is oriented, we write σ_p for the ***sign*** at p, chosen to be respectively 1 or -1 depending whether the complex orientation of T_pM agrees or disagrees with the chosen orientation of M.

The Atiyah–Bott–Berline–Vergne (*ABBV*) fixed-point theorem [BV82, AB84], given an oriented T-manifold M, considers the pushforward map $\pi_!^M\colon H_T^*(M) \to H_T^*(\mathrm{pt}) = H^*(BT)$ in Borel equivariant cohomology and states that for any equivariant cohomology class $c \in H_T^*(M)$, one has

$$\pi_!^M c = \sum_{N \subseteq M^T} \frac{\pi_!^N(c|_N)}{e^T(\nu_M N)} \in H^*(BT), \tag{2.2}$$

where N ranges over components of the fixed point set M^T and e^T is the equivariant Euler class. Part of the statement is the non-obvious fact that the right-hand side, a priori only lying in some localization of $H^*(BT)$, indeed lies in $H^*(BT)$ itself. The phrase "all ABBV identities" in Conjecture 1.1 is an abstraction of the collection of all these statements, over all equivariant cohomology classes c, where we replace M with every isotypic submanifold M^ρ. The precise statement is (A.1). For now, we focus on the GKM case.

Definition 2.8. A stably complex T^k-manifold M^{2n} is called a ***GKM manifold*** when

- the fixed point set M^{T^k} is discrete and
- for each fixed point p, no two weights of the isotropy representation $T^k \curvearrowright T_pM$ are collinear in $\mathrm{Hom}(T^k, S^1)$.

In this case, if $\rho\colon H \longrightarrow \mathrm{Aut}\, V_\rho$ is a representation of a subgroup H of codimension one, the components N of a nonempty M^ρ are 2-spheres N consisting of two points p, q of M^T and a complement N° foliated by orbits of a circle action $\alpha\colon T \twoheadrightarrow T/H \xrightarrow{\sim} S^1$. The normal bundle $\nu_M N^\circ$ to this complement N° is a trivial bundle whose every fibre carries an H-representation we may write as $\bigoplus_{j=1}^{n-1} \mathbb{C}_{\bar\beta_j}$ for some nonzero $\bar\beta_j \in \mathrm{Hom}(H, S^1)$. The isotropy representations at the poles p and q can be written

$$T_pM \cong \mathbb{C}_{\alpha_p} \oplus \bigoplus_{j=1}^{n-1} \mathbb{C}_{\beta_{j,p}}, \qquad T_qM \cong \mathbb{C}_{\alpha_q} \oplus \bigoplus_{j=1}^{n-1} \mathbb{C}_{\beta_{j,q}},$$

where each of α_p and α_q is α or $-\alpha$ and $\beta_{j,p}, \beta_{j,q} \in \mathrm{Hom}(T, S^1)$ restrict, for each j, to $\bar\beta_j \in \mathrm{Hom}(H, S^1)$. Thus

$$\beta_{j,p} \equiv \beta_{j,q} \pmod{\alpha}. \tag{2.3}$$

Select an orientation of N at random. Both tangent spaces T_pN and T_qN are isomorphic to $\mathbb{C}_\alpha$ as real T-representations, but the orientation inherited from N may or may not agree with the complex orientations. Remembering that we defined $\sigma_p \in \{\pm 1\}$ to be 1 just in the case of agreement and applying (2.2) to $c_0 = 1 \in H^0(N)$, we get

$$0 = \int_{S^2} 1 = \frac{1}{e^T\big(\nu_N\{p\}\big)} + \frac{1}{e^T\big(\nu_N\{q\}\big)} = \frac{1}{\alpha} + \frac{1}{-\alpha} = \frac{1}{\sigma_p\alpha_p} + \frac{1}{\sigma_q\alpha_q}. \tag{2.4}$$

When we collect abstract isotropy data in the form of isotropy representations at fixed points, we also forget the 2-spheres N (edges of the GKM graph), and to find the minimal relations abstract isotropy data must satisfy if it comes from a GKM action, we need to identify "potential 2-spheres" in the data. Taking (2.3) as the identifying condition for a potential 2-sphere, we are motivated to define the sets P_ρ in the following definition. The crucial condition (2.5) then follows from summing (2.4) over endpoints of such potential 2-spheres.

Definition 2.9. Let $k \leqslant n$ be natural numbers and M^{2n} a compact, oriented, stably complex T^k-manifold with isolated fixed points. The ***isotropy data of*** M is $(X_p, \sigma_p)_{p\in M^T}$, where X_p is the multiset of weights of T_pM and $\sigma_p \in \{\pm 1\}$ is 1 just if the orientation of T_pM agrees with the orientation of $\bigoplus_{\alpha\in X_p} \mathbb{C}_\alpha$. ***Abstract isotropy data*** is simply a finite family $(X_p, \sigma_p)_{p\in P}$ such that each X_p is a multiset of n nontrivial elements of $\mathrm{Hom}(T^k, S^1)$ and each σ_p is 1 or -1.

Given abstract isotropy data $(X_p, \sigma_p)_{p\in P}$, for each codimension-one closed subgroup H of $T = T^k$ and each $(n-1)$-dimensional representation $\rho\colon H \longrightarrow V_\rho$ with trivial invariant subspace V_ρ^H, we write $P_\rho \subseteq P$ for the set of indices p such that $X_p \subseteq \mathrm{Hom}(T, S^1)$ can be decomposed as $\{\alpha_p\} \amalg Y$, where $H = \ker \alpha_p$ and the restriction of Y to H is the multiset of weights of ρ. We will call abstract isotropy data ***GKM*** when each X_p is a *set* no two elements of which are collinear.

The condition on GKM abstract isotropy data ultimately extracted from (2.4) is this:

$$\text{For all } \rho, \qquad \sum_{p \in P_\rho} \frac{1}{\sigma_p \alpha_p} = 0. \tag{2.5}$$

Now we can state the central problem.

Problem 2.10. Given GKM abstract isotropy data $(X_p, \sigma_p)_{p\in P}$ satisfying (2.5), does there exist a compact, oriented, stably complex GKM T-manifold M whose isotropy data is $(X_p, \sigma_p)_{p\in P}$?

We state a generalization, as yet unanswered, in Appendix A.

3. Two dimensions

The statement of Problem 2.10 in case $n = k = 1$ is particularly simple but gives intuition for the general proof and provides a useful example of the meaning of the signs σ.

Theorem 3.1. *Let P be a finite set, for each $p \in P$ let $X_p = \{\alpha_p\}$ contain a single element of* $\operatorname{Hom}(S^1, S^1)$, *and let $\sigma_p \in \{\pm 1\}$. Partition P into sets P_ℓ consisting of those p for which* $\ker \alpha_p$ *is the group of ℓ^{th} roots of unity, and assume that for all $\ell \geqslant 1$,*

$$\sum_{p \in P_\ell} \frac{1}{\sigma_p \alpha_p} = 0.$$

Then there exists a compact, oriented, stably complex S^1-surface M and an identification of P with M^{S^1} such that for each $p \in M^{S^1}$ the weight of the isotropy action of T on T_pM is α_p and the given orientation of T_pM agrees with the complex orientation if and only if $\sigma_p = 1$.

Proof. For each $\ell \geqslant 1$ and all $p, q \in P_\ell$ we have $\alpha_p = \pm\alpha_q$, so if we write $s\colon S^1 \xrightarrow{\text{id}} S^1$ for the standard generator, (2.5) implies there are as many $p \in P$ with $\sigma_p \alpha_p = \ell s$ as there are $q \in P$ with $\sigma_q \alpha_q = -\ell s$. Use this observation to partition P into pairs $\{p, q\}$. Then M will be a disjoint union of manifolds M' with isotropy data $\big((\alpha_p, \sigma_p), (\alpha_q, \sigma_q)\big)$ corresponding to these, to be constructed in the following examples. ∎

Example 3.2 ($\sigma_p/\sigma_q = 1$). Consider the action of S^1 on $S^2 \cong \mathbb{CP}^1$ induced by $S^1 \xrightarrow{\ell} S^1 \hookrightarrow \mathbb{C}^\times$ and the multiplication of $\mathbb{C}^\times$ With the standard complex structure on $\mathbb{CP}^1$, the tangent representation at the north pole 0 is ℓs and that at the south pole ∞ is $-\ell s$. If we give $\mathbb{CP}^1$ the standard orientation induced from the basis $1, i$ of $\mathbb{C}$, then $\sigma_0 = \sigma_\infty = 1$. If we give it the opposite orientation, then $\sigma_0 = \sigma_\infty = -1$. Either way $\sigma_0\alpha_0 + \sigma_\infty\alpha_\infty = 0$.

Example 3.3 ($\sigma_p/\sigma_q = -1$). The tangent bundle TS^2 to a sphere S^2 is stably trivial, as the normal bundle ν of the inclusion in $\mathbb{R}^3$ is trivial and $\underline{\mathbb{R}}^3 = TS^2 \oplus \nu$. In coordinates, ν and TS^2 can be seen as the subsets of $S^2 \times \mathbb{R}^3$ given respectively by pairs (p, v) such that $p \in S^2$ and $v \perp p$ and by pairs (p, ap) with $a \in \mathbb{R}$. Stabilizing again allows us to define a stable complex structure on S^2 distinct from the standard stable complex structure on $\mathbb{CP}^1$. Explicitly, we have a real bundle isomorphism

$$TS^2 \oplus \nu \oplus \underline{\mathbb{R}} \xrightarrow{\sim} \underline{\mathbb{R}}^3 \oplus \underline{\mathbb{R}} \xrightarrow{\sim} (\underline{\mathbb{C}}^2)_{\mathbb{R}}$$

which on the fibre over each point $p \in S^2$ takes

$$(v, ap, b) \longmapsto \big(x(v + ap) + iy(v + ap), z(v + ap) + ib\big)$$

Pulling back the standard constant real frame on $(\underline{\mathbb{C}^2})_{\mathbb{R}}$ gives a frame on $TS^2 \oplus \underline{\mathbb{R}^2}$ which on the fiber $p^\perp \oplus \mathbb{R}p \oplus \mathbb{R}$ over $p = (x, y, z) \in S^2$ is

$$\big((1,0,0) - xp, 0, 0\big), \qquad \big((0,1,0) - yp, 0, 0\big), \qquad (\vec{0}, p, 0), \qquad (\vec{0}, 0, 1).$$

In particular, at both the north and south poles $p_\pm = (0, 0, \pm 1)$, the first two basis vectors are $\big((1,0,0), 0, 0\big)$ and $\big((0,1,0), 0, 0\big)$.

On the other hand, the standard orientation of S^2 is given at $p_\pm$ by the basis $(\pm 1, 0, 0), (0, 1, 0)$ of $T_{p_\pm} S^2$. Thus $\sigma_{p_+} = 1$ and $\sigma_{p_-} = -1$, so, with the same action as in Example 3.2, we have $\alpha_{p_+} = \alpha_{p_-} = \ell s$. But still $\sigma_{p_+}\alpha_{p_+} + \sigma_{p_-}\alpha_{p_-} = \ell s - \ell s = 0$.

Remark 3.4. In fact, these are in a strong sense the only examples: the only closed, connected, oriented surfaces a circle acts on nontrivially are the torus and the sphere, and circle actions on tori do not admit fixed points.

4. Four dimensions

In the case $n = k = 2$ we are able to provide a more interesting construction. We will realize GKM abstract isotropy data satisfying (2.5) by a skeletal construction starting with a 0-skeleton whose points are in correspondence with the index set of the given abstract isotropy data. The 1-skeleton will be modelled by a 2-regular graph.

Definition 4.1. An ***graph*** consists of a set of vertices and a set of edges between them; multiple edges between the same pair of vertices are allowed. Given an graph Γ, write $\mathscr{V}(\Gamma)$ for the set of vertices of Γ and $\mathscr{E}(\Gamma)$ for the set of orientations of edges of Γ—thus each edge of Γ appears twice in $\mathscr{E}(\Gamma)$, once with either orientation. If $e \in \mathscr{E}(\Gamma)$ starts at $p \in \mathscr{V}(\Gamma)$ and ends at q, we write $e\colon p \to q$, and $\bar{e}\colon q \to p$ for the same edge with the opposite orientation.

We need two lemmas, the first of which we generalize substantially in the next section and the other of which is *sui generis*.

Lemma 4.2. *Given GKM abstract isotropy data* $(X_p, \sigma_p)_{p \in P}$ *satisfying* (2.5), *there exists a 2-regular graph* Γ *with vertex set* P *and labels* $\alpha(e) \in \mathrm{Hom}(T, S^1)$ *for each oriented edge* e *such that*

- *for each* $e\colon p \to q$, *we have* $\sigma_p \alpha(e) + \sigma_q \alpha(\bar{e}) = 0$,
- *if the two oriented edges emanating from* p *are* e, e', *then* $X_p = \{\alpha(e), \alpha(e')\}$, *and*
- *if we write* $X_p = \{\alpha(e), \beta(e)\}$ *and* $X_q = \{\alpha(\bar{e}), \beta(\bar{e})\}$, *then* $\beta(\bar{e}) \equiv \beta(e) \pmod{\alpha(e)}$.

The resulting graph is what is called a *GKM graph*, as we elaborate on in Definition 2.8, and in fact this lemma will be expanded to that case in Lemma 5.7. If the X_p are bases of $\mathrm{Hom}(T, S^1)$ it is what is called a *torus graph*.

Proof. Take P itself as $\mathscr{V}(\Gamma)$ and construct the edges as follows. For each element p of a fixed P_ρ as in Definition 2.9, note that α_p can be one of precisely two elements $\pm\alpha$. From the identity (2.5) we see that there are exactly as many $p \in P_\rho$ such that $\sigma_p \alpha_p = \alpha$ as those such that $\sigma_p \alpha_p = -\alpha$. Choose a bijection between these two subsets of P_ρ and add to $\mathscr{E}(\Gamma)$ an edge e between each pair of points p and q matched by this bijection, setting $\alpha(e) = \alpha$. By construction, we have $\sigma_p \alpha(e) + \sigma_q \alpha(\bar{e}) = 0$ and $\beta(e) \equiv \beta(\bar{e}) \pmod{\alpha}$. Repeat this process for each ρ.

It remains to check Γ is 2-regular. Given $p \in P = \mathscr{V}(\Gamma)$, if $X_p = \{\alpha, \beta\}$, then $p \in P_\rho$ for $\rho = \beta|_{\ker \alpha}$, so there is an edge at p corresponding to α. As the elements of X_p are not collinear, the resulting subgroups $\ker \alpha$ are distinct, so exactly two edges of Γ emanate from p. □

As we will identify P with a subset of our final T-manifold, we will henceforth identify the vertices p of the lemma with the corresponding $p \in P$. The submanifolds corresponding to an edge of this graph will be provided by the following construction.

Lemma 4.3. *Let T be a two-dimensional torus. Given a pair of homomorphisms $\alpha, \beta \in \mathrm{Hom}(T, S^1)$ and an integer k there exists a T-equivariant Hermitian line bundle $\xi\colon E \longrightarrow \mathbb{C}\mathrm{P}^1$ such that the weight of the induced action of T on the tangent space $T_{[1,0]}\mathbb{C}\mathrm{P}^1$ is α and that on $T_{[0,1]}\mathbb{C}\mathrm{P}^1$ is $-\alpha$, while the weight of the induced action of T on $E_{[1,0]}$ is β and that at $E_{[0,1]}$ is $\beta + k\alpha$.*

Proof. Let $\mathbb{C}^\times$ be an abstract algebraic torus, to be used as an auxiliary. Given $\rho \in \mathrm{Hom}(T, S^1)$ and $k \in \mathbb{Z}$ write $\mathbb{C}_{\rho,k}$ for the one-dimensional complex representation of $T \times \mathbb{C}^\times$ given by $(t,z) \cdot v := \rho(t) z^k v$.

Form the $(T \times \mathbb{C}^\times)$-representation $\mathbb{C}_{0,-1} \oplus \mathbb{C}_{\alpha,-1} \oplus \mathbb{C}_{\beta,k}$; projecting out the last coordinate makes this the total space of a $(T \times \mathbb{C}^\times)$-equivariant line bundle over $\mathbb{C}_{0,-1} \oplus \mathbb{C}_{\alpha,-1} \cong \mathbb{C}^2$. If we restrict this line bundle to $\mathbb{C}^2 \backslash \{\vec{0}\}$ and quotient the $\mathbb{C}^\times$-action out from both total and base space of this restricted bundle, we get a T-equivariant line bundle $E \to \mathbb{C}\mathrm{P}^1$. One may check that the T-action on $\mathbb{C}\mathrm{P}^1$ has respective weights α and $-\alpha$ at $[1,0]$ and $[0,1]$ and the other weights of the T-action on the fibres $E_{[1,0]}$ and $E_{[0,1]}$ are respectively β and $\beta + k\alpha$. □

Remark 4.4. The line bundle E contains a natural "unit disc" subbundle DE determined by imposing the restriction $|w| \leqslant \sqrt{|z_0|^2 + |z_1|^2}$ on points $(z_0, z_1, w) \in \mathbb{C}^2 \backslash \{\vec{0}\} \times \mathbb{C}_{\beta,k}$ before quotienting by the $\mathbb{C}^\times$-action. It also induces an orientation on E and a fiberwise complex structure. The weights of the action at $[1,0]$ and $[0,1] \in E$ will differ by a sign from the weights with respect to the stable complex structure we are about to construct.

Now the pieces are in place.

Theorem 4.5. *Let $n = k = 2$. Given GKM abstract isotropy data $(X_p, \sigma_p)_{p \in P}$ satisfying (2.5), there exists a four-dimensional compact stably complex GKM T-manifold M whose isotropy data is $(X_p, \sigma_p)_{p \in P}$. Moreover, if the X_p are all bases of $\mathrm{Hom}(T, S^1)$, then the action is* ***locally standard*** *in the sense that around every point of M there is a local T-equivariant homeomorphism to a neighborhood in $\mathbb{C}^2$ with the standard T-action.*

Proof. Given GKM abstract isotropy data $(X_p, \sigma_p)_{p \in P}$ satisfying (2.5), create a graph Γ as in Lemma 4.2; note in particular that this 2-regular graph is a union of cycles. For each edge $e\colon p \to q$, we write $a_e := \sigma_p \alpha_e$ (note that by Definition 2.8 we always have $a_e + a_{\bar{e}} = 0$), and we write β_e for the single element of $X_p \backslash \{\alpha_e\}$. Consistently orient each cycle of Γ; this distinguishes a subset of oriented edges $\mathscr{E}^+ \subsetneq \mathscr{E}(\Gamma)$ such that the end of each edge of $\mathscr{E}^+$ is the beginning of another. For each oriented edge $e\colon p \to q$, observe that $\beta_{\bar{e}}$ is equal to $\beta_e + m a_e$ for some $m \in \mathbb{Z}$ by Definition 2.8; thus we may use Lemma 4.3 to construct, for each $e \in \mathscr{E}^+$, a complex line bundle $\xi_e\colon E_e \longrightarrow \mathbb{C}\mathrm{P}^1_e := \mathbb{C}\mathrm{P}^1$ with associated weights a_e, β_e at $[1,0]$ and $-a_e, \beta_{\bar{e}}$ at $[0,1]$. We identify $0 \in E_e|_{[1,0]}$ with p and $0 \in E_e|_{[0,1]}$ with q.

Each line bundle ξ_e has a well-defined closed unit disc bundle DE_e by Remark 4.4. Recall that for each pair $e\colon p \to q$ and $f\colon q \to r$ in $\mathscr{E}^+$ we have $a_{\bar{e}} = \beta_f$ by construction. Now we glue together DE_e and DE_f as follows. Select disjoint closed T-invariant neighborhoods $D_{\bar{e}}$ of $[0,1] \in \mathbb{C}\mathrm{P}^1_e$ and

D_f of $[1,0] \in \mathbb{CP}^1_f$, which we may equivariantly identify with closed unit discs $D_{a_{\bar{e}}} \subsetneq \mathbb{C}_{a_{\bar{e}}}$ and $D_{a_f} \subsetneq \mathbb{C}_{a_f}$ in one-dimensional T-representations. Then we have equivariant identifications

$$\begin{aligned} DE_e|_{D_{\bar{e}}} &\cong D_{a_{\bar{e}}} \times D_{\beta_{\bar{e}}}, \\ DE_f|_{D_f} &\cong D_{a_f} \times D_{\beta_f}, \end{aligned} \tag{4.1}$$

and since $\sigma_q \alpha_{\bar{e}} = a_{\bar{e}} = \beta_f$ and $a_f = \beta_{\bar{e}}$, there exists an orientation-preserving equivariant diffeomorphism $DE_e|_{D_{\bar{e}}} \xrightarrow{\sim} DE_f|_{D_f}$ given in terms of the trivializing coordinates by $(z,w) \longmapsto (w,z)$. Thus the interior of the quotient space

$$M_1 \quad := \quad \coprod_{e \in \mathscr{E}^+} DE_e \Big/ \big(e,(z,w)\big) \sim \big(f,(w,z)\big) \quad \text{for } p \xrightarrow{e} q \xrightarrow{f} r$$

naturally inherits the structure of a complex T-manifold.

The remainder of the construction proceeds in parallel for each component of M_1, so we may as well assume M_1 is connected. Note that the T-orbit space $DE_e|_{D_e}/T$ of each of the polydiscs we plumbed with is diffeomorphic to a closed rectangle $(D^2/S^1) \times (D^2/S^1) = [0,1]^2$, and the T-orbit space of the restriction of DE_e to $\mathbb{CP}^1_e \backslash (D_e \cup D_f)$ is diffeomorphic to a rectangle $(0,1) \times [0,1]$. The entire orbit space M_1/T is a closed annulus, smooth except at the points $(0,0)$ and $(1,1)$ in each identification rectangle $[0,1]^2$. Thus a small T-invariant neighborhood of ∂M_1 meets the interior $(M_1)^\circ$ in an open set which is a principal T-bundle over an open annulus. Since any such bundle is globally trivializable, we may smoothly and equivariantly identify this intersection with $S^1 \times (1-\varepsilon, 1) \times T$. Writing $D_{(1-\varepsilon,\infty]}$ for the complement in $S^2 \cong \mathbb{R}^2 \cup \{\infty\}$ of the closed disc of radius $1-\varepsilon$, we set

$$M := M_1^\circ \underset{S^1 \times (1-\varepsilon,1) \times T}{\cup} D_{(1-\varepsilon,\infty]} \times T.$$

It remains to find a T-equivariant stable complex structure on M such that the isotropy weights at each $p \in P$ are X_p. We will identify this as the class of an T-equivariant complex structure on $TM \oplus \underline{\mathbb{R}}^2$, where the T-action on the stabilizing summand is trivial. We do this one level at a time on the equivariant skeleton, first on neighborhoods of the fixed points $p \in P$, then on the edges DE_e for $e \in \mathscr{E}^+$, and finally on the 2-cell $D_{(1-e,\infty]} \times T$.

For the 0-skeleton, each $p \in P$ is the source of one directed edge $e \colon p \to q$ in $\mathscr{E}^+$, note that the natural T-equivariant almost complex structure on the tangent bundle of the unit polydisc $D_{\alpha_e} \times D_{\beta_e} \subsetneq \mathbb{C}_{\alpha_e} \times \mathbb{C}_{\beta_e}$ has the correct weights, but the complex structure on the plumbing locus $DE_e|_{D_e} \cong D_{a_e} \times D_{\beta_e}$ inherited from DE_e will agree only if $\sigma_p = 1$. Using the equivariant homeomorphism $D_{a_e} \approx D_{\alpha_e}$ which is complex conjugation if $\sigma_p = -1$ and the identity if $\sigma_p = 1$, we equip the plumbing locus with the complex structure of $D_{\alpha_e} \times D_{\beta_e}$. If $\sigma_p = 1$, stabilize this bundle by adding $\underline{\mathbb{C}}$; otherwise, add the conjugate $\overline{\underline{\mathbb{C}}}$. Note this structure induces a equivariant complex structure on the bundles TD_e and $E_e|_{D_e} \cong D_{\alpha_e} \times \mathbb{C}_{\beta_e}$ as well. For brevity, we will call these three the ***0-skeleton structures***.

To find an equivariant stable complex structure on the equivariant 1-skeleton $(M_1)^\circ$, it is enough to find one on each equivariant edge DE_e. For this, recall that this edge is the total space of a disc bundle $\pi \colon DE_e \longrightarrow \mathbb{CP}^1_e$ and that since the kernel of the tangent map $\pi_* \colon TDE_e \longrightarrow T\mathbb{CP}^1$ can be identified with $\pi^* E_e$, we have a short exact sequence

$$0 \to \pi^* E_e \longrightarrow TDE_e \oplus \underline{\mathbb{R}}^2 \longrightarrow \pi^*(T\mathbb{CP}^1_e \oplus \underline{\mathbb{R}}^2) \to 0 \tag{4.2}$$

of T-equivariant complex vector bundles over DE_e. Using the identification $DE_e|_{D_e} \cong D_{\alpha_e} \times D_{\beta_e}$ from (4.1), this exact sequence restricts over $DE_e|_{D_e}$ to the evidently equivariantly split sequence

$$0 \to D_{\alpha_e} \times D_{\beta_e} \times \mathbb{C}_{\beta_e} \longrightarrow D_{\alpha_e} \times \mathbb{C}_{\alpha_e} \times D_{\beta_e} \times \mathbb{C}_{\beta_e} \times \mathbb{R}^2 \longrightarrow D_{\alpha_e} \times \mathbb{C}_{\alpha_e} \times D_{\beta_e} \times \mathbb{R}^2 \to 0,$$

equipped with the diagonal action. We have a similar identification near the other pole of $\mathbb{C}P^1_e$, over $DE_e|_{D_{\bar{e}}}$, and since $DE_e|_{D_e}$ and $DE_e|_{D_{\bar{e}}}$ are closed, we may extend these splittings to an equivariant splitting of (4.2). Note that this decomposition is what we want on the level of T-actions, but the orientations on the $\mathbb{C}_{\alpha_e}$ fibers may not agree with the 0-skeleton structure on $DE_e|_{D_e}$. If we can determine equivariant complex structures on the bundles E_e and $T\mathbb{C}P^1_e \oplus \underline{\mathbb{R}}^2$ such that their pullbacks, when restricted over D_e, agree with the 0-skeleton structures, then the splitting will induce an equivariant complex structure on $TDE_e \oplus \underline{\mathbb{R}}^2$ agreeing with the 0-skeleton structure as well.

For E_e, we may just take the original equivariant complex structure provided by Lemma 4.3. For $T\mathbb{C}P^1_e \oplus \underline{\mathbb{R}}^2$, we start with the 0-skeleton structure on $(TD_e \amalg TD_{\bar{e}}) \oplus \underline{\mathbb{R}}^2$ and recall from Definition 2.2 that a equivariant complex structure can be identified with an equivariant section of the bundle

$$C = \mathrm{Fr}(T\mathbb{C}P^1_e \oplus \underline{\mathbb{R}}^2)/\mathrm{GL}(2,\mathbb{C}) \longrightarrow \mathbb{C}P^1_e.$$

The quotient group $T/(\ker \alpha_e)$ acts freely on the restriction γ of this bundle to $\mathbb{C}P^1_e \backslash \{p, q\}$, with quotient a bundle

$$\overline{\gamma} \colon Q \to I$$

over an open interval, and equivariant sections of γ correspond bijectively to sections of $\overline{\gamma}$. Our desired section J is already defined over D_e and $D_{\bar{e}}$, so the corresponding section j of $\overline{\gamma}$ is defined over the two half-open intervals $I_e = D_e/T$ and $I_{\bar{e}} = D_{\bar{e}}/T$, which sit in I as $(0, \varepsilon]$ and $[1-\varepsilon, 1)$ do in $(0, 1)$. As $\mathrm{GL}(4,\mathbb{R})$ has two components, corresponding to positive vs. negative determinant, and $\mathrm{GL}(2,\mathbb{C})$ only one component, it follows the fibers $\mathrm{GL}(4,\mathbb{R})/\mathrm{GL}(2,\mathbb{C})$ of γ and $\overline{\gamma}$ have two components, corresponding to orientations of $T\mathbb{C}P^1_e \oplus \underline{\mathbb{R}}^2$, and hence the total spaces do as well. Moreover, the 0-skeleton structures corresponding to the sections over $DE_e \backslash \{p\}$ and $DE_{\bar{e}} \backslash \{q\}$ lie in the same component of C owing to our choice of $\underline{\mathbb{C}}$ or $\overline{\underline{\mathbb{C}}}$ as stabilizing factor, so corresponding sections from I_e and $I_{\bar{e}}$ to Q do as well. Since I is contractible, Q is a trivial bundle, and hence sections of Q correspond to maps $I \longrightarrow \mathrm{GL}(4,\mathbb{R})/\mathrm{GL}(2,\mathbb{C})$. As I_e and $I_{\bar{e}}$ are closed in I and their images lie in the same component of Q, we may connect the existing sections to a full section $j \colon I \longrightarrow Q$ as hoped. By construction, the resulting J has isotropy weight α_e at p and $\alpha_{\bar{e}}$ at q.

Having done this for all $e \in \mathscr{E}^+$ gives a equivariant complex structure on $TM_1 \oplus \underline{\mathbb{R}}^2$, since for a sequence of edges $p \xrightarrow{e} q \xrightarrow{f} r$, by construction the structures on $E_e|_{D_{\bar{e}}}$ and $E_f|_{D_f}$ agree and likewise that on $TD_{\bar{e}} \oplus \underline{\mathbb{R}}^2$ agrees with that on $TD_f \oplus \underline{\mathbb{R}}^2$. Our final task is to extend this to a equivariant complex structure on the rank-six real vector bundle $TM \oplus \underline{\mathbb{R}}^2 \longrightarrow M$. Under the identifications we used to construct M, this amounts to constructing a section over over the equivariant two-cell $D_{(1-\varepsilon,\infty]} \times T$ agreeing with one already chosen on the equivariant annulus $S^1 \times (1-\varepsilon, 1] \times T$. The bundle of complex structures over $D_{(1-\varepsilon,\infty]} \times T$ is a trivial bundle with fiber $\mathrm{GL}(6,\mathbb{R})/\mathrm{GL}(3,\mathbb{C})$, so the problem reduces to extending a nonequivariant map from an annulus to this fiber over an entire open disc. But this is always possible as $\pi_1\big(\mathrm{GL}^+(6,\mathbb{R})/\mathrm{GL}(3,\mathbb{C})\big) = 0$. $\square$

5. GKM graphs

In this section we answer Problem 2.10 in the affirmative. We use the notation for graphs from Definition 4.1.

Definition 5.1 ([Dar15, Def. 2.6][Dar18]). A ***GKM graph*** is a triple (Γ, α, σ) comprising

- an unoriented n-regular graph Γ,
- an function $\sigma\colon \mathscr{V}(\Gamma) \longrightarrow \{\pm 1\}$ called an ***orientation***, and
- an ***axial function*** $\alpha\colon \mathscr{E}(\Gamma) \longrightarrow \operatorname{Hom}(T^k, S^1)$ for some $k \leqslant n$

such that

- for each $e\colon p \to q$ we have $\alpha(\bar{e}) = -\frac{\sigma(p)}{\sigma(q)}\alpha(e)$,[1]
- for each $p \in \mathscr{V}(\Gamma)$, the elements of $X_p := \{\alpha(e) : e \text{ begins at } p\}$ are pairwise linearly independent in $\operatorname{Hom}(T^k, S^1) \cong \mathbb{Z}^k$, and
- for each $e\colon p \to q$ in $\mathscr{E}(\Gamma)$, the images in $\operatorname{Hom}(T^k, S^1)/\langle \alpha(e)\rangle$ of X_p and X_q are equal.

An ***oriented torus graph*** is a GKM graph such that $k = n$ and each X_p is a basis of $\operatorname{Hom}(T, S^1)$.

GKM graphs fit into a family of combinatorial abstractions of actions along with abstract isotropy data.

Construction 5.2. Given a GKM manifold (respectively, locally standard torus action on a stably complex manifold) M, one extracts the corresponding GKM graph $\Gamma(M) := (\Gamma_M, \alpha_M, \sigma_M)$ (respectively, oriented torus graph) as follows. First, set $\mathscr{V}(\Gamma_M) = M^T$. For each codimension-one subtorus $H < T$ realized as an isotropy type, M^H is a disjoint union of 2-spheres S_e, each of which contains two elements of M^T and inherits a H-equivariant stable complex structure. We assign a pair of oriented edges $e\colon p \leftrightarrow q : \bar{e}$ for each such 2-sphere S_e. The isotropy representation $T \to T/H \rightharpoondown T_pS_e$ is an irreducible summand of the isotropy representation of T on T_pM, and can be viewed as an element $\alpha_M(e) \in \operatorname{Hom}(T, S^1)$, prescribing our axial function.[2] We set $\sigma_M(p) = 1$ if the orientation on T_pM defined by M agrees with the orientation on $\bigoplus_{e\colon p\to q} \mathbb{C}_{\alpha_M(e)}$ and $\sigma_M(p) = -1$ otherwise.

Construction 5.3. Given a GKM graph (Γ, α, σ), note that the local data $\Delta(\Gamma) := \big(X_p, \sigma(p)\big)_{p\in\mathscr{V}(\Gamma)}$ constitute GKM abstract isotropy data satisfying (2.5). The GKM condition on the abstract isotropy data corresponds to the X_p being sets, the clause $X_p \equiv X_q \ \big(\mathrm{mod}\,\langle\alpha(e)\rangle\big)$ is exactly the congruence condition, and that $\sigma(q)\alpha(\bar{e}) = -\sigma(p)\alpha(e)$ guarantees (2.5).

Notation 5.4. From now on we will identify a nonzero weight $\alpha \in J := \operatorname{Hom}(T, S^1)\backslash\{1\}$ with its associated one-dimensional representation $\mathbb{C}_\alpha$ and a multiset X of nonzero weights with the representation $\bigoplus_{\alpha\in X}\mathbb{C}_\alpha$. Write R_+T for the semigroup of representations with no trivial summands, under direct sum. We make the trivial observation that if we write the direct sum operation on representations **multiplicatively**, then we can write elements of R_+T as monomials on elements

[1] Note that this is more general than the original definition [GKM98, GZ01], which requres that $\alpha(\bar{e}) = -\alpha(e)$.

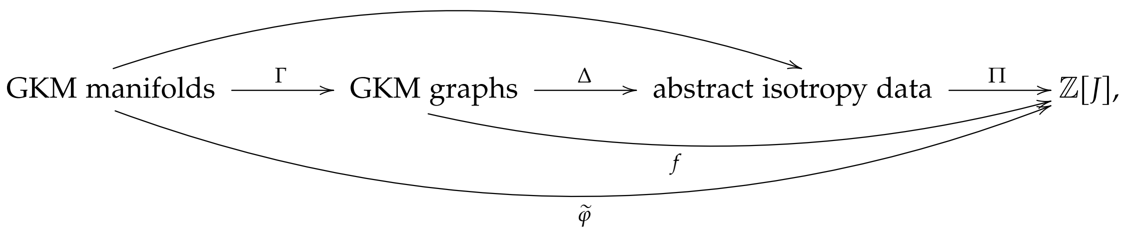
[2] There is a priori an arbitrary sign involved in the identifications $S^1 \cong T/H \cong \operatorname{Aut} T_pS_e$, but there is a natural choice derived from the inherited stable complex structure on S_e.

of J, identifying X_p and $\prod_{\alpha \in X_p} \alpha$. Identifying a list with a formal sum of its members, we forget the labelling and view abstract isotropy data $(X_p, \sigma_p)_{p \in P}$ as elements $\sum_{p \in P}(\prod_{\alpha \in X_p} \alpha, \sigma_p)$ of the free monoid $\mathbb{N} \cdot \big(R_+T \times \{\pm 1\}\big)$.

Construction 5.5. Recall that one construction of $\mathbb{Z}$ from the natural numbers $\mathbb{N}$ is a Grothendieck group construction: integers are equivalence classes of pairs (a, b) of naturals under the relation that $(a, b) \sim (c, d)$ when $a + d = c + b$. Equivalently, start with the free commutative monoid $\mathbb{N} \cdot \{\pm 1\}$ on the letters $\{1\}$ and $\{-1\}$. Then quotienting by the equivalence relation is the map $a\{1\} + b\{-1\} \longmapsto \alpha - b \colon \mathbb{N} \cdot \{\pm 1\} \longrightarrow \mathbb{Z}$. Two elements of $\mathbb{N} \cdot \{\pm 1\}$ have the same image under the map if and only if it is possible to obtain one side from the other by adding a nonnegative multiple of $\{1\} + \{-1\}$.

Repeating this construction in parallel for each element $V \in R_+T$ gives a map Π of additive monoids from the monoid $\mathbb{N} \cdot (R_+T \times \{\pm 1\})$ of abstract isotropy data to the free abelian group $\mathbb{Z} \cdot R_+T = \mathbb{Z}[J]$ on R_+J, given on basis elements by $(V, \sigma) \longmapsto \sigma V$.[3]

Now we have a chain of maps



(5.1)

whereas Problem 2.10 is to find a section of $\Delta \circ \Gamma$. It is known [HO72] that the kernel of $\widetilde{\varphi}$ is given by equivariant bordism, so that $\widetilde{\varphi}$ induces an injection from equivariant bordism classes of GKM manifolds to $\mathbb{Z}[J]$. Moreover, Alastair Darby showed $\operatorname{im} \widetilde{\varphi}$ is all of $\operatorname{im} f$.

Theorem 5.6 (Darby [Dar15, Cor. 2.27, 4.5][Dar18])**.** *For each GKM graph (Γ, α, σ) there exists a stably complex GKM manifold M such that $\widetilde{\varphi}(M) = f(\Gamma)$ in $\mathbb{Z}[J]$. If (Γ, α, σ) is an oriented torus graph, then M is locally standard.*

From the factorization (5.1) and Theorem 5.6, it is then enough to find a section of the process labeled (X, σ) taking a GKM graph to GKM abstract isotropy data satisfying (2.5) and then find GKM manifolds whose abstract isotropy data corresponds to $\ker f$. This is provided by the following lemma and example.

Lemma 5.7. *Let GKM abstract isotropy data $(X_p, \sigma_p)_{p \in P}$ satisfying (2.5) be given. Then there exists a GKM graph (Γ, α, σ) whose associated abstract isotropy data is $(X_p, \sigma_p)_{p \in P}$.*

Proof. The proof of Lemma 4.2 applies, changing only the number of elements of X_p and hence the valence of the resulting graph. □

[3] If we view abstract isotropy data as an element of the domain as in Notation 5.4, this map is given by

$$(X_p, \sigma_p)_{p \in P} \longmapsto \sum_{p \in P} \sigma_p \prod_{\alpha \in X_p} \alpha$$

Example 5.8. Consider the action of T^n on S^{2n} obtained by suspending the standard action on the unit sphere $S^{2n-1} \subsetneq \mathbb{C}^n$. If we write $t_j \colon T^n \longrightarrow S^1$ for the projections, we see the associated GKM graph has two vertices p, q joined by n edges $e_j \colon p \to q$ such that $\alpha(e_j) = \alpha(\bar{e}_j) = t_j$. We have $\sigma(p) = 1 = -\sigma(q)$, and so $f(\Gamma_M) = \prod t_j - \prod t_j = 0$.

More generally, given a GKM graph (Γ, α, σ) with exactly two vertices p, q, and n oriented edges, as each edge $e \colon p \to q$ must satisfy $\sigma(q)\alpha(\bar{e}) = -\sigma(p)\alpha(e)$ by the first condition, from the third condition we see we must have $\alpha(e) = \alpha(\bar{e})$ for all e and hence $\sigma(q) = -\sigma(p)$. We then get

$$f(\Gamma) = \sigma(p) \prod \alpha(e) + \sigma(q) \prod \alpha(e) = 0.$$

This graph is realized as the GKM graph of the unit sphere S^{2n} in the T^k-representation $\mathbb{R} \oplus \bigoplus_e \mathbb{C}_{\alpha(e)}$, where T^k acts trivially on the $\mathbb{R}$ factor.

Taken together, these facts resolve Problem 2.10.

Theorem A. *Given GKM abstract isotropy data* $(X_p, \sigma_p)_{p \in P}$ *satisfying* (2.5)*, there exists a compact, oriented, stably complex GKM* T*-manifold* M *whose abstract isotropy data is* $(X_p, \sigma_p)_{p \in P}$*.*

Proof. By Lemma 5.7, there is a GKM graph Γ' such that $\Delta(\Gamma') = (X_p, \sigma_p)_{p \in P}$. By Theorem 5.6, there is a GKM manifold M' such that $\widetilde{\varphi}(M') = f(\Gamma')$, or rearranging, $(\Delta \circ \Gamma)(M')$ and $(X_p, \sigma_p)_{p \in P}$ have the same Π-image. Per Construction 5.5, this means the two agree up to addition of a nonnegative multiple of $(V, 1) + (V, -1)$ to one side or the other for every $V \in R_+T$.

For each V such that the coefficient of V in $\Pi(X_p, \sigma_p)_{p \in P}$ is greater, construct a disjoint union M_V of $2n$-spheres as in Example 5.8, and replace M' with $M' \amalg \coprod_V M_V$. For each V such that the coefficient of V in $\widetilde{\varphi}(M')$ is greater, we do away with pairs of fixed points p, q with respective isotropy data $(V, 1)$ and $(V, -1)$ via equivariant surgery. To see one can do this smoothly, put an equivariant Hermitian metric on M' so that the exponential equivariantly identifies neighborhoods U_p and U_q of the fixed points in M' with ε-balls in their tangent spaces T_pM and T_qM. Puncture the ε-balls at their origins, identifying them each with the T-space $S^{2n-1}(V) \times (0, \varepsilon)$, where the first factor denotes the unit sphere in the unitary representation V and T acts trivially on the radial coordinate, and glue the two punctured balls via the orientation-reversing equivariant diffeomorphism $(v, r) \sim (v, \varepsilon - r)$. Doing this an appropriate number of times for each discrepant V finally yields a manifold M whose isotropy data is $(X_p, \sigma_p)_{p \in P}$. □

A. The realization challenge in general

In this appendix we present the localization identities in full generality.

A.1. Strata

Given a stably complex T-manifold M, its ***orbit-type strata*** are defined to be the connected components of the sets of points with the same stabilizer $H \leqslant T$. Given a complex representation $\rho \colon H \longrightarrow \operatorname{Aut} V_\rho$ of a closed subgroup H of the torus T with trivial invariant subspace $(V_\rho)^H$, we define the corresponding ***isotropy stratum*** M_ρ to be the set of points p in M whose stabilizer is equal to H and such that the isotropy representation of H on $T_pM/(T_pM)^H$ is isomorphic to ρ. This ρ is also called the ***isotropy label*** of the isotropy stratum. If a point p is in the closure of M_ρ, then the stabilizer K of p contains H. If p has isotropy label $\lambda \colon K \longrightarrow \operatorname{Aut} V_\lambda$, then under $\lambda|_H$, the space V_λ decomposes into V_ρ and $(\dim K - \dim H)$ trivial summands.

Given a fixed point $p \in M^T$, a closed subgroup H of T, and a representation $\rho\colon H \longrightarrow \operatorname{Aut} V_\rho$ with trivial invariant subspace $(V_\rho)^H$ we write $p \in P_\rho$ if there is a T-invariant subspace $W \in T_pM$ such that $H = \ker(T \to \operatorname{Aut} W)$ and $T_pM/W \cong V_\rho$ as a T-representation. It follows that we have a decomposition of T-representations

$$T_pM \cong \bigoplus \mathbb{C}_{\alpha_h} \oplus \bigoplus \mathbb{C}_{\beta_i},$$

where $\alpha_h\colon T \to T/H \to S^1$ descend to span $\operatorname{Hom}(T/H, S^1)$ and $\beta_i \in \operatorname{Hom}(T, S^1)$. This decomposition descends to a decomposition

$$V_\rho \cong \bigoplus \mathbb{C}_{\beta_i}|_H.$$

It is possible the restrictions $\beta_i|_H$ will be equal for multiple i (and the β_i may already not be distinct). Write $\{\bar\beta_1, \ldots, \bar\beta_s\}$ for the set of such restrictions and for each $j \in \{1, \ldots, s\}$ let I_j be the subset of $\{1, \ldots, \dim V_\rho\}$ such that $\beta_i|_H = \bar\beta_j$. If $N \subseteq M_\rho$ contains $p \in M^T$ in its closure, then the decomposition of V_ρ as an H-representation determines an H-equivariant decomposition

$$\nu_M \overline{N} \cong \bigoplus_{j=1}^{s} \nu_{\bar\beta_j}$$

of the normal bundle as a sum of isotypic subbundles $\nu_{\bar\beta_j}$ such that for each $q \in N$,

$$(\nu_{\bar\beta_j})_q \cong \bigoplus_{i \in I_j} \mathbb{C}_{\bar\beta_j}.$$

A.2. Localization identities

This particularly applies to characteristic classes $c(E) \in H^*(M)$ of a T-equivariant bundle $E \to M$, for then there is always an equivariant extension $c^T(E) \in H^*_T(M)$, given as the value of c on the non-equivariant bundle $ET \otimes_T E \longrightarrow ET \otimes_T M$; pullback along the inclusion $E \to ET \otimes_T E$ takes $c^T(E)$ to $c(E)$. The same then obviously follows for monomials in the characteristic classes of bundles. We will particularly be interested in monomials in the Chern classes of direct factors of TN and $\nu_M N$ for N components of an isotypic submanifold M^ρ as in Definition 2.5 in the case M^T is discrete.

The equivariant Euler class is easy to describe completely. Recall that the composition

$$\alpha \longmapsto \mathbb{C}_\alpha \longmapsto (ET \underset{T}{\otimes} \mathbb{C}_\alpha \to ET \underset{T}{\otimes} *) \longmapsto c_1(ET \underset{T}{\otimes} \mathbb{C}_\alpha)$$

is an additive group isomorphism $\operatorname{Hom}(T, S^1) \xrightarrow{\sim} H^2(BT)$ which we notate as if it were the identity. An arbitrary oriented T-representation V is isomorphic as an oriented representation to a direct sum $\bigoplus \mathbb{C}_{\alpha_j}$, where the $\alpha_j \in \operatorname{Hom}(T, S^1)$ are each determined up to a sign and the expression is unique up to reordering and expressing two weights by their opposites. Particularly, the product $\prod \alpha_j \in H^*(BT)$ is well-defined. We thus have

$$e^T(V) = e(ET \underset{T}{\otimes} V) = \sigma \cdot e\big(\bigoplus (ET \underset{T}{\otimes} \mathbb{C}_{\alpha_j})\big) = \sigma \prod e(ET \underset{T}{\otimes} \mathbb{C}_{\alpha_j}) = \sigma \prod c_1(ET \underset{T}{\otimes} \mathbb{C}_{\alpha_j}) = \sigma \prod \alpha_j,$$

where $\sigma = 1$ or -1 depending whether the given orientations of V and $\bigoplus \mathbb{C}_{\alpha_j}$ agree or not. For the Chern classes, we will start with the total Chern class, getting

$$c^T(V) = c(ET \underset{T}{\otimes} V) = c\big(\bigoplus (ET \underset{T}{\otimes} \mathbb{C}_{\alpha_j})\big) = \prod c(ET \underset{T}{\otimes} \mathbb{C}_{\alpha_j}) = \prod \big(1 + c_1(ET \underset{T}{\otimes} \mathbb{C}_\alpha)\big) = \sum_{\ell \in \mathbb{N}} \sigma_\ell(\alpha_1, \ldots, \alpha_n).$$

and reading off $c_\ell^T(V) = \sigma_\ell(\alpha_1, \ldots, \alpha_n)$.

For each p in the component $N \subseteq M_\rho$, assume we have $\alpha_h(p)$, isotypic components $\nu_{\bar\beta_j(p)}$, and indexing as specified as in Appendix A.1. Then

$$0 = \int_N \prod_\ell c_\ell^T(TN)^{h_\ell} \prod_{j=1}^s \prod_\ell c_\ell^T(\nu_{\bar\beta_j})^{m_{j,\ell}} = \sum_{p \in N \cap M^T} \frac{\prod_\ell c_\ell^T(TN)_p^{h_\ell} \prod_{j=1}^s \prod_\ell c_\ell^T(\nu_{\bar\beta_j})_p^{m_{j,\ell}}}{e^T(\nu_N\{p\})}$$
$$= \sum_{p \in N \cap M^T} \frac{\prod_\ell \sigma_\ell(\alpha_h(p))^{h_\ell} \prod_{j=1}^s \prod_\ell \sigma_\ell(\beta_i(p) : i \in I_j)^{m_{j,\ell}}}{\sigma_p \prod \alpha_i(p)}$$

so long as the polynomial on the left-hand side has total degree less than $\dim N$.

A.3. Abstract isotropy data in general

Given abstract isotropy data $(X_p, \sigma_p)_{p\in P}$ and a closed subgroup H of T, and representation $\rho\colon H \longrightarrow \operatorname{Aut} V_\rho$ with trivial invariant subspace $(V_\rho)^H$ we write $p \in P_\rho$ if there exists a decomposition $X_p = \{\alpha_h\} \amalg \{\beta_i\}$ such that $\alpha_h\colon T \to T/H \to S^1$ descend to span $\operatorname{Hom}(T/H, S^1)$ and $\bigoplus \mathbb{C}_{\beta_i}|_H \cong V_\rho$. As $p \in P_\rho$ varies, write these as $\alpha_h(p)$ and $\beta_i(p)$. Further, write $\{\bar\beta_1, \ldots, \bar\beta_s\}$ for the set of restrictions of the $\beta_i(p)$ to H and for each $j \in \{1, \ldots, s\}$ let I_j be the subset of $\{1, \ldots, \dim V_\rho\}$ such that $\beta_i(p)|_H = \bar\beta_j(p)$.

Then the general ABBV conditions are the following:

$$\text{For all } \rho, \qquad \sum_{p \in P_\rho} \frac{\prod_\ell \sigma_\ell(\alpha_h(p))^{h_\ell} \prod_{j=1}^s \prod_\ell \sigma_\ell(\beta_i(p) : i \in I_j)^{m_{j,\ell}}}{\sigma_p \prod \alpha_i(p)} = 0 \tag{A.1}$$

for every product of symmetric polynomials such that the total degree in the α_h and β_i of the numerator is less than $n - k$.

Finally, the precise statement of the realization conjecture is as follows:

Problem A.1. Given abstract isotropy data satisfying (A.1), does there exist a compact, oriented, stably complex T-manifold M whose isotropy data is $(X_p, \sigma_p)_{p\in P}$?

References

[AB84] Michael F. Atiyah and Raoul Bott. The moment map and equivariant cohomology. *Topology*, 23(1):1–28, 1984. doi:10.1016/0040-9383(84)90021-1.

[BV82] Nicole Berline and Michèle Vergne. Classes caractéristiques équivariantes. Formule de localisation en cohomologie équivariante. *C. R. Acad. Sci. Paris*, 295(2):539–541, Nov 1982. URL: http://gallica.bnf.fr/ark:/12148/bpt6k62356694/f77.

[Dar15] Alastair Darby. Torus manifolds in equivariant complex bordism. *Topology Appl.*, 189:31–64, 2015. arXiv:1409.2720, doi:10.1016/j.topol.2015.03.014.

[Dar18] Alastair Darby. Equivariant complex bordism of GKM-manifolds, July 2018. Presented at the Glances@Manifolds conference, Jagiellonian University, Kraków.

[GKM98] Mark Goresky, Robert Kottwitz, and Robert MacPherson. Equivariant cohomology, Koszul duality, and the localization theorem. *Invent. Math.*, 131(1):25–83, 1998. URL: http://math.ias.edu/~goresky/pdf/equivariant.jour.pdf, doi:10.1007/s002220050197.

[GZ01] Victor Guillemin and Catalin Zara. 1-skeleta, Betti numbers, and equivariant cohomology. *Duke Math. J.*, 107(2):283–349, 2001. arXiv:9808135, doi:10.1215/S0012-7094-01-10724-2.

[HO72] Gary Hamrick and Erich Ossa. Unitary bordism of monogenic groups and isometries. In *Proceedings of the Second Conference on Compact Transformation Groups*, pages 172–182. Springer, 1972. doi:10.1007/BFb0070041.

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