Reflections on exactness

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Abstract

The notion of exactness in an abelian category generalizes to notions of exactness and coexactness, due to Moore, in any pointed category. For about sixty years, various authors computing cohomology of various things have insisted on coexactness. This brief note illustrates these definitions in a few well-known algebraic categories and elaborates conditions under which exactness and coexactness are equivalent notions.

1. Introduction

The notion of exactness from the category of abelian groups generalizes well to abelian categories, but already fails to make sense in other categories of algebraic objects. In the category of unital rings, for instance, the image of a ring map is a (unital) subring of the target, but the kernel is an ideal, which fails to be a subring unless the map is identically zero.

One can repair this failure in categories with a zero object by defining, but one finds exactness splits into two dual notions, exactness and coexactness, originally discussed in the context of commutative or cocommutative connected Hopf algebras [MS68, p. 762].

These definitions rely on new notions of kernel and cokernel, image and coimage, which may be somewhat counterintuitive. The image of a morphism $f\colon A \longrightarrow B$ is defined as the kernel of its cokernel, usually not usually agree with the set-theoretic image $f(A) = \{f(a) : a \in A\}$, which is not usually the kernel of anything. For example, in the category of groups, the categorical image is the normal closure $\langle bf(a)b^{-1} : a \in A,\, b \in B\rangle$ of the set-theoretic image, whereas in the category of augmented unital k-algebras, it is the sum $k + \big(f(A)\big)$ of the image of the unit and the two-sided ideal generated by the set-theoretic image. One sees after a moment of reflection that the only reason the classical version of exactness works in categories of modules is that there all monomorphisms and epimorphisms are regular.

It was coexactness that was relevant to the author's thesis, and he remembers trying to justify to his advisor the presence of the irritating *co*. This note may be considered a belated answer. In correspondence about these concepts, Larry Smith suggested another reasonable pair of definitions of exactness and coexactness which we will call *weak*. Weak exactness and weak coexactness turn out to be equivalent to one another, but not generally to exactness or coexactness.

Acknowledgments. This note began as an email to Larry Smith, who has always been a gracious and entertaining correspondent and is responsible for two of the definitions. The other inspiration is his advisor, Loring Tu.

2. Categorical generalities

Let $(\mathscr{C}, *)$ be a ***pointed category***, meaning $*$ is both an initial and a final object in $\mathscr{C}$. The object $*$ is called the ***zero object***. For any two objects $X, Y \in \mathscr{C}$, we denote again by $*$ the unique ***trivial morphism*** $X \to * \to Y$. Given an arrow $f\colon X \to Y$, recall that, if they exist,

- the ***kernel*** $\ker f$ is the equalizer $\operatorname{Ker} f \longrightarrow X$ of f and $*\colon X \longrightarrow Y$; or equivalently the final morphism $k\colon K \longrightarrow X$ such that $f \circ k = *$;
- the ***cokernel*** $\operatorname{coker} f$ is the coequalizer $Y \longrightarrow \operatorname{Coker} f$ of f and $*\colon X \longrightarrow Y$, or equivalently the initial morphism $c\colon Y \longrightarrow C$ such that $c \circ f = *$;
- the ***image*** $\operatorname{im} f$ is $\ker(\operatorname{coker} f)\colon \operatorname{Im} f \longrightarrow Y$;
- the ***coimage*** $\operatorname{coim} f$ is $\operatorname{coker}(\ker f)\colon X \longrightarrow \operatorname{Coim} f$.

Because $\operatorname{coker}(f) \circ f = *$, one has always an ***image factorization***

$$X \xrightarrow{f_{\mathrm{im}}} \operatorname{Im} f \overset{\operatorname{im} f}{\rightarrowtail} Y$$

of f and dually, since $f \circ \ker(f) = *$, a ***coimage factorization***

$$X \overset{\operatorname{coim} f}{\twoheadrightarrow} \operatorname{Coim} f \xrightarrow{f_{\mathrm{coim}}} Y.$$

If f_{im} is epic, we say f ***induces an epimorphism to its image***, and that if f_{coim} is monic, that it ***induces a monomorphism from its coimage***.

There exists the following theorem about these factorizations.

Proposition 2.1 ([Mac78, Lemma VIII.1, p. 189])**.** *Let $\mathscr{C}$ be a pointed category with all equalizers and all monomorphisms m* normal *in the equivalent senses that they are the kernel of some morphism. Then every arrow of $\mathscr{C}$ induces an epimorphism to its image.*

Dually, if $\mathscr{C}$ has all coequalizers and all epimorphisms are normal *in the sense of being cokernels, then every arrow of $\mathscr{C}$ induces a monomorphism from its coimage.*

Of course in many categories of interest, like that of augmented unital k-algebras, neither of these hypotheses is true. But the category $k\text{-}\mathsf{Hopf}_0^*$ of *commutative* connected Hopf algebras over a field does have the former property and the category of *co*commutative connected Hopf algebras has the latter [MS68, p. 756].

These factorizations appear together the below diagram along with two further morphisms.

$$\begin{tikzcd}
\operatorname{Ker} f \arrow[dr, tail, "\ker f"'] & & \operatorname{Im} f \arrow[dr, tail, "\operatorname{im} f"] & & \\
& X \arrow[ur, "f_{\mathrm{im}}"] \arrow[dr, two heads, "\operatorname{coim} f"'] & & Y \arrow[dr, two heads, "\operatorname{coker} f"] & \\
& & \operatorname{Coim} f \arrow[uu, dotted, shift left, "\ell"] \arrow[uu, dotted, shift right, "r"'] \arrow[ur, "f_{\mathrm{coim}}"'] & & \operatorname{Coker} f
\end{tikzcd} \tag{2}$$

To explain the new arrow r, note first that $* = \operatorname{coker}(f) \circ f = \operatorname{coker}(f) \circ f_{\mathrm{coim}} \circ \operatorname{coim}(f)$. Since $\operatorname{coim}(f)$ is epic, one also has $* = \operatorname{coker}(f) \circ f_{\mathrm{coim}}$, meaning f_{coim} factors as $\operatorname{im} f \circ r$ for a unique

$r\colon \operatorname{Coim}(f) \longrightarrow \operatorname{Im}(f)$. Dually, to explain q, note that $* = f \circ \ker(f) = \operatorname{im}(f) \circ f_{\mathrm{im}} \circ \ker(f)$, but since $\operatorname{im}(f)$ is monic, we also have $* = f_{\mathrm{im}} \circ \ker(f)$, implying f_{im} factors as $\ell \circ \operatorname{coim}(f)$ for a unique $\ell\colon \operatorname{Coim}(f) \longrightarrow \operatorname{Im}(f)$.

We claim that ℓ and r are the same map $\underline{f}\colon \operatorname{Coim}(f) \longrightarrow \operatorname{Im}(f)$. Indeed, one has

$$\operatorname{im}(f) \circ r \circ \operatorname{coim}(f) = f_{\mathrm{coim}} \circ \operatorname{coim}(f) = f = \operatorname{im}(f) \circ f_{\mathrm{im}} = \operatorname{im}(f) \circ \ell \circ \operatorname{coim}(f),$$

but $\operatorname{im}(f)$ is mono while $\operatorname{coim}(f)$ is epi.

Lemma 2.3. *In the factorization diagram (2), the map $\underline{f}$ is epic if and only if f induces an epimorphism to its image, and is monic if and only if f induces a monomomorphism from its coimage.*

Now consider a sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z \tag{4}$$

in $\mathscr{C}$ and assume the kernel $\ker g\colon \operatorname{Ker}(g) \to Y$ exists. If the composite $g \circ f$ is $*$, then by the definition of kernel, f factors through $\ker g$, say as

$$X \xrightarrow{\bar{f}} \operatorname{Ker}(g) \xrightarrow{\ker g} Y. \tag{5}$$

Dually, if we assume the cokernel $\operatorname{coker} f\colon X \to \operatorname{Coker}(f)$ exists, then by the definition of cokernel, g factors through $\operatorname{coker} f$, say as

$$X \xrightarrow{\operatorname{coker} f} \operatorname{Coker}(f) \xrightarrow{\bar{g}} Y. \tag{6}$$

Definition 2.7 ([MS68, p. 762])**.** We say the sequence (4) is ***exact*** at Y if the factor map $\bar{f}$ of (5) is an epimorphism, and one says the sequence is ***coexact*** at Y if the factor map $\bar{g}$ of (6) is a monomorphism.

Again assuming $g \circ f = *$ and the kernels and cokernels exist, note that then

$$g \circ \operatorname{im}(f) = \bar{g} \circ \operatorname{coker}(f) \circ \operatorname{im}(f) = *,$$

and consequently $\operatorname{im}(f)\colon \operatorname{Im}(f) \longrightarrow Y$ factors through $\ker g$, yielding the arrow $e\colon \operatorname{Im}(f) \longrightarrow \operatorname{Ker}(g)$ in the diagram

$$\begin{tikzcd}
 & \operatorname{Im}(f) \arrow[dr, tail, "\operatorname{im} f"] \arrow[dd, tail, "e"] & \\
X \arrow[ur, "f_{\mathrm{im}}"] \arrow[dr, "\bar{f}"'] & & Y. \\
 & \operatorname{Ker}(g) \arrow[ur, tail, "\ker g"'] &
\end{tikzcd} \tag{8}$$

As $\operatorname{im} f$ and $\ker g$ are monomorphisms, so also is e. The right triangle of (8) commutes by definition, and both compositions $X \longrightarrow Y$ are f. The left triangle commutes as well because

$$\ker(g) \circ \bar{f} = f = \operatorname{im}(f) \circ f_{\mathrm{im}} = \ker(g) \circ e \circ f_{\mathrm{im}}$$

and $\ker g$ is monic. Dually, since $\operatorname{coim}(g) \circ f = *$, we find $\operatorname{coim} g$ factors through coker) via some epimorphism c. The two fit together in the following diagram, in which χ is defined to be the composition $\operatorname{coker}(f) \circ \ker(g)$.

$$
\begin{tikzcd}
& \operatorname{Im} f \arrow[dr, "\operatorname{im} f"] \arrow[dd, "e"] & & \operatorname{Coim} g \arrow[dr, "g_{\mathrm{coim}}"] & \\
X \arrow[ur, "f_{\mathrm{im}}"] \arrow[dr, "\bar{f}"'] & & Y \arrow[ur, "\operatorname{coim} g"] \arrow[dr, "\operatorname{coker} f"] & & Z \\
& \operatorname{Ker} g \arrow[ur, "\ker g"] \arrow[rr, "\chi"'] & & \operatorname{Coker} f \arrow[uu, "c"] \arrow[ur, "\bar{g}"'] &
\end{tikzcd}
\tag{9}
$$

Definition 2.10 ([Smi15]). We say the sequence (4) is ***weakly exact*** if e is an isomorphism and ***weakly coexact*** if c is an isomorphism.

Curiously, these definitions are not just dual, they are identical.

Proposition 2.11. *A sequence in a pointed category is weakly exact if and only if it is weakly coexact.*

First proof. Fix an object Y in a pointed category $\mathscr{C}$ and recall [Mac78, p. 189] that

$$\operatorname{coker} \dashv \ker$$

forms a Galois connection (an adjunction of preorders) between the slice category $\mathscr{C}/Y$ whose objects are arrows to Y in $\mathscr{C}$ and the slice category $Y/\mathscr{C}$ whose objects are arrows from Y. If e in (9) is an isomorphism, so that $\operatorname{im} f \cong \ker g$ in $\mathscr{C}/Y$, then the cokernels

$$\operatorname{coker}(f) = \operatorname{coker}\big(\ker(\operatorname{coker} f)\big) = \operatorname{coker}(\operatorname{im} f) \cong \operatorname{coker}(\ker g) = \operatorname{coim}(g)$$

will be isomorphic in $Y/\mathscr{C}$, and the isomorphism will be precisely c. Dually, if c is an isomorphism, then writing $\operatorname{coker} f \cong \operatorname{coim} g$ we see the kernels

$$\operatorname{im}(f) = \ker(\operatorname{coker} f) \cong \ker(\operatorname{coim} g) = \ker\big(\operatorname{coker}(\ker g)\big) = \ker(g)$$

will be isomorphic, the isomorphism given by e. □

Second proof. Weak exactness of (6) is equivalent to $\ker g$ factoring through $\operatorname{im} f$. Indeed, right-multiplying the equation $\operatorname{im} f = \ker(g) \circ e$ by an inverse e^{-1} shows $\ker g$ factors through $\operatorname{im} f$. Conversely, the existence of a factorization $\ker g = \operatorname{im}(f) \circ e'$ would imply e' is monic and that $\operatorname{im} f = \ker(g) \circ e = \operatorname{im}(f) \circ e' \circ e$. Left-cancelling $\operatorname{im} f$, this would mean $e' \circ e = \mathrm{id}$. Right-multiplying by e' to get $e' \circ e \circ e' = e'$ and then left-cancelling e' since it is monic, we see $e' = e^{-1}$.

On the other hand, $\ker g$ factors through $\operatorname{im} f = \ker(\operatorname{coker} f)$ if and only if

$$\chi = \operatorname{coker}(f) \circ \ker(g) = *$$

if and only $\operatorname{coker}(f)$ factors through $\operatorname{coim}(g) = \operatorname{coker}(\ker g)$. But by the dual of the argument in the preceding paragraph, $\operatorname{coker}(f)$ factors through $\operatorname{coim}(g)$ precisely if c admits an inverse. □

As implied by the names, the weak notions follow from the others.

Proposition 2.12. *If a sequence in a pointed category is exact, then it is weakly exact, and if it is coexact, then it is weakly coexact.*

Proof. We first note that e is the kernel of χ in (9). Indeed, $\operatorname{coker}(f) \circ \ker(g) \circ e = \operatorname{coker}(f) \circ \operatorname{im}(f) = *$, and if we are given another morphism h with $\operatorname{coker}(f) \circ \ker(g) \circ h = *$, then $\ker(g) \circ h$ must factor through $\operatorname{im} f$, say as

$$\ker(g) \circ h = \operatorname{im}(f) \circ j = \ker(g) \circ e \circ j,$$

and cancelling $\ker g$, we find $h = e \circ j$. Exactness means that $\bar{f}$ is an epimorphism, so e must be as well. But an epic equalizer is an isomorphism,[1] so the sequence is weakly exact.

The proof for coexactness is dual. □

This proof in fact shows e will be an isomorphism precisely if it is epic. If f_{im} is epic as well, then $\bar{f} = e \circ f_{\mathrm{im}}$ will be too. A converse also holds.

Proposition 2.13. *Let $(\mathscr{C}, *)$ be a pointed category. Consider a sequence $\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot$ with trivial composition $* = g \circ f$. Coexactness of the sequence will imply exactness if and only f induces an epimorphism to its image, and exactness of the sequence will imply coexactness if and only if g induces a monomorphism from its coimage.*

Proof. If the sequence is coexact, it is weakly coexact by Proposition 2.12, hence weakly exact by Lemma 2.3. This means $e = \mathrm{id}$ in (9), so that $\bar{f} = f_{\mathrm{im}}$; but the sequence is by definition exact if $\bar{f}$ being epic, while f induces an epimorphism to its image if f_{im} is. The proof of the dual statement is dual. □

Example 2.14. In the category of modules over a unital commutative ring k, every monomorphism is a kernel and every epimorphism a cokernel, so by Proposition 2.1, every morphism induces both an epimorphism to its image and a monomorphism from its coimage. Then Proposition 2.13 says coexactness is equivalent to exactness for all sequences.

At this point it becomes necessary to disabuse oneself of the notion these conditions might always be equivalent.

3. Examples

In this section we elaborate on some examples distinguishing these notions. Let k be a commutative ring with unity. The category of augmented unital k-algebras with zero object k is equivalent to the category of nonunital k-algebras, with zero object 0; transitioning back and forth is simply removing or adding back on a copy of k with multiplication defined so that $1 \in k$ becomes the unity.

We let Alg denote any of the commonly considered categories of k-algebras and k-algebra isomorphisms. For example, it could be the category of connected commutative graded k-algebras, the category of ungraded augmented unital associative algebras, the category of Lie algebras, or the category of all nonunital algebras. The category Alg is pointed by k in the augmented unital case and by 0 in the nonunital case. We will consider the latter for ease of notation, which amounts to considering reduced cohomology in the original case of interest.

We will discuss kernels, cokernels, images, coimages, in Alg, which always exist, as well as epimorphisms and monomomorphisms.

[1] Indeed, suppose e is the equalizer of $v, w \colon A \rightrightarrows B$. Since $v \circ \mathrm{id}_A = w \circ \mathrm{id}_A$, it follows id_A factors through e, say as $\mathrm{id}_A = e \circ e'$. Then $e' = e' \circ e \circ e'$ and, since e' is epic, $e' = e^{-1}$.

Kernels: A composition $K \xrightarrow{\varkappa} A \xrightarrow{f} B$ is zero if and only if one has a containment $\varkappa(K) \subseteq f^{-1}(0)$ of sets, so it follows $\ker f$ is the inclusion of the ideal $f^{-1}(0)$, the traditional ring-theoretic kernel.

Monomorphisms: If $f\colon A \longrightarrow B$ is monic, then left-cancelling f from $f \circ \ker f = 0 = f \circ 0$, we see $f^{-1}(0) = 0$, so monomorphisms are injections. Conversely, if f is an injection, it is left-cancellable on the level of functions, hence a monomorphism.

Cokernels: A composition $A \xrightarrow{f} B \xrightarrow{c} C$ is zero if and only if $f(A) \subseteq c^{-1}(0)$. Since $c^{-1}(0)$ is an ideal, this is equivalent to the two-sided ideal $\big(f(A)\big)$ generated by $f(A)$ being contained in $c^{-1}(0)$. It follows that $\operatorname{coker} f$ is the quotient map from B to $B /\!\!/ A = B/\big(f(A)\big)$.

Epimorphisms: If $f\colon A \longrightarrow B$ is epic, then right-cancelling f from $\operatorname{coker}(f) \circ f = 0 = 0 \circ f$, we see $B = \big(f(A)\big)$. We will see in Example 3.1 that this necessary condition is not sufficient. Surjections are epimorphisms, as are localizations in the event the domain is unital, but general epimorphisms do not admit a simple description.

Coimages: Given $f\colon A \longrightarrow B$, its coimage is the cokernel of the inclusion $f^{-1}(0) \hookrightarrow A$, namely the quotient map $A \longrightarrow A/\big(f^{-1}(0)\big)$ onto the traditional ring-theoretic coimage. The second map $f_{\mathrm{coim}}\colon A/\big(f^{-1}(0)\big) \longrightarrow B$ in the coimage factorization is an injection by definition, hence a monomorphism.

Images: Given $f\colon A \longrightarrow B$, its image is the kernel of the quotient map $B \longrightarrow B/\big(f(A)\big)$, namely the inclusion $\big(f(A)\big) \hookrightarrow B$. The first map $f_{\mathrm{im}}\colon A \longrightarrow \big(f(A)\big)$ in the image factorization is not typically an epimorphism without further restrictions on the type of algebra in question.

Example 3.1. Let A be a k-algebra and f is the inclusion of A in $B = A \otimes_k k[x,y]$. Then $B = (A)$, so $f_{\mathrm{im}} = f$, which is not an epimorphism because the identity and the A-algebra map sending x to y and y to x are distinct but both restrict to the identity on A.

Proposition 3.2. *The notions of coexactness, weak coexactness, and weak exactness of a sequence*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in $(\mathsf{Alg}, 0)$ *each are equivalent to* $\big(f(A)\big) = g^{-1}(0)$, *but exactness holds if and only if additionally* $A \longrightarrow \big(f(A)\big)$ *is an epimorphism.*

Proof. Since f_{coim} is always a monomomorphism but f_{im} is not always an epimorphism, this follows from Proposition 2.11, Proposition 2.12, and Proposition 2.13. ☐

To see this explicitly, consider a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ in Alg, so that the diagram (9) becomes

$$\begin{tikzcd}
 & \big(f(A)\big) \arrow[dr, hook] \arrow[dd, tail, "e"] & & B/g^{-1}(0) \arrow[dr, "g_{\mathrm{coim}}"] & \\
A \arrow[ur, "f_{\mathrm{im}}"] \arrow[dr, "\bar f"'] & & B \arrow[ur, two heads] \arrow[dr, two heads] & & C. \\
 & g^{-1}(0) \arrow[ur, hook] \arrow[rr] & & B/\big(f(A)\big) \arrow[uu, two heads, "c"] \arrow[ur, "\bar g"'] &
\end{tikzcd} \tag{3}$$

Coexactness is injectivity of $\bar{g}\colon B/\big(f(A)\big) \longrightarrow C$, while weak coexactness is $c : B/\big(f(A)\big) \longrightarrow B/g^{-1}(0)$ being an isomorphism, which is equivalent since $g_{\mathrm{coim}}\colon B/g^{-1}(0) \longrightarrow C$ is an injection by definition. But this is the same as $e\colon g^{-1}(0) \longrightarrow \big(f(A)\big)$ being an isomorphism. On the other hand, assuming this, exactness is the same as $A \longrightarrow \big(f(A)\big)$ being epic, which is not generally the case.

The difference becomes even more stark when we specialize and consider instead a category $(\mathsf{Alg}_0, 0)$ of positively-graded k-algebras. We will have to reconsider epimorphisms and images.

Epimorphisms: If $f\colon A \longrightarrow B$ is epic, then we have seen $B = \big(f(A)\big)$. With the positive grading, the converse also holds, because in fact f must be surjective. In case B is commutative, this is an application of the graded Nakayama lemma [NS02, Prop. A.1.1] but the argument holds more generally: We have $B_1 = B_1 \cap \big(f(A)\big) = f(A_1)$ for degree reasons. Suppose inductively that $B_{<n} = \bigoplus_{j<n} B_j$ is contained in $f(A)$. Since $B = \big(f(A)\big)$, we know B_n is spanned by homogeneous degree-n elements of

$$f(A_n) + B_{<n}f(A) + f(A)B_{<n} + B_{<n}f(A)B_{<n} + f(A)B_{<n}f(A)B_{<n} + B_{<n}f(A)B_{<n}f(A) + \cdots,$$

but we have assumed $B_{<n}$ lies in $f(A)$, concluding the induction.

Images: Given $f\colon A \longrightarrow B$, the first map $f_{\mathrm{im}}\colon A \longrightarrow \big(f(A)\big)$ in the image factorization is not typically an epimorphism; as we have seen, it must then be a surjection, which is the case if and only if $\big(f(A)\big)$ is an ideal of B.

Examining Equation (3) again with the understanding that epimorphisms are surjections, we have shown the following.

Proposition 3.4. *The notions of coexactness, weak coexactness, and weak exactness of a sequence*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in $(\mathsf{Alg}_0, 0)$ *(or equivalently, in connected, nonnegatively-graded) k-algebras, with zero object k) each are equivalent to $\big(f(A)\big) = g^{-1}(0)$, but exactness holds if and only if additionally $f(A) = \big(f(A)\big)$ in B.*

We will be more brief about the category $(\mathsf{Grp}, 1)$ of groups and group homomorphisms, with zero object the trivial group, recall that the monomorphisms in Grp are the injections and, less trivially, the epimorphisms are the surjections ([Mac78, Exer. 5, p. 21][2]). The kernel of a group homomorphism $\phi\colon G \longrightarrow H$ is the inclusion $\operatorname{Ker}\phi \hookrightarrow G$ of the set-theoretic kernel, and the coimage is the surjection $G \longrightarrow G/\operatorname{Ker}(\phi)$.

The second map ϕ_{coim} in the coimage factorization $G \to G/\operatorname{Ker}(\phi) \to H$ is the composite of the isomorphism onto the set-theoretic image given by the first isomorphism theorem and its inclusion in H, hence an injection, hence a monomorphism. Thus by Proposition 2.13, coexactness and weak coexactness are equivalent in $(\mathsf{Grp}, 1)$.

As for coexactness, the cokernel of $\phi\colon G \to H$ is the initial surjection out of H annihilating $\phi(G)$, namely the one quotienting from H *normal closure* $\langle\!\langle \phi(G) \rangle\!\rangle$ generated by $h\phi(g)h^{-1}$ for $h \in H$ and $g \in G$. Then $\operatorname{im}\phi = \ker(\operatorname{coker}\phi)$ is the inclusion $\langle\!\langle \phi(G) \rangle\!\rangle \hookrightarrow H$. The first map in the image factorization, $G \twoheadrightarrow \phi(G) \hookrightarrow \langle\!\langle \phi(G) \rangle\!\rangle = \operatorname{Im}\phi$, is epimorphic if and only if it is surjective, which

[2] or http://ncatlab.org/toddtrimble/published/epimorphisms+in+the+category+of+groups

happens if and only if the set-theoretic image of ϕ is normal in H. This is not generically the case; consider for example the case that ϕ is itself the inclusion of a non-normal subgroup. By Proposition 2.13 again, then, weak exactness does not imply exactness in $(\mathsf{Grp}, 1)$.

Proposition 3.5. *The notions of coexactness, weak coexactness, and weak exactness of a sequence*

$$G \xrightarrow{\phi} H \xrightarrow{\psi} K$$

in $(\mathsf{Grp}, 1)$ *each are equivalent to* $\langle\!\langle \phi(G) \rangle\!\rangle = \psi^{-1}(0)$*, but exactness holds if and only if additionally* $\phi(G)$ *is already normal in* H*.*

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