Notes

John McCarthy

June 26, 2019

Abstract

Contents

Part I

Stale Memes

Chapter 1

Bundles

1.1 Fibre Bundles

1.1.1 Definitions

Definition 1.1.1 (Fibre Bundle)**.** *Let M, F be smooth manifolds. A* fibre bundle over M with fibre F *is a smooth manifold E and a smooth surjection $\pi : E \to M$ such that:*

1. *For every $x \in M$, The* fibre *$\pi^{-1}(x)$ of x, denoted E_x, is diffeomorphic to F.*
2. *There exists an open cover $\{U_\alpha\}$ of M and diffeomorphisms $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ such that for every α the following diagram commutes,*

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[rr, "\varphi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times F \arrow[dl, "\mathrm{pr}_1"] \\ & U_\alpha & \end{tikzcd}$$

where pr_1 is projection onto the first factor. The bundle $\pi^{-1}(U_\alpha)$ with its fibre bundle structure induced from that of E is written $E|_{U_\alpha}$.

To be perfectly precise, one should denote a fibre bundle by a vector (E, M, π, F). Since the letters used to denote fibre bundles are largely fixed, they will often be denoted by $\pi : E \to M$, $E \to M$, or simply E.

A collection $\{(U_\alpha, \varphi_\alpha)\}$ satisfying the second condition of Definition 1.1.1 is called a *fibre bundle atlas* for E. A single pair $(U_\alpha, \varphi_\alpha)$ is called a *local trivialisation* for E over the set $U_\alpha \subseteq M$.

One should think of a local trivialisation of a fibre bundle E as some kind of local chart for the fibre bundle considered as a smooth manifold. Indeed if one composes the local trivialisations φ_α with the charts on the U_α coming from M, and charts on F, a genuine atlas for the manifold E is obtained. Because the maps φ_α are required to be diffeomorphisms from the beginning, this smooth structure is of course the assumed smooth structure on E.

Definition 1.1.2 (Trivial Fibre Bundle)**.** *A fibre bundle $\pi : E \to M$ is* trivial *if there exists a fibre bundle atlas consisting of a single local trivialisation.*

Using the terminology of Definition 1.1.2, a fibre bundle is *locally trivial.* That is, there exists an open cover $\{U_\alpha\}$ such that the fibre bundles $E|_{U_\alpha}$ are trivial.

Definition 1.1.3 (Section)**.** *Let $\pi : E \to M$ be a fibre bundle over M with fibre F. A* section *of E is a smooth map $s : M \to E$ such that $s(x) \in E_x$ for every $x \in M$. That is, $\pi \circ s = \mathbf{1}$. The set of all sections of a fibre bundle E is denoted $\Gamma(E)$.*

Example 1.1.4 (The Trivial Fibre Bundle with Fibre F)**.** Let M, F be smooth manifolds. Then $E := M \times F$ with $\pi = \mathrm{pr}_1$ is a fibre bundle over M with fibre F. This is the *trivial fibre bundle over M with fibre F*. One refers to this as *the trivial bundle* when the base manifold M and the fibre F are understood.

1.1.2 Bundle Maps

Definition 1.1.5 (Fibre Bundle Homomorphism)**.** *Let $\pi : E \to X$ and $\pi' : S \to Y$ be fibre bundles over manifolds X and Y. A smooth map $f : E \to S$ is a* fibre bundle homomorphism *if there exists a smooth map $\tilde{f} : X \to Y$ such that the following diagram commutes.*

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi"'] & S \arrow[d, "\pi'"] \\ X \arrow[r, "\tilde{f}"] & Y \end{tikzcd}$$

Note the map $\tilde{f}$ above is actually determined by f. Given a point $x \in X$, let $p \in \pi^{-1}(x)$ be any point in the preimage. Then by commutativity of the diagram we know $\pi' \circ f(p) = \tilde{f} \circ \pi(p)$. But $\pi(p) = x$, so we just have $\pi' \circ f(p) = \tilde{f}(x)$. Note the right side of this equality does not depend on p, so we get a well-defined function $\tilde{f} = \pi' \circ f \circ \pi^{-1}$. The function f is said to *cover* $\tilde{f}$.

Do not however get the impression that the requirement of the existence of $\tilde{f}$ is superfluous. One requires the commutativity of the above diagram for some $\tilde{f}$ to ensure that the map $f : E \to S$ is *fibre-preserving*, which is the real property that makes a smooth map $f : E \to S$ a fibre bundle homomorphism.

In most cases, the the base space of the fibre bundles E and S is the same. If the map $\tilde{f}$ is the identity $\mathbf{1} : M \to M$ then we get the following specialisation.

Definition 1.1.6 (Fibre Bundle Homomorphism over M)**.** *Let $\pi : E \to M$ and $\pi' : S \to M$ be two fibre bundles over a manifold M. Then a* fibre bundle homomorphism over M *is a fibre bundle homomorphism covering the identity $\mathbf{1} : M \to M$. In particular, it is a map $f : E \to S$ such that the following diagram commutes.*

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "\pi"'] & & S \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

Whenever one defines a morphism between two objects, there should be an associated notion of isomorphism. In this case we have the following.

Definition 1.1.7 (Fibre Bundle Isomorphism)**.** *Let $\pi : E \to M$ and $\pi' : S \to M$ be two fibre bundles over a manifold M. Then a* fibre bundle isomorphism *is a fibre bundle homomorphism $f : E \to S$ that is also a diffeomorphism.*

From the perspective of Definition 1.1.7, a trivial fibre bundle is one that is fibre bundle isomorphic to the trivial bundle. In particular, every fibre bundle is locally isomorphic to the trivial bundle.

1.1.3 Transition Functions

Let $\pi : E \to M$ be a fixed fibre bundle. Suppose we have a fibre bundle atlas $\{(U_\alpha, \varphi_\alpha)\}$ for E. On an overlap $U_{\alpha\beta} := U_\alpha \cap U_\beta$ there are two ways of trivialising the fibre bundle $E|_{U_{\alpha\beta}}$. One could either apply φ_α or φ_β. This gives rise to the following commutative diagram.

$$\begin{tikzcd} U_{\alpha\beta} \times F \arrow[dr, "\mathrm{pr}_1"'] & E|_{U_{\alpha\beta}} \arrow[l, "\varphi_\alpha"'] \arrow[r, "\varphi_\beta"] \arrow[d, "\pi"] & U_{\alpha\beta} \times F \arrow[dl, "\mathrm{pr}_1"] \\ & U_{\alpha\beta} & \end{tikzcd}$$

Since φ_α and φ_β are diffeomorphisms, we can consider the map $\varphi_\beta \circ \varphi_\alpha^{-1} : U_{\alpha\beta} \times F \to U_{\alpha\beta} \times F$. By commutativity of the above diagram, points in $\{x\} \times F$ get mapped to points in $\{x\} \times F$. That is, given any point $(x, f) \in U_{\alpha\beta} \times F$, we have

$$\varphi_\beta \circ \varphi_\alpha^{-1}(x, f) = (x, g_{\beta\alpha}(x)(f))$$

for some element of F, possibly depending on x, that we are calling $g_{\beta\alpha}(x)(f)$.

Since $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a diffeomorphism, the map sending $f \mapsto g_{\beta\alpha}(x)(f)$ is a diffeomorphism from F to itself. Call this map $g_{\beta\alpha}(x)$. Then we are saying that $g_{\beta\alpha}(x) \in \mathrm{Diff}(F)$ for every $x \in U_{\alpha\beta}$. Again since $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a diffeomorphism, these maps $g_{\beta\alpha}(x)$ vary smoothly with x, so we obtain a smooth map $g_{\beta\alpha} : U_{\alpha\beta} \to \mathrm{Diff}(F)$ that assigns to each $x \in U_{\alpha\beta}$ a diffeomorphism of F. This map $g_{\beta\alpha}$ describes how the trivialisations $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) differ on this intersection.

Definition 1.1.8. *Let E be a fibre bundle with fibre F and suppose $\{(U_\alpha, \varphi_\alpha)\}$ is a fibre bundle atlas for E. The collection of smooth maps $\{g_{\alpha\beta}\}$ constructed above, where α, β vary over the $U_{\alpha\beta}$ such that $U_{\alpha\beta} \neq \emptyset$, are called the* transition functions *of E with respect to the open cover $\{U_\alpha\}$.*

Remark 1.1.9. The standard notation for the transition function going from considering E with respect to the local trivialisation $(U_\alpha, \varphi_\alpha)$ to with respect to (U_β, φ_β) is $g_{\beta\alpha}$. This is the opposite of what one might expect. The reason for this will be illuminated soon in the case of principal G-bundles, and in the case of vector bundles. Writing α on the right allows one to keep track of which local sections to apply $g_{\beta\alpha}$ to in a sensible way.

The transition functions for a fibre bundle describe the way the local trivialisations are glued together to produce the global structure of the bundle. Although the bundle is locally trivial, if the gluing on intersections of local trivialisations is non-trivial then the bundle may not be globally trivial. We will make this idea of transition functions specifying gluing in the statement of Theorem 1.1.12.

Definition 1.1.10 (The Cocycle Condition)**.** *Let M, F be manifolds.. Suppose $\mathcal{U} := \{U_\alpha\}$ is an open cover for M. A collection of maps $\{g_{\alpha\beta}\}$ with $g_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{Diff}(F)$ will be said to satisfy the* cocycle condition *if for every non-empty triple overlap $U_{\alpha\beta\gamma}$ the maps $g_{\alpha\beta}, g_{\beta\gamma}$, and $g_{\alpha\gamma}$ satisfy*

$$g_{\alpha\beta}(x) \circ g_{\beta\gamma}(x) = g_{\alpha\gamma}(x)$$

for every $x \in U_{\alpha\beta\gamma}$. The collection of maps $\{g_{\alpha\beta}\}$ will be called a cocycle *with values in* $\mathrm{Diff}(F)$ *with respect to the open cover $\mathcal{U}$.*

In particular, the transition functions for a fibre bundle E form a cocycle. In this case we will refer equivalently refer to the transitions as a cocycle *for E* with respect to $\mathcal{U}$ with values in $\mathrm{Diff}(F)$.

Lemma 1.1.11. *Suppose E is a fibre bundle and $\{g_{\alpha\beta}\}$ is a cocycle for E with respect to an open cover $\mathcal{U}$. Then for every $x \in U_{\alpha\beta} \neq \emptyset$,*

1. $g_{\alpha\alpha}(x) = \mathbf{1}(x)$
2. $g_{\alpha\beta}(x) = (g_{\beta\alpha}(x))^{-1}$

Proof. (1) follows from setting $\beta, \gamma = \alpha$ in the cocycle condition, and (2) follows from (1). ☐

Given a fibre bundle E over a manifold M, a fibre bundle atlas gives a cocycle $\{g_{\alpha\beta}\}$ that describes how one should pass between overlapping local trivialisations. The key result in the basic theory of fibre bundles is that one may start with a cocycle for some open cover and construct a fibre bundle from it (Theorem 1.1.12). Furthermore, if one applies this construction to a cocycle for a fibre bundle E, one recovers the bundle up to isomorphism (Corollary 1.1.15).

Theorem 1.1.12 (Fibre Bundle Construction Theorem)**.** *Let M, F be smooth manifolds and $\mathcal{U} := \{U_\alpha\}_{\alpha \in A}$ be an open cover for M indexed by a set A. Let $\{g_{\alpha\beta}\}$ be a cocycle on M with respect to $\mathcal{U}$ where each $g_{\alpha\beta}$ maps into $\mathrm{Diff}(F)$. Then there is a fibre bundle E with a fibre bundle atlas $\{(U_\alpha, \varphi_\alpha)\}$ such that the transition functions for this atlas are the cocycle $\{g_{\alpha\beta}\}$.*

Proof. Let A have the discrete topology. Then $M \times F \times A$ is a topological space. Let $X := \{(x, f, \alpha) \in M \times F \times A \mid x \in U_\alpha\}$. Then X is a disjoint union of the open sets $U_\alpha \times F \times \{\alpha\}$ for each α. Define an equivalence relation on X by

$$(x, p, \alpha) \sim (y, q, \beta) \text{ if and only if } x = x' \text{ and } q = g_{\beta\alpha}(x)(p).$$

Since the $g_{\alpha\beta}$ satisfy the cocycle conditions, $\sim$ is an equivalence relation on X. Let $E := X/\sim$ be the quotient space with the induced topology.

Define a map $\pi : E \to M$ by $\pi([x, p, \alpha]) = x$. Then π is well-defined and surjective. Since the corresponding map from X to M is continuous, in the quotient topology π will be continuous.

Now let $x \in M$. Then $\pi^{-1}(x) = \{[x, p, \alpha] \mid p \in F, x \in U_\alpha\}$. Note that if we fix $p \in F$ then $[x, p, \alpha] = [x, g_{\alpha\beta}(x)p, \beta]$ whenever $x \in U_{\alpha\beta}$. Since the $g_{\alpha\beta}(x)$ are diffeomorphisms of F for each (α, β), there is a well-defined bijection $\pi^{-1}(x) \to F$ sending $[x, p, \alpha]$ to p. Furthermore, this map is clearly continuous in the quotient topology, and indeed is open. Thus for each $x \in M$ we have a homeomorphism from $\pi^{-1}(x)$ to F.

Define a map $U_\alpha \times F \to \pi^{-1}(U_\alpha)$ by $(x, f) \mapsto [x, f, \alpha]$. Then this is a continuous bijection with continuous inverse $[x, f, \alpha] \mapsto (x, f)$.

□

One can define an equivalence relation on cocycles over the same open cover of M by saying $\{g_{\alpha\beta}\}$ is equivalent to $\{h_{\alpha\beta}\}$ if there exists smooth functions $\lambda_\alpha : U_\alpha \to \mathrm{Diff}(F)$ with the property that

$$g_{\beta\alpha}(x) \circ \lambda_\alpha(x) = \lambda_\beta(x) \circ h_{\beta\alpha}(x)$$

for every α, β such that $U_{\alpha\beta} \neq \emptyset$, and for every $x \in U_{\alpha\beta}$.

Definition 1.1.13. *Let $\mathcal{V} = \{V_\alpha\}$ and $\mathcal{W} = \{W_\beta\}$ be open covers of a manifold M. The* common refinement *of $\mathcal{V}$ and $\mathcal{W}$ is the open cover $\mathcal{U} = \{V_\alpha \cap W_\beta\}$ of all possible intersections. This will be denoted $\mathcal{V} \cap \mathcal{W}$.*

Proposition 1.1.14. *Two fibre bundles E and S with fibre F are isomorphic if and only if there exists an open cover $\mathcal{U}$ of M trivialising both E and S such that the transition functions with respect to this cover are equivalent.*

Proof. ($\Longrightarrow$) Suppose E and S are isomorphic as fibre bundles. Let $\mathcal{U}_1$ be a trivialising cover for E, and $\mathcal{U}_2$ be a trivialising cover for S. Then the common refinement $\mathcal{U} := \mathcal{U}_1 \cap \mathcal{U}_2$ is still an open cover of M, and by restriction of local trivialisations, both E and S are trivial over $\mathcal{U} =: \{U_\alpha\}$. Denote the fibre bundle atlas for E by $\{(U_\alpha, \varphi_\alpha)\}$ and for S by $\{(U_\alpha, \phi_\alpha)\}$.

Since E and S are isomorphic, there exists a diffeomorphism $f : E \to S$ preserving fibres. Let $(x, f) \in U_\alpha \times F$. We may then apply φ_α^{-1} to obtain a point $\varphi_\alpha^{-1}(x, f) \in E|_{U_\alpha}$. Then we may apply f to obtain an element $f \circ \varphi_\alpha^{-1}(x, f) \in S|_{U_\alpha}$. Finally we may apply ϕ_α to obtain $\phi_\alpha \circ f \circ \varphi_\alpha^{-1}(x, f) \in U_\alpha \times F$.

Because these maps preserve fibres, this element is of the form $(x, \lambda_\alpha(x)(f))$ for some map $\lambda_\alpha(x) \in \mathrm{Diff}(F)$. In particular we obtain a smooth function $\lambda_\alpha : U_\alpha \to \mathrm{Diff}(F)$.

Now on a non-empty intersection $U_{\alpha\beta}$ we have (with a slight abuse of notation)

$$g_{\beta\alpha}(x) \circ \lambda_\alpha(x) = \varphi_\beta \circ \varphi_\alpha^{-1} \circ \varphi_\alpha \circ f \circ \phi_\alpha^{-1} = \varphi_\beta \circ f \circ \phi_\alpha^{-1}.$$

On the other hand we have

$$\lambda_\beta(x) \circ h_{\beta\alpha}(x) = \varphi_\beta \circ f \circ \phi_\beta^{-1} \circ \phi_\beta \circ \phi_\alpha^{-1} = \varphi_\beta \circ f \circ \phi_\alpha^{-1}.$$

Thus the two cocycles $\{g_{\alpha\beta}\}$ and $\{h_{\alpha\beta}\}$ are equivalent.

($\Longleftarrow$) Suppose now that E and S have equivalent cocycles of transition functions with respect to some trivialising open cover $\mathcal{U}$. Define an isomorphism $f : E \to S$ as follows. If $\pi(p) = x \in U_\alpha$, then write $f(p) = \phi_\alpha^{-1} \circ \lambda_\alpha \circ \varphi_\alpha$, where here by composition with λ_α we mean the map $(x, f) \mapsto (x, \lambda_\alpha(x)(f))$ for $(x, f) \in U_\alpha \times F$.

Because the cocycles $\{g_{\alpha\beta}\}$ and $\{h_{\alpha\beta}\}$ are equivalent via these $\{\lambda_\alpha\}$, this map is well-defined on overlaps. By composition it is smooth, and one may write down an inverse by considering the maps $x \mapsto (\lambda_\alpha(x))^{-1}$. By definition it preserves fibres. Thus E and S are fibre bundle isomorphic. $\square$

Corollary 1.1.15 (Fibre Bundle Reconstruction Theorem)**.** *If $\pi : E \to M$ is an F-fibre bundle over M with trivialising open cover $\mathcal{U}$, and transition functions $\{g_{\alpha\beta}\}$ with respect to this open cover, then the fibre bundle E' constructed from these transition functions using Theorem 1.1.12 is isomorphic to E.*

1.1.4 Pullback Bundles

Let $\pi : E \to M$ be a fibre bundle with fibre F, and suppose $f : N \to M$ is a smooth map from another manifold N. Let $f^*E := \{(x, p) \in N \times E \mid f(x) = \pi(p)\} \subset N \times E$. Define a map $\pi' : f^*E \to N$ by $\pi'(x, p) = x$. Define another map $g : f^*E \to E$ by $g(x, p) = p$.

Equip the set f^*E with the subspace topology induced by $N \times E$. Further, note that f^*E is a smooth submanifold of $N \times E$, and with respect to this smooth structure π and g are continuous.

Definition 1.1.16 (Pullback Bundle)**.** *Let $\pi : E \to M$ be a fibre bundle with fibre F. Suppose $f : N \to M$ is a smooth map from another manifold N into M. Then the bundle f^*E defined above is called the* pullback *of E by f.*

Let $\mathcal{U}$ be a trivialising open cover of M for the bundle E. Then the collection $f^{-1}(\mathcal{U})$ is an open cover of N. On overlaps $f^{-1}(U_\alpha) \cap f^{-1}(U_\beta) = f^{-1}(U_\alpha \cap U_\beta)$, define functions $(f^*g)_{\alpha\beta}$ by $(f^*g)_{\alpha\beta}(x) = g_{\alpha\beta}(f(x))$. Then this open cover with these transition functions is a trivialisation of f^*E. Indeed one could define the pullback by taking these transition functions and using the fibre bundle construction theorem.

Corollary 1.1.17. *The pullback of a trivial bundle is trivial.*

Proof. A pullback bundle is trivialised over the preimage of the trivialising sets of the original bundle. Let $E \to N$ be a trivial fibre bundle and $f : M \to N$ a smooth map. Then E admits a global trivialisation, so f^*E admits a global trivialisation over $f^{-1}(N) = M$. $\square$

Corollary 1.1.18. *If $f : X \to Y$ and $g : Y \to Z$ are smooth maps, and $E \to Z$ is a fibre bundle, then $(g \circ f)^*E = f^*g^*E$.*

Proof. They have exactly the same transition functions. $\square$

The main theorem in the theory of pullbacks of fibre bundles is the following.

Theorem 1.1.19. *Suppose $f_0, f_1 : N \to M$ are homotopic maps, and that $\pi : E \to M$ is a fibre bundle with fibre F. Then the bundles f_0^*E and f_1^*E are homotopic.*

Proof. $\square$

Corollary 1.1.20 (Classification of Smooth Fibre Bundles over Contractible Manifolds)**.** *Any fibre bundle over a contractible manifold is trivial.*

Proof. Let M be contractible, and let $*$ denote the manifold that is a single point. Then there are maps $f : M \to *$ and $g : * \to M$ such that $g \circ f \equiv \mathbf{1}_M$.

Let $E \to M$ be a fibre bundle. Then $E = \mathbf{1}^*E \cong (g \circ f)^*E \cong f^*g^*E$. But $g^*E \to *$ is a fibre bundle over a point, which is trivial. Thus f^*g^*E must also be trivial by Corollary 1.1.17. $\square$

1.1.5 Fibred Products

Let $\pi : E \to M$ and $\pi' : S \to M$ be two fibre bundles over a manifold M, with fibres F and K respectively.

Consider the set $E \times S$. This is not itself a fibre bundle over M, since one has (in some sense) two copies of M, one for E and one for S. Given a point $(x, y) \in E \times S$, we have two natural projections onto M. Firstly, we can take $\pi(x, y) := \pi(x)$. Secondly we have $\pi'(x, y) := \pi'(y)$.

Definition 1.1.21. *Let $E \times_M S$ be the subspace of $E \times S$ such that*

$$E \times_M S := \{(x, y) \in E \times S \mid \pi(x) = \pi'(y)\}.$$

Then $E \times_M S$ is called the fibred product *of E and S.*

Proposition 1.1.22. *The fibred product of E and S is a smooth fibre bundle over M with fibre $F \times K$, and projection map $p : E \times_M S \to M$ given by $p(x, y) := \pi(x, y) = \pi'(x, y)$.*

Proof. The definition of a fibre bundle implies the projection map π is a submersion. Thus the condition that $\pi(x) = \pi'(y)$ means that $E \times_M S$ is a smooth submanifold of $E \times S$. Furthermore, the maps π and π' are clearly smooth with respect to this smooth structure (being smooth as maps on $E \times S$). Given a point $x \in M$, we have $p^{-1}(x) = \{(a, b) \in E \times S \mid \pi(a) = \pi'(b) = x\}$. For any $a \in E$ such that $\pi(a) = x$, one may choose any $b \in F_x$, and this defines a point $(a, b) \in p^{-1}(x)$. Thus the fibre is $F \times K$.

Take trivialisations of E and S, which we will assume to be over the same open cover $\{U_\alpha\}$. Say they have maps $\varphi_\alpha : E|_{U_\alpha} \to U_\alpha \times F$ and $\phi_\alpha : S|_{U_\alpha} \to U_\alpha \times K$.

Define maps $\psi_\alpha : E \times_M S|_{U_\alpha} \to U_\alpha \times F \times K$ by $\psi_\alpha(a, b) = (p(a, b), \mathrm{pr}_2 \circ \varphi_\alpha(a), \mathrm{pr}_2 \circ \phi_\alpha(b))$. Clearly these maps are smooth, and clearly they are bundle maps. Since $\pi(a) = \pi'(b)$, we have a well-defined inverse, which is also clearly smooth.

Thus $E \times_M S$ is a fibre bundle over M. □

1.1.6 Fibre Subbundles

Definition 1.1.23 (Fibre Sub-bundle)**.** *Let $\pi : E \to M$ be a fibre bundle with fibre F and let $S \subset E$ be a submanifold. Then S is called a* fibre sub-bundle of E *if S is such that $\pi|_S : S \to M$ gives S the structure of a fibre bundle with some fibre $K \subseteq F$ a submanifold of F.*

Without any additional structure on E or F there is not a lot that can be said about general fibre sub-bundles.

1.1.7 G-Bundles and Reduction of Structure Group

Definition 1.1.24 (G-Bundle)**.** *Let G be a Lie group. A* G-bundle over M with fibre *F is:*

1. *A fibre bundle P over M with fibre F.*
2. *A smooth left action $G \times F \to F$ on the fibre F, denoted $(g, f) \mapsto g \cdot f$.*
3. *a fibre bundle atlas $\{(U_\alpha, \varphi_\alpha)\}$ for which there exists functions $\tilde{g}_{\alpha\beta} : U_{\alpha\beta} \to G$ with the property that for $(x, f) \in U_{\alpha\beta} \times F$, $g_{\alpha\beta}(x)(f) = \tilde{g}_{\alpha\beta}(x) \cdot f$.*

The last condition in Definition 1.1.24 is saying that the transition functions for the fibre bundle P are given by maps into a group G, that is not necessarily all of $\mathrm{Diff}(F)$, but may be identified with a subgroup by the left action. This group G is thus specifying the structure of P, and one may also refer to a G-bundle as a fibre bundle with *structure group G*. As a slight abuse of notation, the transition functions for a G-bundle are always taken to be the maps $\tilde{g}_{\alpha\beta} : U_{\alpha\beta} \to G$, with the understanding that G is being considered as a subgroup of $\mathrm{Diff}(F)$ given by the smooth action of G on F.

If $F = \mathbb{K}^n$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}$, then $\mathrm{Diff}(\mathbb{K}^n)$ is a complicated group, but the subgroup $G = \mathrm{GL}(n, \mathbb{K}) \subset \mathrm{Diff}(\mathbb{K}^n)$ is much simpler to work with. Examples of $\mathrm{GL}(n, \mathbb{K})$-bundles include all vector bundles, as well as any associated frame bundles.

1.2 Vector bundles

1.2.1 Definitions

Definition 1.2.1 (Vector Bundle). *A* $\mathbb{K}$-vector bundle of rank n *over a manifold* M *is the following data:*

1. *A* $\mathrm{GL}(n, \mathbb{K})$*-bundle* $\pi : E \to M$ *with fibre* $\mathbb{K}^n$*.*
2. *A smooth section* $0 : M \to E$ *called the* zero section*.*
3. *For each* $\lambda \in \mathbb{K}$*, a fibre bundle homomorphism* $s_\lambda : E \to E$*, and*
4. *A smooth bundle map* $+ : E \times_M E \to E$*, such that on each fibre,* $s_\lambda|_{E_x}$ *and* $+|_{E_x}$ *satisfy the properties of scalar multiplication and addition in a vector space, with* $0(x)$ *acting as the origin of the fibre* E_x*.*
5. *A trivialisation* $\{(U_\alpha, \varphi_\alpha)\}$ *such that the maps* φ_α *are linear isomorphisms of vector spaces when restricted to fibres, with respect to the vector space structure induced by the map* $+$ *and the maps* $s_\lambda, \lambda \in \mathbb{K}$*.*

Remark 1.2.2. A vector bundle can be defined in a number of equivalent ways. The above definition was chosen to emphasize the fact that a vector bundle is a fibre bundle with fibre $\mathbb{K}^n$, that has some extra structure on it. Equivalently one might simply require that the fibres have the structure of a vector space by themselves. Then imposing the condition that the maps φ_α are diffeomorphisms that are linear isomorphisms on fibres automatically implies that addition and scalar multiplication are smooth (being smooth locally), and the requirement of E to be a $\mathrm{GL}(n, \mathbb{K})$-bundle implies the existence of a global 0 section.

Another definition is to simply define the vector space structure on each fibre via the local vector bundle isomorphisms φ_α and the standard vector space structure on $\mathbb{K}^n$. On overlaps the transition functions lie in $\mathrm{GL}(n, \mathbb{K})$ (and are hence linear with respect to the defined vector space structure), which means the operations of addition and scalar multiplication are globally well-defined with respect to this fibrewise vector space structure.

In the rest of these notes, we will not reference the maps s_λ or $+$ again. In practice, the other definitions of a vector bundle are conceptually more useful, despite all three being easily equivalent.

We say that two sections s_1, s_2 of a vector bundle E over M are linearly independent on some open set U if for every $x \in U$, the vectors $s_1(x), s_2(x) \in E_x$ are linearly independent.

Definition 1.2.3 (Local Frame). *Let* $\pi : E \to M$ *be a* $\mathbb{K}$*-vector bundle of rank* n *over a manifold* M*. Let* $(U_\alpha, \varphi_\alpha)$ *be a local trivialisation for* E*. Define* $e_i \in \Gamma(E|_{U_\alpha})$ *by* $e_i(x) := \varphi_\alpha^{-1}(x, e_i)$*, where the* e_i *in* (x, e_i) *is the ith standard basis vector of* $\mathbb{K}^n$*. The collection* $\{e_i\}$ *of local sections of* E *over the set* U_α *is called the* local frame *of* E *with respect to this trivialisation.*

Because φ_α is a linear isomorphism on fibres, the collection of vectors $\{e_i(x)\}$ form a basis at each fibre E_x for each $x \in U_\alpha$. Thus the collection $\{e_i\}$ forms a basis of the $C^\infty(M)$-module $\Gamma(E|_{U_\alpha})$. That is to say, any section $s \in \Gamma(E|_{U_\alpha})$ can be decomposed into the form $s = s^i e_i$ for some $s^i \in C^\infty(M)$, simply by decomposing $s(x)$ at each $x \in U_\alpha$ in terms of the basis $\{e_i(x)\} \subset E_x$.

Using $\{e_i\}$ to refer to the local frame and $\{\boldsymbol{e}_i\}$ to refer to the standard basis of the fibre space is an abuse of notation. However, it is often useful to blur the lines between considering a section $s \in \Gamma(E|_{U_\alpha})$ as $s = s^i e_i$ on the vector bundle or $s = s^i \boldsymbol{e}_i$ on $U_\alpha \times \mathbb{K}^n$. Since the local trivialisation is an isomorphism of vector bundles, there are no problems with this abuse of notation, and it will be exploited without comment.

Proposition 1.2.4. *A vector bundle E of rank n over a manifold M is trivial if and only if E admits n global globally linearly independent sections.*

Proof. ($\Longrightarrow$) Suppose E is trivial, with trivialisation $\varphi : E \to M \times \mathbb{R}^n$.

Define sections $s_i : M \to M \times \mathbb{R}^n$ by $s_i(x) = (x, \boldsymbol{e}_i)$. Then these define global sections $\tilde{s}_i = \varphi^{-1} \circ s$. Since φ^{-1} is a linear isomorphism on each fibre, the s_i being linearly independent implies the $\tilde{s}_i$ are linearly independent.

($\Longleftarrow$) Suppose E admits n global globally linearly independent sections $\{s_i\}$. Fixing the standard basis of $\mathbb{R}^n$, define a map $\psi : M \times \mathbb{R}^n \to E$ by

$$(x, c_1\boldsymbol{e}_1 + \cdots + c_n\boldsymbol{e}_n) \mapsto c_1 s_1(x) + \cdots + c_n s_n(x).$$

This is a fibrewise linear isomorphism as the s_i are fibrewise linearly independent, and is clearly a diffeomorphism, and so defines a map $\varphi = \psi^{-1}$ from E to $M \times \mathbb{R}^n$, a trivialisation of E. $\square$

In particular, a vector bundle E is *locally* trivial over a set U_α if it admits n linearly independent sections over U_α, since this occurs if and only if $E|_{U_\alpha}$ is globally trivial. Thus, because of Proposition 1.2.4, one could take the existence of local frames as a definition of the local triviality of a vector bundle. Indeed this is the approach taken in some books. The existence of local frames will play a central role in Section 2.1, when connections on vector bundles are investigated.

In the case where E has rank 1, Proposition 1.2.4 says that a line bundle is trivial if it admits a global non-vanishing section. This gives one possible intuition for what non-trivial vector bundles look like. If a line bundle is twisted (i.e. non-trivial), a global section is forced to cross over the zero-section in order to join back up with itself after the twist.

1.2.2 Bundle Maps

The bundle maps in the category of vector bundles are simply the bundle maps for the vector bundles, considered as fibre bundles, that respect the fibre-wise linear structure.

Definition 1.2.5 (Vector Bundle Homomorphism)**.** *Let $\pi : E \to X$ and $\pi' : F \to Y$ be vector bundles over manifolds X and Y. A* vector bundle homomorphism *$f : E \to F$ is* *a fibre bundle homomorphism such that* $f|_{E_x} : E_x \to F_{\tilde{f}(x)}$ *is linear, where* $\tilde{f} : X \to Y$ *is the map induced by* f*.*

Definition 1.2.6 (Vector Bundle Homomorphism over M)**.** *A* vector bundle homomorphism over M *is a vector bundle homomorphism of bundles* E, F *over* M *that covers the identity.*

Vector bundle homomorphisms over a base space have a simple interpretation. At each point $x \in M$, f is specifying a linear map from E_X to F_x, and these linear maps are varying smoothly.

Definition 1.2.7 (Vector Bundle Isomorphism)**.** *Let* $\pi : E \to M$ *and* $\pi' : F \to M$ *be vector bundles over a manifold* M*. Then a* vector bundle isomorphism *from* E *to* F *is a fibre bundle isomorphism that is also a vector bundle homomorphism.*

Since the fibrewise linear maps $f|_{E_x} : E_x \to F_x$ are bijections for a fibre bundle isomorphism (why?), they are linear isomorphisms. In particular, if E and F are isomorphic vector bundles then they must have the same rank. This is not the case for general vector bundle homomorphisms. For example one may consider the map $M \times \mathbb{R}^2 \to M \times \mathbb{R}$ given by $(m, x, y) \mapsto (m, x)$. This is a homomorphism of vector bundles of different ranks over M.

One is tempted to denote the groups of homomorphisms and endomorphism of vector bundles over a manifold M by $\mathrm{Hom}(E, F)$ and $\mathrm{End}(E)$. In fact we will soon see that such homomorphisms can always be realised as sections of vector bundles, whose fibres at each $x \in M$ are the vector spaces $\mathrm{Hom}(E_x, F_x)$ and $\mathrm{End}(E_x)$. These vector bundles will be denoted by the above terms, and the groups of homomorphisms themselves will subsequently be denoted $\Gamma(\mathrm{Hom}(E, F))$ and $\Gamma(\mathrm{End}(E))$.

In addition to this, if $E \to M$ is a vector bundle, we will denote by $\mathrm{Aut}(E)$ the space of vector bundle isomorphisms from E to itself. Because $\mathrm{GL}(n, \mathbb{K})$ is not a vector space, we will not be able to identify this group with the sections of some vector bundle. It can however be identified with the sections of a bundle called $\mathrm{Ad}(F(E))$, where $F(E)$ denotes the principal $\mathrm{GL}(n, \mathbb{K})$-frame bundle associated to E. This will be expanded upon in Section 1.4.

1.2.3 Transition Functions

In the definition of a vector bundle, we have required that the local trivialisations are linear isomorphisms on fibres. In particular this implies the transition functions $\{g_{\alpha\beta}\}$ for a vector bundle E with respect to some open cover $\mathcal{U}$ have values in $\mathrm{GL}(n, \mathbb{K})$. Thus we have:

Lemma 1.2.8. *A rank* n $\mathbb{K}$*-vector bundle* E *is a* $\mathrm{GL}(n, \mathbb{K})$*-bundle over* M*.*

We have seen in the case of vector bundles a local trivialisation comes with a local frame of sections. The transition functions for a vector bundle are defined in terms of the bundle isomorphisms from $U_{\alpha\beta} \times \mathbb{K}^n$ to itself, where $\{U_\alpha\}$ is some trivialising open set for a vector bundle E.

Let $\{a_i\}$ correspond to the local frame of E on U_α, induced by φ_α, and $\{b_j\}$ correspond to the local frame of E on U_β, induced by φ_β. Let s be a general section on $U_{\alpha\beta}$. Suppose we have $s = u^i a_i$ and $s = v^j b_j$ for some $u^i, v^j \in C^\infty(U_{\alpha\beta})$. That is,

$$\varphi_\alpha(s) = (x, u^i(x)e_i), \tag{Eq. 1.1}$$

$$\varphi_\beta(s) = (x, v^j(x)e_j), \tag{Eq. 1.2}$$

where $\{e_j\}$ is the standard basis of $\mathbb{K}^n$. Then from (Eq. 1.1), $s = \varphi_\alpha^{-1}(x, u^i(x)e_i)$, so $\varphi_\beta(s) = (x, g_{\beta\alpha}(x)(u^i(x)e_i))$. On the other hand, from (Eq. 1.2), $\varphi_\beta(u) = (x, v^j(x)e_j)$, so we have

$$v^j(x)e_j = g_{\beta\alpha}(x)(u^i(x)e_i).$$

In particular if we let $g_{\beta\alpha}(x)e_i := g(x)_i^j e_j$, for some $g(x)_i^j \in \mathbb{K}$, then we have $v^j(x)e_j = u^i(x)g(x)_i^j e_j$, so $v^j(x) = u^i(x)g(x)_i^j$. That is, we have

$$v^j = g_i^j u^i \tag{Eq. 1.3}$$

as the coefficients of s in the two frames $\{b_j\}$ and $\{a_i\}$. Thus (Eq. 1.3) is the expression for how coefficients change between frames.

On the other hand, if we let $u^i = \delta_k^i$ for some fixed k, then we have $s = \delta_k^i a_i = a_k$. On the other hand by (Eq. 1.3) we have $v^j = \delta_k^i g_i^j = g_j^k$. That is, $s = g_k^j b_j$. So we conclude

$$a_k = b_j g_k^j. \tag{Eq. 1.4}$$

This is the formula for how the basis sections of the frame transform under a change of coordinates.

Notice that if we interpret $g_{\beta\alpha}(x) \in \mathrm{GL}(n, \mathbb{K})$ as the matrix $g(x)_i^j$, then (Eq. 1.3) is the matrix multiplication

$$g_{\beta\alpha}(x)u(x) = v(x)$$

where $u(x)$ is the column vector with coefficients $u^i(x)$, (i.e. $s(x)$ in the basis $\{a_i(x)\}$), and $v(x)$ is the column vector with coefficients $v^j(x)$, (i.e. $s(x)$ in the basis $\{b_j(x)\}$). Thus the matrix $g_{\beta\alpha}(x)$ is really the change of basis matrix from $\{a_i(x)\}$ to $\{b_j(x)\}$.

These computations will be used in Section 2.1.3 in order to determine how a connection form changes under a transition function.

1.2.4 Čech Cohomology

The fibre bundle construction theorem (Theorem 1.1.12) can be applied to the case of vector bundles, to obtain:

Theorem 1.2.9 (Vector Bundle Construction Theorem). *Let M be a smooth manifold, and $\mathcal{U}$ an open cover for M. Let $\{g_{\alpha\beta}\}$ be a cocycle on M with respect to $\mathcal{U}$ with values in $\mathrm{GL}(n, \mathbb{K})$. Then there is a rank n $\mathbb{K}$-vector bundle E with transition functions given by the $g_{\alpha\beta}$.*

Proposition 1.1.14 carries over to the case of vector bundles also, by noting that all the maps defined are linear in the case of vector bundles.

Because of Theorem 1.2.9, Proposition 1.1.14 implies there is a bijection between isomorphism classes of vector bundles with trivialising open cover $\mathcal{U}$ and equivalence classes of cocycles $\{g_{\alpha\beta}\}$ with respect to this cover. This set of equivalence classes is called the Čech cohomology with values in $\mathrm{GL}(n, \mathbb{K})$ with respect to an open cover $\mathcal{U}$ of M.

Definition 1.2.10 (Čech Cohomology of an Open Cover)**.** *Let M be a manifold and $\mathcal{U}$ be an open cover of M. Let $Z^1(\mathcal{U}, \mathrm{GL}(n, \mathbb{K}))$ denote the set of cocycles for the open cover $\mathcal{U}$ with values in $\mathrm{GL}(n, \mathbb{K})$. Let $\sim$ be the equivalence relation on cocycles specified previously. Then the set*

$$\check{\mathrm{H}}^1(\mathcal{U}, \mathrm{GL}(n, \mathbb{K})) := Z^1(\mathcal{U}, \mathrm{GL}(n, \mathbb{K}))\Big/\sim$$

is called the first Čech cohomology set for M with respect to the open cover $\mathcal{U}$ with values in $\mathrm{GL}(n, \mathbb{K})$.

Note that $\check{\mathrm{H}}^1(\mathcal{U}, \mathrm{GL}(n, \mathbb{K}))$ is a *pointed set.* There is a distinguished equivalence class of cocycles corresponding to the trivial vector bundle. One representative for this class is just the collection $\{g_{\alpha\beta}\}$ where $g_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}(n, \mathbb{K})$ is defined by $g_{\alpha\beta}(x) = \mathbf{1}_n$ for all $x \in U_{\alpha\beta}$. We will now construct the Čech cohomology for M, without reference to a particular open cover $\mathcal{U}$.

Suppose that $\mathcal{V}$ is a refinement of an open cover $\mathcal{U}$, and that $\{g_{\alpha\beta}\}$ is a cocycle with respect to $\mathcal{U}$. Then we may define a new cocycle with respect to $\mathcal{V}$ simply by restriction on overlaps. That is, since every $V_\alpha \in \mathcal{V}$ is contained inside some $U_\alpha \in \mathcal{U}$, we have $V_{\alpha\beta} \subseteq U_{\alpha\beta}$, so we can restrict. Obviously the cocycle condition is still satisfied for this restriction, so we obtain a new cocycle, which we will call $\{h_{\alpha\beta}\}$. Now if $\{g'_{\alpha\beta}\}$ is equivalent to $\{g_{\alpha\beta}\}$, then by restricting the maps λ_α from U_α to V_α, we get that $\{h'_{\alpha\beta}\}$ is equivalent to $\{h_{\alpha\beta}\}$.

Thus we have a well-defined restriction map

$$r_{\mathcal{U}\mathcal{V}} : \check{\mathrm{H}}^1(\mathcal{U}, \mathrm{GL}(n, \mathbb{K})) \to \check{\mathrm{H}}^1(\mathcal{V}, \mathrm{GL}(n, \mathbb{K})).$$

Lemma 1.2.11. *This map $r_{\mathcal{U}\mathcal{V}}$ satisfies three key properties:*

1. *This map sends the class of the trivial bundle with respect to $\mathcal{U}$ to the class of the trivial bundle with respect to $\mathcal{V}$. That is, it is a homomorphism of pointed sets.*

2. *The map $r_{\mathcal{U}\mathcal{U}}$ is the identity.*

3. *If $\mathcal{W}$ is a refinement of $\mathcal{V}$, then $r_{\mathcal{V}\mathcal{W}} \circ r_{\mathcal{U}\mathcal{V}} = r_{\mathcal{U}\mathcal{W}}$.*

Now write $\mathscr{U}$ for the set of all open covers of M. We may define a partial ordering on $\mathscr{U}$ by saying $\mathcal{U} \leq \mathcal{V}$ if $\mathcal{V}$ is a refinement of $\mathcal{U}$, making $\mathscr{U}$ into a directed set. Then because the maps $r_{\mathcal{U}\mathcal{V}}$ satisfy the above three properties, we obtain a directed system

$$\langle \check{\mathrm{H}}^1(\mathcal{U}, \mathrm{GL}(n, \mathbb{K})), r_{\mathcal{U}\mathcal{V}} \rangle$$

in the category of pointed sets. Thus there is a well-defined direct limit

$$\check{\mathrm{H}}^1(M, \mathrm{GL}(n, \mathbb{K})) := \varinjlim \check{\mathrm{H}}^1(\mathcal{U}, \mathrm{GL}(n, \mathbb{K})).$$

Definition 1.2.12 (Čech Cohomology)**.** *The pointed set* $\check{\mathrm{H}}^1(M, \mathrm{GL}(n, \mathbb{K}))$ *is called the* first Čech cohomology set for M with values in $\mathrm{GL}(n, \mathbb{K})$*.*

To understand what this set represents, we will need to investigate the mechanism behind how a direct limit is obtained. The direct limit of the directed system is defined as

$$\varinjlim \check{\mathrm{H}}^1(\mathcal{U}, \mathrm{GL}(n, \mathbb{K})) := \bigsqcup_{\mathcal{U} \in \mathscr{U}} \check{\mathrm{H}}^1(\mathcal{U}, \mathrm{GL}(n, \mathbb{K})) \Big/ \sim$$

where $\sim$ is the equivalence relation defined as follows. If $\{g_{\alpha\beta}\} \in \check{\mathrm{H}}^1(\mathcal{U}, \mathrm{GL}(n, \mathbb{K}))$ and $\{h_{\alpha\beta}\} \in \check{\mathrm{H}}^1(\mathcal{V}, \mathrm{GL}(n, \mathbb{K}))$, then $\{g_{\alpha\beta}\} \sim \{h_{\alpha\beta}\}$ if and only if there is a common refinement $\mathcal{W}$ of $\mathcal{U}$ and $\mathcal{V}$ such that

$$r_{\mathcal{U}\mathcal{W}}(\{g_{\alpha\beta}\}) = r_{\mathcal{V}\mathcal{W}}(\{h_{\alpha\beta}\}),$$

where this equality is *inside the* Č*ech cohomology set* $\check{\mathrm{H}}^1(\mathcal{W}, \mathrm{GL}(n, \mathbb{K}))$. Note that properties (2) and (3) of Lemma 1.2.11 are what makes $\sim$ an equivalence relation.

To expand on what this equivalence relation is saying, two cocycles coming from different open covers are equivalent if there is a common refinement of these open covers so that on the common refinement, the cocycles are equivalent in the sense of Proposition 1.1.14. But this is precisely the statement that the vector bundles defined by $\{g_{\alpha\beta}\}$ and $\{h_{\alpha\beta}\}$ are isomorphic. Furthermore, the distinguished element in this direct limit is the class of cocycles that define the trivial bundle with respect to any open cover of M. Thus we have established:

Theorem 1.2.13. *The isomorphism classes of rank* n $\mathbb{K}$*-vector bundles over a manifold* M *are in bijection with the pointed set*

$$\check{\mathrm{H}}^1(M, \mathrm{GL}(n, \mathbb{K})).$$

Furthermore, the distinguished element in this pointed set corresponds to the isomorphism class of the trivial bundle.

Remark 1.2.14. It is an unfortunate fact that there is in general no good way of turning the Čech cohomology with values in $\mathrm{GL}(n, \mathbb{K})$ into a group. We will see in Section 3.3 that for Abelian groups G, one can make sense of the Čech cohomology *groups* $\check{\mathrm{H}}^k(M, G)$ with values in G. In particular when $n = 1$, $\mathrm{GL}(n, \mathbb{K})$ is an abelian group for $\mathbb{K} = \mathbb{R}, \mathbb{C}$, so we obtain a group structure on the isomorphism classes of line bundles over the manifold M. This group structure is actually given by the tensor product of line bundles, with inverse given by tensoring with the dual.

Remark 1.2.15. This entire section could have been formulated in the case of general fibre bundles over M, rather than vector bundles. One simply replaces $\mathrm{GL}(n, \mathbb{K})$ by $\mathrm{Diff}(F)$ for the fibre F, and obtains the Čech cohomology with values in $\mathrm{Diff}(F)$. This pointed set would classify all fibre bundle isomorphism classes of F-fibre bundles over M.

Remark 1.2.16. By choosing different subgroups of $\mathrm{GL}(n, \mathbb{K})$, we can obtain isomorphism classes of vector bundles with different properties. For example, choosing $\mathrm{SL}(n, \mathbb{K})$ would give isomorphism classes of *orientable vector bundles*. Choosing $O(n)$ would give isomorphism classes of vector bundles admitting a metric, and $U(n)$ would give isomorphism classes of complex vector bundles admitting a Hermitian metric. Note that since every vector bundle admits such metrics, the groups $\check{\mathrm{H}}^1(M, \mathrm{GL}(n, \mathbb{C}))$ and $\check{\mathrm{H}}^1(M, U(n))$ would be isomorphic, as would $\check{\mathrm{H}}^1(M, \mathrm{GL}(n, \mathbb{K}))$ and $\check{\mathrm{H}}^1(M, O(n, \mathbb{K}))$.

Exactly how these cases are related depends on the reductions of structure group possible for vector bundles over M, considered as $\mathrm{GL}(n, \mathbb{K})$-bundles.

1.2.5 Examples of Vector Bundles

Example 1.2.17 (The Möbius Bundle)**.** Let $\{U, V\}$ be an open cover of S^1 such that each open set is slightly larger than a semi-circle. Then the intersection $U \cap V$ is a disconnected set with two components, which we will call A and B. Define a map $g : U \cap V \to \mathrm{GL}(1, \mathbb{R}) = \mathbb{R}\backslash\{0\}$ by $g(x) = 1$ if $x \in A$ and $g(x) = -1$ if $x \in B$. Then (noticing that $(-1)^2 = 1$) all three compatibility conditions are satisfied by this single transition function g. In particular the cocycle condition is vacuous (because there are only two open sets in the cover). Thus we can define a line bundle E from these transition functions, called the Möbius bundle. The effect of this transition function is, when one takes a piece of paper and constructs a Möbius band, *exactly* the twisting and gluing that occurs during this process.

The above example should be the model for any image you might wish to hold in your head of a non-trivial vector bundle.

Example 1.2.18 (The Tangent Bundle)**.** Let M be a smooth manifold and $\{(U_\alpha, \psi_\alpha)\}$ an atlas for M. On a non-trivial overlap $U_{\alpha\beta}$ define $g_{\alpha\beta}(x) := d(\psi_\alpha \circ \psi_\beta^{-1})_x : \mathbb{R}^n \to \mathbb{R}^n$ for each $x \in U_{\alpha\beta}$, where d is just the differential of smooth maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. These are the Jacobians of the transition functions for the atlas. By the chain rule the transition functions $\{g_{\alpha\beta}\}$ form a cocycle, and the vector bundle associated to this cocycle is called the *tangent bundle* of M. Sections of the tangent bundle are vector fields on M.

The tangent bundle, or equivalently the cotangent bundle, is the single most important example of a vector bundle on a manifold. We will see later in Section 3.2.3 that the tangent bundle here is precisely the bundle of sections corresponding to the locally free sheaf of tangent vector fields over a manifold.

With the help of the construction theorem for vector bundles (Theorem 1.2.9), we now have a simple method of constructing many new vector bundles from given ones.

Definition 1.2.19. *Let E and F be vector bundles over M with transition functions $\{g_{\alpha\beta}\}$ and $\{h_{\alpha\beta}\}$ with respect to some open cover $\mathcal{U}$ of M.*

1. *Define $E \oplus F$ to be the vector bundle over M with fibres $E_x \oplus F_x$ and transition functions*
$$k_{\alpha\beta} := \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & h_{\alpha\beta} \end{pmatrix}.$$
Note that the underlying fibre bundle structure of $E \oplus F$ is that of the fibred product $E \times_M F$, equipped with some vector bundle structure induced by E and F.

2. *Define E^* to be the vector bundle over M with fibres $(E_x)^*$ and transition functions*
$$k_{\alpha\beta} := ({g_{\alpha\beta}}^{\mathsf{T}})^{-1}$$
the inverse of the transpose of $g_{\alpha\beta}$.

3. *Define $E \otimes F$ to be the vector bundle over M with fibres $E_x \otimes F_x$ and transition functions*
$$k_{\alpha\beta} := g_{\alpha\beta} \otimes h_{\alpha\beta}.$$

4. *Define $\bigwedge^k E$ to be the sub-bundle of $\bigotimes^k E$ defined by the subspace of antisymmetric tensors $\bigwedge^k E_x \subset \bigotimes^k E_x$ at each x.*

5. *Define* $\mathrm{Sym}(k, E)$ *to be the sub-bundle of $\bigotimes^k E$ defined by the subspace of symmetric tensors* $\mathrm{Sym}(k, E_x) \subset \bigotimes^k E_x$ *at each x.*

6. *Define* $\mathrm{Hom}(E, F)$ *to be the bundle $E^* \otimes F$, noting that on each fibre we have an isomorphism $E_x^* \otimes F_x \cong \mathrm{Hom}(E_x, F_x)$. Then sections of* $\mathrm{Hom}(E, F)$ *define bundle homomorphisms from E to F, and bundle homomorphisms come from sections of* $\mathrm{Hom}(E, F)$

7. *Define* $\mathrm{End}(E)$ *to be the bundle $E^* \otimes E$. Similarly, endomorphisms of E are the same as sections of* $\mathrm{End}(E)$.

Remark 1.2.20. In the definition of E^*, we take the transition functions to be the inverse *transpose* of the original transition functions. This is the technically correct definition, but requires some interpretation. If $\{e_i\}$ is a basis for a vector space we tend to write the dual basis as, for example, $\{\varepsilon^j\}$, with *upper* indices. With respect to this convention, the transition functions should really just be $g_{\alpha\beta}^{-1}$, but since the definition of a vector bundle requires that we expression local frames with lower indices, the true transition functions are ${g_{\alpha\beta}^{-1}}^{\mathsf{T}}$.

In each case above one should check (and it is straight-forward to do so) that the new transition functions as defined indeed satisfy the required cocycle condition. We can now produce more interesting examples of vector bundles.

Example 1.2.21 (The Cotangent Bundle)**.** Let M be a smooth manifold. Then the bundle $T^*M := (TM)^*$ is called the *cotangent bundle* of M.

Example 1.2.22 (Tensor Bundles)**.** Let M be a smooth manifold. Then the bundle of (p, q)-tensors on M is the bundle $\mathscr{T}^{p,q}(M) := \bigotimes^p TM \otimes \bigotimes^q T^*M$. Sections of $\mathscr{T}^{p,q}(M)$ are p-contra-q-covariant tensor fields on M.

Example 1.2.23 (Determinant Bundle)**.** The *determinant bundle* of a vector bundle E of rank n is the line bundle $\bigwedge^n E$.

Note that the determinant bundle really is a line bundle, because if V is an n-dimensional vector space, then $\bigwedge^n V$ has a single basis element $e_1 \wedge \cdots \wedge e_n$ where $\{e_1, \ldots, e_n\}$ is a basis of V.

To say a few words about why the determinant bundle is useful, the *Chern class* $c_1(E)$ is a characteristic class that may be (reasonably) easily defined for a line bundle, and one can then define the Chern class of any vector bundle as the Chern class of its determinant bundle. In fact, the definition of the Chern class depends only on the cocycle (the cohomology class in Čech cohomology) that defines the vector bundle, and the transition functions of the determinant line bundle are essentially the same as those of the vector bundle, so this seems reasonable.

Furthermore, the determinant line bundle is related to the *orientability* of a vector bundle.

Definition 1.2.24. *A vector bundle E is said to be* orientable as a vector bundle *if for some $\{U_\alpha\}$ a trivialising open cover the determinants of the transition functions $g_{\alpha\beta}$ have constant sign (either positive or negative) over all α, β.*

By Proposition 1.2.4, a vector bundle is orientable if and only if its determinant bundle is trivial.

In particular, the transition functions for the tangent bundle TM of a manifold M are the Jacobians of the transition functions for the smooth atlas on M, so TM being orientable as a bundle is equivalent to M being orientable as a manifold. An example of a non-orientable vector bundle is the Möbius strip.

Note that orientability of a vector bundle is not the same as triviality. The tangent bundle to the two-sphere TS^2 is a non-trivial vector bundle (by the Hairy Ball Theorem and Proposition 1.2.4), but S^2 is an orientable manifold, so TS^2 is an orientable bundle.

1.2.6 Metrics

A metric on a smooth vector bundle is a smoothly varying choice of inner product on each fibre. The inner product varies smoothly if for every pair of sections $s, t \in \Gamma(E)$ we have $g(s, t) \in C^\infty(M)$, where $g(s, t)(x) = g_x(s(x), t(x)) \in \mathbb{R}$. This is in fact a complete definition of a metric on a vector bundle. To give a more invariant description of what it means to have a smoothly varying inner product, we will present another definition.

Note that this inner product must be *real-valued*. In the case where E is a real vector bundle there is no trouble, but if E is a complex vector bundle, the natural definition as a section of $\mathrm{Sym}(2, E^*)$ would allow the inner product to be $\mathbb{C}$-valued. To remedy this situation, we will view $\mathbb{C}^\infty(M)$ as the space of sections of the trivial bundle $M \times \mathbb{R}$.

Definition 1.2.25 (Metric)**.** *Given a vector bundle E, a* metric *on E is a fibre bundle homomorphism $g : E \times_M E \to M \times \mathbb{R}$ such that $g|_{(E\times_M E)_x} : (E \times_M E)_x \to \{x\} \times \mathbb{R}$ is a positive-definite bilinear map of vector spaces.*

Thus the requirement that the inner product g vary smoothly is precisely that it is a smooth fibre bundle homomorphism.

Definition 1.2.26 (Riemannian Metric)**.** *A* Riemannian metric *on a manifold M is a metric on TM.*

In this case we really can define a Riemannian metric to be a smooth section of $\mathrm{Sym}(2, T * M)$ that is positive-definite on fibres.

Lemma 1.2.27. *The trivial bundle admits a metric.*

Proof. Let $E := M \times \mathbb{K}^n$ be the trivial rank n $\mathbb{K}$-vector bundle over M. For each x let the inner product on E_x just be the standard inner product on $\mathbb{K}^n$ (considered as a vector space over $\mathbb{R}$ if $\mathbb{K} = \mathbb{C}$). □

Lemma 1.2.28. *Every vector bundle admits a metric.*

Proof. Let $\pi : E \to M$ be a smooth vector bundle and suppose $\{(U_\alpha, \varphi_\alpha)\}$ is a trivialisation for E. Then $E|_{U_\alpha}$ is trivial for each α, and so admits a metric g_α. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to the open cover $\{U_\alpha\}$ of M. Then for each $x \in M$ define an inner product g_x on E_x by

$$g_x := \sum_\alpha \rho_\alpha(x) g_{\alpha x}.$$

This pieces together to give a global inner product g. □

Corollary 1.2.29. *Every smooth manifold admits a Riemannian metric.*

In addition to real-valued metrics on vector bundles, we also have a notion of Hermitian metrics on $\mathbb{C}$-vector bundles.

Definition 1.2.30 (Hermitian Metric)**.** *Given a complex vector bundle E, a* Hermitian metric *on E is a fibre bundle homomorphism $h : E \times_M E \to M \times \mathbb{C}$ such that $h|_{(E\times_M E)_x} : (E \times_M E)_x \to \{x\} \times \mathbb{C}$ is a positive-definite sesquilinear map of vector spaces.*

Equivalently, a Hermitian Metric is a choice of Hermitian inner product h_x on every fibre E_x of E that varies smoothly in the sense that if $s, t \in \Gamma(E)$ then $h(s, t)$ is a smooth $\mathbb{C}$-valued function on M. By the exact same argument to Lemma 1.2.28, every $\mathbb{C}$-vector bundle admits a Hermitian metric.

1.2.7 Sub-Bundles and Quotient Bundles

Given a vector bundle E, a *sub-bundle* $F \subset E$ is a subset such that $\pi|_F : F \to M$ is also a vector bundle.

Lemma 1.2.31. *Suppose F is a rank k sub-bundle of the rank n vector bundle E over M. Given a local trivialisation $(U_\alpha, \varphi_\alpha)$ of E, there exist a local frame of sections $s_1, \ldots, s_n$ such that for every $x \in U_\alpha$, $F_x = \mathrm{Span}_{\mathbb{K}}\{s_1(x), \ldots, s_k(x)\}$.*

Proof. □

Let $\{(U_\alpha, \varphi_\alpha)\}$ be a trivialisation for E where each local trivialisation is of the form specified in Lemma 1.2.31. Then the transition functions $\{g_{\alpha\beta}\}$ with respect to this trivialisation have the form

$$g_{\alpha\beta}(x) = \begin{pmatrix} h_{\alpha\beta}(x) & A(x) \\ 0 & k_{\alpha\beta}(x) \end{pmatrix},$$

where for each $x \in U_{\alpha\beta}$, $h_{\alpha\beta}(x) \in \mathrm{GL}(k, \mathbb{K})$, $k_{\alpha\beta}(x) \in \mathrm{GL}(n-k, \mathbb{K})$, and A is some matrix.

Proposition 1.2.32. *The sub-bundle F has transition functions $\{h_{\alpha\beta}\}$.*

Proof. □

Having defined sub-bundles, we would of course like to define the corresponding notion of quotient bundles. To present an invariant way of doing this, we will introduce the notion of an exact sequence of vector bundles.

Definition 1.2.33 (Exact Sequence of Vector Bundles)**.** *Let F, E, and S be vector bundles over a manifold M, and suppose $f : F \to E$ and $g : E \to S$ are vector bundle homomorphisms. The sequence*

$$F \xrightarrow{f} E \xrightarrow{g} S$$

of vector bundles is called exact *at E if for every $x \in M$, the sequence*

$$F_x \xrightarrow{f|_{F_x}} E_x \xrightarrow{g|_{E_x}} S_x$$

is an exact sequence of vector spaces.

A sequence is called *exact* if it is exact at every term.

Definition 1.2.34 (Quotient Bundle)**.** *A* quotient bundle *of E by F is the vector bundle Q and a map $j : E \to Q$ such that if $i : F \hookrightarrow E$ is the inclusion map, then the following sequence is exact.*

$$0 \longrightarrow F \xhookrightarrow{i} E \xrightarrow{j} Q \longrightarrow 0$$

Lemma 1.2.35 (Existence)**.** *Suppose* $\{g_{\alpha\beta}\}$ *is a collection of transition functions for* E *of the form in Lemma 1.2.31. Then the sub-bundle* Q *of* E *with transition functions* $\{k_{\alpha\beta}\}$ *over the same trivialising set is a quotient bundle of* E *by* F*.*

Proof. □

Lemma 1.2.36 (Uniqueness)**.** *Any two quotient bundles of* E *by* F *are isomorphic.*

Proof. □

By virtue of the existence and uniqueness lemmas for quotient bundles, we denote the quotient of E by F as E/F.

1.2.8 Kernels and Cokernels

In the theory of vector spaces the kernel of a linear map is a vector subspace of the domain, and the image is a vector subspace of the target. This is no longer true in the category of vector bundles. The kernel of a homomorphism of vector bundles need not be a sub-bundle

Example 1.2.37. Let $E = [0, 2\pi) \times \mathbb{R}^2$ be the trivial bundle over $[0, 2\pi)$ and let e_1, e_2 be the standard basis of $\mathbb{R}^2$. Define a homomorphism $f : E \to E$ by sending $(\theta, ae_1 + be_2) \mapsto (\theta, ae_1 + bR_\theta e_2)$ where R_θ is rotation counter-clockwise by θ. Then for $\theta = 0$ the map f is the identity, but when $\theta = \pi/2$, $R_\theta e_2 = -e_1$, so the kernel consists of those elements $ae_1 + be_2 \in \mathbb{R}^2$ such that $a = b$. This is a one-dimensional subspace, so the kernel is one-dimensional. This jump in dimension means the fibrewise kernels cannot possibly piece together to form a sub-bundle of E.

In addition, the image of a homomorphism of vector bundles is not necessarily a vector bundle. The above example shows why again, since the image jumps from being two-dimensional to one-dimensional at $\theta = \pi/2$. In particular this implies that cokernels of vector bundle homomorphisms (targets quotiented by images) are not in general vector bundles.

A category which has kernels and cokernels (along with several other properties) is called an Abelian category, and such a category is open to attack with techniques from modern algebra. As such it would be desirable if the category of vector bundles over a manifold were Abelian. The most obvious solution is to simply add in all the kernels and cokernels to the category, thereby making the category Abelian by brute force. This is the category of coherent sheaves on a manifold, which we will say no more about.

The other option is to restrict ourselves to those vector bundle homomorphisms for which kernels and cokernels are well-defined. It turns out that the characterisation of such homomorphisms is as nice as one could hope for.

Proposition 1.2.38. *Let* $f : E \to F$ *be a vector bundle homomorphism of locally constant rank between vector bundles* $\pi : E \to M$ *and* $\pi' : F \to M$ *or ranks* n *and* m *respectively. Then* $\ker(f)$ *is a sub-bundle of* E *and* $\operatorname{im}(f)$ *is a sub-bundle of* F*, and in particular coker*$(f) := F/\operatorname{im}(f)$ *is a vector bundle.*

Proof. Define $\ker f := \{p \in E \mid f(p) = 0 \in F_{\pi'(f(p))}\}$. Let $\{e_i\}$ be a local frame of E and $\{f_j\}$ on U and let be a local frame of F on some neighbourhood of $f(U)$, where U is chosen so that f is of constant rank. In this frame the homomorphism f takes the form $f(e_i) = A_i^j f_j$ where A is an $n \times m$ matrix of smooth functions. Since f has constant rank, this matrix has constant rank. Thus the kernel has constant rank, and since the coefficients of A are smooth, the set $\ker A := \{s^i(x)e_i(x) \mid A_i^j(x)s^i(x)f_j(x) = 0\}$ is a smooth sub-bundle of E over U. Since this set coincides with $\ker f$ over U, and since smoothness is a local property, we conclude that $\ker f$ is a smooth sub-bundle of E.

In the above notation, the set $\operatorname{im}(A) := \{A_i^j(x)s^i(x)f_j(x) \mid s^i(x)e_i(x) \in E|_U\}$ is a smooth sub-bundle of F locally, since A is of constant rank with smooth coefficients, so $\operatorname{im}(f)$ is a smooth sub-bundle of F. $\square$

1.3 Principal Bundles

1.3.1 Principal G-Bundles

Definition 1.3.1 (Principal Bundle)**.** *Let G be a Lie group. A* principal G-bundle *over a smooth manifold M is a G-bundle P with fibre G (considered as a smooth manifold!), and a smooth right action $P \times G \to P$ of G on P, denoted $(p, g) \mapsto p \cdot g$, such that:*

1. *The local trivialisations $\varphi_\alpha : P|_{U_\alpha} \to U_\alpha \times G$ are G-equivariant with respect to the action of G on P, and right multiplication in the G-factor of $U_\alpha \times G$.*

2. *The action of G on P is free.*

The group G is usually called the structure group *of P.*

Note that interpreted globally, the first condition of Definition 1.3.1 implies that the action of G on P preserves fibres. That is, for every $g \in G$, if R_g denotes right multiplication by g, the following diagram commutes:

$$\begin{tikzcd} P \arrow[rr, "R_g"] \arrow[dr, "\pi"'] & & P \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

One might ask why we require the action of G on P to be a right action rather than a left action. Since such a bundle P is a G-bundle, it admits transition functions with values in the group G. In a local trivialisation, these transition functions are conventionally taken to act on the *left*. Thus if we wish for the action of G on P to be well-defined on overlaps we should require it be a right action, hence commuting with the transition functions.

We state two simple observations with can be checked easily.

Lemma 1.3.2. *Let $p \in P$. Then $P_{\pi(p)} = \{p \cdot g \mid g \in G\}$.*

Lemma 1.3.3. *The fibres of a principal G-bundle P are G-torsors. That is, if $x \in M$, for any $p, q \in P_x$ there exists a unique $g \in G$ such that $pg = q$. In particular, if we fix any $p \in P_x$ we can define a multiplication on P_x by letting $q \cdot r := pgh$ where $q = pg$ and $r = ph$. With respect to this multiplication P_x is a Lie group isomorphic to G with identity p, and isomorphism $\varphi : pg \mapsto g$.*

The interpretation of fibres of P as G-torsors can be a useful way of thinking about principal bundles. In a sense, a principal bundle is *almost* a Lie group bundle. The fibres are just waiting to become isomorphic to G as Lie groups, and all they need is a choice of a single element to call the identity.

A key difference between vector bundles and principal G-bundles is that a principal bundle need not admit *any* smooth sections.

Proposition 1.3.4. *A principal G-bundle $P \to M$ is trivial if and only if it admits a global section.*

Proof. ($\implies$) Suppose P is trivial. Then (up to isomorphism) $P = M \times G$, so we can take a section $s : M \to M \times G$ to be defined by $s(x) = (x, e)$ where $e \in G$ is the identity.

($\impliedby$) Suppose P admits a global section s. Let $\varphi_x : P_x \to G$ be the isomorphism on each fibre given by taking $s(x) \in P_x$ to be the identity, as in Proposition 1.3.3. Then the map $\varphi : P \to M \times G$ defined by $p \mapsto (\pi(p), \varphi_{\pi(p)}(p))$ is a global trivialisation of P. □

The required condition on principal G-bundle homomorphisms is what one would expect. If $P \to M$, $Q \to N$ are two principal G-bundles over base manifolds M and N, then a principal G-bundle homomorphism from P to Q is a fibre bundle homomorphism that is G-equivariant with respect to the right actions of G on P and Q.

As in the case of fibre bundles and vector bundles we have a principal G-bundle construction theorem. All that one needs to check is that there is a smooth right action of G on the fibre bundle $P \to M$ constructed by the fibre bundle construction theorem. This can simply be defined locally by the right multiplication on G, and is well-defined since right multiplication commutes with the left multiplication of the transition functions.

1.3.2 Frame Bundles

Let $E \to M$ be a vector bundle of rank n. There is a natural way to construct a principle $\mathrm{GL}(n, \mathbb{K})$-bundle out of E, that contains all the information about E and any associated vector bundles.

On each fibre E_x, consider the collection $\mathcal{F}(E)_x$ of frames in the vector space E_x. That is, $\mathcal{F}(E)_x$ consists of all ordered bases of E_x. This submanifold of $(E_x)^n$ admits a natural free transitive right action of $\mathrm{GL}(n, \mathbb{K})$. Fixing some auxillary reference basis, if an element of $\mathcal{F}(E)_x$ is written as a matrix of column vectors, then there is a unique matrix $A \in \mathrm{GL}(n, \mathbb{K})$ taking this element of $\mathcal{F}(E)_x$ to any other element.

Define a principal $\mathrm{GL}(n, \mathbb{K})$-bundle $\mathcal{F}(E) \to M$ by taking $\mathcal{F}(E) \subset \underbrace{E \times_M \cdots \times_M E}_{ntimes}$ to be the submanifold (which is a fibre bundle with respect to $\pi|_{\mathcal{F}(E)}$) whose elements consist of $\mathcal{F}(E)_x$ for every $x \in M$, and right action given by the natural $\mathrm{GL}(n, \mathbb{K})$-action on $\mathcal{F}(E)_x$ for every x. This action is clearly smooth, being a restriction of the natural right action of $\mathrm{GL}(n, \mathbb{K})$ on the nth fibred product of E with itself.

Definition 1.3.5 (Frame Bundle)**.** *Let $E \to M$ be a rank n $\mathbb{K}$-vector bundle over a manifold M. Then the bundle $\mathcal{F}(E) \to M$ described above is called the* frame bundle *associated to E.*

As one might expect, the (somewhat) easier way to describe $\mathcal{F}(E)$ is via transition functions. Since E is a $\mathrm{GL}(n, \mathbb{K})$-bundle, one can construct a principal $\mathrm{GL}(n, \mathbb{K})$-bundle with the same transition functions as E. This is the frame bundle described above.

1.3.3 Associated Fibre Bundles

Let P be a principal G-bundle over a manifold M and let F be a smooth manifold. Suppose $\rho : G \to \mathrm{Diff}(F)$ is a homomorphism of groups. We may define a right action of G on the set $P \times F$ by $(p, f) \cdot g = (p \cdot g, \rho(g^{-1})(f))$. Denote the quotient by this group action by $P \times_\rho F$.

We claim that this quotient is a fibre bundle (in fact a G-bundle) with fibre F, and projection given by $\pi'([p, f]) := \pi(p)$ where $\pi : P \to M$ is the projection for P (this is clearly well-defined).

Proposition 1.3.6. *The quotient space $P \times_\rho F$ is a G-bundle with fibre F.*

Definition 1.3.7. *The G-bundle $P \times_\rho F$ is called the* associated fibre bundle to P with fibre F with respect to the representation ρ*.*

This definition in terms of a quotient is not particularly illuminating. In particular it is very difficult to imagine what the fibre bundle $P \times_\rho F$ looks like. The following proposition will give a nice characterisation of the associated bundle.

Proposition 1.3.8. *Suppose $\{g_{\alpha\beta}\}$ is a cocycle for a principal G-bundle P with respect to an open cover $\mathcal{U}$ of a manifold M. If $P \times_\rho F$ is an associated fibre bundle with fibre F, then over the open cover $\mathcal{U}$, $P \times_\rho F$ has transition functions $\{\rho \circ g_{\alpha\beta}\}$.*

That is, the transition functions for $P \times_\rho F$ are just the ρ of the transition functions for P. Essentially, one takes the same gluing data but swaps out one fibre for another. In particular, $P \times_\rho F$ is trivial over the same sets as P, so the associated fibre bundle cannot be more twisted than P. It can of course be less twisted, for if ρ is the trivial representation then $P \times_\rho F$ is just the trivial bundle $M \times F$.

A key property of associated fibre bundles (that will be important when connections on principal bundles are discussed) is that sections of associated bundles can be identified with equivariant maps into the fibre space.

Proposition 1.3.9. *Let $\pi : P \to M$ be a principal G-bundle and $P \times_\rho F$ an associated bundle with fibre F. Then sections of the associated bundle $P \times_\rho F$ are precisely G-equivariant maps $P \to F$ from P into the fibre. These maps are equivariant with respect to the right action of G on P and the right action $x \mapsto \rho(g^{-1})(x)$ of G on F.*

Proof. Suppose $s : M \to P \times_\rho F$ is a section of an associated bundle. Then s is a map that assigns to each $x \in M$ an equivalence class $[p, f] \subset P \times F$ where $(p', f') \in [p, f]$ if $(p', f') = (pg, \rho(g^{-1}(f)))$ for some $g \in G$. Let $p \in P$. Then define $\tilde{s}(p) := f$ where $f \in F$ is such that $s(\pi(p)) = [p, f]$ (check this does actually define a map). Then $\tilde{s} : P \to F$ is smooth since s is smooth into the quotient $P \times_\rho F$.

If $g \in G$ and $\tilde{s}(p) = f$ then $\tilde{s}(pg) = f' \in F$ such that $s(\pi(pg)) = [pg, f']$. But $s(\pi(pg)) = s(\pi(p)) = [p, f]$, and if $[p, f] = [pg, f']$ then by the definition of these equivalence classes we must have $f' = \rho(g^{-1})(f)$. Thus $\tilde{s}(pg) = \rho(g^{-1})\tilde{s}(p)$ and $\tilde{s}$ is equivariant in the appropriate sense.

On the other hand, suppose $\tilde{s} : P \to F$ is a smooth G-equivariant map. Define a map $s : M \to P \times_\rho F$ by $s(x) := [p, \tilde{s}(p)]$ where p is any element of $\pi^{-1}(x)$. Since $\tilde{s}(p)$ depends smoothly on p, this is smooth into the quotient. Clearly s is a section with respect to the projection π' for $P \times_\rho F$. We need to check that s is well-defined. Thus let $q \in \pi^{-1}(x)$ with $q \neq p$. Then $q = pg$ for some $g \in G$, but then $s(x) = [q, \tilde{s}(q)] = [pg, \tilde{s}(pg)] = [pg, \rho(g^{-1})(\tilde{s}(p))] = [p, \tilde{s}(p)]$. Thus s is well-defined. $\square$

Part of the power of principal bundles lies in the technology of associated vector bundles. Suppose now that P is the frame bundle $\mathcal{F}(E)$ of some vector bundle $E \to M$. Then we have seen that from E we may construct many different kinds of vector bundles in natural ways. Each such vector bundle has a fibre of a certain dimension, with transition functions that can easily be written in terms of those of E (and hence $\mathcal{F}(E)$) via simple representations ρ.

For example, the representation that gives E itself is simply the identity $\mathbf{1} : \mathrm{GL}(n, \mathbb{K}) \to \mathrm{GL}(n, \mathbb{K})$, the transition functions for $E \oplus E$ are given by the representation $\rho : \mathrm{GL}(n, \mathbb{K}) \to \mathrm{GL}(2n, \mathbb{K})$ defined by $\rho(A) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$, etc.

Thus all the associated vector bundles of a given vector bundle E can be constructed as associated bundles of its frame bundle $\mathcal{F}(E)$, with representations ρ appropriately chosen. In this sense the frame bundle contains all of the information about E and all of its associated bundles.

We will see later that a connection on $\mathcal{F}(E)$ in the sense of principal bundles will determine connections on all possible associated vector bundles. In the case where the connection on $\mathcal{F}(E)$ comes from a connection E, the following diagram will commute:

$$
\begin{tikzcd}
\text{Connection on E} \arrow[rr, "\text{induces}"] \arrow[dr, "\text{induces}"'] & & \text{Connection on } \mathcal{F}(E) \arrow[dl, "\text{induces}"] \\
& \text{Connection on associated vector bundle} &
\end{tikzcd}
$$

1.4 Gauge Transformations

In this section we will discuss Gauge transformations of vector bundles and principal bundles.

Let $E \to M$ be a vector bundle. Then we can consider the vector bundle $\mathrm{End}(E)$ of all vector bundle homomorphisms from E to itself. The sections of $\mathrm{End}(E)$ consist of assignments of linear maps $E_x \to E_x$ for every $x \in M$. If we restrict ourselves to sections that give linear isomorphisms $E_x \to E_x$ for every x then we get the automorphisms $\mathrm{Aut}(E) \subset \Gamma(\mathrm{End}(E))$ of the vector bundle E.

Of course, these sections are not all the sections of a vector bundle associated to E, for its fibre would have to be $\mathrm{GL}(n, \mathbb{K})$, which is rather a Lie group. The question we will answer is the following: *What* $\mathrm{GL}(n, \mathbb{K})$*-bundle is the* $\mathrm{GL}(n, \mathbb{K})$*-bundle whose sections are precisely those in* $\mathrm{Aut}(E) \subset \Gamma(\mathrm{End}(E))$*?*.

Certainly we know at least one possible choice, namely the frame bundle $\mathcal{F}(E)$. We will see that this is not quite the correct answer to the question momentarily.

1.4.1 Bundle of Gauge Transformations

Let $P \to M$ be a principal G-bundle. The representation $\mathrm{Ad} : G \to \mathrm{Aut}(G)$ given by conjugation gives an associated bundle $\mathrm{Ad}\, P := P \times_{\mathrm{Ad}} G$. Sections $s \in \Gamma(\mathrm{Ad}\, P)$ may be identified with G-equivariant maps $P \to G$, with respect to the natural right action of G on P, and the conjugation (by the inverse) action of G on itself.

Proposition 1.4.1. *Sections of* $\mathrm{Ad}\, P$ *are precisely principal* G*-bundle isomorphisms* $P \to P$.

Proof. Let $s \in \Gamma(\mathrm{Ad}\, P)$. Then s may be identified with a G-equivariant map $s : P \to G$. Define a map $\tilde{s} : P \to P$ by $\tilde{s}(p) := ps(p)$. Since s is smooth this is a smooth map $P \to P$, and preserves fibres. Thus we need to check $\tilde{s}$ is equivariant. Let $g \in G$. Then $\tilde{s}(pg) = pgs(pg) = pgg^{-1}s(p)g = ps(p)g$.

Suppose $f : P \to P$ is a principal G-bundle isomorphism. Then f preserves the fibres of P, so the image $f(p)$ of some element $p \in P$ can be written $f(p) = pf_p$ for some $f_p \in G$. Define a map $\tilde{f} : P \to G$ by $\tilde{f}(p) := f_p$. Clearly this map is smooth, so we need to show it is equivariant in the required sense. That is, we need to show $\tilde{f}(pg) = g^{-1}\tilde{f}(p)g$. Let $q = pg \in P$. Then on the one hand $f(q) = qf_q = pgf_q$, and on the other hand $f(q) = f(pg) = f(p)g = pf_pg$. Since the action of G is free and transitive on fibres, we must have $gf_q = f_pg$, or $f_q = g^{-1}f_pg$. Changing notation, we obtain $\tilde{f}(pg) = g^{-1}\tilde{f}(p)g$ as desired. □

Thus we see that for a principal G-bundle P, automorphisms are sections of $\mathrm{Ad}\, P$ (in particular, they are NOT sections of P itself!). Furthermore, the bundle $\mathrm{Ad}\, P$, while a fibre bundle with fibre G, does not have to look like P itself. Namely, if G is an Abelian Lie group then the bundle $\mathrm{Ad}\, P$ will always be trivial, even if P is not.

Definition 1.4.2 (Gauge Transformations)**.** *The bundle* $\mathrm{Ad}\, P$ *is called the* bundle of gauge transformations*. The group* $\mathscr{G} := \mathscr{G}_P := \mathrm{Aut}(P) = \Gamma(\mathrm{Ad}\, P)$ *of bundle automorphisms of* P *is called the* gauge group *or* group of gauge transformations *of* P*.*

Formally $\mathscr{G}$ is an infinite-dimensional Lie group, and we can attempt to determine its Lie algebra. With some thought one can deduce that the Lie algebra of $\mathscr{G}$ should be the sections of the Lie algebra bundle $\operatorname{ad} P := P \times_{\mathrm{ad}} \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G and $\operatorname{ad} : G \to \operatorname{Aut}(\mathfrak{g})$ is the adjoint representation of G on its Lie algebra.

1.4.2 Gauge Transformations and the Frame Bundle

We will now look at the interpretation of gauge transformations for the case of the frame bundle $\mathcal{F}(E)$ of a vector bundle $E \to M$. In this situation the frame bundle is a principal $\mathrm{GL}(n, \mathbb{K})$-bundle over M whose fibres consist of all ordered bases for the fibres of E.

Consider the group $\operatorname{Aut}(E)$ of vector bundle automorphisms of E.

Proposition 1.4.3. *A vector bundle automorphism $f \in \operatorname{Aut}(E)$ of E is precisely a gauge transformation for the frame bundle $\mathcal{F}(E)$.*

Proof. Suppose $f : E \to E$ is a vector bundle automorphism. Then f induces a linear isomorphism $E_x \to E_x$ for each $x \in M$. If $(e_1, \ldots, e_n)$ is a frame of the vector space E_x, then since f is a linear isomorphism, $(f(e_1), \ldots, f(e_n))$ is again a frame, and this map depends smoothly on the frame. Thus f may be interpreted as a fibre bundle map $\mathcal{F}(E) \to \mathcal{F}(E)$. We need to check this map is equivariant on $\mathcal{F}(E)$. Let $(e_1, \ldots, e_n) \in \mathcal{F}(E)_x$ and $A_i^j \in \mathrm{GL}(n, \mathbb{K})$. Then $f(e_j A_j^i) = f(e_j) A_j^i$ so the map f is $\mathrm{GL}(n, \mathbb{K})$-equivariant.

On the other hand, suppose $f : \mathcal{F}(E) \to \mathcal{F}(E)$ is a gauge transformation of the frame bundle. Let $p \in E$. Then in any frame $p = p^i e_i$ and one can define $f(p) := p^i f(e_i)$. Clearly this is smooth and linear, and does not depend on the particular choice of frame. Thus f may be interpreted as a vector bundle automorphism $E \to E$. $\square$

Thus if $E \to M$ is a vector bundle the *gauge transformations of* E are precisely the gauge transformations of the frame bundle $\mathcal{F}(E)$. We also observe that the sections of $\operatorname{ad} P$ can be identified with vector bundle *homomorphisms* $E \to E$, and in fact

Lemma 1.4.4. *If $E \to M$ is a vector bundle of rank n then $\operatorname{End}(E) \cong \mathcal{F}(E) \times_{\mathrm{ad}} \mathfrak{gl}(n, \mathbb{K})$.*

Proof. The transition functions for $\mathcal{F}(E) \times_{\mathrm{ad}} \mathfrak{gl}(n, \mathbb{K})$ are given by $x \mapsto \operatorname{ad}(g_{\alpha\beta}(x))$ for $x \in U_{\alpha\beta}$. That is, if in a local trivialisation we take a point $(x, A) \in U_{\alpha\beta} \times \mathfrak{gl}(n, \mathbb{K})$ then applying the transition functions takes us to $(x, g_{\alpha\beta}(x) A g_{\alpha\beta}(x)^{-1})$.

Now on the trivialisation U_α fix $\mathfrak{gl}(n, \mathbb{K}) \cong \operatorname{Mat}(n, \mathbb{K}) \cong (\mathbb{K}^n)^* \otimes \mathbb{K}^n$ and write $A = A_i^j \varepsilon^i \otimes e_j$ for a basis $\{e_j\}$ of $\mathbb{R}^n$ and its dual basis $\{\varepsilon^i\}$. Then with respect to this isomorphism we have $\operatorname{ad}(g_{\alpha\beta}) \mapsto g_{\alpha\beta}^{-1} \otimes g_{\alpha\beta}$. That is, if $g_{\alpha\beta} = (g_i^j)$ then

$$\begin{aligned}
(g_{\alpha\beta}^{-1}) \otimes g_{\alpha\beta}(A_i^j \varepsilon^i \otimes e_j) &= A_i^j (g_{\alpha\beta}^{-1})(\varepsilon^i) \otimes g_{\alpha\beta}(e_j) \\
&= A_i^j (g^{-1})_k^i \varepsilon^k \otimes g_j^l e_l \\
&= g_j^l A_i^j (g^{-1})_k^i \varepsilon^k \otimes e_l \\
&= \operatorname{ad}(g_{\alpha\beta})(A_i^j \varepsilon^i \otimes e_j).
\end{aligned}$$

In particular, with the convention of taking the dual frame to have upper indices, these are precisely the transition functions of $\operatorname{End}(E)$. Since the isomorphisms $\mathfrak{gl}(n, \mathbb{K}) \cong$

$\mathrm{Mat}(n, \mathbb{K}) \cong (\mathbb{K}^n)^* \otimes \mathbb{K}^n$ are all canonical, this shows that $\mathrm{End}(E)$ is canonically isomorphic to $\mathcal{F}(E) \times_{\mathrm{ad}} \mathfrak{gl}(n, \mathbb{K})$. $\square$

Chapter 2

Connections and Curvature

2.1 Linear Connections

2.1.1 Why Do We Need Connections?

Suppose one has a vector bundle E and a section $s \in \Gamma(E)$. A natural question to ask is how one might take a derivative of s. Three natural definitions spring to mind. The first is to define the derivative from first principles. This runs into problems immediately.

In $\mathbb{R}^n$ if one has a vector field $X \in \Gamma(\mathbb{R}^n)$, we can take a directional derivative of X, at x, in the direction v, denoted $dX(v)(x)$, using the following expression:

$$dX(v)(x) := \lim_{t\to 0} \frac{X(x+tv) - X(x)}{t}.$$

Notice that $dX(v)(x) \in \mathbb{R}^n$ for each x, so we obtain another vector field. When we pass to a manifold, we run into trouble. Firstly, there is no notion of adding two points on a manifold in the way we have written $x+tv$. This is easily rectified of course. Given a tangent vector $v \in T_xM$, we can take a path γ such that $\gamma(0) = x$ and $\gamma'(0) = v$, and then consider

$$\lim_{t\to 0} \frac{X(\gamma(t)) - X(x)}{t}. \qquad \text{(Eq. 2.1)}$$

Note this expression may depend on the particular path γ chosen, but we are making progress. Of course this statement is still not well-defined. For each t, $X(\gamma(t)) \in T_{\gamma(t)}M$, the tangent space at the point $\gamma(t)$. But $X(x) \in T_xM$, the tangent space at x. Unless $t = 0$, these are different vector spaces, so there is no way of subtracting these two vectors. In $\mathbb{R}^n$ there is no problem, because one may transport everything to the origin and perform computations there, and then transport back. Without a linear structure, there is no obvious way of doing the same thing on a manifold. This same problem of course arises for any vector bundle over a manifold, not just the tangent bundle.

The second natural definition to consider is simply to take a local trivialisation U of a vector bundle E, and write a section $s = s^i e_i$ for the local frame $\{e_i\}$. Then one could define a derivative by saying $Ds := ds^i \otimes e_i$. When passing between trivialisations, the coefficients of a section in a local frame transform by $s^i \mapsto s^j g^i_j$ for the transition matrix

g. Thus the definition is not well-defined, unless the transition functions g^i_j are constant, so that $d(s^j g^i_j) = ds^j g^i_j$. This is in general not the case.

The third natural definition, and perhaps the most obvious of all, is to consider a section $s \in \Gamma(E)$ as a map $s : M \to E$, and take its differential $ds : TM \to TE$. This is of course a perfectly fine notion of derivative. There are perhaps two complaints. Firstly, it seems to use nothing about the map s being a section, except that the differential satisfies $d\pi \circ ds = \mathbf{1}$. Secondly, it uses nothing at all about E being a vector bundle over M.

Our aim is to describe a notion of derivative of a section that acts similarly to a directional derivative of vector fields. In particular, one might hope that given a section, and a vector field, the result would be another section describing the change of the section in the direction of the vector field. Let $X \in \Gamma(TM)$ be such a vector field. Then we can consider $ds(X) \in \Gamma(TE)$. The problem with the differential is that $ds(X)$ is not a section of E itself.

We can however use an interesting property of vector bundles at this point. The *vertical sub-bundle* V of the tangent bundle TE is the sub-bundle defined by $d\pi(v) = 0$ for all $v \in \Gamma(V)$. This is a canonically defined sub-bundle of TE. When E is a vector bundle, the vertical subspace of the tangent space at $p \in E$ is canonically identified with the fibre $E_{\pi(p)}$, in the same way that the tangent space to a vector space is canonically identified with the vector space itself.

We could therefore obtain a section of E that is a directional derivative of s in the direction of X if we could 'take the vertical part' of the vector field $ds(X) \in \Gamma(TE)$. However, there is no canonical projection onto this vertical sub-bundle. Thus we can *define* a connection to be a projection onto this vertical sub-bundle, satisfying suitable linearity conditions.

What has just been described is an Ehresmann connection on a vector bundle. This theme will be expanded upon in Section 2.2, where we will construct the Ehresmann connection for general fibre bundles, and show how the vector bundle case naturally produces the notions we will present in this section.

The rest of this section on linear connections will focus on the correct way of rectifying the issues presented by the first two natural but flawed definitions of a connection. In particular, we will develop along the following three paths:

1. Look at what properties a derivative of sections *should* satisfy, and then define a connection as something satisfying these properties.

   This is the approach of Definition 2.1.1 that we give in the next section.

2. Fix an isomorphism between any two fibres of the vector bundle, depending on any path between their base points, and define a derivative in the direction of a vector field at each fibre according to (Eq. 2.1).

   This is the notion of *parallel transport*, which we will investigate in Section 2.1.6. Indeed this is the motivation for the term *connection*, as the above recipe describes how to connect fibres of the vector bundle.

3. Attempt to define a derivative naively in terms of local trivialisations, observe what goes wrong, and then define connections as objects that rectify what goes wrong.

   This naturally leads to the consideration of local connection forms, which we will investigate in Section 2.1.3. In particular, these local forms transform in precisely the right way to counteract the extra dg term that is picked up when passing between local trivialisations.

2.1.2 The Invariant Definition of a Connection

Definition 2.1.1 (Linear Connection). *A connection on a vector bundle $E \to M$ is a $\mathbb{K}$-linear map*

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$$

such that for all $f \in C^\infty(M)$, $s \in \Gamma(E)$,

$$\nabla(fs) = df \otimes s + f\nabla(s).$$

Proposition 2.1.2. *In a local trivialisation (U, φ) the exterior derivative is a connection.*

Proof. Let $s \in \Gamma(E|_U)$ be a local section of E. Then we know $s = s^i e_i$ for the local frame e_i induced by $\varphi : E|_U \to U \times \mathbb{R}^n$. Let $ds := (ds^i) \otimes e_i$. Then the claim is that d acting in this way is a connection on $E|_U$.

Let $f \in C^\infty(U)$. Then $fs = (fs^i)e_i$, so $d(fs) = ((df)s^i + f ds^i) \otimes e_i = df \otimes s + f ds$. □

Corollary 2.1.3. *Connections exist. Given a vector bundle E of rank n over a manifold M, there exists a connection on E.*

Proof. Let $\{U_\alpha\}$ be a trivialising open cover for E. Then on each U_α we have a connection d_α that is just the exterior derivative d on U_α with its local frame. Let $\{\rho_\alpha\}$ be a smooth partition of unity subordinate to $\{U_\alpha\}$, and define

$$\nabla := \sum_\alpha \rho_\alpha d_\alpha.$$

This is a well defined operator on sections of E, for if $s \in \Gamma(E)$ then whenever $\rho_\alpha \neq 0$, we have $s = s^i e_i$ for some e_i a local frame of U_α, thus we may apply d_α to s in this frame.

To verify the Leibniz rule we have

$$\begin{aligned}\nabla(fs) &= \sum_\alpha \rho_\alpha d_\alpha(fs) \\ &= \sum_\alpha \rho_\alpha df \otimes s + \rho_\alpha f d_\alpha(s) \\ &= df \otimes s + f\nabla(s).\end{aligned}$$

□

A connection is a map from $\Gamma(E)$ to $\Omega^1(M)\otimes_{C^\infty(M)}\Gamma(E) = \Gamma(T^*M\otimes_{\mathbb{R}} E)$. We know that if s is a section of E and u is a section of the vector bundle $\mathrm{Hom}(E, T^*M \otimes_{\mathbb{R}} E)$, then $u(s)$ is a section of $T^*M \otimes_{\mathbb{R}} E$. This motivates the question: *Is a connection a section of the bundle* $\mathrm{Hom}(E, T^*M \otimes_{\mathbb{R}} E)$*?*

Remark 2.1.4. That $\Omega^1(M) \otimes_{C^\infty(M)} \Gamma(E) = \Gamma(T^*M \otimes_{\mathbb{R}} E)$ is a question of sheaf isomorphisms. This is proved in Section 3.1.3.

This answer to this question is clearly no. The bundle $\mathrm{Hom}(E, T^*M \otimes_{\mathbb{R}} E)$ is a real vector bundle over M, and is thus $\Gamma(\mathrm{Hom}(E, T^*M \otimes_{\mathbb{R}} E))$ is a module over $C^\infty(M)$ (by linearity on each fibre). This means, for every $s \in E$, we must have $u(fs) = fu(s)$. But we have just seen that for a connection we have $\nabla(fs) = df \otimes s + f\nabla s$. This can be seen as a result of the following lemma. The proof is that presented in the book of Taubes [Tau11].

Lemma 2.1.5. *Let E and F be vector bundles and $K : E \to F$ a bundle map such that $K(fs) = fK(s)$ for all $f \in C^\infty(M)$ and $s \in \Gamma(E)$. Then there exists a unique section u of* $\mathrm{Hom}(E, F)$ *such that $K(s)_x = u_x(s_x)$ for all $s \in \Gamma(E)$, $x \in M$.*

Proof. □

Notice that the term $df \otimes s$ in the definition of a connection (Definition 2.1.1) does not depend on the particular connection ∇. Indeed, if ∇^1 and ∇^2 are two connections on E, then we have $(\nabla^1 - \nabla^2)(fs) = f(\nabla^1 - \nabla^2)s$ for all $f \in C^\infty(M)$ and $s \in \Gamma(E)$. By Lemma 2.1.5, $\nabla^1 - \nabla^2$ may be identified uniquely with a section of $\mathrm{Hom}(E, T^*M \otimes_{\mathbb{R}} E)$, which we also refer to as $\nabla^1 - \nabla^2$. Thus we arrive at the following result:

Proposition 2.1.6. *The difference of two connections is an element of* $\Gamma(\mathrm{Hom}(E, T^*M \otimes_{\mathbb{R}} E))$*.*

Let the set of all connections on a vector bundle E be denoted by $\mathscr{A}$. The above result has the following corollary.

Corollary 2.1.7. *Given a connection ∇ and a section $a \in \Gamma(\mathrm{Hom}(E, T^*M \otimes_{\mathbb{R}} E))$, $\nabla + a$ is also a connection. In particular, the set $\mathscr{A}$ of all connections on a vector bundle E is an infinite-dimensional affine space modelled on the vector space* $\Gamma(\mathrm{Hom}(E, T^*M \otimes_{\mathbb{R}} E))$*.*

By 'affine space modelled on $\Gamma(\mathrm{Hom}(E, T^*M \otimes_{\mathbb{R}} E))$' we mean, given a choice of fixed connection ∇, $\mathscr{A}$ becomes a vector space isomorphic to the real vector space $\Gamma(\mathrm{Hom}(E, T^*M \otimes_{\mathbb{R}} E))$, with origin ∇. In particular, $\mathscr{A}$ is infinite dimensional.

To simplify notation in what follows, if E is a vector bundle, then $\Gamma(\bigwedge^k T^*M \otimes_{\mathbb{R}} E)$ will be written $\Omega^k(E)$. Note that $\Gamma(\bigwedge^k T^*M \otimes_{\mathbb{R}} E) = \Omega^k(M) \otimes_{C^\infty(M)} \Gamma(E)$. The sections in $\Omega^k(E)$ are called *E-valued k-forms on M*. One could equivalently refer to such sections as *k-form-valued sections of E*.

With this convention, we see a connection is a map $\nabla : \Omega^0(E) \to \Omega^1(E)$ satisfying a Leibniz rule. Furthermore, if ∇^1 and ∇^2 are two connections on E, then $\nabla^1 - \nabla^2 \in \Omega^1(\mathrm{End}(E))$.

Remark 2.1.8. Note that fibrewise, we have $\mathrm{Hom}(E, T^*M \otimes E)_x \cong (E^* \otimes T^*M \otimes E)_x \cong (T^*M \otimes E^* \otimes E)$. But $E^* \otimes E \cong \mathrm{End}(E)$, so the difference of two connections is actually a section of the bundle $T^*M \otimes \mathrm{End}(E)$. This observation is useful in that we see the difference of two connections looks like a sum $\omega^i \otimes u_i$ where the $\omega^i \in \Omega^1(M)$ and the $u_i \in \Gamma(\mathrm{End}(E))$.

Remark 2.1.9. Much as in Section A.1 for matrices, we may construct a local frame of the vector bundle $\mathrm{End}(E)$ that acts as the basis E^i_j of matrices acting on a vector space V. Let $\{e_i\}$ be a local frame for E over a trivialising set U. Then one may define a dual frame $\{\epsilon^i\}$ of E^* by the expression $\epsilon^i(e_j) = \delta^i_j$. Equivalently, one could take the local trivialisation on E^* defined by the cocycle for E^* obtained from E as in Section 1.2.5. That these two local frames agree is a good exercise in the expressions of these objects in local coordinates. In any case, one can define $E^i_j := \epsilon^i \otimes e_j \in \Gamma(\mathrm{End}(E)|_U)$.

Remark 2.1.10. Suppose $A = \omega^i \otimes u_i$ is a section of $T^*M \otimes \mathrm{End}(E)$. Then in light of Section A.1 and Remark 2.1.9, we know that $u_i = U^j_{ik} E^i_j$, where E^i_j is the standard matrix basis frame for $\mathrm{End}(E)$ given in terms of the local frame $\{e_i\}$ of E. Then we have $A = \omega^i \otimes U^j_{ik} E^i_j = (\omega^i U^j_{ik}) \otimes E^i_j$. That is, with respect to the standard basis, A has matrix coefficients $A^j_k = \omega^i U^j_{ik}$. These matrix coefficients are one-forms by virtue of the ω^i, so we see that A is actually a *matrix of one-forms*. This is the correct way of thinking of sections of $T^*M \otimes \mathrm{End}(E)$, or indeed of $\bigwedge^k T^*M \otimes \mathrm{End}(E)$ for any k.

2.1.3 The Local Description of a Connection

The preceding results about differences of connections give us a powerful local characterisation of a connection. Let E be a vector bundle and $\{U_\alpha\}$ be a trivialising cover for E. Suppose ∇ is a connection on E. Then in particular ∇ is a connection on $E|_{U_\alpha}$. But we already know of an essentially canonical connection on U_α, the exterior derivative d. Thus there exists some A_α such that $\nabla|_{U_\alpha} = d + A_\alpha$. This means, given some section $s \in \Gamma(E|_{U_\alpha})$,

$$\nabla s = ds + A_\alpha s \qquad \text{(Eq. 2.2)}$$

This notation is in fact quite confusing. The section s is not a function on M, so how do we apply d to it? Also, A is not simply a matrix that one may apply to the vector s.

We will now investigate this expression in detail in the local frame on a trivialising open set U. Here we will call the matrix of one-forms simply A.

Let $\{e_i\}$ be the local frame on U. Then we have $s = s^i e_i$ for some smooth functions $s^i \in C^\infty(U)$. The exterior derivative applied to s will mean the exterior derivative considered as a connection on U. That is, $ds := (ds^i) \otimes e_i$. If we write s as a vector in the local frame $\{e_i\}$ we have

$$s = \begin{pmatrix} s^1 \\ \vdots \\ s^n \end{pmatrix}$$

and ds corresponds to the element-wise exterior derivative in this vector of functions. Now we have A^i_j is a matrix of one-forms which we wish to apply to s. First let us determine how the matrix A looks when applied to a vector in the frame, e_i say. We have

$$\nabla e_i = de_i + Ae_i.$$

Now $de_j = 0$ for all j, because the coefficients of e_j in the frame are all constant. Furthermore if we consider A as a matrix of one forms, and e_i as a column vector with a 1 in the ith row, then the multiplication Ae_i is the following:

$$\begin{pmatrix} A_1^1 & \cdots & A_n^1 \\ \vdots & \ddots & \vdots \\ A_1^n & \cdots & A_n^n \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} A_i^1 \\ \vdots \\ A_i^n \end{pmatrix} = A_i^j \otimes e_j.$$

Thus we have that As is the vector $As = A(s^i e_i) = (s^i A_i^j) \otimes e_j$, and further this may be interpreted as the matrix multiplication

$$As = \begin{pmatrix} A_1^1 & \cdots & A_n^1 \\ \vdots & \ddots & \vdots \\ A_1^n & \cdots & A_n^n \end{pmatrix} \begin{pmatrix} s^1 \\ \vdots \\ s^n \end{pmatrix} = \begin{pmatrix} s^i A_i^1 \\ \vdots \\ s^i A_i^n \end{pmatrix}.$$

It is important to realise that we are interpreting the $\mathrm{End}(E)$-valued one-form A here as a matrix, but the process we are actually doing is taking a section s, and looking at $A = \omega^i \otimes u_i$ for some differential forms ω^i and endomorphisms u_i, and then taking $As = \omega^i \otimes (u_i(s))$. If one writes A in the basis of endomorphisms E_j^i and compares the answers, one will see that $\sum_{i,j} A_j^i \otimes E_j^i(s)$ is precisely the vector $(s^j A_i^j) \otimes e_j$ that we have found doing matrix multiplication.

It is very instructive to sit down and confirm and convince yourself that this is the case. It will help immensely in feeling comfortable with connections and curvature.

In any case, we have the full formula for the connection ∇ applied to a vector s:

$$\begin{aligned} \nabla(s) &= (d + A)s \\ &= ds^i \otimes e_i + s^i A_i^j \otimes e_j \\ &= ds^i \otimes e_i + s^j A_j^i \otimes e_i \\ &= (ds^i + s^j A_j^i) \otimes e_i \end{aligned}$$

Definition 2.1.11 (Connection form)**.** *Let U be a trivialising set for a vector bundle E and suppose ∇ is a connection on E. Then the* $\mathrm{End}(E)$*-valued one-form* $A \in \Omega^1(\mathrm{End}(E|_U))$ *defined by* $A := \nabla|_U - d$ *is called the* local connection one-form for ∇ on the trivialising set U.

The above definition is perhaps one of the most important in the entire theory of vector bundles. As a fair warning for the rest of these notes, and for the entire world of differential geometry, the characters ∇ and A are used interchangeably to refer to a connection on a vector bundle. The letter A will be used to refer to the connection as well as its local connection form in any given trivialisation. Only when one is interested in the difference between connection forms on different trivialisations will care be taken in specifying which connection one-form is being used.

This will become confusing if you do not keep track of what is going on. It is vitally important to point out that the connection forms defined locally do *not* piece together to form a global $\mathrm{End}(E)$-valued one-form.

Suppose U_α, U_β are two trivialising open sets for E with $U_{\alpha\beta} \neq 0$, and let ∇ be a connection on E. Then we have local connection forms A_α and A_β defined as above. Suppose $\{e_i\}$ is the frame on U_α and $\{f_j\}$ is the frame on U_β.

Let $g^i_j := (g_{\beta\alpha})^i_j$. Then we know from Section 1.2.3 that the frames transform by (Eq. 1.4). That is,

$$e_i = f_j g^j_i \qquad \text{(Eq. 2.3)}$$

Now in the local frame on U_α we have

$$\nabla e_i = (A_\alpha)^j_i \otimes e_j.$$

Using (Eq. 2.3), this becomes

$$\nabla(f_l g^l_i) = (A_\alpha)^j_i \otimes (f_k g^k_j).$$

By the Leibniz rule for the connection, this becomes

$$dg^l_i \otimes f_l + g^l_i \nabla f_l = g^k_j (A_\alpha)^j_i \otimes f_k.$$

Now using the connection with respect to U_β we have

$$\nabla f_l = (A_\beta)^k_l \otimes f_k,$$

so we obtain

$$dg^l_i \otimes f_l + g^l_i (A_\beta)^k_l \otimes f_k = g^k_j (A_\alpha)^j_i \otimes f_k.$$

Changing the index in the first term, we find

$$dg^k_i + g^l_i (A_\beta)^k_l = g^k_j (A_\alpha)^j_i$$

for all k.

To turn this index expression into matrix multiplication, we need to rearrange the repeated indices to go from lower index to higher index, after which we obtain

$$dg_{\beta\alpha} + A_\beta g_{\beta\alpha} = g_{\beta\alpha} A_\alpha.$$

Multiplying on the left by $g_{\beta\alpha}^{-1}$, we conclude

$$A_\alpha = g_{\beta\alpha}^{-1} dg_{\beta\alpha} + g_{\beta\alpha}^{-1} A_\beta g_{\beta\alpha} \qquad \text{(Eq. 2.4)}$$

This formula for how the connection form changes under a change of basis is one of the key ways of interpreting a connection. In particular it shows that the connection forms do not piece together to make a global form. If the local forms pieced together to make a global form, they would transform according to the change of basis rule on overlaps, but the extra $g_{\beta\alpha}^{-1}dg_{\beta\alpha}$ term prevents this.

Remark 2.1.12. Suppose one attempted to define a connection naively by taking the exterior derivative of the coefficients of a section in a local frame (this is precisely what the derivative of a vector field in $\mathbb{R}^n$ is). Doing this calculation one would observe that on an overlap, one would differentiate the transition functions $g_{\beta\alpha}$, and by the Leibniz rule, pick up a term of the form $g_{\beta\alpha}^{-1}dg_{\beta\alpha}$. This is not a coincidence. A connection form transforms according to (Eq. 2.4) precisely to cancel out this factor of $g_{\beta\alpha}^{-1}dg_{\beta\alpha}$ that one picks up when changing coordinates.

We conclude this section by observing that a collection of local forms satisfying (Eq. 2.4) piece together to give a connection with local action $d + A_\alpha$. This is a straight forward verification that the resulting connection satisfies a Leibniz rule, and is well-defined on overlaps. The former is due to the exterior derivative in $d + A_\alpha$, and the latter is satisfied because of (Eq. 2.4). To sum up:

Theorem 2.1.13. *Let $E \to M$ be a vector bundle with local trivialisation $\{(U_\alpha, \varphi_\alpha)\}$. Suppose $\{A_\alpha\}$ is a collection of* $\mathrm{End}(E)$*-valued one-forms with $A_\alpha \in \Omega^1(\mathrm{End}(E|_{U_\alpha}))$ such that*

$$A_\alpha = g_{\beta\alpha}^{-1}dg_{\beta\alpha} + g_{\beta\alpha}^{-1}A_\beta g_{\beta\alpha}$$

on overlaps $U_{\alpha\beta}$, where $\{g_{\alpha\beta}\}$ are the transition functions with respect to the trivialisation. Then the linear operator ∇ defined by $\nabla|_{U_\alpha} := d + A_\alpha$ is a connection on E.

2.1.4 Gauge Transformations

A gauge transformation of a vector bundle E is an element of $\mathrm{Aut}(E)$, a vector bundle isomorphism from E to itself. By composition, the the gauge transformations of a bundle E form a group $\mathscr{G}$.

A gauge transformation $u \in \mathscr{G}$ can be identified with a section of of the bundle $Ad(F(E))$. That is, a gauge transformation is a smooth assignment of a linear isomorphism $E_x \to E_x$ for each fibre E_x of the vector bundle x. The composition of bundle isomorphisms in $\mathscr{G}$ is precisely the fibrewise composition of linear isomorphism under this identification.

Given a gauge transformation $u \in \mathscr{G}$, one obtains a gauge transformation of the bundle of E-valued one-forms, $T^*M \otimes E$, which is simply the identity on the T^*M part of the fibre $T_x^*M \otimes E_x$. We will also call this u. Let ∇ be a connection on E. Then we have a commutative diagram

$$\begin{array}{ccc} \Omega^0(E) & \xrightarrow{\nabla} & \Omega^1(E) \\ \big\downarrow u & & \big\downarrow u \\ \Omega^0(E) & \xrightarrow{u\cdot\nabla} & \Omega^1(E) \end{array}$$

where $u \cdot \nabla(s) := u(\nabla(u^{-1}(s)))$.

Lemma 2.1.14. *The map $u \cdot \nabla$ is a connection on E.*

Proof. Let $f \in C^\infty(M)$ and $s \in \Omega^0(E)$. Then because $u, u^{-1} \in \mathscr{G}$,

$$\begin{aligned} u \cdot \nabla(fs) &= u(\nabla(u^{-1}(fs))) \\ &= u(\nabla(fu^{-1}(s))) \\ &= u(df \otimes u^{-1}(s)) + u(f\nabla(u^{-1}(s))) \\ &= df \otimes s + fu \cdot \nabla. \end{aligned}$$

Thus $u \cdot \nabla$ satisfies the Leibniz rule, and is a connection on E. □

If we consider the space of all connections as the affine space $\mathscr{A}$, then we have $u \cdot \nabla \in \mathscr{A}$ whenever $\nabla \in \mathscr{A}$.

Lemma 2.1.15. *The map $\mathscr{G} \times \mathscr{A} \to \mathscr{A}$ defined by $(u, \nabla) \mapsto u \cdot \nabla$ is a left action of $\mathscr{G}$ on $\mathscr{A}$.*

Proof. Clearly $\mathbf{1} \cdot \nabla = \nabla$. Given $u, v \in \mathscr{G}$, $s \in \Omega^0(E)$, we have

$$\begin{aligned} (uv) \cdot \nabla(s) &= (uv)(\nabla((uv)^{-1}(s))) \\ &= u(v\nabla v^{-1}(u^{-1}(s))) \\ &= u \cdot (v \cdot \nabla)(s). \end{aligned}$$

□

One might be tempted to now take a quotient $\mathscr{A}/\mathscr{G}$. This presents many problems, that we will not investigate at this time. In particular, one must specify a topology on $\mathscr{A}$, and even when this is done, the quotient space will not be Hausdorff, and may have other pathological properties. Defining such a quotient in a suitable way is a question in the realm of the theory of *moduli spaces*.

We will conclude this brief discussion by mentioning how the connection form changes under a gauge transformation. On a trivialisation U, for which the connection has local form A, a gauge transformation is equivalent to a smooth map $u : U \to \mathrm{GL}(n, \mathbb{K})$. Let φ be the trivialisation map on U. Define a new trivialisation $(U, \tilde{\varphi})$ by defining $\tilde{\varphi} := \varphi \circ u$. If $\{e_i\}$ is the local frame with respect to φ and $\{f_j\}$ is the local frame with respect to

$\tilde{\varphi}$, then $u^{-1}(e_i) = f_i$. Let $\nabla(f_i) = B_i^j \otimes f_j$ for a local connection form B with respect to the trivialisation $\tilde{\varphi}$. Then

$$\begin{aligned} u \cdot \nabla(e_i) &= u(\nabla(u^{-1}(e_i))) \\ &= u(\nabla(f_i)) \\ &= u(B_i^j \otimes f_j) \\ &= B_i^j \otimes u(f_j) \\ &= B_i^j \otimes e_j \end{aligned}$$

Thus the connection form for $u \cdot \nabla$ with respect to the local trivialisation (U, φ) is the same as the connection form for ∇ with respect to the local trivialisation $(U, \varphi \circ u)$. This if we can show that the transition function g from (U, φ) to $(U, \tilde{\varphi})$ is u, (Eq. 2.4) will give us the transformation law.

Now the composition $\tilde{\varphi} \circ \varphi^{-1}$ takes (x, v) to $(x, u(v))$. To see this, let $\{\alpha_i\}$ be the standard basis of $\mathbb{K}^n$. Then $\varphi^{-1}(x, v^i \alpha_i) = v^i e_i$, so $u \circ \varphi^{-1}(x, v^i \alpha_i) = v^i u(e_i) = v^i u_i^j e_j$. Then $\tilde{\varphi} \circ \varphi^{-1}(x, v^i \alpha_i) = (x, v^i u_i^j \alpha_j) = (x, u(v^i \alpha_i))$, so the transition function from φ to $\tilde{\varphi}$ is precisely u. Then by (Eq. 2.4) we have

$$A = u^{-1} du + u^{-1} B u.$$

To sum up:

Proposition 2.1.16. *Let ∇ be a connection on a vector bundle E, and $u \in \mathscr{G}$ be a gauge transformation of E. If ∇ is has local connection form A and $u \cdot \nabla$ has local connection form B, then*

$$B = uAu^{-1} + du\, u^{-1}.$$

Later we will see this same formula appear in the more general case of gauge transformations on principal G-bundles. In the case where G is a linear group (as in the case of vector bundles), the Maurer-Cartan form of G is $g^{-1}dg$. Indeed, in this general case the $u^{-1}du$ term obtained above will be replaced by the Maurer-Cartan form of the Lie group.

That the Maurer-Cartan form is $g^{-1}dg$ as opposed to $dg\, g^{-1}$ explains why we sometimes choose to express the change of formula in the opposite way to what one might expect (i.e. choosing to write $A =$ instead of $B =$, even though we started with A).

2.1.5 Directional Derivatives

Given a section $s \in \Omega^0(E)$ and a connection ∇ on E, we have $\nabla(s) \in \Omega^1(E)$. In particular $\nabla(s) = \omega^i \otimes s_i$ for some one-forms $\omega^i \in \Omega^1(M)$ and some sections $s_i \in \Omega^0(E)$.

Given a vector field $X \in \Gamma(TM)$, we can contract $\nabla(s)$ with this vector field to obtain $\nabla_X(s)$. In particular this is done by writing $\nabla_X(s) = \omega^i(X) s_i$. Notice that $\nabla_X(s) \in \Omega^0(E)$.

Definition 2.1.17 (Covariant Derivative)**.** *The* covariant deriative *of a section $s \in \Omega^0(E)$ in the direction of $X \in \Gamma(TM)$ is the section*

$$\nabla_X s \in \Gamma(E).$$

The covariant derivative of s in the direction of X may also be referred to as the *directional derivative* of s in the direction of X.

Lemma 2.1.18. *The covariant derivative is $C^\infty(M)$-linear in the X argument.*

Proof. This follows immediately from observing that $\omega(fX + gY) = f\omega(X) + g\omega(Y)$ for vector fields X, Y, smooth functions f, g, and a one-form ω. $\square$

Definition 2.1.19 (Horizontal Section)**.** *A section $s \in \Omega^0(E)$ is called* horizontal *if and only if $\nabla_X s = 0$ for all $x \in \Gamma(TM)$. Equivalently, s is horizontal if and only if $\nabla s = 0 \in \Omega^1(E)$.*

A horizontal section is also called a *covariantly constant* section. That the two definitions given are equivalent follows from Lemma 2.1.18. The terminology 'horizontal' will be explained more thoroughly when we investigate Ehresmann connections in Section 2.2. For now, one may interpret it as saying they are horizontal much the same way as a constant function is a horizontal line.

Lemma 2.1.20. *The covariant derivative in the direction of a vector field X at a point $p \in M$ depends only on the value of X at p.*

Proof. This is essentially just the statement that a differential form is $C^\infty(M)$ linear. In particular ∇s is an E-valued one-form, and when contracting with a vector field X, $\nabla s(p)$ acts like an E_p-valued linear functional on T_pM, so clearly only depends on X_p, the element of T_pM that is put into it. $\square$

Definition 2.1.21 (Covariant Derivative Along a Curve)**.** *Let $\gamma : (a, b) \to M$ be a curve, and suppose $s \in \Omega^0(E)$ is a section of E. Let X be a vector field defined on a neighbourhood of γ such that $X_{\gamma(t)} = \dot{\gamma}(t)$ for every $t \in (a, b)$. Define an operator*

$$\frac{D}{dt}$$

by

$$\frac{Ds}{dt}(t_0) := \nabla_X s(\gamma(t_0))$$

for all $t_0 \in (a, b)$.

Remark 2.1.22. Such an extension X of $\dot{\gamma}$ to a neighbourhood of γ clearly exists (since one may define it on local trivialisations and use a partition of unity). By Lemma 2.1.20 the definition above is well-defined. That is, it does not depend on the particular extension X of $\dot{\gamma}$.

Lemma 2.1.23. *The covariant derivative of a section $s \in \Omega^0(E)$ along a curve γ only depends on the values of s along the curve γ.*

Proof. Suppose γ is contained entirely within a single trivialising set U, and A is the corresponding local connection form. Then the expression for $\frac{Ds}{dt}(t_0)$ for $t_0 \in [0,1]$ is

$$\frac{Ds}{dt}(t_0) = (ds^i(\dot{\gamma}(t_0)) + s^j(t_0)A^i_j(\dot{\gamma}(t_0)))e_i.$$

The second term of this expression only depends on the value of s at t_0. Furthermore, $\dot{\gamma}(t_0)$ is defined to be $d\gamma_{t_0}(\frac{\partial}{\partial t}(t_0))$, so $ds^i(\dot{\gamma}(t_0)) = d(s^i \circ \gamma)_{t_0}(\frac{\partial}{\partial t}(t_0))$. But since we are composing s^i with γ, this term then only depends on the value of s^i along the curve γ. $\square$

Remark 2.1.24. Because of Lemma 2.1.23, one may take the covariant derivative along a curve γ of a section defined *only* along the curve γ. In particular one must take an extension of the section s to a neighbourhood of γ, and the lemma implies the result does not depend on the choice of extension. This will be useful when we give define parallel transport in the next section.

Definition 2.1.25 (Horizontal Section Along a Curve)**.** *Let $s \in \Omega^0(E|_\gamma)$ be a section defined along a curve γ in M. The section s is called* horizontal along the curve γ *if $\frac{Ds}{dt} = 0$, where $\frac{D}{dt}$ is the covariant derivative along γ.*

Again, the section s may equivalently be called *covariantly constant along* γ. To link with the previous definition of horizontal sections, a section s of the whole vector bundle E is horizontal if and only if it is horizontal along every curve.

2.1.6 Parallel Transport and Holonomy

Parallel Transports

Let $\gamma : [0,1] \to M$ be a smooth path in a smooth manifold M. Let $\xi_0 \in E_{\gamma(0)}$.

Given a connection ∇ on the vector bundle E, there is a unique element $\xi_1 \in E_{\gamma(1)}$ called the *parallel transport of ξ_0 along γ.*

Let s be a section of E along γ. Then we may consider the differential equation

$$\frac{Ds}{dt} = 0; \quad s(\gamma(0)) = \xi_0. \qquad \text{(Eq. 2.5)}$$

Suppose that the image of γ is contained within a single trivilising set U for the vector bundle E, and that A is the connection one-form with respect to this trivialisation. Suppose $s = s^i e_i$ where $\{e_i\}$ is the local frame of U, and $s^i \in C^\infty(U)$. Then we saw in Lemma 2.1.23 that

$$\frac{Ds}{dt} = (ds^i(\dot{\gamma}) + s^j A^i_j(\dot{\gamma}))e_i.$$

Thus in the local trivialisation U, (Eq. 2.5) is the following system of ordinary differential equations in the coefficients s^i of the section s.

$$ds^i(\dot\gamma) + s^j A^i_j(\dot\gamma) = 0, \quad s(\gamma(0)) = \xi_0. \qquad \text{(Eq. 2.6)}$$

Note that $\dot\gamma(t) \in T_{\gamma(t)}M$ is defined to be

$$\dot\gamma(t) := d\gamma_t \left(\left.\frac{\partial}{\partial t}\right|_t \right).$$

Thus $ds^i(\dot\gamma) = ds^i \circ d\gamma \left(\frac{\partial}{\partial t}\right) = d(s^i \circ \gamma)\left(\frac{\partial}{\partial t}\right)$. By definition this is denoted $\frac{ds^i}{dt}$, so we may write the equations (Eq. 2.6) in the more suggestive form

$$\frac{ds^i}{dt} + s^j A^i_j(\dot\gamma) = 0, \quad s(\gamma(0)) = \xi_0.$$

Now existence and uniqueness theorems for solutions to systems of ordinary differential equations state that for any given initial condition $s(\gamma(0)) = \xi_0$, there is a unique smooth solution s, a section along γ. Define $\xi_1 := s(\gamma(1))$.

Definition 2.1.26. *The element $\xi_1 \in E_{\gamma(1)}$ constructed above is called the* parallel transport *of ξ_0 along γ.*

Remark 2.1.27. The above construction was restricted to the case where the image of γ was contained within a single trivialisation. If γ passes between trivialisations, then on each trivialisation it passes through there is a unique s, so the construction above applied to each trivialisation must agree on overlaps by uniqueness. Thus one obtains a well-defined parallel transport along any γ.

Remark 2.1.28. Suppose $\gamma_1, \gamma_2 : [0,1] \to M$ are two smooth curves such that $\gamma_1(1) = \gamma_2(0)$. Define the parallel transport of $\xi_0 \in E_{\gamma_1(0)}$ along the concatenated curve $\gamma_2 \cdot \gamma_1$ by first taking the parallel transport of ξ_0 along γ_1, and then parallel transporting the resulting vector along γ_2. This gives a notion of parallel transport along piecewise smooth curves in M.

Now the equations (Eq. 2.5) are linear, so if s is the solution for ξ_0, and s' is the solution for ξ_0', then $s + s'$ is the solution for $\xi_0 + \xi_0'$. Similarly the solution to $c\xi_0$ for $c \in \mathbb{K}$ is cs. In particular the map $\xi_0 \mapsto \xi_1$ is linear. Thus the path γ defines a linear homomorphism from $E_{\gamma(0)}$ to $E_{\gamma(1)}$. If one reverses the path γ by defining $\delta(t) := \gamma(1-t)$, then the parallel transport of ξ_1 along δ is ξ_0. Thus this linear homomorphism is invertible. In particular it is an element of $\mathrm{GL}(n,\mathbb{K})$, which we call the *parallel transport map associated to γ*.

We will denote the parallel transport map along γ from p to q by $P^\nabla_{p,q,\gamma} : E_p \to E_q$. Thus $P^\nabla_{p,q,\gamma}\xi_0 = \xi_1$, where $\xi_0 \in E_p$, and $\gamma : [0,1] \to M$ is a curve from p to q. Furthermore in this notation we have $P^\nabla_{p,q,\gamma} = (P_{q,p,\gamma^-})^{-1}$. Because we often refer to the map $\gamma : [0,1] \to M$ and the image $\gamma([0,1]) \subset M$ interchangeably, we will be slack in writting γ^- for the reversed path. Only when γ is a loop is it critical to keep track of which direction around γ we are going from p to q. We will also use the fact that if γ

does not pass through the point p twice, that $P^{\nabla}_{p,p,\gamma} = \mathbf{1}$. If γ does pass through p twice, one should be precise about the curve γ and its orientation.

These parallel transport maps are where the name *connection* comes from. Given a curve γ, the parallel transport map defines a way of connecting $E_{\gamma(0)}$ and $E_{\gamma(1)}$. The following theorem gives the relation between parallel transport and connections, when considered as separate concepts. In particular, it implies one may recover a connection from its parallel transport.

Theorem 2.1.29. *Let ∇ be a connection on a vector bundle E. Let $s \in \Omega^0(E)$ be a section and $X \in \Gamma(TM)$ be a vector field. Let $p \in M$ and suppose $\gamma : (-\varepsilon, \varepsilon) \to M$ is an integral curve for X near p. Then*

$$\nabla_X s(p) = \frac{d}{dt}\left(P^{\nabla}_{\gamma(t),p,\gamma} s(\gamma(t))\right)_{t=0}.$$

Proof. Pick a basis $\{e_i(p)\}$ of E_p, and use the parallel transport defined by ∇ to extend this to a frame along the curve γ. Note that the parallel transport maps from the curve γ are isomorphisms, so linear independence is preserved and this really does define a frame along γ. In this basis we have $s|_{\gamma} = s^i e_i$ where $s^i \in C^{\infty}(\gamma)$. Now along γ we have

$$\nabla_X s|_{\gamma} = \frac{Ds}{dt} = \frac{D}{dt}(s^i e_i) = ds^i(X) e_i + s^i \frac{D}{dt}(e_i).$$

Since we defined the e_i to be horizontal along γ, we have

$$\nabla_X s(\gamma(t)) = ds^i(X)(\gamma(t)) e_i(\gamma(t)). \qquad \text{(Eq. 2.7)}$$

On the other hand,

$$\begin{aligned}
\frac{d}{dt}\left(P^{\nabla}_{\gamma(t),p,\gamma} s(\gamma(t))\right)_{t=0} &= \frac{d}{dt}\left(P^{\nabla}_{\gamma(t),p,\gamma} s^i(\gamma(t)) e_i(\gamma(t))\right)_{t=0} \\
&= \frac{d}{dt}\left(s^i(\gamma(t)) P^{\nabla}_{\gamma(t),p,\gamma} e_i(\gamma(t))\right)_{t=0} \\
&= \frac{d}{dt}\left(s^i(\gamma(t))\right)_{t=0} P^{\nabla}_{\gamma(0),p,\gamma} e_i(\gamma(0)) \\
&\quad + s^i(\gamma(0)) \frac{d}{dt}\left(P^{\nabla}_{\gamma(t),p,\gamma} e_i(\gamma(t))\right)_{t=0}.
\end{aligned}$$

Now the e_i were defined as the parallel transports of $e_i(p)$, so $P^{\nabla}_{\gamma(t),p,\gamma} e_i(\gamma(t)) = e_i(p)$ for all t. But then the derivative of this second term is zero. Furthermore since $\gamma(0) = p$ and $P^{\nabla}_{p,p,\gamma} = \mathbf{1}$, the first term becomes

$$\frac{d}{dt}\left(P^{\nabla}_{\gamma(t),p,\gamma} s(\gamma(t))\right)_{t=0} = \frac{d}{dt}\left(s^i(\gamma(t))\right)_{t=0} e_i(p).$$

But this is precisely $\nabla_X s(\gamma(0))$, according to the expression for the covariant derivative derived above (Eq. 2.7). $\square$

Note that the derivative in the statement of Theorem 2.1.29 may be equivalently written in the form

$$\frac{d}{dt}\left(P^{\nabla}_{\gamma(t),p,\gamma}s(\gamma(t))\right)_{t=0} = \lim_{h\to 0}\frac{P^{\nabla}_{\gamma(h),p,\gamma}s(\gamma(h)) - s(\gamma(0))}{h},$$

where the subtraction here is now well-defined, occuring in the single vector space E_p. This expression is perhaps the best evidence that a connection is the correct generalisation of directional derivatives of vector fields.

If care is taken, it is possible to *define* a connection via parallel transports. The difficulty comes in identifying the correct statement of how the parallel transport maps should depend smoothly on each of their arguments. If this is done, then Theorem 2.1.29 gives the definition of the corresponding covariant derivative.

Holonomy

In the case of the trivial connection on $\mathbb{R}^n$, the parallel transport along any curve is constant. On the sphere, however, parallel transport is non-trivial. This can for example be observed by performing parallel transport of a vector around the boundary of a quarter of a hemisphere. In this case, the vector will be rotated by $\pi/2$ once it has been transported back to its starting point.

The difference in these two situations is that in the case of the sphere, the connection has *curvature*. The notion of the curvature of a connection will be investigated momentarily, but first, based on the above examples of $\mathbb{R}^n$ and the sphere, we will now define the holonomy of a connection.

Fix a point $p \in M$. Let $\gamma : [0,1] \to M$ be a smooth curve such that $\gamma(0) = \gamma(1) = p$. Then the parallel transport along γ defines an isomorphism of E_p to itself. We call this element the *holonomy of the connection* ∇ *at the point* p *around the loop* γ, and denote it by

$$\mathrm{Hol}^{\nabla}(p,\gamma) \in \mathrm{GL}(n,\mathbb{K}).$$

Definition 2.1.30 (Holonomy)**.** *Define the set*

$$Hol^{\nabla}(p) := \{A \in \mathrm{GL}(n,\mathbb{K}) \mid A = Hol^{\nabla}(p,\gamma) \textit{ for some } \gamma : [0,1] \to M\}.$$

This set is called the holonomy of ∇ *around (or at)* p.

Proposition 2.1.31. *The holonomy of a connection* ∇ *around a point* $p \in M$ *is a group.*

Proof. The group operation is defined by parallel transport around concatenated curves $\gamma_2 \cdot \gamma_1$, as defined in Remark 2.1.28. The identity element is the element corresponding to the constant loop. As mentioned previously, the inverse of an element corresponding to γ_1 is the holonomy around the reversed loop γ_1^-. Associativity follows from the definition of parallel transport for piecewise smooth curves. □

Lemma 2.1.32. *Suppose* M *is a path connected manifold. Then* $Hol^{\nabla}(p)$ *and* $Hol^{\nabla}(q)$ *are conjugate in* $\mathrm{GL}(n,\mathbb{K})$ *for any* $p, q \in M$.

Proof. Let $\gamma : [0,1] \to M$ be a path from p to q. Then for each δ a loop around p, $\gamma \cdot \delta \cdot \gamma^{-1}$ is a loop around q. Thus if $g \in \mathrm{GL}(n, \mathbb{K})$ denotes the parallel transport map associated to γ, we have $g\mathrm{Hol}^{\nabla}(p)g^{-1} \subseteq \mathrm{Hol}^{\nabla}(q)$. Now given any $h \in \mathrm{Hol}^{\nabla}(q)$, we have some loop κ such that h is the holonomy around κ. Define a new loop η around p by $\eta := \gamma^{-1} \cdot \kappa \cdot \gamma$. Then $g\mathrm{Hol}^{\nabla}(p, \eta)g^{-1} = \mathrm{Hol}^{\nabla}(q, \gamma \cdot \gamma^{-1} \cdot \kappa \cdot \gamma \cdot \gamma^{-1}) = \mathrm{Hol}^{\nabla}(q, \kappa)$. Thus we can always find a loop η around p such that $\gamma \cdot \eta \cdot \gamma^{-1}$ has the same holonomy as κ. In particular this implies $\mathrm{Hol}^{\nabla}(q) \subseteq g\mathrm{Hol}^{\nabla}(p)g^{-1}$. □

By virtue of Lemma 2.1.32, the holonomy of a connection around any two points is isomorphic (for a connected manifold). Thus one will often refer simply to the holonomy of the connection, with the understanding that the holonomy for any two points are conjugate. There is a distinguished subgroup of the holonomy group around any point p.

Definition 2.1.33. *Let E be a vector bundle over M with connection ∇. Denote by $Hol_0^{\nabla}(p)$ the subgroup of the holonomy around p defined by contractible loops around p.*

For a simply connected manifold M, $\mathrm{Hol}_0^{\nabla}(p) = \mathrm{Hol}^{\nabla}(p)$ for any $p \in M$. If one has a loop γ based at p, and a contractible loop δ, the loop $\gamma \cdot \delta \cdot \gamma^{-1}$ is contractible. This can be observed by contracting the δ component of the curve to a point while leaving γ and γ^{-1}, and then contracting the loop left by $\gamma \cdot \gamma^{-1}$. Thus $\mathrm{Hol}_0^{\nabla}(p)$ is a normal subgroup of $\mathrm{Hol}^{\nabla}(p)$. There is a natural homomorphism from $\pi_1(M)$ to $\mathrm{Hol}^{\nabla}(p)/\mathrm{Hol}_0^{\nabla}(p)$ taking $[\gamma]$ to the coset $P_{p,p,\gamma}^{\nabla} \cdot \mathrm{Hol}_0^{\nabla}(p)$.

Once we have investigated curvature, we will see how the holonomy gives a geometric interpretation of the curvature of a connection at a point p. The curvature at p is in a precise sense the limit of the holonomy around p as one takes smaller and smaller loops γ.

In the case where the connection is *flat*, we will see that the holonomy depends only on the homotopy class of the loop γ. Then in the case where the manifold is connected, $\mathrm{Hol}_0^{\nabla}(p)$ is trivial, so the natural homomorphism described above is into $\mathrm{GL}(n, \mathbb{K})$, or the structure group of the vector bundle. Thus one may phrase the study of representations of the fundamental group into *linear* groups in terms of the study of connections on vector bundles. In fact, the connection ∇ is flat *if and only if* $\mathrm{Hol}_0^{\nabla}(p)$ is trivial.

Later we will investigate connections on principal G-bundles, in which case the corresponding notion of holonomy will allow one to extend this discussion to obtain representations of $\pi_1(M)$ into an arbitrary Lie group G.

2.1.7 Curvature

A connection ∇ can be thought of as a map $d_A : \Omega^0(E) \to \Omega^1(E)$ satisfying the Leibniz rule, where we are labelling $d_A := \nabla$ by some letter A. Of course, we have chosen to use d here to be suggestive. In the same way that the exterior derivative $d : \Omega^0(M) \to \Omega^1(M)$ may be extended to a differential operator on the full algebra $\Omega^\bullet(M)$, we can extend ∇ to an operator d_A on $\Omega^\bullet(E)$.

Definition 2.1.34. *Given a connection ∇, define the kth* exterior covariant derivative

$$d_A^k : \Omega^k(E) \to \Omega^{k+1}(E)$$

on pure tensors by

$$d_A^k(\omega \otimes s) = d\omega \otimes s + (-1)^k \omega \wedge \nabla s$$

where $\omega \in \Omega^k(M)$ and $s \in \Gamma(E)$. Extend by linearity to all sections in $\Omega^k(E)$. Here $\omega \wedge \nabla s$ means wedging the form ω with the one-form part of ∇s.

As in the case of the exterior derivative, the index k is usually suppressed, and we take d_A^0 to be ∇. In the rest of these notes, we will at times refer to applications of the connection ∇ to sections of $\Omega^k(E)$, and to sections of any associated bundles upon which ∇ induces a connection, as d_A. This is a slight abuse of notation, but we will justify what we mean by d_A in every case in which we apply it. The main property of d_A, that it satisfies a Leibniz rule, will remain constant throughout, so it is often constructive to blur the line (though not always).

In the case of the exterior derivative, $d^{k+1} \circ d^k = 0$ for any k. In general this is not true for the exterior covariant derivative. The obstruction to this property is called the *curvature* of the connection.

Definition 2.1.35. *The curvature F_A of a connection A is the map $(d_A)^2 : \Omega^0(E) \to \Omega^2(E)$.*

The curvature is a measure of the extent to which the complex $(\Omega^\bullet(E), d_A)$ fails to be a *chain* complex.

In general the curvature map is non-zero, and we will soon see a local characterisation of F_A as a two-form, in terms of the one-forms defined in Section 2.1.3. Before this, we have the following fundamental result about the curvature of a linear connection.

Theorem 2.1.36. *The curvature of a connection A is $C^\infty(M)$-linear.*

Proof. Let $s \in \Gamma(E)$ and $f \in C^\infty(M)$. Then

$$\begin{aligned}
F_A(fs) &= d_A(df \otimes s + f d_A(s)) \\
&= d_A(df \otimes s) + d_A(f d_A(s)) \\
&= d^2 f \otimes s - df \wedge d_A(s) + df \wedge d_A(s) + f d_A^2(s) \\
&= f F_\nabla(s).
\end{aligned}$$

□

Here we have used the following lemma.

Lemma 2.1.37. *Let $f \in C^\infty(M)$ and $s \in \Omega^0(E)$. Then for any connection A on E, we have*

$$d_A(f d_A(s)) = df \wedge d_A(s) + f d_A^2(s).$$

Proof. We have $s \in \Omega^0(E)$ is a general section of E. Thus $d_A(s) \in \Omega^1(E)$ is an E-valued one-form. Suppose $d_A(s) = \omega^i \otimes s_i$ for some one-forms ω^i and sections s_i.

$$\begin{aligned}
d_A(f d_A(s)) &= d_A(f(\omega^i \otimes s_i)) \\
&= d_A((f\omega^i) \otimes s_i) \\
&= d(f\omega^i) \otimes s_i + f\omega^i \wedge d_A(s_i) \\
&= (df \wedge \omega^i + f d\omega^i) \otimes s_i + f\omega^i \wedge d_A(s_i) \\
&= df \wedge \omega^i \otimes s_i + f(d\omega^i \otimes s_i + \omega^i \wedge d_A(s_i)) \\
&= df \wedge d_A(s) + f d_A^2(s).
\end{aligned}$$

□

From this theorem, we obtain the curvature is a $C^\infty(M)$-linear mapping from $\Omega^0(E)$ to $\Omega^2(E)$. By Lemma 2.1.5, this implies that $F_A \in \Gamma(\mathrm{End}(E) \otimes \Omega^2(M)) = \Omega^2(\mathrm{End}(E))$, so F_A is an $\mathrm{End}(E)$-valued 2-form on M. In light of Remark 2.1.10, the curvature is (locally) a matrix of two-forms.

Remark 2.1.38. The letter A is the standard label for connections and their local connection forms due to a coincidence coming from physics. In classical electromagnetism, the electromagnetic potential tensor is denoted by the letter A. The corresponding electromagnetic field strength tensor is then denoted by F. Remarkably, the theory of classical electromagnetism, including all of Maxwell's equations of electromagnetism, can be naturally interpreted in the language of connections and curvature of vector bundles. In particular, the electromagnetic potential is in a natural way a connection on a principal $U(1)$-bundle over spacetime, and the electromagnetic field strength F is its curvature. One can then write down the Yang-Mills functional in the case of connections on principal $U(1)$-bundles. The Yang-Mills equations (the Euler-Lagrange equations for this functional) together with the Bianchi identity (Proposition 2.1.52) completely recover Maxwell's equations of electromagnetism.

This phenomenon explains the labels A and F_A for connections and their curvature.

2.1.8 Operations on $\mathrm{End}(E)$-valued Differential Forms

Before we continue, we will define some operations on $\mathrm{End}(E)$-valued differential forms. These will come up when analysing the properties of the curvature in local trivialisations.

First, suppose $A \in \Omega^k(\mathrm{End}(E))$ is an $\mathrm{End}(E)$-valued k-form. Then we may write $A = \eta^i \otimes N_i$ for some $\eta^i \in \Omega^k(M)$ and $N_i \in \Omega^0(\mathrm{End}(E))$. Further, suppose $B = \mu^j \otimes M_j$ is an element of $\Omega^l(\mathrm{End}(E))$, an $\mathrm{End}(E)$-valued l-form.

Define the wedge product $A \wedge B$ by

$$A \wedge B := (\eta^i \wedge \mu^j) \otimes (N_i M_j).$$

This is well defined since the η^i and μ^j are differential forms on M, for which the wedge product makes sense, and the N_i and M_j are endomorphisms of E, so we may compose them to get new endomorphisms $N_i M_j$.

This wedge product is perhaps the most natural way of defining a multplication operation on $\mathrm{End}(E)$-valued differential forms, but there is another useful operations that one may perform:

Define the commutator $[A, B]$ of A and B by

$$[A, B] := (\eta^i \wedge \mu^j) \otimes [N_i, M_j]$$

where the commutator on the right is that of endomorphisms, defined by $[N_i, M_j] = N_i M_j - M_j N_i$.

First we remark that the commutator $[A, B]$ satisfies

$$[A, B] = (-1)^{\deg(A)\deg(B)+1}[B, A].$$

This can be seen from combining the $(-1)^{\deg(A)\deg(B)}$ term one gets when interchanging the forms η^i and μ^j, as well as the -1 from the commutator of endomorphisms. In addition we have

Lemma 2.1.39.

$$[A, B] = A \wedge B - (-1)^{\deg(A)\deg(B)} B \wedge A$$

Proof.

$$\begin{aligned}
[A, B] &= (\eta^i \wedge \mu^j) \otimes [N_i, M_j] \\
&= (\eta^i \wedge \mu^j) \otimes N_i M_j - (\eta^i \wedge \mu^j) \otimes M_j N_i \\
&= (\eta^i \wedge \mu^j) \otimes N_i M_j - (-1)^{\deg(A)\deg(B)} (\mu^j \wedge \eta^i) \otimes M_j N_i \\
&= A \wedge B - (-1)^{\deg(A)\deg(B)} B \wedge A.
\end{aligned}$$

□

For one-forms the above expression implies $\frac{1}{2}[A, A] = A \wedge A$. In the literature, the curvature of a connection may be denoted using either of these expressions, and the above discussion gives the relation between them. The form in terms of $[\cdot, \cdot]$ is more common when discussing how connections on vector bundles arise from connections on associated principal bundles. For example in the case of the adjoint Lie algebra bundle, the commutator of endomorphisms used in the definition of $[\cdot, \cdot]$ is precisely the Lie bracket on the fibres of the adjoint bundle.

2.1.9 Local Curvature Form

Definition 2.1.40. *Let d_A be a connection on a bundle E. Then the curvature F_A corresponds to a form we will also denote $F_A \in \Omega^2(\mathrm{End}(E))$. This is the* curvature form *of the connection d_A.*

Proposition 2.1.41 (Cartan's Structure Equation)**.** *Suppose d_A has local connection form A on some trivilising set U. Then the curvature F_A locally has the form $\Omega_A = dA + A \wedge A$, where $\Omega_A \in \Omega^2(\mathrm{End}(E)|_U)$.*

Proof. Let $\{e_i\}$ be the local frame on a trivialisation U, and A the local connection form. Then we have

$$\begin{aligned}
F_A(e_i) &= d_A(A_i^j \otimes e_j) \\
&= dA_i^j \otimes e_j - A_i^j \wedge d_A(e_j) \\
&= dA_i^k \otimes e_k - A_i^j \wedge A_j^k \otimes e_k \\
&= (dA_i^k + A_j^k \wedge A_i^j) \otimes e_k.
\end{aligned}$$

That is, $(\Omega_A)_i^k = dA_i^k + A_j^k \wedge A_i^j$. In light of Sections 2.1.3 and A.1, this wedge product corresponds to the matrix multiplication of A with itself, where the coefficient multiplication is wedging of one-forms. This is precisely the wedge product of $\mathrm{End}(E)$-valued forms defined in the previous section. In particular we have

$$\Omega_A = dA + A \wedge A.$$

□

To say that the curvature is locally the $\mathrm{End}(E)$-valued two-form Ω_A means that given some $s = s^i e_i$ written in the local frame on U, we have

$$d_A^2(s) = \Omega_A s = dAs + A \wedge As,$$

where the term on the right is interpreted as genuine matrix multiplication.

Remark 2.1.42. Since F_A is a global $\mathrm{End}(E)$-valued two-form on M, the local curvature forms Ω_A should be compatible on overlaps, in contrast to the local connection forms A. Indeed using (Eq. 2.4) for how A transforms under a change of basis, if we have local forms $A_\alpha = A$ and $A_\beta = B$, with $g_{\beta\alpha} = g$, then

$$\begin{aligned}
\Omega_A &= dA + A \wedge A \\
&= d(g^{-1}dg + g^{-1}Bg) + (g^{-1}dg + g^{-1}Bg) \wedge (g^{-1}dg + g^{-1}Bg) \\
&= dg^{-1} \wedge dg + dg^{-1} \wedge Bg + g^{-1}dBg - g^{-1}B \wedge dg + g^{-1}dg \wedge g^{-1}dg + \\
&\qquad g^{-1}dg \wedge g^{-1}Bg + g^{-1}B \wedge dg + g^{-1}B \wedge Bg \\
&= g^{-1}(dB + B \wedge B)g + dg^{-1} \wedge dg + dg^{-1} \wedge Bg + g^{-1}dg \wedge g^{-1}dg + g^{-1}dg \wedge g^{-1}Bg.
\end{aligned}$$

Now because $0 = d(gg^{-1}) = dgg^{-1} + gdg^{-1}$, we have $dg = -gdg^{-1}g$. Substituting this into the above expression, we have

$$\begin{aligned}
\Omega_A &= g^{-1}\Omega_B g - dg^{-1} \wedge gdg^{-1}g + dg^{-1}g \wedge dg^{-1}g + dg^{-1} \wedge Bg - dg^{-1} \wedge Bg \\
&= g^{-1}\Omega_B g \\
&= g_{\alpha\beta}\Omega_B g_{\alpha\beta}^{-1}
\end{aligned}$$

This is the change of basis formula for $\mathrm{End}(E)$-valued forms, so the local curvature forms Ω_A piece together into the global form F_A, as expected. We may refer to the curvature form as either F_A or Ω_A in the future. Doing local calculations it is suggestive to use Ω_A, due to its dependence on the local connection forms A, which don't themselves piece together into a global form.

Remark 2.1.43. Because $A \wedge A = \frac{1}{2}[A, A]$, one will often see Cartan's Structure Equation written in the form $\Omega_A = dA + \frac{1}{2}[A, A]$ in the literature. This notation is related to the expression for the curvature of a connection in associated bundles to principal bundles, where the bracket $[\cdot, \cdot]$ corresponds to a Lie bracket.

Remark 2.1.44. One will often see Cartan's Structure Equation written in the form $dA - A \wedge A$ in the literature. This expression is not the natural wedge product on $\mathrm{End}(E)$-valued forms as defined in the previous section. In fact this minus sign follows from not keeping track of indices, and interpreting $-A_i^j \wedge A_j^i$ as a matrix multiplication. In these cases, $dA - A \wedge A$ is shorthand notation for $dA_i^k - A_i^j \wedge A_j^k$, and should not be interpreted as matrix multiplication (i.e. composition of $\mathrm{End}(E)$ components of $\mathrm{End}(E)$-valued forms).

2.1.10 Induced Connections

Given a vector bundle $E \to M$, and a connection d_A on E, one obtains connections, also typically labelled d_A, on vector bundles associated to E. In this section we will review how one obtains such connections.

Dual Bundles

First we will consider the case of the dual bundle E^*. One may define a connection d_A on E^* by the formula

$$d\langle \ell, s\rangle = \langle d_A(\ell), s\rangle + \langle \ell, d_A(s)\rangle, \qquad \text{(Eq. 2.8)}$$

where $\ell \in \Omega^0(E^*)$, $s \in \Omega^0(E)$, and the pairing $\langle \cdot, \cdot\rangle$ is the natural one between E^* and E. Note that since $d_A(\ell)$ and $d_A(s)$ are section-valued one-forms, the pairing here means pairing *on the section part*, while leaving the one-form part alone. We will see what this means momentarily.

To check that this formula does indeed define a connection on E^*, we need only check that it satisfies the Leibniz rule. We have

$$\begin{aligned}
\langle d_A(f\ell), s\rangle &= d\langle f\ell, s\rangle - \langle f\ell, d_A(s)\rangle \\
&= (df)\langle \ell, s\rangle + f d\langle \ell, s\rangle - f\langle \ell, d_A(s)\rangle \\
&= \langle df \otimes \ell, s\rangle + f\langle d_A(\ell), s\rangle \\
&= \langle df \otimes \ell + f d_A(\ell), s\rangle.
\end{aligned}$$

Since this holds for all sections $s \in \Omega^0(E)$, we conclude that $d_A(f\ell) = df \otimes \ell + f d_A(\ell)$.

Definition 2.1.45. *The connection d_A defined on E^* above is called the* dual connection *to d_A on E.*

We can easily compute the local connection form for the dual connection d_A. Let $\{e_i\}$ be a local frame for E on a trivialising set U, and let $\{\epsilon^j\}$ be the corresponding dual frame on E^* over U. Suppose we write $d_A(\epsilon^j) := B_i^j \otimes \epsilon^i$ for some one-forms B_i^j. Then using (Eq. 2.8) we have

$$\begin{aligned} d\langle \epsilon^j, e_i \rangle &= \langle B_k^j \otimes \epsilon^k, e_i \rangle + \langle \epsilon^j, A_i^k \otimes e_k \rangle \\ &= B_k^j \langle \epsilon^k, e_i \rangle + A_i^k \langle \epsilon^j, e_k \rangle \\ &= B_k^j \delta_i^k + A_i^k \delta_k^j \\ &= B_i^j + A_i^j. \end{aligned}$$

Now $\langle \epsilon^j, e_i \rangle = \delta_i^j$, which is a constant function (either one or zero, depending on i and j), so the left-hand-side of this expression is zero. Thus we obtain

$$B_i^j = -A_i^j.$$

In particular, we have the following expression for how the dual connection acts on a section $s = s_i \epsilon^i \in \Omega^0(E^*)$:

$$d_A(s_i \epsilon^i) = (ds_i - s_j A_i^j) \otimes \epsilon^i \tag{Eq. 2.9}$$

Given this expression for the action of the connection d_A on the dual bundle, one can now proceed, just as in Section 2.1.7, to extend d_A to $\Omega^\bullet(E^*)$, and for example compute the curvature.

Remark 2.1.46. In the sense of Section 2.1.3, the expression in (Eq. 2.9) is not the true action of the connection form on E^*. In particular it is not of the form $\nabla s = ds - As$. This is notational, due to the convention of writing the dual basis $\{\epsilon^j\}$ with upper indices. With this convention (Eq. 2.9) says that $d_A(s) = ds - sA$ (in matrix notation).

If one were to write the dual basis as $\{\epsilon_j\}$ (considering E^* as a standalone vector bundle), satisfying the same relations $\langle \epsilon_i, e_j \rangle = \delta_{ij}$, then the genuine connection form B_i^j on E^* would be given by $-A_j^i$, the negative *transpose* of A.

We will now compute the curvature on the dual bundle in two ways. Firstly, writing the dual basis as $\{\epsilon^i\}$ we have

$$\begin{aligned} d_A^2(\epsilon^i) &= d_A(-A_j^i \otimes \epsilon^j) \\ &= -dA_j^i \otimes \epsilon^j + A_j^i \wedge A_k^j \otimes \epsilon^k \\ &= (-dA_j^i + A_k^i \wedge A_j^k) \otimes \epsilon^j. \end{aligned}$$

Since the curvature is $C^\infty(M)$-linear, if we denote by $\tilde{\Omega}_A$ the local curvature form in this basis, we have the matrix expression

$$d_A^2(s) = -sdA + sA \wedge A.$$

Again, since we are interpreting s as a row vector (being a section of the dual bundle), our expression varies from that found in Section 2.1.9. If we instead write our dual basis as $\{\epsilon_j\}$ and use the connection form as described in Remark 2.1.46, we obtain

$$\begin{aligned} d_A^2(\epsilon_i) &= d_A(-A_i^j \otimes \epsilon_j) \\ &= -dA_i^j \otimes \epsilon_j + A_i^j \wedge A_j^k \otimes \epsilon_k \\ &= -(dA_i^j + A_k^j \wedge A_i^k) \otimes \epsilon_j \\ &= -(\Omega_A)_i^j \otimes \epsilon_j. \end{aligned}$$

Writing a dual section as $s = s^i \epsilon_i$, we now obtain the matrix expression

$$d_A^2(s) = -\Omega_A s.$$

These varying expressions for the connection form and curvature form on the dual bundle are all based on notational conventions. It is useful to see where each comes into play, so as to make sure one uses the write expression when doing computations. In particular, one should be wary of writing that the curvature of the dual bundle is $-\Omega_A$ unless the dual sections are being expressed in the form $s^i \epsilon_i$.

Nevertheless, the notion of the induced connection on the dual bundle is invariantly defined by (Eq. 2.8), so the differences between these notational conventions is only a secondary consideration.

Direct Sum Bundles

Given two vector bundles E, F over a manifold M, with connections d_A and d_B, one may define a connection on the direct sum bundle $E \oplus F$. Sections $s \in \Omega^0(E \oplus F)$ uniquely split into sections $s_1 \in \Omega^0(E)$ and $s_2 \in \Omega^0(F)$, such that

$$s = s_1 + s_2.$$

Thus one obtains the induced connection $d_{A \oplus B}$ on $E \oplus F$ by the expression

$$d_{A \oplus B}(s) := d_A(s_1) + d_B(s_2).$$

To see that this is a connection, we again need to verify the Leibniz rule.

$$\begin{aligned} d_{A \oplus B}(fs) &= d_A(fs_1) + d_B(fs_2) \\ &= df \otimes s_1 + f d_A(s_1) + df \otimes s_2 + f d_B(s_2) \\ &= df \otimes (s_1 + s_2) + f d_{A \oplus B}(s_1 + s_2) \\ &= df \otimes s + f d_{A \oplus B}(s). \end{aligned}$$

The standard notation for this connection is $d_{A \oplus B} = d_A \oplus d_B$. In this sense the local connection form on a common trivialisation for E and F with local forms A and B is $A \oplus B$. That is, if $\{e_1, \dots, e_n, e'_1, \dots, e'_m\}$ is the local frame of $E \oplus F$ on this trivialisation, where the e_i correspond to the rank n bundle E, and the e'_j correspond to the rank m bundle F, we have

$$d_{A \oplus B}(e_j) = A_j^i \otimes e_i, \quad d_{A \oplus B}(e'_j) = B_j^i \otimes e_i.$$

Then if $s = a^i e_i + b^j e'_j$, letting $s_1 = a^i e_i$ and $s_2 = b^j e'_j$ we have

$$d_{A \oplus B}(s) = ds + As_1 + Bs_2.$$

Tensor Product Bundles

Given two vector bundles E and F, with connections d_A and d_B, one obtains an induced connection $d_{A\otimes B}$ on the tensor product bundle $E\otimes F$. This is defined according to the expression

$$d_{A\otimes B} := d_A \otimes \mathbf{1} + \mathbf{1} \otimes d_B. \tag{Eq. 2.10}$$

Then for a (pure) section $s\otimes t \in \Omega^0(E\otimes F)$, we have

$$d_{A\otimes B}(s\otimes t) = d_A(s)\otimes t + s\otimes d_B(t).$$

Note that this second term on the right is actually a section of $E\otimes T^*M\otimes F$. Since there is a canonical isomorphism $E\otimes T^*M\otimes F \cong T^*M\otimes E\otimes F$, one interprets the section $s\otimes d_B(t)$ by declaring that the one-form part of $d_B(t)$ is written first, as according to this isomorphism. If this is understood, then $d_{A\otimes B}$ defines a map from sections of $E\otimes F$ to sections of $T^*M\otimes E\otimes F$. To see that this defines a connection, we have

$$\begin{aligned}
d_{A\otimes B}(fs\otimes t) &= d_A(fs)\otimes t + fs\otimes d_B(t) \\
&= df\otimes(s\otimes t) + f(d_A(s)\otimes t + s\otimes d_B(t)) \\
&= df\otimes(s\otimes t) + f d_{A\otimes B}(s\otimes t).
\end{aligned}$$

Thus $d_{A\otimes B}$ satisfies the Leibniz rule. Indeed this is why we require the tensor product to be of the form given in (Eq. 2.10). The expression $d_A\otimes d_B$ for example does not satisfy the Leibniz rule.

Just as in Section 2.1.7, we can extend $d_{A\otimes B}$ to $\Omega^\bullet(E\otimes F)$ by the expression

$$d_{A\otimes B}(\omega\otimes s\otimes t) := d\omega\otimes s\otimes t + (-1)^{\deg\omega}\omega\wedge d_{A\otimes B}(s\otimes t). \tag{Eq. 2.11}$$

Using this, we can now determine the curvature of the connection $d_{A\otimes B}$ in terms of d_A and d_B. Let $\{e_i\}$ and $\{f_j\}$ be local frames for E and F over a common trivialising set U, which also trivialises $E\otimes F$. Then one has a local frame $\{e_i\otimes f_j\}$ of $E\otimes F$ in this trivialisation. If we say d_A has local connection form A, and d_B has local connection form B, then

$$\begin{aligned}
F_{A\otimes B}(e_i\otimes f_j) &= d^2_{A\otimes B}(e_i\otimes f_j) \\
&= d_{A\otimes B}(A_i^k\otimes e_k\otimes f_j + B_j^k\otimes e_i\otimes f_k) \\
&= dA_i^k\otimes e_k\otimes f_j - A_i^k\wedge d_{A\otimes B}(e_k\otimes f_j) + dB_j^k\otimes e_i\otimes f_k - B_j^k\wedge d_{A\otimes B}(e_i\otimes f_k) \\
&= dA_i^k\otimes e_k\otimes f_j + dB_j^k\otimes e_i\otimes f_k \\
&\quad - A_i^k\wedge(A_k^l\otimes e_l\otimes f_j + B_j^l\otimes e_k\otimes f_l) - B_j^k\wedge(A_i^l\otimes e_l\otimes f_k + B_k^l\otimes e_i\otimes f_l) \\
&= dA_i^k\otimes e_k\otimes f_j + A_k^l\wedge A_i^k\otimes e_l\otimes f_j + dB_j^k\otimes e_i\otimes f_k + B_k^l\wedge B_j^k\otimes e_i\otimes f_l \\
&\quad - A_i^k\wedge B_j^l\otimes e_k\otimes f_l - B_j^k\wedge A_i^l\otimes e_l\otimes f_k \\
&= (\Omega_A)_i^k\otimes e_k\otimes f_j + (\Omega_B)_j^k\otimes e_i\otimes f_k - A_i^l\wedge B_j^k\otimes e_l\otimes f_k + A_i^l\wedge B_j^k\otimes e_l\otimes f_k \\
&= F_A(e_i)\otimes f_j + e_i\otimes F_B(f_j).
\end{aligned}$$

That is, the curvature of the tensor product connection of two vector bundles E and F with connections d_A and d_B is given by

$$F_{A\otimes B} = F_A \otimes \mathbf{1} + \mathbf{1} \otimes F_B. \tag{Eq. 2.12}$$

Note that the induced connections on dual bundles and tensor product bundles mean that given a connection d_A on a vector bundle E, one obtains induced connections on all tensor bundles associated to E. These include dual bundles, bundles of p-co-q-contravariant tensors on E, symmetric tensor powers of E, antisymmetric tensor power of E, and so on.

This vast array of information is better formulated in terms of a single connection on the principal frame bundle associated to the vector bundle E. The connections on all associated tensor bundles can then be seen as coming from the principal connection in a natural way.

The Endomorphism Bundle

The computations of the previous subsections allow us to write down a nice formula for the induced connection on the endmorphism bundle $\mathrm{End}(E) \cong E^* \otimes E$, and on the bundles of $\mathrm{End}(E)$-valued differential forms.

Let E be a vector bundle over M with connection d_A. Let $\{e_i\}$ be a local frame for E on a trivialising set U, and $\{\epsilon^j\}$ be the corresponding dual frame. Suppose A is the local connection form on U.

In what follows, we will refer to the induced connection on $E^* \otimes E$ simply by d_A.

Proposition 2.1.47. *Let $\alpha \in \Omega^k(\mathrm{End}(E))$ be a global $\mathrm{End}(E)$-valued k-form. Then on a local trivialisation such that A is the connection form for d_A and $\alpha|_U := a = a^i_j \otimes \epsilon^j \otimes e_i$ is the local form of α, we have*

$$d_A(\alpha) = da + [A, a].$$

Proof. By (Eq. 2.11) and (Eq. 2.10), we have

$$\begin{aligned}
d_A(a^i_j \otimes \epsilon^j \otimes e_i) &= da^i_j \otimes \epsilon^j \otimes e_i + (-1)^{\deg a} a^i_j \wedge d_A(\epsilon^j \otimes e_i) \\
&= da^i_j \otimes \epsilon^j \otimes e_i + (-1)^{\deg a} a^i_j \wedge (-A^j_k \otimes \epsilon^k \otimes e_i + A^k_i \otimes \epsilon^j \otimes e_k) \\
&= da^i_j \otimes \epsilon^j \otimes e_i + (-1)^{\deg a+1} a^i_j \wedge A^j_k \otimes \epsilon^k \otimes e_i \\
&\quad\; + (-1)^{\deg a} a^i_j \wedge A^k_i \otimes \epsilon^j \otimes e_k \\
&= da^i_j \otimes \epsilon^j \otimes e_i - (-1)^{\deg a} a^i_j \wedge A^j_k \otimes \epsilon^k \otimes e_i \\
&\quad\; + (-1)^{\deg a} (-1)^{\deg a \cdot 1} A^k_i \wedge a^i_j \otimes \epsilon^j \otimes e_k \\
&= da^i_j \otimes \epsilon^j \otimes e_i + A^i_k \wedge a^k_j \otimes \epsilon^j \otimes e_i - (-1)^{\deg a} a^i_k \wedge A^k_j \otimes \epsilon^j \otimes e_i \\
&= (da^i_j + [A, a]^i_j) \otimes \epsilon^j \otimes e_i.
\end{aligned}$$

That is, in matrix notation,

$$d_A(a) = da + [A, a].$$

□

Remark 2.1.48. If α has local representative $a_\alpha := \alpha|_{U_\alpha}$ and similarly one has a_β, then if $g = g_{\beta\alpha}$ we have $a_\beta = g a_\alpha g^{-1}$. In particular using this expression and (Eq. 2.4) we can see that the local expression given in Proposition 2.1.47 for $d_A(\alpha)$ pieces together to give a well-defined global $\mathrm{End}(E)$-valued $(k+1)$-form. This is expected, since the form $d_A(\alpha)$ is globally well-defined by assertion.

To justify the idea that this connection on $\mathrm{End}(E)$ that we have constructed is indeed natural, we will prove the following lemma, which indicates that the connection d_A on $\mathrm{End}(E)$ is about as nice as one could hope. In addition this lemma will give us a nice characterisation of the action of $\mathscr{G}$ on $\mathscr{A}$ as described in Section 2.1.4.

Lemma 2.1.49. *The connection d_A induced on* $\mathrm{End}(E)$ *by d_A on E, according to* (Eq. 2.10)*, satisfies the Leibniz rule*

$$d_A(u(s)) = d_A(u)s + u d_A(s).$$

Proof. Let $u = u_i^j \epsilon^i \otimes e_j$ and $s = s^k e_k$. Then

$$\begin{aligned} d_A(u(s)) &= d_A(u_i^j s^k \delta_k^i e_j) \\ &= d(u_i^j s^i) \otimes e_j + u_i^j s^i d_A(e_j) \\ &= du_i^j s^i \otimes e_j + u_i^j ds^i \otimes e_j + u_i^j s^i A_j^k \otimes e_k. \end{aligned}$$

$$\begin{aligned} d_A(u)s &= d_A(u_i^j \epsilon^i \otimes e_j)(s) \\ &= (d_A(u_i^j \epsilon^i) \otimes e_j)(s) + (u_i^j \epsilon^i \otimes d_A(e_j))(s) \\ &= (du_i^j \otimes \epsilon^i \otimes e_j)(s) + (u_i^j d_A(\epsilon^i) \otimes e_j)(s) + u_i^j \epsilon^i(s^k e_k) A_j^l \otimes e_l \\ &= du_i^j \epsilon^i(s^k e_k) \otimes e_j - (u_i^j A_k^i \otimes \epsilon^k \otimes e_j)(s) + u_i^j s^i A_j^l \otimes e_l \\ &= du_i^j s^i \otimes e_j - u_i^j A_k^i s^k \otimes e_j + u_i^j s^i A_j^k \otimes e_k. \end{aligned}$$

$$\begin{aligned} u d_A(s) &= (u_i^j \epsilon^i \otimes e_j)((ds^k + s^l A_l^k) \otimes e_k) \\ &= u_i^j \delta_k^i (ds^k + s^l A_l^k) \otimes e_j \\ &= u_i^j ds^i \otimes e_j + u_i^j s^k A_k^i \otimes e_j. \end{aligned}$$

Adding the second two expressions we see that it equals the first. □

Remark 2.1.50. In Section 2.1.4 we observed that the gauge group $\mathscr{G}$ of a vector bundle naturally acts on the space $\mathscr{A}$ of connections. Further, we determined the explicit action in terms of local connection forms. The induced connection on $\mathrm{End}(E)$ allows us to write down a more invariant formula for the action of $\mathscr{G}$ on $\mathscr{A}$. Let $u \in \mathscr{G}$ and $d_A \in \mathscr{A}$. Then

$$\begin{aligned} u \cdot d_A(s) &= u(d_A(u^{-1}(s))) \\ &= u(d_A(u^{-1})(s) + u^{-1}d_A(s)) \\ &= u(-u^{-1}d_A(u)u^{-1}(s)) + d_A(s) \\ &= d_A(s) - d_A(u)u^{-1}(s). \end{aligned}$$

Thus we have the action of $\mathscr{G}$ on $\mathscr{A}$ is $u \cdot d_A = d_A - d_A(u)u^{-1}$.

2.1.11 More Properties of Curvature

We have seen that given a connection d_A, and an $\mathrm{End}(E)$-valued one-form $a \in \Omega^1(E)$, the sum $d_A + a$ is again a connection on E. How does F_A compare to F_{A+a}?

Proposition 2.1.51. *If $a \in \Omega^1(\mathrm{End}(E))$, then*

$$F_{A+a} = F_A + d_A(a) + a \wedge a.$$

Proof. Suppose d_A has local connection form A. Then we have that d_{A+a} has local connection form $A + a$. Then

$$\begin{aligned} \Omega_{A+a} &= d(A + a) + (A + a) \wedge (A + a) \\ &= dA + da + A \wedge A + A \wedge a + a \wedge A + a \wedge a \\ &= \Omega_A + da + [A, a] + a \wedge a \\ &= \Omega_A + d_A(a) + a \wedge a. \end{aligned}$$

Since this is true locally, and Ω_{A+a} pieces together to form F_{A+a}, we have

$$F_{A+a} = F_A + d_A(a) + a \wedge a$$

as desired. □

Proposition 2.1.52 (Bianchi Identity)**.**

$$d_A F_A = 0$$

Proof. Suppose d_A has local connection form A, and local curvature form Ω_A. Then using the expression for the commutator $[A, \Omega_A]$ in terms of the wedge product on $\mathrm{End}(E)$-valued forms,

$$\begin{aligned} d_A(\Omega_A) &= d\Omega_A + [A, \Omega_A] \\ &= d(dA + A \wedge A) + A \wedge (dA + A \wedge A) + (-1)^{2+1}(dA + A \wedge A) \wedge A \\ &= d(A \wedge A) + A \wedge dA + A \wedge A \wedge A - dA \wedge A - A \wedge A \wedge A \\ &= dA \wedge A - A \wedge dA + A \wedge dA - dA \wedge A \\ &= 0. \end{aligned}$$

But then we have $d_A F_A = 0$, since it is zero locally. □

Remark 2.1.53. The Bianchi identity rears its head in a number of different forms in differential and pseudo-Riemannian geometry. For example, one will often see the Bianchi identity stated as

$$dF_A = F_A \wedge A - A \wedge F_A.$$

This is of course equivalent to the above expression, by the formula for the induced connection on $\mathrm{End}(E)$-valued forms. However, given Cartan's structure equation for the curvature, one may then derive the Bianchi identity in this form directly, by taking the exterior derivative:

$$\begin{aligned} dF_A &= d(dA + A \wedge A) \\ &= dA \wedge A - A \wedge dA \\ &= (dA + A \wedge A) \wedge A - A \wedge (dA + A \wedge A) \\ &= F_A \wedge A - A \wedge F_A. \end{aligned}$$

In Proposition 2.1.52 we have stated the Bianchi identity in a more invariant form, albeit one that requires determining the correct formula for the induced connection on $\mathrm{End}(E)$.

2.1.12 Curvature and Directional Derivatives

2.1.13 Curvature and Holonomy

2.2 Ehresmann Connections

2.2.1 Definitions

Suppose E is a vector bundle over M, and ∇ is a connection on E. This differential operator ∇ defines a notion of parallel, or *horizontal* sections of E. A section $s \in \Gamma(E)$ is *horizontal* if $\nabla s = 0$. That is, for every $X \in \Gamma(TM)$, $\nabla_X s = 0$; s is covariantly constant in the direction of X.

An Ehresmann connection specifies this notion of parallelism axiomatically, without reference to a covariant derivative ∇.

Definition 2.2.1 (Vertical Bundle)**.** *Let $\pi : E \to M$ be a fibre bundle over M, with fibre F. At each $p \in E$ consider the subspace $V_p := \ker(d\pi_p : T_pE \to T_{\pi(p)}M)$ of T_pE. This is called the* vertical subspace *of T_pE. The bundle $V \subset TE$ of vertical subspaces is called the* vertical bundle *of E. This is a smooth sub-bundle because π is.*

The vertical bundle of a fibre bundle E is canonically defined due to the existence of the projection map $\pi : E \to M$. Indeed V has a concrete visual interpretation as the subspace of tangent vectors to E pointing *along the fibre* at each point. Because the bundle locally varies along the fibre, but also along the base manifold, tangent vectors to E do not have to sit entirely within the vertical bundle V.

One would hope for some kind of complement to the vertical bundle that describes the tangent vectors pointing "along the base space." Unfortunately, there is no canonical choice of such a complement. A choice of such a complementary bundle is precisely what an Ehresmann connection is.

Definition 2.2.2 (Ehresmann Connection)**.** *Let E be a fibre bundle over a manifold M, with fibre F. An* Ehresmann connection *on E is a horizontal sub-bundle $H \subset TE$ such that for every $p \in E$, $T_pE = H_p \oplus V_p$.*

Recall that a sub-bundle of a vector bundle is required to be smoothly varying. Let $X \in \Gamma(TE)$. Then by the direct sum splitting we have $X(x) = X_h(x) + X_v(x)$ for some unique $X_h(x) \in H_p$ and $X_v(x) \in V_p$. Define projections $h : TE \to TE$ and $v : TE \to TE$ by $h(X)(x) := H_x(x)$ and $v(X)(x) := X_v(x)$.

For the horizontal distribution H to vary smoothly means that for any vector field $X \in \Gamma(TE)$, the horizontal projection $X_h \in \Gamma(TE)$, defined by the direct sum decomposition $X = X_h + X_v$, is also a smooth vector field.

Note that Ehresmann connections certainly exist. Since Riemannian metrics always exist (Corollary 1.2.29) one may choose a Riemannian metric on E considered as a smooth manifold, and then the orthogonal bundle $V^{\perp}$ defines a horizontal sub-bundle of TE.

The map $v : TE \to TE$ is fibrewise linear map that takes in tangent vectors to E and produces tangent vectors. Thus it may be interpreted as a TE-valued one-form on E. That is, $v \in \Omega^1(TE)$. Indeed v is actually a V-valued one-form, $v \in \Omega^1(V)$. Then we have that $\ker(v_p : T_pE \to V_p) = H_p$, so $\ker(v) = H$ is just the connection on E.

Definition 2.2.3 (Connection Form)**.** *The vertical-bundle-valued one-form $v \in \Omega^1(V)$ is the* connection form *of the Ehresmann connection H.*

Note that this connection form v satisfies the property that $v^2 = v$. Given that this additional property is satisfied, one may go in the other direction.

Proposition 2.2.4. *Let $v \in \Omega^1(V)$ be a vertical-bundle-valued one-form on E such that $v^2 = v$. Then $H := \ker(v)$ is an Ehresmann connection on E.*

Proof. Let $X(p) \in T_pE$. Define $X_h(p) := X(p) - v_p(X)$. Then $v_p(X_h(p)) = v_p(X(p)) - v_p^2(X(p)) = 0$, so $X_h(p) \in \ker(v_p)$ and this defines a direct-sum decomposition $T_pE = \ker(v_p) \oplus V_p$. Since v is a smooth one-form, the horizontal projection $h : TE \to TE$ is smooth, being the map $X \mapsto X - v(X)$. Thus H is an Ehresmann connection for E, with connection form v. □

2.2.2 Linear Connections as Ehresmann Connections

2.2.3 Horizontal Lifts and Holonomy

2.3 Principal Connections

2.3.1 Definitions

A principal connection on a principal G-bundle is simply an Ehresmann connection that is suitably equivariant with respect to the right action of G on the bundle.

Definition 2.3.1 (Principal Connection)**.** *Let $\pi : P \to M$ be a principal G-bundle. A* principal connection *on P is an Ehresmann connection H satisfying*

$$H_{p \cdot g} = d(R_g)_p H_p$$

for all $p \in P$ and $g \in G$.

2.3.2 Connection Forms

2.3.3 Curvature

2.4 Relations between Principal Connections and Linear Connections

2.4.1 Induced Connections on the Frame Bundle

2.4.2 Induced Connections on Associated Vector Bundles

2.5 Flat Connections and Representations

2.5.1 Flat Connections

2.5.2 Representations of the Fundamental Group

2.5.3 Projectively Flat Connections

2.5.4 Projective Representations of the Fundamental Group

Chapter 3

Sheaves

3.1 Sheaves of R-Modules

3.1.1 The Étale Space of a Sheaf

Definition 3.1.1 (Sheaf)**.** *Let K be a principal ideal domain. Let M be a manifold. A* sheaf *over M is a topological space $\mathcal{S}$ and a continuous surjection $\pi : \mathcal{S} \to M$ such that*

1. *π is a local homeomorphism,*
2. *$\pi^{-1}(x)$ has the structure of a K-module for every $x \in M$,*
3. *for each $\lambda \in K$, the map $s_\lambda : \mathcal{S} \to \mathcal{S}$ defined by $s_\lambda(p) := \lambda p$ is continuous, and*
4. *if $\mathcal{S} \circ \mathcal{S} := \{(s_1, s_2) \in \mathcal{S} \times \mathcal{S} \mid \pi(s_1) = \pi(s_2)\}$, then the map $+ : \mathcal{S} \circ \mathcal{S} \to \mathcal{S}$ defined by $(s_1, s_2) \mapsto s_1 + s_2$ is continuous.*

3.1.2 Presheaves

3.1.3 Complete Presheaves

3.2 Locally Free Sheaves

3.2.1 Definitions

3.2.2 Vector Bundles

3.2.3 The Tangent Bundle

3.3 Cech Cohomology

3.3.1 Fine Sheaves

3.3.2 Cech Cohomology Groups

Chapter 4

Complex Geometry

4.1 Complex Manifolds

4.1.1 Complex Structure

Definition 4.1.1 (Complex Manifold)**.** *A* complex manifold *is a topological space M such that*

1. *M is Hausdorff and paracompact,*
2. *M admits an open cover $\{U_\alpha\}$ for which there exists open sets $V_\alpha \subseteq \mathbb{C}^n$ for some n and homeomorphisms $\varphi_\alpha : U_\alpha \to V_\alpha$, and*
3. *for each $U_\alpha \cap U_\beta := U_{\alpha\beta} \neq \emptyset$, the maps*

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_{\alpha\beta}) \to \varphi_\alpha(U_{\alpha\beta})$$

are biholomorphisms of open subsets of $\mathbb{C}^n$.

Definition 4.1.2 (Complex Tangent Bundle)**.** *Let M be a complex manifold. The* complex tangent bundle *of M is the complex vector bundle*

$$TM \otimes (\mathbb{C} \times M).$$

Definition 4.1.3 (Canonical Bundle)**.** *Let M be a complex manifold. The* canonical bundle *of M is the line bundle $K := \det T^*M$.*

4.1.2 Almost Complex Structure

4.1.3 Dolbeault Cohomology

4.1.4 Riemann-Roch

Theorem 4.1.4 (Riemann-Roch)**.** *Let Σ be a Riemann surface of genus g and $\mathcal{L}$ a holomorphic line bundle over Σ. Then*

$$\dim H^0(M, \mathcal{L}) - \dim H^1(M, \mathcal{L}) = \deg \mathcal{L} - g + 1.$$

Definition 4.1.5. *Let $\mathcal{E}$ be a holomorphic vector bundle of rank n over a compact complex manifold M. The* holomorphic Euler characteristic *of E is the integer*

$$\chi(\mathcal{E}) := \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{C}} H^i(M, \mathcal{E}).$$

Theorem 4.1.6 (Hirzebruch-Riemann-Roch)**.** *Let M be a compact complex manifold and $\mathcal{E}$ be a holomorphic vector bundle over M. Then*

$$\chi(\mathcal{E}) = \int_M \mathrm{Ch}(\mathcal{E}) \, \mathrm{Td}(M)$$

Let Σ be a Riemann surface of genus g and $\mathcal{E} = \mathcal{L}$ a holomorphic line bundle. The Hirzebruch-Riemann-Roch theorem then says

$$\dim H^0(M, \mathcal{L}) - \dim H^1(M, \mathcal{L}) = \int_\Sigma \mathrm{Ch}(\mathcal{L}) \, \mathrm{Td}(\Sigma). \qquad \text{(Eq. 4.1)}$$

Now $\mathrm{Ch}(\mathcal{L}) = \exp(\mathrm{c}_1(\mathcal{L}))$ for a line bundle $\mathcal{L}$, and further we have

$$\exp(\mathrm{c}_1(\mathcal{L})) = 1 + \mathrm{c}_1(\mathcal{L})$$

because the higher powers vanish on a surface Σ. Here 1 is the generator of $H^0(\Sigma, \mathbb{Z})$, interpreted as the constant function 1.

On the other hand, $\mathrm{Td}(\Sigma) = 1 + \frac{\mathrm{c}_1(\Sigma)}{2}$ for a surface Σ, so the right-hand-side of (Eq. 4.1) becomes

$$\int_\Sigma \left(\frac{\mathrm{c}_1(\Sigma)}{2} + \mathrm{c}_1(\mathcal{L}) \right) = \frac{1}{2}(2 - 2g) + \deg \mathcal{L} = \deg \mathcal{L} - g + 1.$$

Thus in the case of line bundles on Riemann surfaces, the Hirzebruch-Riemann-Roch theorem (Theorem 4.1.6) recovers the classical Riemann-Roch theorem (Theorem 4.1.4).

4.2 Symplectic Manifolds

4.2.1 Symplectic Vector Spaces

Let V be a finite-dimensional real vector space. Let $\omega : V \times V \to \mathbb{R}$ be an antisymmetric bilinear form on V. That is, $\omega(v, w) = -\omega(w, v)$ for all $v, w \in V$. Define $\ker \omega$ to be the collection of $v \in V$ such that $\omega(v, w) = 0$ for all $w \in V$. Note taking v in the first or second position is equivalent, since ω is anti-symmetric.

Such a bilinear form ω is called *non-degenerate* if $\ker \omega = 0$. That is, for every $v \in V$ such that $v \neq 0$, $\omega(v, w) = 0$ if and only if $w = 0$. Equivalently, if we define a map $\lambda : V \to V^*$ by $\lambda(v)(w) := \omega(v, w)$, then ω is non-degenerate if and only if this map is a linear isomorphism. Indeed $\ker \omega$ is just the kernel of this map λ considered as a regular linear map.

Definition 4.2.1 (Symplectic Vector Space)**.** *Let V be a finite-dimensional real vector space. A* symplectic form *ω on V is an element of $\bigwedge^2 V^*$ that is non-degenerate as a bilinear map $\omega : V \times V \to \mathbb{R}$. The pair (V, ω) is called a* symplectic vector space*.*

Example 4.2.2. Let $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ be a basis for $\mathbb{R}^{2n}$. Suppose $\{\epsilon^1, \ldots, \epsilon^n, \xi^1, \ldots, \xi^n\}$ is the dual basis. Define a form $\omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ by

$$\omega := \sum_{i=1}^{n} \epsilon^i \wedge \xi^i.$$

The form ω is clearly antisymmetric and bilinear. Now with respect to the standard basis ω has the matrix

$$\omega = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix}.$$

This matrix is clearly invertible, so ω is non-degenerate. The pair $(\mathbb{R}^{2n}, \omega)$ is called the standard symplectic vector space, and ω is called the standard symplectic form on $\mathbb{R}^{2n}$.

Consider $\mathbb{R}^{2n}$ as $\mathbb{C}^n$ and let $h : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be the standard Hermitian form on $\mathbb{C}^n$. Then as a bilinear form h splits into real and imaginary parts $h = g + i\omega$. The symmetric bilinear form g is the standard inner product on $\mathbb{R}^{2n}$ and the anti-symmetric bilinear form ω is the standard symplectic form on $\mathbb{R}^{2n}$.

Definition 4.2.3. *Let (V, ω) be a symplectic vector space, and suppose $U \subset V$ is a subspace. Define $U^\perp$ to be the subspace*

$$U^\perp := \{v \in V \mid \omega(u, v) = 0 \textit{ for all } u \in V\}.$$

Then $U^\perp$ is called the symplectic complement *of U.*

Lemma 4.2.4. *Let (V, ω) be a symplectic vector space. Then there is a basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ of V such that $\omega(e_i, f_j) = \delta_{ij}$.*

Proof. Let $0 \neq e_1 \in V$. Then there exists $f_1 \in V$ such that $\omega(e_1, f_1) = 1$. Let $U_1 = \text{Span}\{e_1, f_1\}$. Suppose $u \in U_1 \cap U_1{}^\perp$. Then $u = ae_1 + bf_1$. Since $u \in U_1{}^\perp$, $\omega(u, e_1) = 0$. That is, $a = 0$. Similarly $\omega(u, f_1) = 0$, so $b = 0$. But then $U_1 \cap U_1{}^\perp = 0$. Now suppose $v \in V$. Then define $\omega(e_1, v) := a$ and $\omega(v, f_1) := b$. We have $v = (ae_1 + be_2) + (v - ae_1 - be_2)$, where the first term is in U_1 and the second is in $U_1{}^\perp$. Thus $V = U_1 \oplus U_1{}^\perp$.

If $U_1{}^\perp$ is the zero vector space, we are done. Suppose not, then let $0 \neq e_2 \in U_1{}^\perp$. Since ω is non-degenerate, there must exist $f_2 \in U_1{}^\perp$ such that $\omega(e_2, f_2) = 1$. Note that f_2 cannot be in U_1, since $\omega(e_2, u) = 0$ for all $u \in U_1$. Then define $U_2 := \text{Span}\{e_2, f_2\}$. Then we obtain $V = U_1 \oplus U_2 \oplus U_2{}^\perp$.

Since V is finite-dimensional, this process must eventually terminate, after which we obtain

$$V = U_1 \oplus U_2 \oplus \cdots \oplus U_n$$

for some n. Then the basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ is the desired basis for V. □

The basis for V formed above is called a *canonical basis* for (V,ω). There may be many such bases, but there is always one.

Corollary 4.2.5. *Let (V,ω) be a symplectic vector space. Then $\dim V$ is even.*

Note that the block matrix for ω with respect to a canonical basis is

$$\omega = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix}.$$

Definition 4.2.6. *Let $f : V \to W$ be a linear isomorphism of two symplectic vector spaces V and W. Then f is a* symplectomorphism *if $\omega_V(u,v) = \omega_W(f(u), f(v))$ for all $u, v \in U$. That is, $f^*\omega_V = \omega_U$.*

Thus Lemma 4.2.4 says that if (V,ω) is a $2n$-dimensional symplectic vector space, then V is symplectomorphic to $\mathbb{R}^{2n}$ with its standard symplectic form.

Lemma 4.2.7. *Let $U \subset V$ be a subspace of a symplectic vector space (V,ω). Then*

1. $\dim U + \dim U^\perp = \dim V$.
2. $(U^\perp)^\perp = U$.
3. *If $U \subset W$, then $W^\perp \subset U^\perp$.*

Proof. For (1), consider the map $f : V \to U^*$ defined by $v \mapsto \omega(v,\cdot)|_U$. Then $\dim\ker f = \dim U^\perp$. Since ω is non-degenerate, f is surjective, so the rank of f is $\dim U^* = \dim U$. Then by the rank theorem $\dim U + \dim U^\perp = \dim V$.

For (2), let $u \in U$. Then since $\omega(u,v) = 0$ for all $v \in U^\perp$, $u \in (U^\perp)^\perp$. But we know $\dim U + \dim U^\perp = \dim U^\perp + \dim (U^\perp)^\perp = \dim V$, so $\dim U = \dim (U^\perp)^\perp$. Thus $U = (U^\perp)^\perp$.

(3) is obvious. ∎

Definition 4.2.8. *Let $U \subset V$ be a subspace. Then U is*

1. symplectic *if $U \cap U^\perp = \{0\}$,*
2. isotropic *if $U \subset U^\perp$,*
3. coisotropic *if $U \supset U^\perp$, or*
4. Lagrangian *if $U = U^\perp$.*

A subspace U is symplectic if and only if $\omega|_U$ is a symplectic form on U. Note that we also have $V = U \oplus U^\perp$ in this case.

A subspace U is isotropic if and only if $\omega|_U$ is the zero form. Note that since $\omega(v,v) = 0$ for any $v \in V$, one-dimensional subspaces of V are isotropic.

A subspace U is coisotropic if and only if $U/U^\perp$ is a symplectic space with respect to the induced form defined by $\omega(u+U^\perp, v+U^\perp) := \omega|_U(u,v)$. This is because the part of U making the form $\omega|_U$ non-degenerate is the $U^\perp$ part, so removing this produces a symplectic space.

A subspace is Lagrangian if and only if it is isotropic and coisotropic.

Lemma 4.2.9. *Suppose $U \subset V$ is Lagrangian. Then* $\dim U = \frac{1}{2}\dim V$.

Proof. Since U is Lagrangian, it is isotropic and coisotropic. Suppose $\dim U > \frac{1}{2}\dim V$. Then since $U \subseteq U^{\perp}$, $\dim U^{\perp} > \frac{1}{2}\dim V$. But then $\dim U + \dim U^{\perp} > \dim V$. Similarly if $\dim U < \frac{1}{2}\dim V$, by coisotropy one must have $\dim U + \dim U^{\perp} < \dim V$. Since neither of these can occur, we must have $\dim U = \frac{1}{2}\dim V$. $\square$

Proposition 4.2.10. *An antisymmetric bilinear form ω on an even-dimensional vector space V is non-degenerate if and only if $\omega^n \in \bigwedge^{2n} V^*$ is non-zero.*

Proof. ($\implies$) Suppose ω is non-degenerate. Let $\{e_i, f_i\}$ be a canonical basis for the symplectic vector space (V, ω), and let $\{\epsilon^i, \xi^i\}$ be the dual basis. Then $\omega^n = n!\epsilon^1 \wedge \xi^1 \wedge \cdots \wedge \epsilon^n \wedge \xi^n$, which is non-zero.

($\impliedby$) Suppose ω is degenerate. Then there is some $v \neq 0$ such that $\omega(v, w) = 0$ for all $w \in V$. Complete v to a basis $\{v, w_2, \ldots, w_{2n}\}$ of V. Then $\omega^n(v, w_2, \ldots, w_{2n}) = 0$. But if $\{\epsilon^i\}$ is the dual basis to $\{v, w_j\}$ then $\omega^n = k\epsilon^1 \wedge \cdots \wedge \epsilon^{2n}$ for some k. Since $v \wedge w_2 \wedge \cdots \wedge w_{2n}$ is the dual basis to $\epsilon^1 \wedge \cdots \wedge \epsilon^{2n}$ inside $\bigwedge^{2n} V^*$, one must have $k = 0$, so $\omega^n = 0$. $\square$

The form $d\text{vol} := \omega^n / n!$ is called the *symplectic volume form* on (V, ω). The normalisation factor is to make the area of the unit parallelepiped in a canonical basis for ω equal to 1.

4.2.2 Symplectic Manifolds

Definition 4.2.11 (Symplectic Manifold)**.** *Let M be a smooth manifold, and $\omega \in \Omega^2(M)$. Then ω is* symplectic *if ω is closed, and $\omega|_p : T_pM \times T_pM \to \mathbb{R}$ is symplectic. The pair (M, ω) is called a* symplectic manifold*, and ω is the symplectic form.*

Note in particular that symplectic manifolds must be even-dimensional, since the dimension of the tangent space at each point is equal to the dimension of the manifold.

Example 4.2.12. Let $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ be the standard coordinates on $\mathbb{R}^{2n}$. Then the form

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy^i$$

gives $\mathbb{R}^{2n}$ the structure of a symplectic manifold. This is the standard symplectic form on $\mathbb{R}^{2n}$.

Example 4.2.13. Let $(z^1, \ldots, z^n)$ be the standard coordinates on $\mathbb{C}^n$. Then the form

$$\omega := \sum_{i=1}^{n} dz^i \wedge d\bar{z}^i$$

gives $\mathbb{C}^n$ a symplectic structure. This is the standard symplectic form on $\mathbb{C}^n$.

Let (M, ω) be a symplectic manifold of dimension $2n$. Since ω is everywhere non-degenerate, it defines a volume form $d\mathrm{vol} := \omega^n/n!$. In particular, M is orientable. As was mentioned earlier, M must also be even dimensional.

Suppose further that M is compact. Then the closed form ω defines a non-zero cohomology class in $H^2(M, \mathbb{R})$. The volume form $d\mathrm{vol}$ also defines a non-zero cohomology class in $H^{2n}(M, \mathbb{R})$. Since $d\mathrm{vol} = \omega^n$, we must have $\omega^k \neq 0$ for every k. In particular $H^{2k}(M, \mathbb{R})$ must be non-trivial for all $k = 1, \ldots, n$.

This gives us a collection of simple obstructions to the existence of a symplectic structure on a smooth manifold. A manifold that fails to satisfy those properties mentioned above cannot admit any symplectic structure.

Example 4.2.14. The Lie group $\mathrm{U}(2) \cong S^3 \times S^1$ is compact, even-dimensional, and orientable. However, by the Künneth formula, the second degree cohomology of $\mathrm{U}(2)$ is zero. Thus $\mathrm{U}(2)$ does not admit any symplectic structures.

Similarly to the linear case, we can define a notion of equivalence of symplectic manifolds.

Definition 4.2.15 (Symplectomorphism)**.** *Let (M, ω) and (N, η) be symplectic manifolds. Let $f : M \to N$ be a diffeomorphism. Then f is a* symplectomorphism *if $f^*\omega_N = \omega_M$.*

Equivalently, f is a symplectomorphism if $df|_{T_pM}$ is a symplectomorphism of symplectic vector spaces, for each $p \in M$.

In the Riemannian case, it is a fantastic result of Nash that any Riemannian manifold of arbitrary dimension (and arbitrary smoothness class C^k for $3 \leq k \leq \infty \leq \omega$) can be isometrically embedded with a map of the same smoothness class into some $\mathbb{R}^N$.

A natural question to ask is whether there is such an embedding theorem for symplectic manifolds. Consider $(\mathbb{R}^{2N}, \omega)$, where ω is the standard symplectic form. This form ω can be written as $-d\theta$ for a one-form $\theta = \sum_{i=1}^N y^i dx^i$. In particular, ω is exact.

Now suppose $\iota : M \hookrightarrow \mathbb{R}^{2N}$ is a compact submanifold of $\mathbb{R}^{2n}$. Then $-d(\iota^*\theta) = -\iota^* d(\theta) = \iota^*\omega \in H^2(M, \mathbb{R})$. But then the induced two-form on M is cohomologically trivial. If one chooses a symplectic form on M that is cohomologically non-trivial, such a form cannot be the pullback of the standard form ω on $\mathbb{R}^{2N}$ for any N. Thus we must conclude there is no symplectic embedding theorem.

4.2.3 Phase Space

In the previous section, we saw that the standard symplectic form ω on $\mathbb{R}^{2n}$ may be written as $-d\theta$ for a one-form θ. In this section we will expand on this observation.

Let M be a smooth manifold, and $(U, (x^1, \ldots, x^n))$ be a coordinate chart. Let $\varphi : T^*M|_U \to U \times \mathbb{R}^n$ be the trivialisation, and $\phi : U \to \mathbb{R}^n$ be the chart. Then one can define coordinates $\psi := (\phi, \mathbf{1}) \circ \varphi$.

Given a point $p \in T^*M|_U$ with $\phi(\pi(p)) = (x^1, \ldots, x^n)$, one has $\psi(p) = (x^1, \ldots, x^n, y_1, \ldots, y_n)$ for some $y_1, \ldots, y_n \in \mathbb{R}$. Indeed if $\{dx^i\}$ is the local frame for the cotangent bundle over U, then $p = y_i dx^i \in T^*_{\pi p}M$.

Definition 4.2.16 (Canonical Coordinates)**.** *The coordinates* $(x^1, \ldots, x^n, y_1, \ldots, y_n)$ *defined above on* $T^*M|_U$ *are called the* canonical coordinates *on* T^*M*.*

Let M be a smooth manifold. Let $N := T^*M$. Define a one-form θ on T^*M as follows. Let $\pi : N \to M$ be the standard projection, and consider $d\pi : TN \to TM$. Let $n \in TN$. Then if $\pi(n) = q$, we may interpret n as a linear functional $n : T_qM \to \mathbb{R}$ on the tangent space at q. Define $\theta|_n := n \circ d\pi|_{T_qN}$. That is, given a covector $n \in T_qM$ and a tangent vector v in $T_n(T^*M)$, project v to a tangent vector $d\pi(v)$ in T_qM, and then evaluate $d\pi(v)$ on the covector n, to obtain a real number.

Definition 4.2.17 (Tautological One-Form)**.** *The differential form* θ *on* $N = T^*M$ *is called the* tautological one-form *of* M*.*

If one considers $\mathbb{R}^{2n}$ as $T^*\mathbb{R}^n$, with the standard global coordinates on $\mathbb{R}^n$, then the one-form θ as described in the previous section is precisely the tautological one-form on $\mathbb{R}^n$.

Definition 4.2.18 (Canonical Symplectic Form)**.** *Let* $\omega := -d\theta$*. Then* ω *is called the* canonical symplectic form*.*

The form ω is clearly symplectic, because in canonical coordinates

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy_i.$$

This construction gives us a wealth of examples of symplectic manifolds. Namely, any smooth manifold produces a symplectic manifold. The cotangent bundle $N = T^*M$ equipped with its canonical symplectic form is often called a phase space. This terminology arises from physics, where M is usually the configuration space of a classical system, whose points describe positions of all the particles in the system. Then the cotangent vectors on this configuration space are the momentums of the particles. A point in the cotangent space thus completely describes the state of the system, and so is called the phase space. The dynamics of the classical system can be naturally interpreted in terms of the canonical symplectic form.

Before moving on, we will observe an interesting property of the tautological one-form.

Proposition 4.2.19. *The tautological one-form on a manifold* M *satisfies* $\xi^*\theta = \xi$ *for every* $\xi \in \Omega^1(M)$*.*

Proof. Let $\xi \in \Omega^1(M)$ be a one-form. Then we may interpret ξ as a section $\xi : M \to T^*M$. Then $\xi^*\theta = \xi$. To see this, let $p \in M$ and $v \in T_pM$. Then

$$\begin{aligned} \xi^*\theta|_p(v) &= \theta_{\xi|_p}(d\xi(v)) \\ &= \xi|_p \circ d\pi \circ d\xi(v) \\ &= \xi|_p \circ d(\pi \circ \xi)(v) \\ &= \xi|_p(v). \end{aligned}$$

□

4.2.4 Darboux's Theorem

In Riemannian geometry, an arbitrary Riemannian manifold (M, g) is not locally isometric to Euclidean space. The obstruction to this local triviality of the Riemannian structure is the Riemannian curvature tensor. The symplectic case is remarkably different, in that any symplectic manifold is locally symplectomorphic to (an open set in) $\mathbb{R}^{2n}$ with its trivial symplectic structure. This is the famous Darboux theorem.

Theorem 4.2.20 (Darboux Theorem)**.** *Let*

4.2.5 Coadjoint Orbits

Definition 4.2.21 (Coadjoint Representation)**.** *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Let* $\mathrm{ad} : G \to \mathrm{GL}(\mathfrak{g})$ *denote the adjoint representation of G on $\mathfrak{g}$. Let $\xi \in \mathfrak{g}^*$. Define the* coadjoint representation *of G on $\mathfrak{g}^*$ by*

$$\langle \mathrm{ad}^*(g)\xi, X\rangle := \langle \xi, \mathrm{ad}(g)X\rangle$$

for all $g \in G$, and $X \in \mathfrak{g}$.

Let $\xi \in \mathfrak{g}^*$. The *coadjoint orbit* of ξ is the orbit of ξ under the coadjoint action of G on $\mathfrak{g}$. If $S_G(\xi)$ denotes the subgroup of G that stabilises ξ under the coadjoint representation, then $G \cdot \xi \cong G\big/S_G(\xi)$.

Proposition 4.2.22. *The coadjoint orbit of any $\xi \in \mathfrak{g}^*$ is a smooth manifold.*

Proof. Since the action of G on $\mathfrak{g}^*$ is smooth, the stabiliser $S_G(\xi)$ of any $\xi \in \mathfrak{g}^*$ is a closed subgroup of G. Hence it is a Lie subgroup, and the quotient $G\big/S_G(\xi)$ admits the structure of a smooth manifold, for any $\xi \in \mathfrak{g}^*$. □

Now let $\zeta \in \mathfrak{g}^*$ be arbitrary. For each such ζ, one may define a bilinear form $B_\zeta : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ by the expression

$$B_\zeta(X, Y) := \langle \zeta, [X, Y]\rangle$$

for all $X, Y \in \mathfrak{g}$. Note that the bilinear form B_ζ is antisymmetric.

Suppose $G \cdot \xi$ is a coadjoint orbit in $\mathfrak{g}^*$. Since $G \cdot \xi \cong G\big/S_G(\xi)$, if $\zeta \in G \cdot \xi$ is arbitrary, then $T_\zeta G \cdot \xi = T_\zeta G \cdot \zeta \cong \mathfrak{g}\big/\mathfrak{s}_G(\zeta)$, where $\mathfrak{s}_G(\zeta)$ denotes the Lie subalgebra of $\mathfrak{g}$ corresponding to the Lie subgroup $S_G(\zeta)$.

Let $X, Y \in T_\zeta G \cdot \xi$. Then there are unique $\tilde{X}, \tilde{Y} \in \mathfrak{g}$ such that $\tilde{X} + \mathfrak{s}_G(\zeta)$ and $\tilde{Y} + \mathfrak{s}_G(\zeta)$ correspond to X and Y. Define a two-form $\omega \in \Omega^2(G \cdot \xi)$ by

$$\omega|_\zeta (X, Y) := B_\zeta(\tilde{X}, \tilde{Y}).$$

Lemma 4.2.23. *The two-form ω is well-defined.*

Proof. It suffices to show $\mathfrak{s}_G(\xi) = \ker B_\xi$, where $\xi \in \mathfrak{g}^*$. Now $X \in \mathfrak{s}_G(\xi)$ if and only if $(\mathrm{ad}\,*)_*(X)\xi = 0$. We have

$$\begin{aligned}
\left\langle \frac{d}{dt}\left(\mathrm{ad}^*(\exp(tX))\xi\right)_{t=0}, Y\right\rangle &= \frac{d}{dt}\left(\langle \mathrm{ad}^*(\exp(tX))\xi, Y\rangle\right)_{t=0} \\
&= \frac{d}{dt}\left(\langle \xi, \mathrm{ad}(\exp(-tX))Y\rangle\right)_{t=0} \\
&= \left\langle \xi, \frac{d}{dt}\left(\mathrm{ad}(\exp(-tX))Y\right)_{t=0}\right\rangle \\
&= \langle \xi, -[X, Y]\rangle.
\end{aligned}$$

Thus $X \in \mathfrak{s}_G(\xi)$ if and only if $B_\xi(X, Y) = 0$ for all $Y \in \mathfrak{g}$. In particular the two-form ω above is well-defined, for all $\zeta \in G \cdot \xi$ and for all $X, Y \in T_\zeta G \cdot \xi$. $\square$

Lemma 4.2.24. *The two-form ω is closed.*

Proof. $\square$

Thus the pair $(G\cdot\xi, \omega)$ is a symplectic manifold for every $\xi \in \mathfrak{g}^*$. The form ω is called the *Kirillov-Kostant form* of the coadjoint orbit ξ. In particular this gives a plentiful supply of examples of symplectic manifolds.

4.2.6 Hamiltonian Vector Fields and the Poisson Bracket

Let (M, ω) be a symplectic manifold and $f \in C^\infty(M)$ be a smooth function on M. Then $df \in \Omega^1(M)$ is a one-form.

Let $X \in \Gamma(TM)$. Let i_X be the contraction operator $i_X : \Omega^k(M) \to \Omega^{k-1}(M)$ defined by

$$i_X\eta(X_2, \ldots, X_k) := \eta(X, X_2, \ldots, X_k)$$

for $\eta \in \Omega^k(M)$ and $X_2, \ldots, X_k \in \Gamma(TM)$, and $i_X|_{\Omega^0(M)} := 0$.

Then we have $i_X\omega \in \Omega^1(M)$ for our symplectic form ω and for any $X \in \Gamma(TM)$. Since ω is non-degenerate, it defines an isomorphism $\Gamma(TM) \to \Omega^1(M)$ given by $X \mapsto i_X\omega$. But then given $f \in C^\infty(M)$, there exists a smooth vector field $X_f \in \Gamma(TM)$ such that $i_{X_f}\omega = -df$ (why we require a negative sign will become clear later).

Definition 4.2.25. *Let (M, ω) be a symplectic manifold and let $f \in C^\infty(M)$. Then the unique vector field $X_f \in \Gamma(TM)$ such that*

$$i_{X_f}\omega = -df$$

is called the Hamiltonian vector field associated to f.

Note that not every vector field $X \in \Gamma(TM)$ is Hamiltonian, because not every one-form η is necessarily exact. If $H^1(M, \mathbb{R}) = 0$ however, that is, M is simply connected, then every vector field is the Hamiltonian vector field of some function.

A natural question one might ask is whether the Hamiltonian vector fields are closed under natural operations one can perform on vector fields. Clearly the Hamiltonian vector fields form a vector subspace of $\Gamma(TM)$ by linearity of d. What about the commutator $[X_f, X_g]$ of two Hamiltonian vector fields?

First we recall a useful formula of Cartan.

Theorem 4.2.26 (Cartan's Magic Formula)**.** *Let $X \in \Gamma(TM)$ and $\omega \in \Omega^k(M)$. Then*

$$\mathcal{L}_X\omega = i_X d\omega + d(i_X\omega).$$

Lemma 4.2.27. *Let $f, g \in C^\infty(M)$ and let $h := \omega(X_f, X_g)$. Then $[X_f, X_g] = X_h$.*

Proof. Using Cartan's magic formula, we have

$$\mathcal{L}_{X_f}(g) = i_{X_f}dg = i_{X_f}(-i_{X_g}\omega) = \omega(X_f, X_g).$$

Then

$$\begin{aligned} d(\omega(X_f, X_g)) &= d(\mathcal{L}_{X_f}g) \\ &= \mathcal{L}_{X_f}(dg) \\ &= -\mathcal{L}_{X_f}(i_{X_g}\omega). \end{aligned}$$

Now $\mathcal{L}_U(i_V\alpha) = i_{[U,V]}\alpha + i_V\mathcal{L}_U(\alpha)$, and by Cartan's magic formula if α is closed and $i_U\alpha$ is closed, then $\mathcal{L}_U\alpha = 0$. Since $i_{X_f}\omega = -df$ is exact, it is closed, so we have

$$d(\omega(X_f, X_g)) = -i_{[X_f,X_g]}\omega$$

as desired. □

Definition 4.2.28. *Let $f, g \in C^\infty(M)$ and suppose $X_f, X_g \in \Gamma(TM)$ are their corresponding Hamiltonian vector fields. Define the* Poisson bracket *of f and g to be the smooth function*

$$\{f, g\} := \omega(X_f, X_g).$$

We recall a useful characterisation of the Lie derivative for differential forms:

Theorem 4.2.29 (Cartan's Magic Formula)**.** *Let $X \in \Gamma(TM)$ and $\omega \in \Omega^k(M)$. Then*

$$\mathcal{L}_X\omega = i_X d\omega + d(i_X\omega).$$

By Cartan's magic formula we can write the Poisson bracket is several equivalent ways:

$$\mathcal{L}_{X_f}(g) = i_{X_f}dg = i_{X_f}(-i_{X_g}\omega) = \omega(X_f, X_g) = \{f, g\}.$$

From the definition of the Poisson bracket, we have that $\{f, g\} = -\{g, f\}$ for $f, g \in C^\infty(M)$, and

4.2.7 Moment Maps and Symplectic Reduction

4.3 Kähler Manifolds

4.4 Hyper-Kähler Manifolds

4.5 Holomorphic Vector Bundles

4.5.1 Dolbeault Operators

4.5.2 Line Bundles on Riemann Surfaces

Smooth Case

Let $L \to \Sigma$ be a smooth $\mathbb{C}$-line bundle over a Riemann surface of genus g. Let $\mathbb{C}_\infty$ (resp. $\mathbb{C}^*_\infty$))denote the sheaf of smooth $\mathbb{C}$-valued (resp. $\mathbb{C}^*$-valued) functions on Σ. Since $\mathrm{GL}(1,\mathbb{C}) \cong \mathbb{C}^*$, smooth complex line bundles are classified up to isomorphism by the group $H^1(\Sigma, \mathbb{C}^*_\infty)$.

Consider the short exact sequence of constant sheaves

$$0 \longrightarrow \underline{\mathbb{Z}} \hookrightarrow \mathbb{C}_\infty \xrightarrow{\exp} \mathbb{C}^*_\infty \longrightarrow 0,$$

where $\exp : \mathbb{C}_\infty \to \mathbb{C}^*_\infty$ sends f to $\exp(2\pi i f)$.

This induces a long exact sequence in sheaf cohomology, which takes the form

$$\begin{array}{l} 0 \longrightarrow H^0(\Sigma,\underline{\mathbb{Z}}) \longrightarrow H^0(\Sigma,\mathbb{C}_\infty) \longrightarrow H^0(\Sigma,\mathbb{C}^*_\infty) \\ \quad \hookrightarrow H^1(\Sigma,\underline{\mathbb{Z}}) \longrightarrow H^1(\Sigma,\mathbb{C}_\infty) \longrightarrow H^1(\Sigma,\mathbb{C}^*_\infty) \\ \quad \hookrightarrow H^2(\Sigma,\underline{\mathbb{Z}}) \longrightarrow H^2(\Sigma,\mathbb{C}_\infty) \longrightarrow H^2(\Sigma,\mathbb{C}^*_\infty) \longrightarrow 0. \end{array}$$

Now $\mathbb{C}_\infty$ is a fine sheaf, so $H^i(\Sigma, \mathbb{C}_\infty) = 0$ for all $i > 0$. Thus we obtain the short exact sequence

$$0 \longrightarrow H^1(\Sigma, \mathbb{C}^*_\infty) \longrightarrow H^2(\Sigma, \underline{\mathbb{Z}}) \longrightarrow 0.$$

Now $H^2(\Sigma, \underline{\mathbb{Z}}) \cong \mathbb{Z}$, from which we conclude

Theorem 4.5.1. *Isomorphism classes of smooth complex line bundles over a Riemann surface are in bijection with the integers.*

Label the map $H^1(\Sigma, \mathbb{C}^*_\infty) \to H^2(\Sigma, \underline{\mathbb{Z}})$ by deg. The image of a line bundle $L \in H^1(\Sigma, \mathbb{C}^*_\infty)$ under deg is called the *degree* of the line bundle. Thus the theorem above states that smooth complex line bundles on a surface are classified by their degree.

Furthermore, the above isomorphism is one of *groups*. The group structure on $H^1(\Sigma, \mathbb{C}^*_\infty)$ is by multiplication of transition functions on overlaps $U_{\alpha\beta}$. In the case of line bundles, this is the same as the tensor product of transition functions, as per the construction of the tensor product bundle, from which we conclude:

Corollary 4.5.2.

$$(\{\textit{Isomorphism classes of smooth complex line bundles on } \Sigma\}, \otimes) \cong (\mathbb{Z}, +)$$

as groups.

The degree as stated here has several interpretations. Firstly, the degree is equal to the integral of the first Chern class $c_1(L)$ of the line bundle, over Σ. Secondly, the degree is the number of zeros, counted with multiplicity, of a section of L which intersects the zero-section transversally. Thirdly, in the case of the sphere S^2, the degree is the winding number of the clutching function defined on (a tubular neighbourhood of) the equator.

Holomorphic Case

Let $\mathcal{L} \to \Sigma$ be a holomorphic line bundle over a Riemann surface Σ of genus g. Then we have a short exact sequence

$$0 \longrightarrow \mathbb{Z} \hookrightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0.$$

Then we obtain a long exact sequence

$$\begin{array}{l} 0 \longrightarrow H^0(\Sigma, \mathbb{Z}) \longrightarrow H^0(\Sigma, \mathcal{O}) \longrightarrow H^0(\Sigma, \mathcal{O}^*) \\ \hookrightarrow H^1(\Sigma, \mathbb{Z}) \longrightarrow H^1(\Sigma, \mathcal{O}) \longrightarrow H^1(\Sigma, \mathcal{O}^*) \\ \hookrightarrow H^2(\Sigma, \mathbb{Z}) \longrightarrow H^2(\Sigma, \mathcal{O}) \longrightarrow H^2(\Sigma, \mathcal{O}^*) \longrightarrow 0 \end{array}$$

Now since Σ has complex dimension one, the group $H^2(\Sigma, \mathcal{O}) = 0$. The map $\mathcal{O} \to \mathcal{O}^*$ is surjective, so the image of the map $H^0(\Sigma, \mathcal{O}^*) \to H^1(\Sigma, \mathbb{Z})$ is 0. Thus we obtain an exact sequence

$$0 \longrightarrow H^1(\Sigma, \mathbb{Z}) \longrightarrow H^1(\Sigma, \mathcal{O}) \longrightarrow H^1(\Sigma, \mathcal{O}^*) \longrightarrow H^2(\Sigma, \mathbb{Z}) \longrightarrow 0.$$

By the first isomorphism theorem, the image of $H^1(\Sigma, \mathbb{Z}) \to H^1(\Sigma, \mathcal{O})$ is isomorphic to the quotient $H^1(\Sigma, \mathcal{O})/H^1(\Sigma, \mathbb{Z})$, so we have a reduction to the short exact sequence

$$0 \longrightarrow H^1(\Sigma, \mathcal{O})/H^1(\Sigma, \mathbb{Z}) \longrightarrow H^1(\Sigma, \mathcal{O}^*) \longrightarrow H^2(\Sigma, \mathbb{Z}) \longrightarrow 0.$$

Now $H^1(\Sigma, \mathbb{Z}) \cong \mathbb{Z}^{2g}$, and $H^1(\Sigma, \mathcal{O}) \cong \mathbb{C}^g$. We also know that $H^2(\Sigma, \mathbb{Z}) \cong \mathbb{Z}$. Thus we obtain the exact sequence

$$0 \longrightarrow \mathbb{C}^g/\mathbb{Z}^{2g} \longrightarrow H^1(\Sigma, \mathcal{O}^*) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Let $\mathcal{L} \in H^1(\Sigma, \mathcal{O}^*)$ be a line bundle. Then the image under the map $H^1(\Sigma, \mathcal{O}^*) \to \mathbb{Z}$ is called the *degree* of $\mathcal{L}$. This is the same as $\deg(L)$ where L is $\mathcal{L}$ interpreted as a smooth bundle. It follows from the above diagram that the group $\mathrm{Jac}(\Sigma)$ of all line bundles of degree zero is isomorphic to the torus

$$\mathrm{Jac}(\Sigma) \cong \mathbb{C}^g/\mathbb{Z}^{2g}.$$

It follows that the set $\mathrm{Pic}(\Sigma)$ of all holomorphic line bundles over Σ up to isomorphism is isomorphic to a countable disjoint union of g-tori. In particular if $\mathrm{Pic}_d(\Sigma)$ denotes the set of all holomorphic line bundles of a fixed degree d, then $\mathrm{Pic}_d(\Sigma) \cong \mathbb{C}^g/\mathbb{Z}^{2g}$ for all $d \in \mathbb{Z}$.

Riemann-Roch for Surfaces

Now $H^0(\Sigma, \mathcal{L})$ is the space of global holomorphic sections of L. Suppose then that $\deg L > g - 1$. Then by (Eq. **??**) we have

$$\dim H^0(\Sigma, \mathcal{L}) - \dim H^1(\Sigma, \mathcal{L}) > 0.$$

Since the dimension is always a non-negative integer, when this is the case we can conclude that

$$\dim H^0(\Sigma, \mathcal{L}) > 0.$$

Also, if $\deg L < g - 1$ we can similarly conclude that $\dim H^0(\Sigma, \mathcal{L}) = 0$. Thus $\deg L$, which is purely topological in nature, gives us holomorphic information about the line bundle L. Its size determines the existence of holomorphic sections. Note that if $\deg L = g - 1$ we cannot conclude anything about the existence of holomorphic sections.

Later we will study the *slope* of a vector bundle on a Riemann surface, and will discover that this (rational) number, defined topologically (it is the degree divided by the rank), also contains important holomorphic information.

4.5.3 Stable Bundles

Chapter 5

Moduli Spaces

5.1 The Yang-Mills equations

5.1.1 The Hodge star

The Inner Product on Differential Forms

Let M be an oriented Riemannian manifold with volume form $d\mathrm{vol}$ and metric g.

The metric g on M induces a metric on $\Omega^k(M)$ for every k as follows:

Suppose $\omega, \eta \in \Omega^k(M)$ are k-forms. In local coordinates we can always split these k-forms into sums of wedge products of one-forms. For example, for ω we might have

$$\omega = \sum_i \omega_1^i \wedge \cdots \wedge \omega_k^i$$

where the $\omega_j^i \in \Omega^1(M)$.

The metric g on M induces an isomorphism $(\cdot)^\flat : \Gamma(TM) \to \Omega^(M)$. This is fibrewise the isomorphism of T_pM and T_p^*M given by $v \mapsto g(v, \cdot)$.

Thus from ω we obtain an element of $\bigwedge^k TM$

$$\omega^\sharp := \sum_i (\omega_1^i)^\sharp \wedge \cdots \wedge (\omega_k^i)^\sharp.$$

Now suppose for a moment that ω and η are pure differential k-forms. That is, $\omega = \omega_1 \wedge \cdots \omega_k$ and $\eta = \eta_1 \wedge \eta_k$ for some $\omega_i, \eta_j \in \Omega^1(M)$. Since the inner product will be bilinear, it suffices to define the inner product just on these such k-forms.

Define

$$\langle \omega, \eta \rangle_g := \det(g((\omega_i)^\sharp, (\eta_j)^\sharp)).$$

First we note that this definition makes sense. The terms $(\omega_i)^\sharp$ and $(\eta_j)^\sharp$ are (at least locally) sections of the tangent bundle, so it makes sense to apply the metric g to them.

This definition produces a smooth function $\langle \omega, \eta \rangle_g$ on M, that satisfies the properties of the inner product (all following from the properties of g and the musical isomorphisms $\flat$ and $\sharp$).

One must of course check that this definition is independent of which one-forms we write ω and η as. It would suffice to consider local charts on M and expand these differential forms in the coordinate one-form basis and compute the change of coordinates there.

The Hodge Star on Differential Forms

Let $\omega \in \Omega^k(M)$ be a k-form. Define the $n-k$-form $\star\omega$ to be the unique differential form such that

$$\eta \wedge \star\omega = \langle \eta, \omega \rangle_g d\mathrm{vol}$$

for all differential k-forms η.

Such an $n-k$-form is indeed unique, because if μ also satisfied the above condition, on a local coordinate chart (U, φ) we would have

$$dx^i \wedge \star\omega = \langle dx^i, \omega \rangle_g d\mathrm{vol} = dx^i \wedge \mu$$

for the entire coordinate one-form basis $\{dx^i\}$. But then $\star\omega = \mu$.

The existence of such a form can be deduced as follows: In local coordinates the k-forms $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ form a basis for the k-forms on the local chart, (U, φ) say. Let $(i_1, \ldots, i_n)$ be any permutation σ of $(1, \ldots, n)$. Then one may define

$$\star(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) := \mathrm{sgn}(\sigma) dx^{i_{k+1}} \wedge \cdots \wedge dx^{i_n}.$$

This defines the Hodge star on the basis of the k-forms, and if we consider $\star$ to be an operator on differential forms that is linear over $C^\infty(M)$, this defines the Hodge star of any differential k-form on U. By uniqueness of the Hodge star of a differential form, the above expressions will be equal on overlaps of coordinates, so we get a well defined Hodge star of any differential k-form defined on all of M.

To see what the Hodge star *does*, we need only look at the simplest examples. In $\mathbb{R}^3$ with the standard coordinate one-forms, we have the standard volume form $dx \wedge dy \wedge dz$, and by the above definition

$$\begin{aligned}
&\star\, dx = dy \wedge dz, \\
&\star\, (dx \wedge dy) = dz, \\
&\star\, (dx \wedge dy \wedge dz) = 1.
\end{aligned}$$

So the Hodge star is the operator that gives you "the rest of the differential form." One would expect that any differential k-form could be wedged with something to produce some multiple of the volume form, and the Hodge star is precisely this $n-k$-form.

The L^2 inner product on differential forms with values in a vector bundle

Suppose E is some vector bundle over M and $\omega \in \Omega^k(E)$. Then we can write $\omega = \omega^i \otimes s_i$ for some k-forms ω^i and sections s_i of E. One can hence define $\star\omega := (\star\omega^i) \otimes s_i$.

Now suppose further that E has a metric. That is, an inner product h on each fibre of E, varying smoothly in the sense that $h(s, s') \in C^\infty(M)$ for all $s, s' \in \Omega^0(E)$.

Suppose now that $a, b \in \Omega^k(E)$ are differential k-forms with values in E. Suppose $a = \omega^i \otimes s_i$ and $b = \eta^j \otimes t_j$, where ω^i, η^j are k-forms on M and $s^i, t_j \in \Omega^0(E)$. Define an operator $h(a \wedge b)$ by

$$h(a \wedge b) := h(s_i, t_j)(\omega^i \wedge \eta^j).$$

Then finally define

$$\langle a, b \rangle_{L^2} := \int_M h(a \wedge \star b).$$

5.1.2 Derivation of the Yang-Mills equations

Suppose M is an oriented Riemannian manifold and E is a smooth vector bundle over M. The trace defines a metric on $\mathrm{End}(E)$. Thus we can define the Yang-Mills functional as

$$\mathcal{YM}(\nabla) := \langle F_\nabla, F_\nabla \rangle_L^2.$$

This will also be written

$$\begin{aligned} \mathcal{YM}(A) &:= \int_M \mathrm{tr}(F_A \wedge \star F_A) \\ &:= |F_\nabla|_{L^2}^2 \\ &:= \int_M |F_\nabla|^2 d\mathrm{vol} \end{aligned}$$

Let $A \in \mathscr{A}$ be a connection. The critical points of the Yang-Mills functional occur when the first variation vanishes. That is, when

$$\frac{d}{dt}(\mathcal{YM}(A + ta))_{t=0} = 0$$

We know from the previous section that $F_{A+ta} = F_A + td_A(a) + t^2 a \wedge a$ so we have

$$\mathcal{YM}(A + ta) = \int_M \mathrm{tr}((F_A + td_A(a) + t^2 a \wedge a) \wedge \star(F_A + td_A(a) + t^2 a \wedge a))$$

Observe that the only terms of order t in this expression are $F_A \wedge \star d_A(a)$ and $d_A(a) \wedge \star F_A$, so we have

$$\frac{d}{dt}(\mathcal{YM}(A + ta))_{t=0} = \int_M \mathrm{tr}(F_A \wedge \star d_A(a)) + \int_M \mathrm{tr}(d_A(a) \wedge \star F_A)$$

for any $a \in \Omega^1(\mathrm{End}(E))$.

But this is just the inner product as defined in the previous section, so we may collect these two terms and take an adjoint to conclude

$$\frac{d}{dt}(\mathcal{YM}(A+ta))_{t=0} = \int_M \operatorname{tr}(d_A^* F_A \wedge \star a).$$

Since this holds for any $a \in \Omega^1(\mathrm{End}(E))$, the critical points of the Yang-Mills equations occur precisely when $d_A^* F_A = 0$.

5.2 Line Bundles on a Riemann Surface

5.2.1 Smooth Case

Let $L \to \Sigma$ be a smooth complex line bundle on a Riemann surface Σ_g of genus g. Let $C^\infty(\mathbb{C})$ (resp. $C^\infty(\mathbb{C}^*)$) denote the sheaf of smooth $\mathbb{C}$-valued (resp. $\mathbb{C}^*$-valued) functions on Σ_g. Since $\mathrm{GL}(1,\mathbb{C}) \cong \mathbb{C}^*$, smooth complex line bundles are classified up to isomorphism by their representative in $H^1(\Sigma, C^\infty(\mathbb{C}^*))$.

Let $\underline{\mathbb{Z}}$ denote the constant $\mathbb{Z}$ sheaf on Σ_g. Consider the short exact sequence sheaf sequence

$$0 \longrightarrow \underline{\mathbb{Z}} \hookrightarrow C^\infty(\mathbb{C}) \xrightarrow{\exp} C^\infty(\mathbb{C}^*) \longrightarrow 0,$$

where $\exp : C^\infty(\mathbb{C}) \to C^\infty(\mathbb{C}^*)$ sends f to $\exp(2\pi i f)$.

This induces a long exact sequence in sheaf cohomology, which takes the form

$$\begin{array}{l} 0 \longrightarrow H^0(\Sigma_g,\underline{\mathbb{Z}}) \longrightarrow H^0(\Sigma_g, C^\infty(\mathbb{C})) \longrightarrow H^0(\Sigma_g, C^\infty(\mathbb{C}^*)) \\ \longrightarrow H^1(\Sigma_g,\underline{\mathbb{Z}}) \longrightarrow H^1(\Sigma_g, C^\infty(\mathbb{C})) \longrightarrow H^1(\Sigma_g, C^\infty(\mathbb{C}^*)) \\ \longrightarrow H^2(\Sigma_g,\underline{\mathbb{Z}}) \longrightarrow H^2(\Sigma_g, C^\infty(\mathbb{C})) \longrightarrow H^2(\Sigma_g, C^\infty(\mathbb{C}^*)) \longrightarrow 0. \end{array}$$

Now $C^\infty(\mathbb{C})$ is a fine sheaf, so $H^i(\Sigma, C^\infty(\mathbb{C})) = 0$ for all $i > 0$. Thus we obtain the short exact sequence

$$0 \longrightarrow H^1(\Sigma_g, C^\infty(\mathbb{C}^*)) \longrightarrow H^2(\Sigma_g,\underline{\mathbb{Z}}) \longrightarrow 0.$$

Now $H^2(\Sigma_g,\underline{\mathbb{Z}}) \cong \mathbb{Z}$, from which we conclude:

Theorem 5.2.1. *Isomorphism classes of smooth complex line bundles over a Riemann surface are in bijection with the integers.*

Label the map $H^1(\Sigma_g, C^\infty(\mathbb{C})) \to H^2(\Sigma_g,\underline{\mathbb{Z}})$ by deg. The image of a line bundle $L \in H^1(\Sigma_g, C^\infty(\mathbb{C}^*))$ under deg is called the *degree* of the line bundle. Thus the theorem above states that smooth complex line bundles on a surface are classified by their degree.

Furthermore, the above isomorphism is one of *groups.* The group structure on $H^1(\Sigma_g, C^\infty(\mathbb{C}^*))$ is by multiplication of transition functions on overlaps $U_{\alpha\beta}$. In the case of line bundles, this is the same as the tensor product of transition functions, as per the construction of the tensor product bundle, from which we conclude:

Corollary 5.2.2.

$$(\{\textit{Isomorphism classes of smooth complex line bundles on } \Sigma_g\}, \otimes) \cong (\mathbb{Z}, +)$$

as groups.

The degree as stated here has several interpretations. Firstly, the degree is equal to the integral of the first Chern class $c_1(L)$ of the line bundle, over Σ. Secondly, the degree is the number of zeros, counted with multiplicity, of a section of L which intersects the zero-section transversally. Thirdly, in the case of the sphere S^2, the degree is the winding number of the clutching function defined on (a tubular neighbourhood of) the equator.

5.2.2 Holomorphic Case

Consider the exponential sequence

$$0 \longrightarrow \underline{\mathbb{Z}} \hookrightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0.$$

This induces a long exact sequence

$$\begin{array}{l} 0 \longrightarrow H^0(\Sigma_g, \underline{\mathbb{Z}}) \longrightarrow H^0(\Sigma_g, \mathcal{O}) \longrightarrow H^0(\Sigma_g, \mathcal{O}^*) \\ \hookrightarrow H^1(\Sigma_g, \underline{\mathbb{Z}}) \longrightarrow H^1(\Sigma_g, \mathcal{O}) \longrightarrow H^1(\Sigma_g, \mathcal{O}^*) \\ \hookrightarrow H^2(\Sigma_g, \underline{\mathbb{Z}}) \longrightarrow H^2(\Sigma_g, \mathcal{O}) \longrightarrow H^2(\Sigma_g, \mathcal{O}^*) \longrightarrow 0. \end{array}$$

Since Σ_g has complex dimension one, the group $H^2(\Sigma_g, \mathcal{O}) = 0$. Since Σ_g is compact, global holomorphic sections in $\mathcal{O}$ or $\mathcal{O}^*$ are constant. Thus $H^0(\Sigma_g, \mathcal{O}) \cong \mathbb{C}$ and $H^0(\Sigma_g, \mathcal{O}^*) \cong \mathbb{C}^*$. But then taking a logarithm, the map $H^0(\Sigma_g, \mathcal{O}) \to H^0(\Sigma_g, \mathcal{O}^*)$ is surjective. Since the sequence is exact, we must have $H^0(\Sigma_g, \mathcal{O}^*) \to H^1(\Sigma_g, \underline{\mathbb{Z}})$ has full kernel. In particular, the image of this map is zero, so we get a reduction of our sequence to

$$0 \longrightarrow H^1(\Sigma_g, \underline{\mathbb{Z}}) \longrightarrow H^1(\Sigma_g, \mathcal{O}) \longrightarrow H^1(\Sigma_g, \mathcal{O}^*) \longrightarrow H^2(\Sigma_g, \underline{\mathbb{Z}}) \longrightarrow 0.$$

By the first isomorphism theorem, the image of $H^1(\Sigma_g, \mathcal{O}) \to H^1(\Sigma_g, \mathcal{O}^*)$ is isomorphic to the quotient $H^1(\Sigma, \mathcal{O})/H^1(\Sigma, \mathbb{Z})$, so we have a reduction to the short exact sequence

$$0 \longrightarrow H^1(\Sigma, \mathcal{O})/H^1(\Sigma, \mathbb{Z}) \longrightarrow H^1(\Sigma, \mathcal{O}^*) \longrightarrow H^2(\Sigma, \mathbb{Z}) \longrightarrow 0.$$

Now $H^1(\Sigma, \mathbb{Z}) \cong \mathbb{Z}^{2g}$, and $H^1(\Sigma, \mathcal{O}) \cong \mathbb{C}^g$. We also know that $H^2(\Sigma, \mathbb{Z}) \cong \mathbb{Z}$. Thus we obtain the exact sequence

$$0 \longrightarrow \mathbb{C}^g/\mathbb{Z}^{2g} \longrightarrow H^1(\Sigma, \mathcal{O}^*) \longrightarrow \mathbb{Z} \longrightarrow 0,$$

Remark 5.2.3. Note that the map $H^1(\Sigma_g, \underline{\mathbb{Z}}) \to H^1(\Sigma_g, \mathcal{O})$ is injective, so $\mathbb{Z}^{2g}$ sits inside $\mathbb{C}^g$ as a lattice. In particular the quotient $\mathbb{C}^g/\mathbb{Z}^{2g}$ is a torus.

Let $L \in H^1(\Sigma_g, \mathcal{O}^*)$ be a line bundle. Then the image under the map $H^1(\Sigma_g, \mathcal{O}^*) \to \mathbb{Z}$ is again called the degree $\deg L$ of L. To show that this is the same notion of degree as before, consider the homomorphism of short exact sequences of sheaves given by

$$\begin{tikzcd}
0 \arrow[r] & \underline{\mathbb{Z}} \arrow[r, hook] \arrow[d, equal] & \mathcal{O} \arrow[r, "\exp"] \arrow[d, hook] & \mathcal{O}^* \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \underline{\mathbb{Z}} \arrow[r, hook] & C^\infty(\mathbb{C}) \arrow[r, "\exp"] & C^\infty(\mathbb{C}^*) \arrow[r] & 0.
\end{tikzcd}$$

This induces a homomorphism of the corresponding long exact cohomology sequences. The relevant part of this homomorphism of sequences is the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & H^1(\Sigma_g, \mathcal{O}) \arrow[r] \arrow[d] & H^1(\Sigma_g, \mathcal{O}^*) \arrow[r] \arrow[d] & H^2(\Sigma_g, \underline{\mathbb{Z}}) \arrow[r] \arrow[d, equal] & 0 \\
\cdots \arrow[r] & 0 \arrow[r] & H^1(\Sigma_g, C^\infty(\mathbb{C}^*)) \arrow[r, "\cong"] & H^2(\Sigma_g, \underline{\mathbb{Z}}) \arrow[r] & 0.
\end{tikzcd}$$

This last square shows that two holomorphic line bundles have the same smooth degree if and only if they have the same holomorphic degree. Thus one simply denotes the degree of a line bundle by $\deg L$, regardless of whether L is being considered as smooth or holomorphic.

It follows from the above diagram that the group $\mathrm{Jac}(\Sigma)$ of all line bundles of degree zero is isomorphic to the torus

$$\mathrm{Jac}(\Sigma_g) \cong \mathbb{C}^g/\mathbb{Z}^{2g}.$$

It follows that the set $\mathrm{Pic}(\Sigma)$ of all holomorphic line bundles over Σ up to isomorphism is isomorphic to a countable disjoint union of g-tori. In particular if $\mathrm{Pic}_d(\Sigma)$ denotes the set of all holomorphic line bundles of a fixed degree d, then $\mathrm{Pic}_d(\Sigma) \cong \mathbb{C}^g/\mathbb{Z}^{2g}$ for all $d \in \mathbb{Z}$.

5.2.3 Flat Case

The group $H^1(\Sigma_g, \underline{\mathbb{C}^*})$ classifies *flat* complex line bundles on a Riemann surface Σ_g. These are the smooth complex line bundles that admit a trivialisation with constant transition functions. Equivalently, flat line bundles are those admitting a flat connection.

There exists an isomorphism from the sheaf cohomology groups $H^i(\Sigma_g, \underline{G})$ for some Abelian group G with the singular cohomology groups $H^i(\Sigma_g, G)$. By the universal coefficients theorem, $H^i(\Sigma_g, G) \cong H^i(\Sigma_g, \mathbb{Z}) \otimes_{\mathbb{Z}} G$. Thus we have $H^1(\Sigma_g, \underline{\mathbb{C}^*}) \cong H^1(\Sigma_g, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^{2g}$.

Now we also may consider the group $\mathrm{U}(1) \subset \mathbb{C}^*$. By the above argument we have $H^1(\Sigma_g, \mathrm{U}(1)) \cong \mathrm{U}(1)^{2g}$. In particular, this is a torus of dimension $2g$. The cohomology group $H^1(\Sigma_g, \mathrm{U}(1))$ classifies flat *unitary* complex line bundles; those with constant transition functions in $\mathrm{U}(1)$. Equivalently, these are the line bundles admitting a flat unitary connection with respect to some auxillary Hermitian metric.

One might ask if this torus $\mathrm{U}(1)^{2g}$ is the same as the torus $\mathrm{Jac}(\Sigma_g)$ constructed in the previous section. Indeed this is the case. A flat unitary line bundle $L \in H^1(\Sigma_g, \underline{\mathrm{U}(1)})$ admits a (flat) unitary connection with respect to some Hermitian metric h. The $(0,1)$-part of this connection gives a Dolbeault operator on the bundle L, which provides it with the structure of a holomorphic line bundle.

Conversely, given a Hermitian metric h on a holomorphic line bundle $L \to \Sigma_g$ with $\deg L = 0$, there exists a unique unitary connection d_A such that the $(0,1)$ part of this connection is the Dolbeault operator of the original holomorphic structure. Since $\deg L = 0$, this connection d_A is flat. Thus L has the structure of a flat unitary line bundle on Σ_g.

Given that this isomorphism exists for the case of flat unitary bundles, it is natural to ask if there is an isomorphism for flat bundles that are not necessarily unitary. In fact the moduli space of flat bundles $(\mathbb{C}^*)^{2g}$ can be identified not with a moduli space of holomorphic line bundles, but with a moduli space of holomorphic *Higgs* line bundles. In turn this moduli space can be identified with the cotangent bundle of $\mathrm{Jac}(\Sigma_g)$.

5.2.4 Classification of Smooth Vector Bundles

As an application of the notion of degree, in this section we will classify all smooth vector bundles on a Riemann surface.

Definition 5.2.4. *Let $E \to \Sigma_g$ be a smooth (or holomorphic) complex vector bundle of rank n on a Riemann surface Σ_g. The* degree *of E, denoted $\deg E$, is defined to be the degree of the smooth (or holomorphic) line bundle $\bigwedge^n E$.*

Lemma 5.2.5. *Let $E \to \Sigma_g$ be a smooth complex vector bundle of rank n on a Riemann surface Σ_g of genus g. Then if $s \in \Gamma(E)$ is a non-vanishing section, $E \cong E' \oplus (\mathbb{C} \times \Sigma_g)$ where E' has rank $n-1$.*

Proof. Since s is non-zero, one can define a sub-bundle $I := \mathrm{Span}\{s\}$. This sub-bundle admits a global non-vanishing section (namely s) so $I \cong \mathbb{C} \times \Sigma_g$. Take any metric on E, and let $E' := I^\perp$. Then E' has rank $n-1$ and $E \cong E' \oplus (\mathbb{C} \times \Sigma_g)$. □

Lemma 5.2.6. *Let $E \to \Sigma_g$ be a smooth complex vector bundle of rank n on a Riemann surface Σ_g. If $n > 1$ there exists a non-vanishing section.*

Proof. Let $s \in \Gamma(E)$ be any section. Then s defines a smooth 2-dimensional submanifold of the smooth $(2n+2)$-dimensional manifold E. The zero section defines another smooth 2-dimensional submanifold. Since $n > 1$, these two 2-dimensional submanfiolds meet inside a space of dimension at least 5. In particular the section s may be smoothly homotopied to a peturbed section s' that no longer intersects the zero section. $\square$

Theorem 5.2.7. *Smooth complex vector bundles on a Riemann surface Σ_g are classified up to isomorphism by their rank and degree.*

Proof. Let $E \to \Sigma_g$ be a smooth complex vector bundle of rank n. If $n = 1$ then E is classified by its degree.

Suppose $n > 1$. Then by Lemma 5.2.6 E has a non-vanishing section s. By Lemma 5.2.5 then $E \cong E' \oplus (\mathbb{C} \times \Sigma_g)$ where E' has rank $n-1$. Repeating this process a total of $n-1$ times we obtain $E \cong L \oplus (\mathbb{C}^{n-1} \times \Sigma_g)$ for some smooth complex line bundle L of some degree $d := \deg L$.

We claim the pair (n, d) classifies E up to isomorphism. First we will show d does not depend on the particular choice of deconstruction of E. In particular we have $d = \deg E$.

To see this, note that $\bigwedge^n E \cong \bigwedge^n (L \oplus (\mathbb{C}^{n-1} \times \Sigma_g))$. Furthermore, note that the isomorphism $\bigwedge(V \oplus W) \cong \bigwedge(V) \otimes \bigwedge(W)$ of exterior algebras of vectors spaces (or of exterior algebra bundles of vector bundles) is graded in the sense that

$$\bigwedge^n (L \oplus (\mathbb{C}^{n-1} \times \Sigma_g)) \cong \bigoplus_{p+q=n} \left(\bigwedge^p L\right) \otimes \left(\bigwedge^q (\mathbb{C}^{n-1} \times \Sigma_g)\right).$$

Since L is rank one, any higher exterior powers of L are the zero vector bundle, so we must have

$$\bigwedge^n E \cong L \otimes \bigwedge^{n-1} (\mathbb{C}^{n-1} \times \Sigma_g) \cong L \otimes (\mathbb{C} \times \Sigma_g) \cong L.$$

But then $\deg E = \deg L = d$.

Now suppose $F \to \Sigma_g$ is another vector bundle of rank n with $\deg F = \deg E$. Then we may similarly deconstruct F as $F \cong L' \oplus (\mathbb{C}^{n-1} \times \Sigma_g)$ for some line bundle L' of degree $\deg E$. Then since they have the same degrees, we have an isomorphism $L \cong L'$ of line bundles. Using the identity on the trivial part, we then obtain an isomorphism $E \cong F$. Clearly we also have that if $E \cong F$ then $\deg E = \deg F$ and the two bundles have the same rank, so we conclude $E \cong F$ if and only if they have the same rank and degree. $\square$

5.3 Stable Bundles

In the previous section the case of moduli spaces for holomorphic line bundles was investigated. When the rank n of a holomorphic vector bundle is greater than one, the situation becomes considerably more complex.

Let $\mathscr{D}$ be the space of Dolbeault operators on a smooth complex vector bundle E over a surface Σ. Then we know $\mathscr{D}$ is an affine space modelled on $\Omega^{0,1}(\Sigma, \mathrm{End}(E))$.

Let $\mathscr{D}^s$ be the subset of $\mathscr{D}$ corresponding to the stable holomorphic structures on E. Then we have

$$\mathscr{N}_{n,d} := \mathscr{D}\big/\mathscr{G}^{\mathbb{C}}.$$

Since $\mathscr{D}$ is an affine space, the tangent space to $\mathscr{D}$ at $\overline{\partial}_E$ is $\Omega^{0,1}(\Sigma, \mathrm{End}(E))$.

Let $\pi : \mathscr{D}^s \to \mathscr{N}_{n,d}$ be the natural projection. Then

$$T_{[\overline{\partial}_E]}\mathscr{N}_{n,d} \cong T_{\overline{\partial}_E}\mathscr{D}^s \Big/ \ker(d\pi : T_{\overline{\partial}_E}\mathscr{D}^s \to T_{[\overline{\partial}_E]}\mathscr{N}_{n,d}).$$

Proposition 5.3.1.

$$T_{[\overline{\partial}_E]}\mathscr{N}_{n,d} \cong H^{0,1}_{\overline{\partial}_E}(\Sigma, \mathrm{End}(E)),$$

where

$$H^{0,1}_{\overline{\partial}_E}(\Sigma, \mathrm{End}(E)) := \frac{\ker(\overline{\partial}_E : \Omega^{0,1}(\Sigma, \mathrm{End}(E)) \to \Omega^{0,2}(\Sigma, \mathrm{End}(E)))}{\mathrm{im}((\overline{\partial}_E : \Omega^{0}(\Sigma, \mathrm{End}(E)) \to \Omega^{0,1}(\Sigma, \mathrm{End}(E)))}.$$

Proof. Let $\overline{\partial}_E(t) := \overline{\partial}_E + t\alpha$ for some $\alpha \in \Omega^{0,1}(\Sigma, \mathrm{End}(E))$. Then α is in the kernel of $d\pi$ at $\overline{\partial}_E \in \mathscr{D}$ precisely when this line $\overline{\partial}_E(t)$ comes from a one-parameter family of gauge transformations $g_t \in \mathscr{G}^{\mathbb{C}}$ such that

$$\overline{\partial}_E(t) = g_t\overline{\partial}_E g_t^{-1}.$$

Then we have

$$\overline{\partial}_E + t\alpha = g_t\overline{\partial}_E g_t^{-1}$$

which implies (when $t \neq 0$) that

$$\alpha = \frac{g_t\overline{\partial}_E g_t^{-1} - g_0\overline{\partial}_E g_0^{-1}}{t},$$

where $g_0 = \mathbf{1} \in \mathscr{G}^{\mathbb{C}}$.

Then we must have

$$\alpha = \lim_{t\to 0}\frac{g_t\overline{\partial}_E g_t^{-1} - g_0\overline{\partial}_E g_0^{-1}}{t},$$

which may be written as

$$\begin{aligned}
\alpha &= \partial_t(g_t\overline{\partial}_E g_t^{-1})_{t=0} \\
&= \partial_t(g_t)_{t=0}\overline{\partial}_E g_0^{-1} + g_0\partial_t(\overline{\partial}_E)_{t=0}g_0^{-1} + g_0\overline{\partial}_E(\partial_t(g_t^{-1})_{t=0}) \\
&= 0 + 0 + \overline{\partial}_E(\partial_t(g_t^{-1})_{t=0}) \\
&= -\overline{\partial}_E(\partial_t(g_t)_{t=0}) \\
&= -\overline{\partial}_E a.
\end{aligned}$$

Here $a := \partial_t(g_t)_{t=0} \in T_e\mathscr{G}^{\mathbb{C}} \cong \Omega^0(\Sigma, \mathrm{End}(E))$. But then we have $\alpha \in \ker(d\pi)$ precisely when α is in the image of $\overline{\partial}_E$. Since $\overline{\partial}_E$ is the zero map on $\Omega^{0,1}(\Sigma, \mathrm{End}(E))$ because Σ has complex dimension one, we are done.

□

Corollary 5.3.2. *The dimension of* $\mathscr{N}_{n,d}$ *is* $1 + n^2(g-1)$.

Proof. Let $[\overline{\partial}_E]$ correspond to a stable holomorphic structure on E. Then by the Hirzebruch-Riemann-Roch theorem (Theorem 4.1.6), we have

$$\chi(\mathrm{End}(E)) = \int_\Sigma \mathrm{Ch}(\mathrm{End}(E))\,\mathrm{Td}(\Sigma)$$

with

$$\chi(\mathrm{End}(E)) = \dim H^0(\Sigma, \mathrm{End}(E)) - \dim H^1(\Sigma, \mathrm{End}(E)).$$

On the other hand, $\mathrm{Ch}(\mathrm{End}(E)) = \mathrm{Ch}(E^*)\,\mathrm{Ch}(E) = (n^2 - \mathrm{c}_1(E))(n^2 + \mathrm{c}_1(E))$ □

5.4 Higgs Bundles

Definition 5.4.1. *Let* Σ *be a Riemann surface. A* Higgs bundle *is a pair* (E, Φ) *where* E *is a rank* n *holomorphic vector bundle* $E \to \Sigma$ *and* $\Phi : E \to E \otimes K$ *is holomorphic* $\mathrm{End}(E)$*-valued 1-form.*

5.4.1 Higgs Line Bundles

When $L \to \Sigma$ is a line bundle, $\mathrm{End}(E)$ is trivial so a Higgs pair (E, Φ) decouples into a holomorphic line bundle L and a holomorphic one-form Φ.

5.5 One-Dimensional Representations of the Fundamental Group

5.5.1 Complex Representations

Chapter 6

Geometric Quantization

6.1 Prequantization and Kähler Polarizations

Given a symplectic manifold (M, ω), it is a general problem of physical motivations to quantize the corresponding classical system described by M. By quantize, one roughly means associate to (M, ω) a Hilbert space (of states) and to each smooth function (observable) $f \in C^\infty(M)$ a (possibly unbounded) operator on this Hilbert space, satisfying various axioms of physical origins.

One method of achieving this is via *geometric quantization*, where the Hilbert space is (the completion of) the space of holomorphic sections of a holomorphic line bundle over M, where M is equipped with a certain complex structure.

Before putting this complex structure on M, one needs a smooth complex line bundle.

Definition 6.1.1. *Let (M, ω) be a compact symplectic manifold with $[\omega] \in H^2(M, \mathbb{Z})$, so that ω is an integral form. A* prequantum line bundle *for (M, ω) is a Hermitian line bundle $L \to M$ with Hermitian connection ∇ such that $c_1(L) = [\omega]$. That is, $[F_\nabla] = [-2\pi i \omega]$.*

Given such a line bundle, to continue the geometric quantization process one needs a complex structure.

Definition 6.1.2. *Let (M, ω) be a compact symplectic manifold. A* Kähler Polarization *of M is an integrable almost complex structure $I : TM \to TM$ such that with respect to I, ω is a Kähler form for M.*

When a Kähler polarization of M exists, we obtain a holomorphic structure on the Hermitian line bundle L. This can be specified by defining the $(0, 1)$ part of the connection ∇ on L, by

$$\nabla^{0,1} := \frac{i}{2}(\mathbf{1} + iI)\nabla.$$

Correspondingly $\nabla^{1,0} := \frac{i}{2}(\mathbf{1} - iI)\nabla$ and $\nabla = \nabla^{1,0} + \nabla^{0,1}$.

To check this does in fact define a complex structure, one needs to check the existence of local solutions to the equation $\nabla^{0,1}s = 0$. Locally $\nabla^{0,1}$ is of the form

$$\nabla^{0,1}(s) = \left(\frac{\partial s}{\partial \bar{z}^i} + A_i s\right) \otimes d\bar{z}^i$$

where A is a $(0,1)$-form with $A = A_i d\bar{z}^i$. The Dolbeault Lemma asserts a solution s exists when $\overline{\partial} A = 0$. But $dA = -2\pi i\omega + d\theta$ is a $(1,1)$-form on the Kähler manifold (M, ω, I), so $\overline{\partial} A = \omega^{0,2} = 0$ and $\nabla^{0,1}$ does define a holomorphic structure on L. Write $\mathcal{L}_I$ for the holomorphic line bundle structure induced by I.

Finally, to this Kähler polarization we associate the vector space $H^0(M, \mathcal{L}_I)$ which acts as the geometric quantization of (M, ω).

6.2 Hitchin's Connection

In the above discussion of geometric quantization, a choice of complex structure I was made. It is important to determine the dependence of the final Hilbert space $H^0(M, \mathcal{L}_I)$ on this choice of complex structure.

Firstly, how does the dimension depend on I? By the Hirzebruch-Riemann-Roch formula, we have

$$\sum_i (-1)^i \dim H^i(M, \mathcal{L}_I) = \int_M \mathrm{Ch}(\mathcal{L}_I)\,\mathrm{Td}(M),$$

where the right side does not depend on the holomorphic structure of $\mathcal{L}_I$.

If one takes a sufficiently high power $\mathcal{L}_I^k$ of the line bundle $\mathcal{L}_I$, then by the Kodaira vanishing theorem $H^j(M, \mathcal{L}_I^k)$ vanishes for $j > 0$. Thus replacing $\mathcal{L}_I$ with $\mathcal{L}_I^k$ and ω with $k\omega$ we have that $\dim H^0(M, \mathcal{L}_I^k)$ does not depend on the choice of I.

At this point, we will consider a parameter space $\mathcal{T}$ of Kähler polarizations for (M, ω, L) a prequantized compact symplectic manifold. Taking $k \in \mathbb{Z}$ large enough that the dimension of $H^0(M, \mathcal{L}_{I_\sigma}^k)$ does not depend on the complex structure I_σ corresponding to $\sigma \in \mathcal{T}$, we obtain a vector bundle $V \to \mathcal{T}$ with fibres the corresponding geometric quantizations of (M, ω) for each polarization. That this object is a vector bundle follows from Kodaira's work on deformations of complex structures, where the vector spaces $H^0(M, \mathcal{L}_{I_\sigma}^k)$ are the kernels of a family of elliptic operators varying smoothly on a family of line bundles varying smoothly over M. Elliptic regularity then implies the kernels form a smooth sub-bundle of the infinite-dimensional bundle $\mathcal{T} \times C^\infty(M, L^k) \to \mathcal{T}$.

To satisfactorily complete the geometric quantization of (M, ω), we would like an identification of the fibres V_σ of this vector bundle $V \to \mathcal{T}$ that does not depend on any more choices. Thus we would like to find a projectively flat connection on V, with parallel transport giving the identification of fibres. Up to a constant, this identification will not depend on the path chosen.

Such a connection is known as a Hitchin connection, and this problem was first studied by Nigel Hitchin in [Hit90] when M is the moduli space $\mathcal{N}$ of stable bundles on a Riemann surface. Here the symplectic structure is from the Atiyah-Bott symplectic form on $\mathcal{N}$ constructed in [AB83]. The prequantum line bundle $L \to \mathcal{N}$ is the so called determinant line bundle on $\mathcal{N}$ defined by Daniel Quillen in [Qui85].

6.3 Geometric Quantization for the Moduli Space of Higgs Bundles

In the following, we will be concerned with generalising Hitchin's work in [Hit90] to the moduli space $\mathcal{M}$ of stable Higgs bundles over a Riemann surface. The geometric quantization of this space presents challenges not present in the case of $\mathcal{N}$, since $\mathcal{M}$ is non-compact. Fortunately, $\mathcal{M}$ does come equipped with a natural $\mathbb{C}^*$ action given by multiplication of the Higgs field, and this action can help to complete the quantization. Additionally, the presence of the $\mathbb{C}^*$ action begs the question of whether a quantization of $\mathcal{M}$ exists that respects this $\mathbb{C}^*$ action.

The cotangent bundle $T^*\mathcal{N}$ to the moduli space of stable bundles sits inside $\mathcal{M}$ as an open dense subset, and Hitchin observed that (at least for $g > 1$) the complement has codimension at least 2. Therefore the line bundle $L \to \mathcal{N}$ can be pulled back to $T^*\mathcal{N}$ and extended over this codimension 2 subspace to obtain a line bundle $L_H \to \mathcal{M}$. This line bundle L_H will act as the prequantum line bundle for the geometric quantization of $\mathcal{M}$.

References

[AB83] Michael Francis Atiyah and Raoul Bott. The yang-mills equations over riemann surfaces. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences*, pages 523–615, 1983. 85

[Hit90] Nigel J Hitchin. Flat connections and geometric quantization. *Communications in mathematical physics*, 131(2):347–380, 1990. 84, 85

[Qui85] Daniel Quillen. Determinants of cauchy-riemann operators over a riemann surface. *Functional Analysis and Its Applications*, 19(1):31–34, 1985. 85

[Tau11] Clifford Henry Taubes. *Differential geometry: Bundles, connections, metrics and curvature*, volume 23. OUP Oxford, 2011. 32