Kodaira Embedding Theorem

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Line bundles and divisors

Let X be a complex manifold. A **divisor** is a finite formal linear combination

$$D = \sum a_i V_i, \quad a_i \in \mathbb{Z}$$

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There is a natural bijection between the following sets

$$\mathsf{Div}(X) \longleftrightarrow \{(L, s) \mid L \in \mathsf{Pic}(X), s \in \Gamma(X, L \otimes \mathcal{M}_X^*)\}.$$

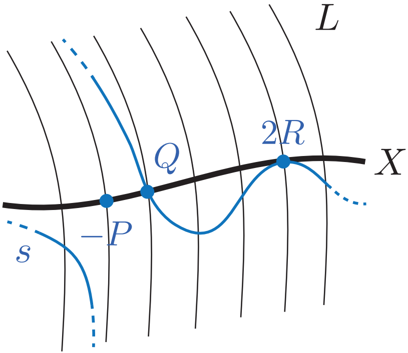
Locally, a couple (L, s) is given by the data

$$L \longleftrightarrow \{(U_\alpha), g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*\},$$
$$s \longleftrightarrow \{s_\alpha \in \mathcal{M}_X^*(U_\alpha), s_\alpha = g_{\alpha\beta} s_\beta\}.$$

Line bundles and divisors

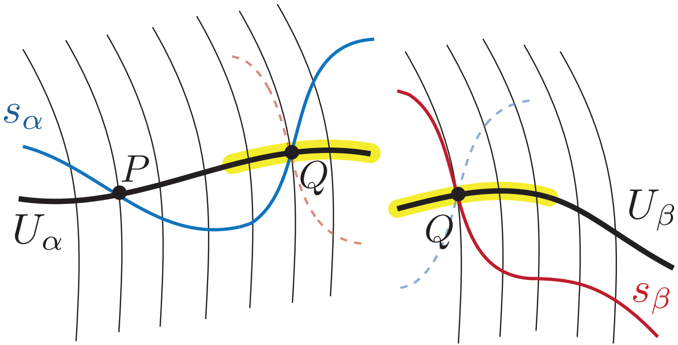
On one side, given a line bundle $L \to X$ and a meromorphic section s, the associated divisor is simply given by its zeros and poles:

$$\left(L = (g_{\alpha\beta}), s = \left(\frac{f_\alpha}{g_\alpha} \right) \right) \mapsto D = (Z(f_\alpha) - Z(g_\alpha)) \, .$$



Line bundles and divisors

Viceversa, assume $D = V \subset X$ is an irreducible hypersurface. It is locally defined by holomorphic sections s_α of the trivial bundle over U_α (i.e. holomorphic functions).
Since $\frac{s_\alpha}{s_\beta} \in \mathcal{O}^*_X(U_\alpha \cap U_\beta)$, the pairs $(U_\alpha \times \mathbb{C}, s_\alpha)$ glue to a global line bundle L, together with a holomorphic section s such that $V = Z(s)$.



Line bundles and maps to projective space

Let X be a compact complex manifold and let $L \to X$ be a holomorphic line bundle on it. We can consider the map

$$\begin{aligned} i_L : X &\dashrightarrow \mathbb{P}(H^0(X, L)^\vee) \\ x &\mapsto [\mathsf{ev}_x : H^0(X, L) \to \mathbb{C}]. \end{aligned}$$

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It is not defined at $x \in X$ if all sections of L vanish at x, so if we want a holomorphic map we need to assume

$$\mathsf{Bs}(L) = \{x \in X \mid s(x) = 0 \text{ for all } s \in H^0(X,L)\} = \emptyset.$$

We say L has no **base points** in this case.
A line bundle L is called **very ample** if i_L is an embedding, and it is called **ample** if a suitable high power L^k, $k \gg 0$, is very ample.

Line bundles and maps to projective space

Locally, if we trivialise L on an open cover $\{U_\alpha\}$ of X using transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$ and we fix a basis

$$H^0(X, L) = \langle s_0, \ldots, s_N \rangle,$$

then the local expression of i_L over the open subset U_α is given by

$$i_L(x) = [s_{0,\alpha}(x) : \ldots : s_{N,\alpha}(x)] \in \mathbb{P}^N,$$

where $s_{i,\alpha} : U_\alpha \to \mathbb{C}$ are the local expressions of the sections s_i. Note that this is well defined for $x \in U_\alpha \cap U_\beta$ as $s_{i,\alpha} = g_{\alpha\beta} s_{i,\beta}$.

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$$i_L^* \mathcal{O}_{\mathbb{P}^N}(1) = L,$$
$$i_L^* H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) = H^0(X, L).$$

Line bundles and maps to projective space

Example

Let $X = \mathbb{P}^n$ and $L = \mathcal{O}_{\mathbb{P}^n}(1)$ the hyperplane bundle. Since $H^0(X, L) = \langle X_0, \dots X_n \rangle$, $i_L : \mathbb{P}^n \to \mathbb{P}^n$ is a linear projective automorphism.
More invariantly, write $\mathbb{P}^n = \mathbb{P}(V)$: then $H^0(X, L) = V^\vee$ and $i_L(x)$ is given by the set of hyperplanes in $\mathbb{P}(V)$ passing through x, so that $i_L : \mathbb{P}(V) \to \mathbb{P}(V^{\vee\vee})$ is just the natural identification.

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Example

Let $X = \mathbb{P}^1$ and $L = \mathcal{O}_{\mathbb{P}^1}(d)$, for $d > 0$. We have $H^0(X, L) = \langle X_0^d, X_0^{d-1} X_1, \ldots X_1^d \rangle$, so that $i_L : \mathbb{P}^1 \to \mathbb{P}^N$ is an embedding of $\mathbb{P}^1$ in $\mathbb{P}^N$ as a curve of degree d.

Positive line bundles

Assume (L, h) is a hermitian line bundle on X. We denote by $\Theta(L, h) \in \wedge^{1,1} T^* X$ its curvature form, which satisfies $c_1(L) = \left[\frac{i}{2\pi}\Theta(L, h)\right]$.
We say L is **positive** if it admits a metric h such that $i\Theta(L, h)$ is a positive $(1, 1)$-form, i.e.

$$i\Theta(L, h)(v, Jv) > 0$$

for each non-zero tangent vector v.

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Fixing local holomorphic coordinates $z_1, \ldots, z_n$ and writing

$$\Theta(L, h) = \frac{i}{2} \sum h_{ij} dz_i \wedge d\bar{z}_j,$$

positivity is equivalent to the hermitian matrix of $\mathbb{C}$-valued smooth functions (h_{ij}) being positive definite at each point $x \in X$.

Positivity of line bundles

Note that if ω is a positive $(1,1)$-form on X, then $\int_C \omega > 0$ for any complex curve $C \subset X$, as $\omega\mid_C$ is locally of the form

$$\omega\mid_C = f(x,y)dx \wedge dy, \quad f(x,y) > 0.$$

One can prove a similar positivity property for any complex submanifold $V \subset X$ by replacing ω with $\omega^{\dim(V)}$ (compare with the algebraic notion of positivity).

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Example

On a smooth curve C a line bundle L corresponding to $D \in \mathrm{Div}(C)$ is positive if and only if $\deg(D) > 0$. Note that $\int_C c_1(L) = \deg(D)$.

Kodaira Embedding Theorem

Theorem

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Clearly any ample line bundle L is positive, since for sufficiently large $k \gg 0$ we have

$$L^k \cong \mathcal{O}_{\mathbb{P}^N}(1)\,|_X,$$

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The other implication is more involved, and it relies on the following ingredients:

- Cohomological characterisation of ampleness;
- Kodaira Vanishing Theorem;
- Blowing up to replace points with divisors (if $\dim X > 1$).

Cohomological characterisation of ampleness

Given a line bundle $L \to X$, in order for $i_L : X \dashrightarrow \mathbb{P}(H^0(X,L)^\vee)$ to be an embedding, we need the following properties:

1. i_L is a well defined **morphism**, i.e. $Bs(L) = \emptyset$. This amounts to the surjectivity of the restriction map

$$H^0(X,L) \to L_x, \quad \text{for all } x \in X.$$

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   Note that such a map comes from taking global sections of the exact sequence of sheaves

$$0 \to L \otimes \mathcal{I}_x \to L \to L_x \to 0,$$

   where $L \otimes \mathcal{I}_x$ is the sheaf of holomorphic sections of L vanishing at x. Hence we want

$$H^1(X, L \otimes \mathcal{I}_x) = 0.$$

Cohomological characterisation of ampleness

2. i_L is **injective**, i.e. for all $x, y \in X$, $x \neq y$, there is a section of L vanishing at x but not at y, or equivalently, the map

$$H^0(X, L) \to L_x \oplus L_y, \quad \text{for all } x \neq y,$$

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is surjective. Again we can fit this map into an exact sequence of sheaves

$$0 \to L \otimes \mathcal{I}_{x,y} \to L \to L_x \oplus L_y \to 0,$$

so that injectivity becomes equivalent to the following vanishing

$$H^1(X, L \otimes \mathcal{I}_{x,y}) = 0.$$

Cohomological characterisation of ampleness

3. i_L is **immersive**, i.e. the differential $di_L : T_xX \to T_{i_L(x)}\mathbb{P}^N$ is injective at every point $x \in X$. Dually, we need the map

$$\begin{aligned} H^0(X, L \otimes \mathcal{I}_x) &\to T_x^*X \otimes L_x \\ s_\alpha &\mapsto ds_\alpha(x) \end{aligned}$$

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to be surjective. Looking at the exact sequence

$$0 \to L \otimes \mathcal{I}_x^2 \to L \otimes \mathcal{I}_x \to L \otimes T_x^* X \to 0,$$

we want

$$H^1(X, L \otimes \mathcal{I}_x^2) = 0.$$

Vanishing Theorems

Theorem (Kodaira Vanishing)

If L is a positive line bundle on a compact complex manifold X with canonical bundle K_X, then

$$H^q(X, L \otimes K_X) = 0 \quad \textit{for } q > 0.$$

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Theorem (Serre Vanishing)

Let L be a positive line bundle on a compact complex manifold X. Then

$$H^q(X, L^k) = 0 \quad \textit{for } q > 0 \textit{ and } k \gg 0.$$

Note that by Serre Vanishing and Riemann-Roch

$$\chi(L^k) = \dim H^0(X, L^k) = \frac{k^n}{n!} \int_X c_1(L)^n + \cdots \gg 0 \text{ if } k \gg 0,$$

so we expect to eventually get "enough sections" to define an embedding.

Kodaira Embedding for curves

Lemma

Let C be a compact Riemann surface of genus g. Then if $\deg(D) \geq 2g+1$, the line bundle $\mathcal{O}_C(D)$ is very ample. In particular, any positive line bundle is ample.

Proof.

We simply use the previous characterisation of ampleness in terms of cohomology groups. Given $x \neq y$ in C, since they are **effective divisors** on C, $\mathcal{O}_C(D) \otimes \mathcal{I}_{x,y} = \mathcal{O}_C(D - x - y)$. Hence

$$H^1(C, D - x - y) = H^1(C, K_C + (D - x - y - K_C)) = 0$$

by Kodaira Vanishing, as

$$\deg(D - x - y - K_C) \geq 2g + 1 - 2 - (2g - 2) = 1 > 0.$$

A similar computation for $D - 2x$ completes the proof. □

Blowing-up

Recall the **blow-up** of $\mathbb{C}^n$ at the origin is given by the incidence variety

$$\mathsf{Bl}_0(\mathbb{C}^n) = \{(z, \ell) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \mid z \in \ell\} \subset \mathbb{C}^n \times \mathbb{P}^{n-1},$$

together with the projection $\pi = p_1$ onto $\mathbb{C}^n$. The fibre $E = \pi^{-1}(0)$ is called the **exceptional divisor**. Note that π is an isomorphism outside the exceptional locus.

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If we consider the second projection $p_2 : \mathsf{Bl}_0(\mathbb{C}^n) \to \mathbb{P}^{n-1}$, we realise that it can be regarded as the **total space of the tautological bundle** $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$. Then E is simply the zero-section and

$$\mathcal{O}_{\mathsf{Bl}_0(\mathbb{C}^n)}(E) = p_2^* \mathcal{O}_{\mathbb{P}^{n-1}}(-1).$$

When we restrict to $E \cong \mathbb{P}^{n-1}$, we have $\mathcal{O}_E(E) \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$.

Blowing-up

Now if X is an arbitrary complex manifold of dimension n and $x \in X$, we can define $\tilde{X} = \mathsf{Bl}_x(X)$ by simply replacing a nhood of x with $\mathsf{Bl}_0(\mathbb{C}^n)$.

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We can endow $\mathcal{O}_{\tilde{X}}(E)$ with a metric by glueing the local metric induced by $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ around E and a flat metric away from E (where $\mathcal{O}_{\tilde{X}}(E)$ is trivial!).

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Lemma

If L is a positive line bundle on X and M is any line bundle, then for all $m > 0$ the line bundle

$$\pi^*(L^k \otimes M) \otimes \mathcal{O}_{\tilde{X}}(-mE)$$

is positive on $\tilde{X}$ for $k \gg 0$.

Idea of the proof if $\dim X > 1$

We will prove that if L is positive, then L^k has no base points if $k \gg 0$.
Given $x \in X$, call $\pi : \tilde{X} \to X$ the blow-up of X at x and let E be the exceptional divisor.
Consider the following diagram:

$$\begin{array}{ccc} H^0(X, L^k) & \longrightarrow & L_x^k \\ \wr\big\downarrow \pi^* & & \big\| \\ H^0(\tilde{X}, \pi^* L^k) & \longrightarrow & H^0(E, \pi^* L^k) \end{array}$$

It is enough to prove that

$$H^0(\tilde{X}, \pi^* L^k) \to H^0(E, \pi^* L^k)$$

is surjective.

Idea of the proof if $\dim X > 1$

Once again, the map $H^0(\tilde{X}, \pi^* L^k) \to H^0(E, \pi^* L^k)$ comes from the exact sequence of sheaves on $\tilde{X}$

$$0 \to \pi^* L^k \otimes \mathcal{I}_E \to \pi^* L^k \to \pi^* L^k \mid_E \to 0,$$

so we are left to prove the vanishing

$$H^1(\tilde{X}, \pi^* L^k \otimes \mathcal{I}_E) = H^1(\tilde{X}, \pi^* L^k \otimes \mathcal{O}_{\tilde{X}}(-E)) = 0.$$

This follows at once from Kodaira Vanishing, as the line bundle

$$K_{\tilde{X}}^{-1} \otimes \pi^* L^k \otimes \mathcal{O}_{\tilde{X}}(-E) = \pi^*(L^k \otimes K_X^{-1}) \otimes \mathcal{O}_{\tilde{X}}(-nE)$$

is positive by the previous Lemma.

Kähler cone and projectivity

If X is a compact Kähler manifold, its projectivity can be deduced from the position of the **Kähler cone** $\mathcal{K}_X \subset H^{1,1}(X) \cap H^2(X, \mathbb{R})$ with respect to the integral lattice $H^2(X, \mathbb{Z})$:

Corollary

A compact Kähler manifold X is projective if and only if $\mathcal{K}_X \cap H^2(X, \mathbb{Z}) \neq \{0\}$.

One can also replace $H^2(X, \mathbb{Z})$ with $H^2(X, \mathbb{Q})$ (a suitable multiple will be integral).

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Example

Any compact Kähler manifold X with $H^{0,2}(X) = 0$ is projective, as $\mathcal{K}_X \subset H^2(X, \mathbb{R})$ is an open cone and hence will intersect the integral lattice.

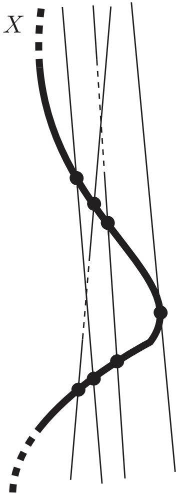
Projectivity of Riemann Surfaces

Corollary

Every compact complex curve is projective and admits an embedding in $\mathbb{P}^3$.

Clearly one has $H^{0,2}(X) = 0$. In fact any positive degree line bundle is ample and hence defines an embedding in $\mathbb{P}^N$ for some $N \gg 0$.

Projecting from any point not on $Sec(X)$ gives an isomorphism from X to a curve in $\mathbb{P}^{N-1}$, as long as $N > 3$ as $\dim Sec(X) \leq 3$ (all the chords are parametrised by a 2-dimensional subvariety of Grass$(1, N)$).



$$Sec(X) = \overline{\bigcup_{p,q \in X} \ell(p,q)} \subseteq \mathbb{P}^N$$