Introduction to Gauge Theory

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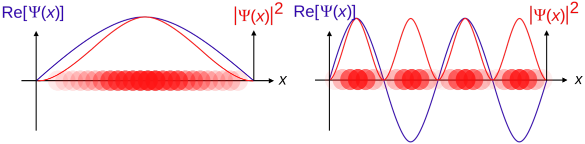
What is a gauge: the phase of an electron wavefunction

- Quantum mechanics models an electron as a *complex* wave function $\psi : \mathbb{R}^{3,1} \to \mathbb{C}$ on spacetime.
- The probability density $|\psi|^2 : \mathbb{R}^{3,1} \to \mathbb{R}$ describes the probable location of the particle, and only this density is observable.



- If $\psi(x) = |\psi(x)|e^{i\varphi(x)}$ then the phase $\varphi(x)$ is not observable and must therefore be a symmetry of the physical theory.
- Under a *gauge transformation* $\varphi(x) \mapsto \varphi(x) + \alpha$, the equations of quantum mechanics should remain invariant. This is a *(global) gauge symmetry*.
- The theory should also remain invariant under a *local* gauge symmetry $\varphi(x) \mapsto \varphi(x) + \alpha(x)$.

What is a gauge theory?

In physics, *a gauge theory* is a *field theory* which admits *global and/or local gauge symmetry*.

- Field theory = physical model which represents some quantity as a field defined at each point of spacetime M.
- (Matter) Field = section of vector bundle $E \to M$ (e.g. electron field, quark field).
- (Gauge) Field = connection on principal G-bundle $P \to M$ over spacetime (e.g. EM field, gluon field).

A physical theory is described by a *Lagrangian*:

- Lagrangian = Function on space of all possible field configurations. Assigns to each possible field configuration a number, the kinetic energy - potential energy.
- Gauge symmetry: an automorphism $g : E \to E$ or $g : P \to P$ which leaves the Lagrangian invariant.

An example of a gauge theory: Electromagnetism

- Spacetime $= M = \mathbb{R}^{3,1}$.
- Electron field $=$ section ψ of a trivial bundle $E = M \times \mathbb{C} \to M$, called a *wave function*.
- Electromagnetic field $=$ connection A on a principal U(1)-bundle $P = M \times \mathrm{U}(1) \to M$.
- "Minimal coupling" $= E$ is an associated bundle to P, and obtain D_A a covariant derivative on E.
- Lagrangian: $\mathcal{L} = \bar{\psi}(\sigma \circ D_A - m)\psi + |F_A|^2, \quad D_A = \frac{d}{dx} + A$.
- Equations of motion: Maxwell's equations of electromagnetism.
- **Key point:** Since the physical system must be invariant under local symmetry (i.e. a change of phase factor $\psi \mapsto e^{i\alpha(x)}\psi$ with $d\alpha \neq 0$), the derivative operator D_A in the Lagrangian must transform precisely to counteract such a change.
- Gauge symmetry: $e^{i\alpha} : M \to \mathrm{U}(1)$, $\psi \mapsto e^{i\alpha(x)}\psi, A \mapsto A - id\alpha$. Leaves Lagrangian invariant.

Yang and Mills: Why do we need principal bundles and vector bundles?

- Dirac observed that fermions (neutrons, protons, etc.) should be viewed as sections of higher rank vector bundles $E \to M$ over spacetime: spinor bundles.
- There exists a symmetry (called isospin symmetry) $\psi \mapsto \rho(S^{-1})\psi$ for $S : M \to \mathrm{SU}(2)$ and $\rho : \mathrm{SU}(2) \to \mathbb{C}^2$ a representation of SU(2) on the the fibre of E ("spin representation").
- One can emulate the same Lagrangian as in the case of electromagnetism, except now A must transform as

$$A \mapsto S^{-1}AS + S^{-1}dS$$

  to preserve the gauge symmetry. Thus A is just a connection on a principal SU(2)-bundle over spacetime!
- This symmetry transformation was written down by Yang and Mills, and the corresponding equations of motion are the *Yang–Mills equations*.

Mathematical gauge theory

In mathematics, *gauge theory* is the study of connections on vector bundles and principal bundles. This includes, for example:

- Studying natural differential equations for connections; e.g. the Yang–Mills equations.
- Classifying connections up to gauge equivalence ("physical equivalence").
- Constructing moduli spaces of connections, and investigating their properties.
- Relating connections to other areas of geometry, such as algebraic geometry.

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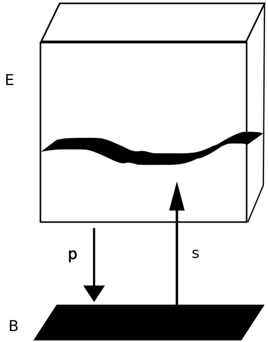
Refresher: Connections on vector bundles

- A section $s : M \to E$ of a vector bundle $E \to M$ is like a generalised function.
- One can attempt to differentiate s in the direction of a vector field X:

$$ds(X)(p) = \lim_{t \to 0} \frac{s(p + tX) - s(p)}{t}.$$



Problems:

1. M is not linear, so need to take a path $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$, $\gamma'(t) = X(\gamma(t))$.
2. $s(\gamma(t))$ is in the fibre over $\gamma(t)$, which is a different vector space to the fibre over $\gamma(0) = p$, so we can't subtract vectors.

Refresher: Connections on vector bundles

Solutions:

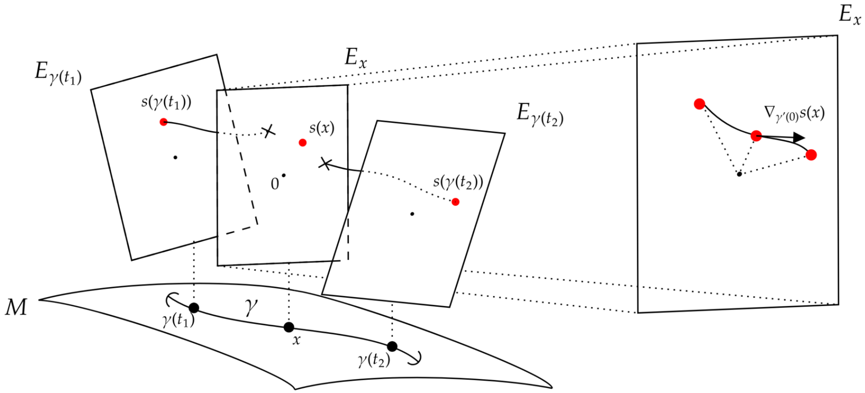
- (*Parallel transport*) Pick an isomorphism $P_t^\gamma : E_{\gamma(t)} \to E_p$ for every $t \in (-\varepsilon, \varepsilon)$ and use this to move $s(\gamma(t))$ to the same fibre as $s(p)$.
- (*Ehresmann connection*) Use the total differential $ds : TM \to TE$ by treating s as a smooth map from M to E. To make $ds(X)$ land back as a section of E, one needs to project onto the *vertical tangent bundle* $V = \ker(d\pi : TE \to TM) = \pi^* E \to E$. There are many choices of projections onto a linear subspace.
- (*Covariant Derivative/Connection*) Abstract the notion of a derivative as a differential operator

  $$\nabla_X : \Gamma(M, E) \to \Gamma(M, E)$$

  satisfying a product rule $\nabla_X(fs) = df(X)s + f\nabla_X s$.

Each of these three approaches requires making a *choice* of connection. They can all be shown to be equivalent.

Example equivalence: Parallel transport to covariant derivative



Curvature

Q: Can we choose a basis of local sections of a vector bundle which are covariantly constant?

$$e_i : U \to E|_U, \quad \nabla_X(e_i) = 0 \quad \forall X \in \mathfrak{X}(M).$$

A: No. Such a frame is called *flat*, and the obstruction is curvature.If a connection is defined over a set U by

$$\nabla(s) = ds + As$$

for $A \in \Omega^1(M) \otimes \operatorname{End} E = \Omega^1(\operatorname{End} E)$, then the curvature is

$$F_A = dA + A \wedge A.$$

Invariantly, one can write
$F_A = \nabla \circ \nabla : \Gamma(M, E) \to \Omega^2(M) \otimes \Gamma(M, E)$.Given a frame e_i', one can solve the differential equation

$$\nabla(f^i e_i') = d(f^i)e_i' + f^i A_i^j e_j'$$

if and only if F_A vanishes.

The space of all connections

- Given two connections ∇, ∇', their difference is $C^\infty(M)$-linear:

  $$\nabla(fs) - \nabla'(fs) = df(X)s + f\nabla(s) - df(X)s - f\nabla'(s) = f(\nabla - \nabla')(s).$$

- There is a unique endomorphism-valued one-form $A \in \Omega^1(M) \otimes \Gamma(M, \operatorname{End} E)$ such that

  $$\nabla' - \nabla = A.$$

- Since the set of all connections is non-empty, it is therefore an infinite-dimensional affine space $\mathcal{A}$ modelled on $\Omega^1(\operatorname{End} E)$.
- The curvature F_A is itself $C^\infty(M)$-linear, so defines a map

  $$\mathcal{A} \to \Omega^2(\operatorname{End}(E)), \quad A \mapsto F_A.$$

Gauge transformations

- A vector bundle automorphism $g : E \to E$ defines an action by conjugation on connections:

$$(g \cdot \nabla)_X(s) := g(\nabla_X(g^{-1}s)).$$

- In terms of the affine space $\mathcal{A}$,

$$g \cdot \nabla - \nabla = -d_\nabla(g)g^{-1}.$$

- The group of all automorphisms, $\mathcal{G}$, acts on $\mathcal{A}$, and the quotient is the *moduli space of connections on* E:

$$\mathcal{A}/\mathcal{G}.$$

- In general this space is not smooth, or even Hausdorff, but one can look for a special class of connections with better analytical properties: e.g. Yang–Mills connections!

Yang–Mills connections

Suppose that M is oriented and fix a Riemannian metric g. The Yang–Mills equations for a connection A are

$$d_A \star F_A = 0$$

where $\star$ is the Hodge star operator on M.These are the "vacuum" version of Yang and Mills's equations of motion, and are the critical points (local and global minima) of the functional

$$\mathrm{YM} : \mathcal{A} \to \mathbb{R}, \quad \mathrm{YM}(A) = \int_M |F_A|^2 \, d\mathrm{vol}_g = \int_M \mathrm{tr}(\star F_A \wedge F_A) \, d\mathrm{vol}_g.$$

- Whilst there is no natural choice of connection on a manifold, Yang–Mills connections have curvature "as small as possible."
- The equations $d\omega = d \star \omega = 0$ for a harmonic differential form should be seen as an analogy to $d_A F_A = d_A \star F_A = 0$. In some sense, studying Yang–Mills connections in a gauge orbit is like studying harmonic representatives of a de Rham cohomology class.

Pet problem: Riemann Surfaces

- Atiyah and Bott studied the Yang–Mills equations on a vector bundle over a compact Riemann surface, $E \to M = \Sigma_g$.
- There exists a natural symplectic form Ω on $\mathcal{A}$, defined by

$$\Omega_A(a, b) = \int_{\Sigma_g} \mathrm{tr}(a \wedge b)$$

  where $a, b \in \Omega^1(\mathrm{End}\, E)$.
- The Yang–Mills equation on Σ_g says that $F_A = \lambda \mathbf{1}_E \otimes d\mathrm{vol}$ for a constant $\lambda \in \mathbb{C}$, and Atiyah–Bott proved that

$$\mu : A \mapsto F_A - \lambda \mathbf{1}_E d\mathrm{vol}$$

  is a moment map for the gauge group $\mathcal{G}$ acting on $\mathcal{A}$.
- They also showed that YM is a perfect equivariant Morse function for the action of $\mathcal{G}$, and used equivariant cohomology to compute the cohomology of $\mathcal{M} = \mu^{-1}(0)/\mathcal{G}$.

Pet problem: Riemann Surfaces

- An old theorem of Narasimhan and Seshadri states that a holomorphic vector bundle $\mathcal{E} \to \Sigma_g$ of rank r is *stable* if and only if it arises from a unitary representation $\pi_1(\Sigma_g) \to \mathrm{U}(r)$.
- The holonomy of a Yang–Mills connection on $\mathcal{E}$ gives such a unitary representation.
- Donaldson reproved the theorem: A holomorphic vector bundle over a compact Riemann surface is *stable* if and only if it admits a Hermitian metric whose Chern connection is a Yang–Mills connection.
- A holomorphic vector bundle $\mathcal{E}$ is *stable* if, for all (proper, non-zero) subbundles $\mathcal{F} \subset \mathcal{E}$, we have

$$\frac{\deg \mathcal{F}}{\operatorname{rk} \mathcal{F}} < \frac{\deg \mathcal{E}}{\operatorname{rk} \mathcal{E}}.$$

- The moduli space of stable holomorphic vector bundles $\mathcal{N}^s$ is a smooth complex projective variety. Thus there is a smooth isomorphism $\mathcal{M} \cong \mathcal{N}^s$ and $\mathcal{M}$ obtains a Kähler structure.

Mathematical gauge theory: a history

- 1970s: Atiyah, Singer, Hitchin study the self-duality equations on a four-manifold. Mathematically describe the moduli space of the BPST instanton, the unique instanton on $\mathbb{R}^4$ up to translation and scaling.
- 1970s: Atiyah and Ward relate SD instantons to certain algebraic vector bundles on $\mathbb{CP}^3$.
- 1980s: Atiyah and Bott, Donaldson, study the Yang–Mills equations on compact Riemann surfaces.
- 1980s: Donaldson, Uhlenbeck and Yau prove Hitchin–Kobayashi correspondence identifying instantons with stable holomorphic vector bundles.
- Late 1980s: Hitchin studies dimensional reductions of SD equations: Higgs bundles and examples of non-symmetric hyper-Kähler manifolds.
- 1980s-1990s: Witten relates Chern–Simons gauge theory to knot invariants. Seiberg–Witten derive new equations and invariants.

Higher dimensions and further gauge theory

The pet problem of compact Riemann surfaces suggests three avenues of investigation in higher dimensions:

1. The theorem of Narasimhan–Seshadri can be generalised to a correspondence between stable holomorphic vector bundles and so-called *Hermitian Yang–Mills connections* over a complex manifold of any dimension.
2. $\mathcal{M}$ has more structure than just a topological space: it is naturally symplectic, and through the NS theorem, Kähler. One can construct more exotic geometric structures on moduli spaces of connections: Higgs bundle moduli spaces over compact Riemann surfaces are hyper-Kähler and come with completely integrable Hamiltonian systems.
3. One can compute the cohomology of moduli spaces of connections in higher dimension, and derive topological, smooth, or geometric invariants of the underlying space M. For example Donaldson invariants or Seiberg–Witten invariants.

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Higher dimensions: Algebraic Geometry

Let (M, ω) be a compact Kähler manifold of any dimension. Let $\mathcal{E} \to M$ be a holomorphic vector bundle with Hermitian metric h. Then one naturally has a Chern connection ∇_h and curvature $F(h)$, with the property that $\nabla_h^{0,1} = \overline{\partial}_E$.We say this metric is *Hermitian Yang–Mills* if

$$n\omega^{n-1} \wedge F(h) = \lambda \mathbf{1}_E \otimes \omega^n$$

where $\lambda \in \mathbb{C}$ is a constant.

Theorem (Hitchin–Kobayashi correspondence)

An irreducible holomorphic vector bundle $\mathcal{E} \to (M, \omega)$ admits a HYM metric if and only if it is stable.

This theorem is a vast generalisation of the NS theorem as proven by Donaldson. It was proven by Donaldson for algebraic surfaces and Uhlenbeck and Yau for arbitrary compact Kähler manifolds.

Hitchin–Kobayashi, the easy direction

If $\mathcal{F} \subset \mathcal{E}$ is a holomorphic subbundle then the Chern connection ∇_h on $\mathcal{E}$ splits up as

$$\nabla_h = \begin{pmatrix} \nabla_{\mathcal{F}} & \gamma \\ -\gamma^* & \nabla_{\mathcal{E}/\mathcal{F}} \end{pmatrix}, \quad F(h) = \begin{pmatrix} F_{\mathcal{F}} - \gamma \wedge \gamma^* & d_A\gamma \\ -d_A\gamma^* & F_{\mathcal{E}/\mathcal{F}} - \gamma^* \wedge \gamma \end{pmatrix},$$

where $\gamma \in \Omega^{0,1}(M, \mathrm{Hom}(\mathcal{E}/\mathcal{F}, \mathcal{F}))$ is the *second fundamental form*.Chern–Weil theory says that $\deg(\mathcal{E}) = \int_M \frac{i}{2\pi} \operatorname{tr}(F(h)) \wedge \omega^{n-1}$. If h is YM then this equals

$$\mathrm{rk}(E) \frac{i\lambda}{2\pi} \mathrm{vol}(M).$$

Thus $\lambda = \frac{-2\pi i}{\mathrm{vol}(M)} \frac{\deg \mathcal{E}}{\mathrm{rk}\, \mathcal{E}}$.Doing the same for the top left block gives

$$\mathrm{rk}(F) \frac{i\lambda}{2\pi} \mathrm{vol}(M) = \deg \mathcal{F} + \|\gamma\|^2.$$

Since $\|\gamma\|^2 > 0$, we get stability for $\mathcal{F} \subset \mathcal{E}$.

Higher dimensions: Algebraic Geometry – Compactifications

The moduli space of stable holomorphic vector bundles $\mathcal{N}^s$ admits a natural algebraic compactification $\mathcal{N}^{ss}$ to the moduli space of *semi-stable* vector bundles. This is in general a singular, projective algebraic variety.

Thus the Hitchin–Kobayashi correspondence allows one to compactify moduli spaces of Yang–Mills connections.

Four dimensions: Topological invariants

In four dimensions, the Yang–Mills equations simplify: If M is of dimension four, then $\star : \Omega^2(M) \to \Omega^2(M)$ and $\star\star = \mathbf{1}$, so

$$\Omega^2(M) = \Omega_+(M) \oplus \Omega_-(M)$$

where $\alpha \in \Omega_\pm(M)$ if $\star\alpha = \pm\alpha$.

Since $d_A F_A = 0$, if $F_A \in \Omega_\pm(\mathrm{End}\, E)$, then

$$d_A \star F_A = \pm d_A F_A = 0$$

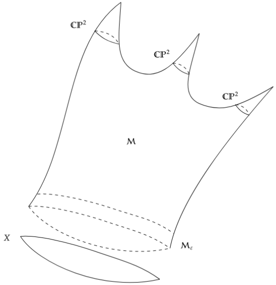
automatically. The equations $\star F_A = \pm F_A$ are the *(anti-)self-duality ((A)SD) equations*.

- (A)SD connections are absolute minima of the Yang–Mills functional YM. The solutions to $d_A \star F_A = 0$ which are not (A)SD represent the higher local minima.

Four dimensions: Topological invariants

The moduli space $\mathcal{M}_{ASD}$ of ASD connections on a bundle P over a simply connected four-manifold M has good analytical properties:

- It is generically smooth, except at a collection of points corresponding reducible connections.
- It is orientable.
- It has a natural compactification by gluing in a copy of M itself at ∞.



Since $\bar{\mathcal{M}}_{ASD}$ is orientable, one can integrate cohomology classes to obtain numbers! Using the universal bundle $\mathbb{E} \to \mathcal{M}_{ASD} \times M$, one can generate cohomology classes on $\mathcal{M}_{ASD}$ from characteristic classes of $\mathbb{E}$, and obtain topological invariants of M.

Other gauge theoretic problems

- Seiberg–Witten equations: better analytical properties than the YM equations, and produce essentially equivalent invariants.
- Hitchin's equations and Higgs bundles: attach an auxilliary field Φ compatible with a connection. This has a strong relation to algebraic geometry and produces examples of hyper-Kähler moduli spaces.
- Dimensional reduction of gauge-theroetic equations: suppose a symmetry exists under some subgroup of the automorphism group of M and study the equations on the quotient: magnetic monopole equations, Hitchin's equations, Nahm equations are all examples.
- Higher dimensional instanton equations: Nearly Kähler instantons, $G2, \mathrm{Spin}(7)$ instantons.
- Analytical properties of moduli spaces in dimension > 4. Compactifications? structure of singular sets? This is not know even in dimension 6.
- Other enumerative invariants: Donaldson–Thomas theory, Vafa–Witten theory.