Divisor-Line Bundle Correspondence

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November 7, 2020

Algebraic Geometry decoded

Let X be a noetherian, integral, separated scheme which is regular in codimension one.
What does this mean? 

- noetherian : underlying topological space is noetherian
- integral : underlying topological space is irreducible and the structure $\mathcal{O}_X$ sheaf is reduced
- separated : the topological space is Hausdorff (roughly)
- regular in codimension one : discretizes functions on X wrt. codimension one closed subsets.

More precisely, if $Y \subset X$ is a codimension closed subset and η be its (unique) generic point, then regular in codimension one means $\mathcal{O}_{X,\eta}$ is a discrete valuation ring.
Example: X a smooth (projective) variety.

Weil Divisors

A Weil Divisor D is a formal $\mathbb{Z}$-linear combination of codimension one irreducible closed subsets of X *i.e.*

$$D = \sum_{Y \subset X \text{ codim } 1} n_Y \, [Y] \tag{1}$$

where $n_Y \in \mathbb{Z}$ and all but a finite number are zero.
The set of Weil divisors $\text{Div}(X)$ form an abelian group with respect to addition.
We say D is effective if $n_Y \geq 0$ for all Y.
If $D = [Y]$, then we say D is a prime divisor.
Define the support of D as

$$\text{Supp}(D) := \bigcup_{n_Y \neq 0} Y \tag{2}$$

Weil Divisors

Example

When X is a curve, $\text{Div}(X) =$ linear combinations of closed points.

Example

When X is a surface, $\text{Div}(X) =$ linear combinations of irreducible curves. We will study the intersection theory of non-singular surfaces later.

Weil Divisors

How should we think of Weil divisors?

If Y is an irreducible closed subset of codimension one, let $\eta \in Y$ be its (unique) generic point. Then $\mathcal{O}_{X,\eta}$ is a discrete valuation ring with fraction field $K = K(X)$.

Let v_Y denote the valuation on K. Then for $f \in K^\times$, $v_Y(f) \in \mathbb{Z}$. If $v_Y(f) > 0$, then Y is a zero of order $v_Y(f)$ and if $v_Y(f) < 0$, then Y is a pole of order -$v_Y(f)$.

Weil Divisors

Lemma

Let $f \in K^\times$ be a non-zero function on X. Then $v_Y(f) = 0$ for all but finitely many prime divisors Y.

So it makes sense to look at the zero-pole locus of functions on X. Define the divisor of $f \in K^\times$, denoted $\text{div}(f)$,

$$\text{div}(f) = \sum_{Y \subset X \text{ codim } 1} v_Y(f)\ [Y] \tag{3}$$

By the lemma, $\text{div}(f) \in \text{Div}(X)$. Any divisor which is equal to the divisor of a function is called a principal divisor.

Equivalence of Weil Divisors

Note $\text{div}(\frac{f}{g}) = \text{div}(f) - \text{div}(g)$ for $f, g \in K^\times$. So we have a homomorphism

$$K^\times \longrightarrow \text{Div}(X) \tag{4}$$

whose image is the set of principal divisors and is a subgroup of X.

Definition

Two divisors D, D' are said to be linearly equivalent if $D - D' = \text{div}(f)$ for some $f \in K^\times$. We write $D \sim D'$ if D and D' are equivalent.

Divisor Class Group

Define the Divisor Class Group as the equivalence classes of Weil Divisors *i.e.*

$$\mathsf{Cl}(X) = \mathsf{Div}(X)/\sim$$

We shall see that $\mathsf{Div}(X)$ is a very interesting invariant of the space X though computing the divisor class group can be quite difficult in general. In the case $X = \mathsf{Spec}\ A$, where A is a Dedekind domain, then $\mathsf{Cl}(X)$ is just the ideal class group of A, and thus is easy to compute.

Divisor Class Group

Example

$\mathrm{Cl}(\mathbb{P}^n_k) = \mathbb{Z}$ generated by the hyperplane $H = \{x_0 = 0\}$.

Lemma

Let $Z \neq X$ be a closed subset and $U = X \backslash Z$. Then if $\mathrm{codim}(Z, X) \geq 2$, then $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ is an isomorphism. If Z an irreducible closed subset of codim. 1, then there is an exact sequence

$$\mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(U) \to 0$$

Example

The quadric affine cone $X = \mathrm{Spec}\ k[x, y, z]/(xy - z^2)$ in $\mathbb{A}^3_k$ has $\mathrm{Cl}(X) = \mathbb{Z}/2\mathbb{Z}$ generated by a ruling of the cone $y = z = 0$.

Line Bundles

Definition

A line bundle L on X is given by the data of an open cover U_i of X and trivialisations $U_i \times \mathbb{C}$ and transition functions $f_{ij} \in \mathcal{O}_X^\times(U_i \cap U_j)$ such that $f_{ij}f_{jk} = f_{ik}$ (cocycle condition), $f_{ii} = 1$ and $f_{ij} = f_{ij}^{-1}$.

In particular, by gluing, L is also a scheme with a morphism $\pi_L : L \longrightarrow X$.

- A section of L is a morphism $s : X \longrightarrow L$ such that $\pi_L \circ s = \mathrm{Id}_X$.
- A line bundle is trivial ($\cong X \times \mathbb{C}$) if and only if there exists a nowhere vanishing section.

Invertible Sheaves

Definition

An invertible sheaf on X is a locally free $\mathcal{O}_X$-module of rank 1.

If $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves, then so is $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}$.
$\mathcal{L}^{-1} := \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ is an invertible sheaf. Then $\mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathcal{O}_X$.

Definition

The Picard Group of X, $\mathrm{Pic}(X)$, is the group of isomorphism classes of invertible sheaves on X under the operation $\otimes$ and identity $\mathcal{O}_X$.

The Line Bundles-Invertible Sheaf correspondence

Given a line bundle $\pi : L \to X$, its sections form a sheaf. More precisely, for $U \subset X$ open, define the sheaf of sections $\mathcal{L}$ by

$$\mathcal{L}(U) := \{s : U \to L : \pi \circ s = \mathrm{id}_U\}$$

Exercise: $\mathcal{L}$ is an invertible sheaf.
Given an invertible sheaf $\mathcal{L}$, you can cook up a line bundle *i.e.* take a trivialising open cover U_i and transition functions given by $f_{ij} = \varphi_{ij}(1)$ where

$$\varphi_{ij} : \mathcal{O}_X(U_i \cap U_j) \xrightarrow{\sim} \mathcal{O}_X(U_j \cap U_i)$$

are the gluing functions of $\mathcal{L}$.

Philosophy

Classically, people would study varieties by looking at them as topological spaces.

In Algebraic Geometry, we study functions on varieties (a dual point of view). Line Bundles are the topological view and invertible sheaves the algebro-geometric view.

The Holy Trinity

$$\begin{tikzcd}
\text{Line Bundles} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow, "?"] & & \text{Invertible Sheaves} \arrow[dl, leftrightarrow, "?"] \\
& \text{Divisors} &
\end{tikzcd}$$

The Divisor-Invertible Sheaf Correspondence

We will assume X is a smooth variety from now on. We will construct an isomorphism of groups

$$\mathsf{Cl}(X) \longrightarrow \mathsf{Pic}(X).$$

Let K_X be the sheaf of rational functions on X. Since X is integral, K_X is just the constant sheaf of the function field $K = K(X)$ (nothing interesting happens outside of a dense open set).
Given a Weil Divisor $D = \sum_Y n_Y \cdot [Y]$, define $\mathcal{O}_X(D)$ as follows: for $U \subset X$, let

$$\mathcal{O}_X(D)(U) := \{f \in K^\times \mid v_Y(f) \geq -n_Y \ \forall \text{ prime div. s.t. } Y \cap U \neq 0\} \cup \{0\} \tag{5}$$

$\mathcal{O}_X(D)$ is an invertible sheaf since it is a subsheaf of K_X.
$D \mapsto \mathcal{O}_X(D)$ is a homomorphism since $\mathcal{O}_X(D + D') \cong \mathcal{O}_X(D) \otimes \mathcal{O}_X(D')$ given by $fg \hookleftarrow f \otimes g$ (and by smoothness).

The Divisor-Invertible Sheaf Correspondence

Theorem

The homomorphism

$$\mathrm{Cl}(X) \longrightarrow \mathrm{Pic}(X)$$

is an isomorphism.

Proof.

We first check injectivity: To do this, we show

$$D \sim D' \iff \mathcal{O}_X(D) \cong \mathcal{O}_X(D').$$

It is enough to show that

$$D \text{ is principal} \iff \mathcal{O}_X(D) \cong \mathcal{O}_X,$$

and this follows from the isomorphism $1 \in \mathcal{O}_X(D) \mapsto f \in \mathcal{O}_X$ where $D = (f)$. Thus the homomorphism is injective.

The Divisor-Invertible Sheaf correspondence

Theorem

$$\mathsf{Cl}(X) \longrightarrow \mathsf{Pic}(X)$$

is an isomorphism.

Proof.

(Continued) We construct an inverse:
Given an invertible sheaf $\mathcal{L}$, take a trivialising open cover U_i and $\varphi_i : \mathcal{L}|_{U_i} \xrightarrow{\sim} \mathcal{O}_X|_{U_i}$.
Let $f_i = \varphi_i(1)$ *i.e.* $\mathcal{L}|_{U_i} \cong f_i^{-1}\mathcal{O}_X|_{U_i}$.
Construct the divisor D as follows: for every prime divisor Y, take the coefficient of Y to be $v_Y(f_i)$ whenever $U_i \cap Y \neq \emptyset$.
If $j \neq i$ and $Y \cap U_j \neq \emptyset$, f_i/f_j is invertible on $U_i \cap U_j$ and so $v_Y(f_i) = v_Y(f_j)$.
Thus we obtain a well defined Weil divisor $D = \sum_Y v_Y(f_i) \cdot [Y]$. These constructions are inverse to each other. $\square$

The Divisor-Line Bundle Correspondence

Given a line bundle L on X and s be a non-zero rational section of L (a section of L defined over an open (=dense) subset of X). Then we can define a Weil divisor div(s) as

$$\text{div}(s) := \sum_{Y} v_Y(s) \cdot [Y]$$

where $v_Y(s)$ is defined as follows:
Let η be the generic point of Y and take U to be a trivializing open set of $\mathcal{L}$ containing η. Then s corresponds to a rational function f on U (up to multiplication by $\mathcal{O}_X(U)^\times$) and so we take

$$v_Y(s) := v_Y(f) = \text{“zeros of } s\text{” - “poles of } s\text{”} \tag{6}$$

We have that div$(s) =$ div(s') if and only if they differ by $f \in \Gamma(X, \mathcal{O}_X^\times)$.

The Divisor-Line Bundle Correspondence

So we have a injective homomorphism

$$\{(L,s)\}/\Gamma(X,\mathcal{O}_X^\times) \longrightarrow \mathsf{Cl}(X)$$

We can construct an inverse:
Let D be a Weil divisor and let $\mathcal{L}(D)$ be the line bundle associated to $\mathcal{O}_X(D)$.
By smoothness, we can find an open cover U_i of X such $D|_{U_i} = \sum_{Y \in U_i} n_Y \cdot [Y]$ is principal, say given by $f_i \in K(U_i) = K(X)$.
Viewing 1 as a rational section of $\mathcal{L}(D)$, we have $\mathrm{div}(1) = D$ (gluing all $\mathrm{div}(f_i)|_{U_i}$ together).
Thus the inverse is given by

$$\begin{aligned} \mathsf{Div}(X) &\longrightarrow \{(L,s)\}/\Gamma(X,\mathcal{O}_X^\times) \\ D &\longmapsto (\mathcal{L}(D),1) \end{aligned} \tag{7}$$

Move Fast and Break Things

Why have we been assuming X smooth? The divisor class group is isomorphic to the Picard group when X is integral, noetherian, separated and locally factorial.

> locally factorial: all local rings are UFD's
>
> Where do we use it? It ensures that locally, a divisor can be viewed as principal.

If we drop integrality, then $\mathrm{Div}(X)$ is a subgroup of $\mathrm{Pic}(X)$.
The Divisor-Line Bundle correspondence suggests another notion of divisor (called a Cartier divisor) given by an open cover and rational functions on each cover.

Intersection Theory for surfaces

Let X be a non-singular surface with C and D curves on X. Then for $P \in C \cap D$, we say C and D meet transversally at P if locally the equations at P of C and D generate the maximal ideal of $\mathcal{O}_{X,P}$.

Theorem (Bertini)

Let X be a closed smooth subvariety of $\mathbb{P}^n_k$ with k algebraically closed. Then there exists a hyperplane H, not containing X such that $H \cap X$ is regular at every point. Furthermore, the set of hyperplanes with this property forms an open dense subset of the complete linear system $|H| := \{H' \geq 0 \mid H' \sim H\}$, viewed as a projective space.

Then Serre Duality for Projective schemes implies that the schemes $H \cap X$ are irreducible and non-singular when $\dim X \geq 2$.

Transversal Position

Lemma

Let $C_1, \ldots, C_r$ be irreducible curves on a non-singular surface X and D a very ample divisor. Then almost all curves $D' \in |D|$ are irreducible, non-singular and intersect each of the C_i transversally.

Proof.

Since D is very ample, we can embed X in a projective space $\mathbb{P}^n$. Then by Bertini's theorem and since $\dim X \geq 2$ applied simultaneouly, almost all $D' \in |D|$ are irreducible, non-singular curves in X and that their intersections $C_i \cap D$ are non-singular *i.e.* they are closed points with multiplicity one and so C_i and D meet transversally. $\square$

Lemma

Let C be an irreducible non-singular curve on X and D any curve meeting C transversally. Then

$$\#(C \cap D) = \deg_C(\mathcal{O}(D) \otimes \mathcal{O}_C)$$

where $\deg$ is the degree of the invertible sheaf $\mathcal{O}(D) \otimes \mathcal{O}_C$.

Proof.

$\mathcal{O}(-D)$ is the ideal sheaf of D on X *i.e.* we have the short exact sequence $0 \to \mathcal{O}(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$. Then by tensoring with $\mathcal{O}_C$, we have

$$0 \to \mathcal{O}(-D) \otimes \mathcal{O}_C \to \mathcal{O}_C \to \mathcal{O}_{C \cap D} \to 0.$$

So $\mathcal{O}(D) \otimes \mathcal{O}_C$ is the invertible sheaf corresponding to the divisor $C \cap D$. Since the intersection is transversal, the degree of the divisor is just the number of points $\#(C \cap D)$. $\square$

Intersection pairing for surfaces

Theorem

There is a unique well defined pairing

$$\mathsf{Cl}(X) \times \mathsf{Cl}(X) \longrightarrow \mathbb{Z}$$

denoted $C \cdot D$ for $C, D \in \mathsf{Div}(X)$ which is symmetric and additive. Moreover if C and D are non-singular curves meeting transversally, then $C \cdot D = \#(C \cap D)$.

Proof.

We will show existence. Fix an ample divisor H on X. Given two divisors C and D on X, we can find an integer n such that $C + nH, D + nH$ and nH are very ample. Then by the lemma, choose non-singular curves $C' \in |C + nH|$, $D' \in |D + nH|$ transversal to C', $E' \in |nH|$ transversal to D' and $F' \in |nH|$ transversal to C' and E'.

Intersection Pairing for surfaces

Proof (continued.)

Then

$$C \cdot D = C' \cdot D' - C' \cdot F' - D' \cdot E' + E' \cdot F'$$

where recall $C' \cdot D' = \deg_{D'}(\mathcal{O}(C') \otimes \mathcal{O}_{D'})$. In the case that both C and D are very ample divisors, we can take $C' \in |C|$ and $D' \in |D|$ and take $C \cdot D = \#(C' \cap D')$. The proof of uniqueness is very similar. $\square$

Example

Let $X = \mathbb{P}^2$. Then Pic $X \cong \mathbb{Z}$ and take h to the generator. Since any two lines are linearly equivalent and two distinct lines meet in one point, we have $h \cdot h = 1$. This determines the intersection pairing by linearity: if C, D are curves of degree n and m respectively then $C \cdot D = nm$.

Self Intersection

If D is any divisor on X, define the self intersection number $D \cdot D$, denoted D^2. We have $C^2 = \deg_C(\mathcal{O}(C) \otimes \mathcal{O}_C)$. Since the ideal sheaf $\mathscr{I}$ of C on X is $\mathcal{O}(-C)$, we have $\mathscr{I}/\mathscr{I}^2 \cong \mathcal{O}(-C) \otimes \mathcal{O}_C$. So its dual $\mathcal{O}(C) \otimes \mathcal{O}_C$ is isomorphic to the normal sheaf

$$\mathcal{N}_{C/X} := \mathcal{H}\mathrm{om}(\mathscr{I}/\mathscr{I}^2, \mathcal{O}_C)$$

and thus $C^2 = \deg_C \mathcal{N}_{C/X}$.

Example

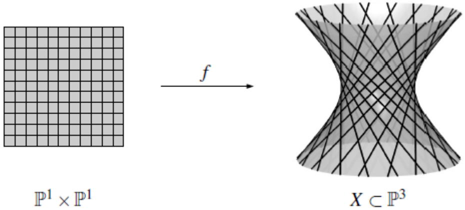
Let X be a non-singular variety and $P \in X$ be a point. Let $\tilde{X}$ be the blow up of X at P. Let E be the exceptional divisor. Then $E^2 = -1$.

How should one think of negative self-intersection divisor? Rigidity and Blow ups.For rigidity, consider the blow up of $\mathbb{A}^2$ at a point and think about where the exceptional divisor can go.

Intersection Theory for surfaces

Example

Let X be the Segre embedding (nonsingular quadric surface) in $\mathbb{P}^3$ given by $xy = zw$ which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Then $\text{Div}(X) = \mathbb{Z} \oplus \mathbb{Z}$. Let ℓ be of type $(1, 0)$ and m be of type $(0, 1)$. Then $\ell^2 = 0 = m^2$ and $\ell \cdot m = 0$ since two lines in the same family are skew and two lines of opposite families meet at a point. Thus if C has type (a, b) and D has type (a', b'), then $C \cdot D = ab' + a'b$.



MMP for surfaces

Theorem (Castelnuovo)

Let X be a smooth projective surface and Y a curve on X with $Y \cong \mathbb{P}^1$ and $Y^2 = -1$. Then there exists a morphism $f : X \to X_0$ with X_0 smooth projective such that X is isomorphic to the blowup of X_0 along $P \in X_0$ and Y is the exceptional divisor.

We can define a minimal surface to be a surface which contains no (-1)-curves. More generally, K_X is n.e.f *i.e.* $K_X \cdot D \geq 0$ for every effective divisor D.

Theorem

Contracting all the (-1)-curves on X to give Y is isomorphic to a (unique) minimal model with K_Y nef or a ruled surface.

Example

Let X be a K3 surface. Then by adjunction, for every $C \cong \mathbb{P}^1$ on X, we have $C^2 \cong = -2$ hence minimal.