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Optimal Dividend Strategies of Two Collaborating Businesses in the Diffusion Approximation Model

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In this paper, we consider the optimal dividend payment strategy for an insurance company, having two collaborating business lines. The surpluses of the business lines are modeled by diffusion processes. The collaboration between the two business lines permits that money can be transferred from one line to another with or without proportional transaction costs while money must be transferred from one line to another to help both business lines keep running before simultaneous ruin of the two lines eventually occur.

Key words: Optimal Dividends Strategy; Diffusion Model; Collaborating Businesses; Stochastic Control
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1. Introduction This paper is concerned with a classical problem of actuarial mathematics, strategies for optimal pay-out of dividends when two collaborating lines of business are taken into account. Since De Finetti (1957) introduced the dividend barrier model, in which all surpluses above a given level are transferred (subject to a discount rate) to a beneficiary, and raised the question of how to optimize this barrier, it has been of particular interest to determine the optimal dividend payment strategy which maximizes discounted dividend payments. Gerber (1972) considered an optimal dividend strategy problem, where the income process of a firm is assumed to be a homogeneous Markov process (discrete or continuous).

A classical model is Cramér-Lundberg model, i.e., a compound Poisson risk model. In 1969, Gerber showed that if the free surplus of an insurance portfolio is modelled by a compound Poisson risk model, it is optimal to pay dividends according to a so-called band strategy, which collapses to a barrier strategy for exponentially distributed claim amounts. Whereas Gerber found this result by taking a limit of an associated discrete problem, this optimal dividend problem was later studied with techniques of modern stochastic control theory (see Azcue and Muler (2014) for summary).

In Gerber and Shiu (2004), the surplus of a company is a Brownian motion with a positive drift and the dividends are paid according to a barrier strategy. The optimal barrier is determined in that paper. Asmussen and Taksar (1996) consider the optimal dividend strategy problem in diffusion approximation model in both of the following cases: (1) the dividend rate is restricted;

(2) the dividend rate is unrestriced. They showed that, in both cases, the optimal strategies are barrier strategies.

All these control problems have been formulated and studied in the one-dimensional framework. However, in recent years there has been an increased interest in risk theory in considering the dynamics of several connected insurance portfolios simultaneously, see e.g. Asmussen and Albrecher (2010) for an overview. Albrecher et al. (2015) extend the optimal dividend problem from univariate risk theory to a two-dimensional setup of two collaborating companies under a compound Poisson model framework. The collaboration consists of paying the deficit ('bailing out') of the partner company if its surplus is negative and if this financial help can be afforded with the current own surplus level.

In this paper, we consider the optimal dividend payment strategy for an insurance company, having two collaborating business lines. The surpluses of the business lines are modeled by diffusion processes. The collaboration between the two business lines permits that money can be transferred from one line to another with or without transaction costs while money must be transferred from one line to another to help both business lines keep running before simultaneous ruin of the two lines eventually occur. In contrast to Albrecher et al. (2015), the continuity of the non-controlled processes prevents the situation where one line cannot afford to save another line while keeping itself out of trouble. So the situation in Albrecher et al. (2015) where one business line continues after another business line has gone ruin, and even failed to pay the last claim, cannot happen here. Therefore ruin is defined by simultaneous ruin.

Although the setup with two risk processes is similar to that of Albrecher et al. (2015), we prefer to think of the two-dimensional problem as arising from capital control in two business lines within one company rather than two cooperating companies. The presentation is fairly generic but the reader can think of two business lines, like e.g. motor insurance and theft insurance. For our diffusion approximation to be reasonable, think of two light-tailed business lines. Each line has his own manager being responsible for the profitability of his business line. The manager of the whole company can allocate capital to the two business lines in order to serve a given objective. Actually, this is a practically realistic setup and capital allocation is an important management discipline nowadays, typically performed with the objective to minimize the amount of total supporting capital it takes to be solvent as a company. Our problem here is the similar decision problem with the classical objective to maximize the dividend payouts obtained from the two business lines until ruin of the whole company.

The main results of the paper are the following. First, we find the optimal dividend strategy when money is transferred between the two business lines with no transaction costs. Second, we prove the optimal value function is a continuous viscosity solution to the corresponding HJB equation in the case with transaction costs. A verification theorem is also proven. Further, we characterize explicitly the solution in the special case where the problem is symmetric in the two business lines.

The technical contribution is a novel characterization (Prop. 8) of the viscosity solution of the HJB equation and the optimal control when transaction costs are taken into account. Consequently, a new verification theorem (Thm. 3) is obtained. This characterization also leads us to find the optimal control in the symmetric case and enlightens us to search the optimal control in general cases.

The remainder of the paper is structured as follows. In Section 2, we present the diffusion model with no transaction costs. In Section 3, we find the optimal value function and the optimal dividend strategy. In Section 4, we prove that, when transaction costs occur, the optimal value function is a continuous viscosity solution to the corresponding HJB equation. A verification theorem is also given. In section 5, we search the optimal dividend strategy in the symmetric case. Conclusions are given in Section 6.

2. The Model We consider an insurance company, having two collaborating business lines, line One and line Two. Let us call $X_1(t)$ the free surplus of Business One and $X_2(t)$ the one of Business Two. We assume that the free surplus of each of the lines of business follows a diffusion process such that

$$X_1(t) = x_1 + \mu_1 t + \sigma_1 W_1(t),$$
$$X_2(t) = x_2 + \mu_2 t + \sigma_2 W_2(t),$$

where x_1 and x_2 are the respective initial surplus levels and W_1 and W_2 are independent standard Brownian motions.

There is a contract of collaboration: If the current surplus of line One hits zero, there is a positive cash flow from line Two to line One to help One remain non-negative, and vice versa. Hence, ruins of the two businesses occur simultaneously at the moment that both surplus processes hit zero. Moreover, both businesses are free to transfer money to each other.

Both businesses use part of their surplus to pay dividends to their shareholders. The dividend payment strategy $L = (L_1, L_2)$ is the total amount of dividends paid by the two businesses up to time t. The associated controlled surplus processes become

$$\bar{X}_1(t) = X_1(t) + C_{21}(t) - C_{12}(t) - L_1(t),$$
$$\bar{X}_2(t) = X_2(t) + C_{12}(t) - C_{21}(t) - L_2(t),$$

where $C_{21}(t)$ corresponds to the cumulative amount transferred from Business Two to One up to time t; $C_{12}(t)$ corresponds to the cumulative amount transferred from Business One to Two up to time t. The ruin time of the businesses is defined as

$$\tau := \inf\{t > 0 : \bar{X}_1(t), \bar{X}_2(t) < 0\}.$$

We call a dividend and transferring strategy $(L, C) = (L_1, L_2, C_{12}, C_{21})$ is admissible, denoted as $(L, C) \in \pi_{(x_1,x_2)}$, if

1. L_1, L_2 are left continuous with right limits and $\mathcal{F}_t$-predictable, where $\mathcal{F}_t$ is the natural filtration generated by $\bar{X}_1$ and $\bar{X}_2$.
2. C_{12}, C_{21} are right continuous with left limits and $\mathcal{F}_t$-adapted.
3. L_1, L_2, C_{12}, C_{21} are non-negative and non-decreasing.
4. $L_1(t) \le X_1(t) + C_{21}(t) - C_{12}(t)$ and $L_2(t) \le X_2(t) + C_{12}(t) - C_{21}(t)$

Let $\mathcal{R}_+^2$ denote the set $\{(x_1, x_2) : x_1 \ge 0, x_2 \ge 0\}$ throughout the paper. For any initial surplus level $(x_1, x_2) \in \mathcal{R}_+^2$, we write the optimal value function as the following

$$V(x_1, x_2) = \sup_{(L,C)\in\pi_{(x_1,x_2)}} V_{L,C}(x_1, x_2),$$

where

$$V_{L,C}(x_1, x_2) = E_{x_1,x_2}\left(a\int_0^\tau e^{-\beta s} dL_1(s) + (1-a)\int_0^\tau e^{-\beta s} dL_2(s)\right).$$

The weights a and $(1-a)$ just reflects that there may be different proportional costs connected with drawing money out of the two business lines.

Remark 1. As V is the optimal value function, we conclude some facts without proof.

$$\begin{aligned} &V(x_1 + \Delta x, x_2) - V(x_1, x_2) \ge a\Delta x, && \Delta x \ge 0,\\ &V(x_1, x_2 + \Delta x) - V(x_1, x_2) \ge (1-a)\Delta x, && \Delta x \ge 0,\\ &V(x_1 + \Delta x, x_2 - \Delta x) = V(x_1, x_2), && \Delta x \in \mathcal{R}. \end{aligned}$$

The corresponding HJB equation is given by

$$\begin{aligned}
0 &= \max\Big(\mathcal{L}(V)(x_1,x_2), a - \tfrac{\partial V}{\partial x_1}(x_1,x_2), (1-a) - \tfrac{\partial V}{\partial x_2}(x_1,x_2),\\
&\quad \tfrac{\partial V}{\partial x_1}(x_1,x_2) - \tfrac{\partial V}{\partial x_2}(x_1,x_2), \tfrac{\partial V}{\partial x_2}(x_1,x_2) - \tfrac{\partial V}{\partial x_1}(x_1,x_2)\Big),\\
0 &= V(0,0),
\end{aligned} \tag{1}$$

where $a \in (0,1)$ and

$$\mathcal{L}(V) = \mu_1 \frac{\partial V}{\partial x_1} + \mu_2 \frac{\partial V}{\partial x_2} + \frac{1}{2}\sigma_1^2 \frac{\partial^2 V}{\partial x_1^2} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 V}{\partial x_2^2} - \beta V.$$

Due to symmetry, without loss of generality, we assume that $a \geq \frac{1}{2}$.

3. Optimal Strategy

PROPOSITION 1. *Let $\bar{X}_1, \bar{X}_2$ be the controlled surplus processes with control (L, C) and initial values x_1, x_2. For any twice continuously differentiable function ψ on $\mathcal{R}_+^2$ and a finite stopping time $\tau^* \leq \tau$, if one of the following two conditions holds, (1) $\bar{X}_1, \bar{X}_2$ are bounded; (2) ψ has bounded first derivatives, then, we have*

$$\begin{aligned}
&e^{-\beta\tau^*}\psi(\bar{X}_1(\tau^*), \bar{X}_2(\tau^*)) - \psi(x_1,x_2)\\
&= \textstyle\int_0^{\tau^*} e^{-\beta s}\mathcal{L}(\psi)(\bar{X}_1(s-), \bar{X}_2(s-))ds + M(\tau^*)\\
&\quad + \textstyle\int_0^{\tau^*} e^{-\beta s}[\psi_{x_1}(\bar{X}_1(s-), \bar{X}_2(s-)) - \psi_{x_2}(\bar{X}_1(s-), \bar{X}_2(s-))]dC_{21}^C(s)\\
&\quad + \textstyle\int_0^{\tau^*} e^{-\beta s}[\psi_{x_2}(\bar{X}_1(s-), \bar{X}_2(s-)) - \psi_{x_1}(\bar{X}_1(s-), \bar{X}_2(s-))]dC_{12}^C(s)\\
&\quad + \textstyle\sum_{\bar{X}_1(s-)\neq \bar{X}_1(s), \bar{X}_2(s-)\neq \bar{X}_2(s), s\leq \tau^*} e^{-\beta s}[\psi(\bar{X}_1(s), \bar{X}_2(s)) - \psi(\bar{X}_1(s-), \bar{X}_2(s-))]\\
&\quad - \textstyle\int_0^{\tau^*} a e^{-\beta s} dL_1(s) - \int_0^{\tau^*}(1-a)e^{-\beta s}dL_2(s)\\
&\quad + \textstyle\int_0^{\tau^*}(a - \psi_{x_1}(\bar{X}_1(s-), \bar{X}_2(s-)))e^{-\beta s}dL_1^C(s)\\
&\quad + \textstyle\sum_{L_1(s+)\neq L_1(s), s<\tau^*} \int_0^{L_1(s+)-L_1(s)}(a - \psi_{x_1}(\bar{X}_1(s)-\alpha, \bar{X}_2(s)))e^{-\beta s}d\alpha\\
&\quad + \textstyle\int_0^{\tau^*}((1-a) - \psi_{x_2}(\bar{X}_1(s-), \bar{X}_2(s-)))e^{-\beta s}dL_2^C(s)\\
&\quad + \textstyle\sum_{L_2(s+)\neq L_2(s), s<\tau^*} \int_0^{L_2(s+)-L_2(s)}((1-a) - \psi_{x_1}(\bar{X}_1(s+), \bar{X}_2(s)-\alpha))e^{-\beta s}d\alpha,
\end{aligned} \tag{2}$$

where M is a martingale and L_i^C, C_{ij}^C are the continuous parts of L_i, C_{ij}.

Proof. Since L_i is nondecreasing and left continuous, we can write

$$L_i(t) = \int_0^t dL_i^C(s) + \sum_{\bar{X}_i(s+)\neq \bar{X}_i(s), s<t} (L_i(s+) - L_i(s)).$$

By Itô's formula for semi-martingales and the optional sampling theorem,

$$\begin{aligned}
&e^{-\beta\tau^*}\psi(\bar{X}_1(\tau^*), \bar{X}_2(\tau^*)) - \psi(x_1,x_2)\\
&= \textstyle\int_0^{\tau^*} e^{-\beta s} d\psi(\bar{X}_1(s), \bar{X}_2(s)) - \beta\int_0^{\tau^*} e^{-\beta s}\psi(\bar{X}_1(s), \bar{X}_2(s))ds\\
&= \textstyle\int_0^{\tau^*} e^{-\beta s}\mathcal{L}(\psi)(\bar{X}_1(s-), \bar{X}_2(s-))ds\\
&\quad + \textstyle\int_0^{\tau^*} e^{-\beta s}[\psi_{x_1}(\bar{X}_1(s-), \bar{X}_2(s-))\sigma_1 dW_1 + \psi_{x_2}(\bar{X}_1(s-), \bar{X}_2(s-))\sigma_2 dW_2]\\
&\quad + \textstyle\int_0^{\tau^*} e^{-\beta s}[\psi_{x_1}(\bar{X}_1(s-), \bar{X}_2(s-)) - \psi_{x_2}(\bar{X}_1(s-), \bar{X}_2(s-))]dC_{21}^C(s)\\
&\quad + \textstyle\int_0^{\tau^*} e^{-\beta s}[\psi_{x_2}(\bar{X}_1(s-), \bar{X}_2(s-)) - \psi_{x_1}(\bar{X}_1(s-), \bar{X}_2(s-))]dC_{12}^C(s)\\
&\quad + \textstyle\sum_{\bar{X}_1(s-)\neq \bar{X}_1(s), \bar{X}_2(s-)\neq \bar{X}_2(s), s\leq \tau^*} e^{-\beta s}[\psi(\bar{X}_1(s), \bar{X}_2(s)) - \psi(\bar{X}_1(s-), \bar{X}_2(s-))]\\
&\quad - \textstyle\int_0^{\tau^*} e^{-\beta s}[\psi_{x_1}(\bar{X}_1(s-), \bar{X}_2(s-))dL_1^C(s) + \psi_{x_2}(\bar{X}_1(s-), \bar{X}_2(s-))dL_2^C(s)]\\
&\quad + \textstyle\sum_{L_1(s+)\neq L_1(s), s<\tau^*} e^{-\beta s}[\psi(\bar{X}_1(s+), \bar{X}_2(s)) - \psi(\bar{X}_1(s), \bar{X}_2(s))]\\
&\quad + \textstyle\sum_{L_2(s+)\neq L_2(s), s<\tau^*} e^{-\beta s}[\psi(\bar{X}_1(s+), \bar{X}_2(s+)) - \psi(\bar{X}_1(s+), \bar{X}_2(s))].
\end{aligned} \tag{3}$$

Define

$$M(t) := \int_0^t e^{-\beta s}[\psi_{x_1}(\bar X_1(s-),\bar X_2(s-))\sigma_1 dW_1 + \psi_{x_2}(\bar X_1(s-),\bar X_2(s-))\sigma_2 dW_2], \tag{4}$$

which is a martingale. Since $X_1(s+) - X_1(s) = -(L_1(s+) - L_1(s))$, we can write

$$\begin{aligned}
&-\int_0^{\tau^*} e^{-\beta s}\psi_{x_1}(\bar X_1(s-),\bar X_2(s-))dL_1^C(s)\\
&+\sum_{L_1(s+)\neq L_1(s), s<\tau^*} e^{-\beta s}[\psi(\bar X_1(s+),\bar X_2(s)) - \psi(\bar X_1(s),\bar X_2(s))]\\
=&-\int_0^{\tau^*} e^{-\beta s}\psi_{x_1}(\bar X_1(s-),\bar X_2(s-))dL_1^C(s)\\
&-\sum_{L_1(s+)\neq L_1(s), s<\tau^*} e^{-\beta s}\int_0^{L_1(s+)-L_1(s)}\psi_{x_1}(\bar X_1(s)-\alpha,\bar X_2(s))d\alpha\\
=&-\int_0^{\tau^*} a e^{-\beta s}dL_1(s) + \int_0^{\tau^*} e^{-\beta s}(a - \psi_{x_1}(\bar X_1(s-),\bar X_2(s-)))dL_1^C(s)\\
&+\sum_{L_1(s+)\neq L_1(s), s<\tau^*} e^{-\beta s}\int_0^{L_1(s+)-L_1(s)}(a - \psi_{x_1}(\bar X_1(s)-\alpha,\bar X_2(s)))d\alpha.
\end{aligned} \tag{5}$$

Similarly,

$$\begin{aligned}
&-\int_0^{\tau^*} e^{-\beta s}\psi_{x_2}(\bar X_1(s-),\bar X_2(s-))dL_2^C(s)\\
&+\sum_{L_2(s+)\neq L_2(s), s<\tau^*} e^{-\beta s}[\psi(\bar X_1(s+),\bar X_2(s+)) - \psi(\bar X_1(s+),\bar X_2(s))]\\
=&-\int_0^{\tau^*} e^{-\beta s}\psi_{x_2}(\bar X_1(s-),\bar X_2(s-))dL_2^C(s)\\
&-\sum_{L_2(s+)\neq L_2(s), s<\tau^*} e^{-\beta s}\int_0^{L_2(s+)-L_2(s)}\psi_{x_2}(\bar X_1(s+),\bar X_2(s)-\alpha)d\alpha\\
=&-\int_0^{\tau^*} (1-a) e^{-\beta s}dL_2(s) + \int_0^{\tau^*} e^{-\beta s}((1-a) - \psi_{x_2}(\bar X_1(s-),\bar X_2(s-)))dL_2^C(s)\\
&+\sum_{L_2(s+)\neq L_2(s), s<\tau^*} e^{-\beta s}\int_0^{L_2(s+)-L_2(s)}((1-a) - \psi_{x_2}(\bar X_1(s+),\bar X_2(s)-\alpha))d\alpha.
\end{aligned} \tag{6}$$

Combining the equalities above, the result follows. □

Define the following function f on $\mathcal{R}_+^2$:

$$f(x_1,x_2) = \begin{cases} aC(e^{\theta_1(x_1+x_2)} - e^{-\theta_2(x_1+x_2)}), & x_1+x_2 < m,\\ a[C(e^{\theta_1 m} - e^{-\theta_2 m}) + x_1 + x_2 - m], & x_1+x_2 \geq m, \end{cases} \tag{7}$$

where θ_1,θ_2,C,m are some positive constants to be determined. We require $\mathcal{L}(f) = 0$ when $x_1+x_2 < m$, which leads to

$$\begin{cases} \mu_1\theta_1 + \mu_2\theta_1 + \frac{1}{2}\sigma_1^2\theta_1^2 + \frac{1}{2}\sigma_2^2\theta_1^2 - \beta = 0,\\ \mu_1\theta_2 + \mu_2\theta_2 + \frac{1}{2}\sigma_1^2(-\theta_2^2) + \frac{1}{2}\sigma_2^2(-\theta_2^2) + \beta = 0, \end{cases} \tag{8}$$

that is $\theta_1, -\theta_2$ are the roots of the equation

$$\left(\frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2\right)\theta^2 + (\mu_1+\mu_2)\theta - \beta = 0, \tag{9}$$

where $\theta_2 > \theta_1 > 0$. We require f be twice continuously differentiable in $\mathcal{R}_+^2$, which leads to

$$\begin{cases} \left.\frac{\partial f}{\partial x_1}\right|_{x_1+x_2=m} = a,\ \left.\frac{\partial f}{\partial x_2}\right|_{x_1+x_2=m} = a,\\ \left.\frac{\partial^2 f}{\partial x_1^2}\right|_{x_1+x_2=m} = 0,\ \left.\frac{\partial^2 f}{\partial x_2^2}\right|_{x_1+x_2=m} = 0,\ \left.\frac{\partial^2 f}{\partial x_1\partial x_2}\right|_{x_1+x_2=m} = 0. \end{cases} \tag{10}$$

Coefficients C,m are determined by

$$\begin{cases} C(\theta_1 e^{\theta_1 m} + \theta_2 e^{-\theta_2 m}) = 1,\\ \theta_1^2 e^{\theta_1 m} - \theta_2^2 e^{-\theta_2 m} = 0, \end{cases} \tag{11}$$

which gives

$$m = \frac{2(\ln\theta_2 - \ln\theta_1)}{\theta_1+\theta_2}, C = \frac{1}{\theta_1 e^{\theta_1 m} + \theta_2 e^{-\theta_2 m}}. \tag{12}$$

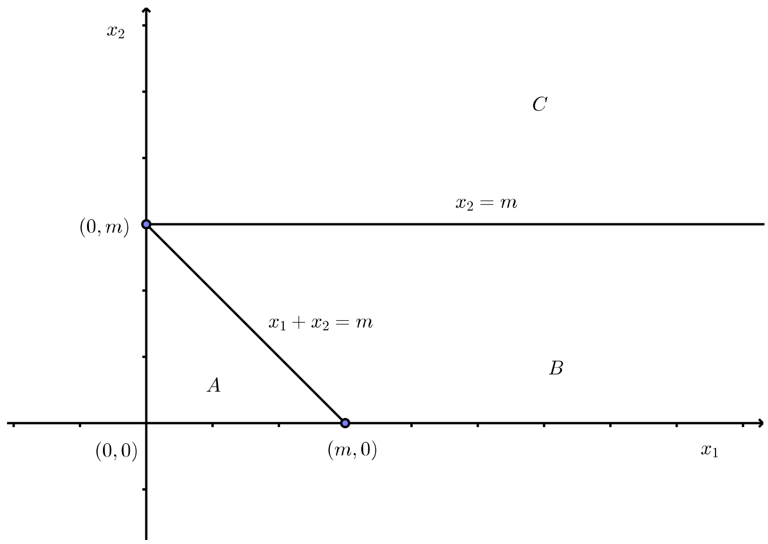

FIGURE 1. Optimal Strategy

PROPOSITION 2. *The function f defined in (7), with coefficients given in (9) and (12) is twice continuously differentiable and has bounded first derivatives and is a solution to the HJB equation (1).*

Proof. Apparently, $\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2} = 0$ and $a - \frac{\partial f}{\partial x_1} = 0$, $(1-a) - \frac{\partial f}{\partial x_2} \le 0$, when $x_1 + x_2 \ge m$. It remains to show that

$$\begin{cases} a - \frac{\partial f}{\partial x_1} \le 0, \ (1-a) - \frac{\partial f}{\partial x_2} \le 0, & x_1 + x_2 < m, \\ \mathcal{L}(f) \le 0, & x_1 + x_2 \ge m. \end{cases}$$

• When $x_1 + x_2 < m$, we have $\frac{\partial^3 f}{\partial x_1^3} = aC(\theta_1^3 e^{\theta_1(x_1+x_2)} + \theta_2^3 e^{-\theta_2(x_1+x_2)}) > 0$. The derivative $\frac{\partial^2 f}{\partial x_1^2}$ is increasing and $\frac{\partial^2 f}{\partial x_1^2}\Big|_{x_1+x_2=m} = 0$, so $\frac{\partial^2 f}{\partial x_1^2} \le 0$. The derivative $\frac{\partial f}{\partial x_1}$ is decreasing and $a - \frac{\partial f}{\partial x_1} \le 0$. Similarly, $(1-a) - \frac{\partial f}{\partial x_2} \le a - \frac{\partial f}{\partial x_2} \le 0$.

• When $x_1 + x_2 \ge m$, we distinguish between $x_2 \le m$ and $x_2 > m$. If $x_2 \le m$, $\mathcal{L}(f)(x_1, x_2) = \mu_1 a + \mu_2 a - \beta f(x_1, x_2) \le \mu_1 a + \mu_2 a - \beta f(m - x_2, x_2) = \mathcal{L}(f)(m - x_2, x_2) = 0$. If $x_2 > m$, $\mathcal{L}(f)(x_1, x_2) = \mu_1 a + \mu_2 a - \beta f(x_1, x_2) \le \mu_1 a + \mu_2 a - \beta f(0, m) = \mathcal{L}(f)(0, m) = 0$.

□

We divide the domain $\mathcal{R}_+^2$ into three parts (see Figure 1).

• $A = \{(x_1, x_2) : x_1 \ge 0, x_2 \ge 0, x_1 + x_2 \le m\}$.
• $B = \{(x_1, x_2) : x_1 > 0, x_2 \in [0, m], x_1 + x_2 > m\}$.
• $C = \{(x_1, x_2) : x_1 \ge 0, x_2 > m\}$.

Denote by (L^*, C^*) a dividend and transferring strategy with initial surplus level (x_1, x_2). Assume (L^*, C^*) is given by the following series of decisions (See Figure 2-4).

1. If $(x_1, x_2) \in C$, Business Two transfers an amount $x_2 - m$ to Business One and we go to 2.
2. If $(x_1, x_2) \in B$, Business One pays directly an amount $x_1 + x_2 - m$ as dividend and we go to 3.
3. If $(x_1, x_2) \in A$, Business One pays the accumulated amount $\max_{s \le t}\{\bar{X}_1(s) + \bar{X}_2(s) - m\}$ up to time t until the surplus processes hit $(0, 0)$.

REMARK 2. If $(x_1, x_2) \in A$, line One only pays dividends when $\bar{X}_1 + \bar{X}_2$ hits $\{x_1 + x_2 = m\}$ and the process $(\bar{X}_1, \bar{X}_2)$ remains in area A.

THEOREM 1. *The strategy (L^*, C^*) gives the optimal strategy and the function f gives the optimal value function.*

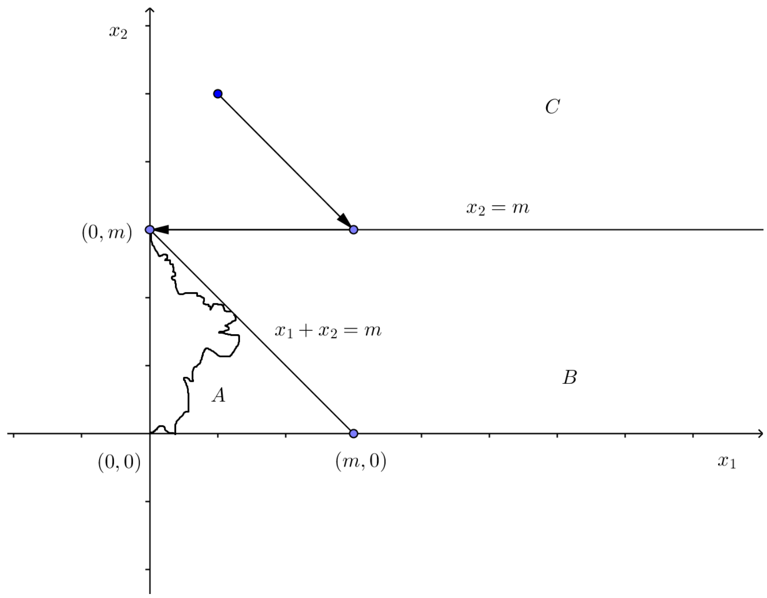

FIGURE 2. Optimal Strategy when initial value in area C

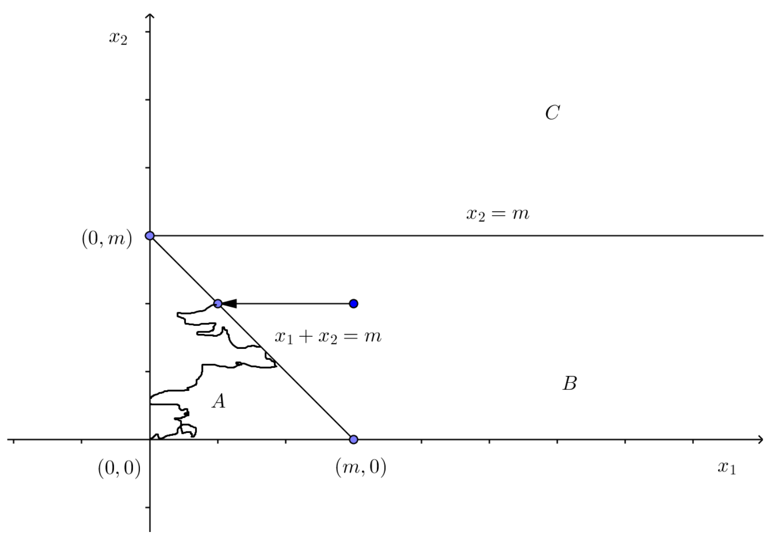

FIGURE 3. Optimal Strategy when initial value in area B

Proof. We first show that for any $(x_1, x_2) \in \mathcal{R}_+^2$ and any admissible control (L, C),

$$f(x_1, x_2) \geq V_{L,C}(x_1, x_2).$$

Let $\bar{X}_1, \bar{X}_2$ be the controlled surplus process with L and C. Then, $\bar{X}_1(s) \neq \bar{X}_1(s-), \bar{X}_2(s) \neq \bar{X}_2(s-)$ only when money are transferred between the two businesses, Hence

$$\bar{X}_1(s) - \bar{X}_1(s-) = \bar{X}_2(s-) - \bar{X}_2(s),$$

and

$$f(\bar{X}_1(s), \bar{X}_2(s)) = f(\bar{X}_1(s-), \bar{X}_2(s-)).$$

By Proposition 1 with stopping time $\tau \wedge t$ and Proposition 2,

$$\begin{aligned} &e^{-\beta(\tau\wedge t)} f(\bar{X}_1(\tau \wedge t), \bar{X}_2(\tau \wedge t)) - f(x_1, x_2) \\ &\leq -\int_0^{\tau\wedge t} a e^{-\beta s} dL_1(s) - \int_0^{\tau\wedge t} (1-a) e^{-\beta s} dL_2(s) + M(\tau \wedge t) \end{aligned} \tag{13}$$

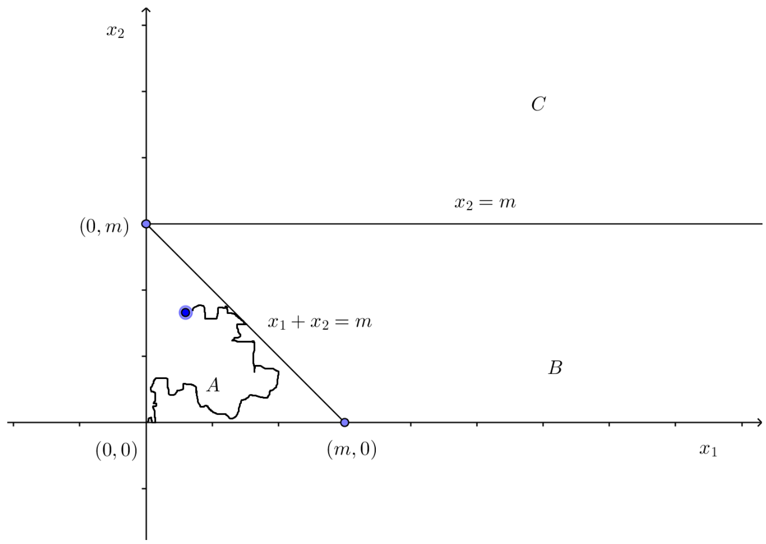

Figure 4. Optimal Strategy when initial value in area A

where M is a zero-expectation martingale. Hence,

$$f(x_1,x_2) \geq E\left[\int_0^{\tau\wedge t} ae^{-\beta s}dL_1(s) + \int_0^{\tau\wedge t}(1-a)e^{-\beta s}dL_2(s)\right]. \tag{14}$$

By the Monotone Convergence Theorem,

$$f(x_1,x_2) \geq E\left[\int_0^{\tau} ae^{-\beta s}dL_1(s) + \int_0^{\tau}(1-a)e^{-\beta s}dL_2(s)\right] = V_{L,C}(x_1,x_2). \tag{15}$$

Let $\bar{X}_1^*, \bar{X}_2^*$ be the controlled surplus process with L^* and C^*. For $(x_1,x_2)\in A$, by Propositions 1 and 2, $\mathcal{L}(f)=0$, and $L_2^*=0$, $L_1^*(s+)-L_1^*(s)=0$, and

$$\begin{aligned} &e^{-\beta(\tau\wedge t)}f(\bar{X}_1^*(\tau\wedge t),\bar{X}_2^*(\tau\wedge t)) - f(x_1,x_2)\\ =\ &M(\tau\wedge t) - \textstyle\int_0^{\tau\wedge t} ae^{-\beta s}dL_1^*(s))\\ &+\textstyle\int_0^{\tau\wedge t}(a-f_{x_1}(\bar{X}_1^*(s-),\bar{X}_2^*(s-)))e^{-\beta s}dL_1^{*C}(s), \end{aligned} \tag{16}$$

where

$$\begin{aligned} &\textstyle\int_0^{\tau\wedge t}(a-f_{x_1}(\bar{X}_1^*(s-),\bar{X}_2^*(s-)))e^{-\beta s}dL_1^{*C}(s)\\ =\ &\textstyle\int_0^{\tau\wedge t}(a-f_{x_1}(\bar{X}_1^*(s),\bar{X}_2^*(s)))e^{-\beta s}dL_1^{*C}(s)\\ =\ &\textstyle\int_0^{\tau\wedge t}(a-f_{x_1}(\bar{X}_1^*(s),\bar{X}_2^*(s)))e^{-\beta s}1_{\{\bar{X}_1^*(s)+\bar{X}_2^*(s)=m\}}dL_1^{*C}(s)\\ =\ &0. \end{aligned}$$

Since $(\bar{X}_1^*(\tau),\bar{X}_2^*(\tau))=(0,0)$, we have

$$\begin{aligned} &f(x_1,x_2)\\ =\ &E[e^{-\beta(\tau\wedge t)}f(\bar{X}_1^*(\tau\wedge t),\bar{X}_2^*(\tau\wedge t))] + E[\textstyle\int_0^{\tau\wedge t} ae^{-\beta s}dL_1^*(s))]\\ =\ &E[e^{-\beta t}f(\bar{X}_1^*(t),\bar{X}_2^*(t))1_{\{t<\tau\}}] + E[\textstyle\int_0^{\tau\wedge t} ae^{-\beta s}dL_1^*(s))]. \end{aligned} \tag{17}$$

Letting t tend to $+\infty$, and using the Bounded Convergence Theorem and the Monotone Convergence Theorem, we have

$$f(x_1,x_2) = V_{L^*,C^*}(x_1,x_2).$$

For $(x_1,x_2)\in B$ and $(x_1,x_2)\in C$, the equality holds trivially. □

REMARK 3. Comparing with the optimal strategy in one dimension, we claim without proof that the optimal strategy for the collaborating business lines is to let line One always pay dividends when the sum of the two surpluses reaches or excesses the threshold m. In the proposed optimal strategy (L^*, C^*), when the sum of the two surpluses excesses the threshold m (Figure 2, 3), the surplus is transferred to line One (Figure 2) and dividends are always paid by line One (Figure 2, 3). When the sum of the two surpluses reaches m, the excess is always paid by line One (Figure 4). Since there is no transactions costs, the transfer of surplus from line Two to Line One can happen freely at any time. Another optimal strategy is to keep the surplus in line Two at zero at any time point by continuously transferring money to and from line One. This solution corresponds to merging the two lines into line One and solving the problem as a one-dimensional problem.

4. Transaction Costs In this section, proportional transaction costs are considered when cash flows between the two businesses, i.e., the controlled surplus processes are given by

$$\begin{aligned}\bar{X}_1(t) &= X_1(t) + C_{21}(t) - kC_{12}(t) - L_1(t),\\ \bar{X}_2(t) &= X_2(t) + C_{12}(t) - kC_{21}(t) - L_2(t),\end{aligned}$$

where $k > 1$. The intuition is that whenever one business line is supported by the other business line with 1 unit, it costs the supporting line $k > 1$ units. Upon the money transfer, $k-1$ units are lost by friction.

REMARK 4. As V is the optimal value function, we conclude some facts without proof.

$$\begin{aligned}&V(x_1+\Delta x, x_2) - V(x_1,x_2) \ge a\Delta x, && \Delta x \ge 0,\\ &V(x_1, x_2+\Delta x) - V(x_1,x_2) \ge (1-a)\Delta x, && \Delta x \ge 0,\\ &V(x_1+\Delta x, x_2 - k\Delta x) - V(x_1,x_2) \le 0, && \Delta x \ge 0,\\ &V(x_1-k\Delta x, x_2+\Delta x) - V(x_1,x_2) \le 0, && \Delta x \ge 0.\end{aligned}$$

The corresponding HJB equation becomes

$$\begin{aligned}0 &= \max\Big(\mathcal{L}(V)(x_1,x_2), a - \tfrac{\partial V}{\partial x_1}(x_1,x_2), (1-a) - \tfrac{\partial V}{\partial x_2}(x_1,x_2),\\ &\quad \tfrac{\partial V}{\partial x_1}(x_1,x_2) - k\tfrac{\partial V}{\partial x_2}(x_1,x_2), \tfrac{\partial V}{\partial x_2}(x_1,x_2) - k\tfrac{\partial V}{\partial x_1}(x_1,x_2)\Big),\\ 0 &= V(0,0).\end{aligned} \tag{18}$$

If we take $k = 1$, the model reduces to the special case without transaction costs discussed in Section 2.

Using a similar argument as in Proposition 1 and the Mean Value Theorem in two variables, we obtain the following result.

PROPOSITION 3. *Let $\bar{X}_1, \bar{X}_2$ be the controlled surplus processes with control (L, C) and initial values x_1, x_2. For any twice continuously differentiable function ψ on $\mathcal{R}_+^2$ and a finite stopping time $\tau^* \le \tau$, if one of the following two conditions holds, (1) $\bar{X}_1, \bar{X}_2$ are bounded; (2) ψ has bounded first derivatives, we have*

$$\begin{aligned}&e^{-\beta\tau^*}\psi(\bar{X}_1(\tau^*), \bar{X}_2(\tau^*)) - \psi(x_1,x_2)\\ =& \int_0^{\tau^*} e^{-\beta s}\mathcal{L}(\psi)(\bar{X}_1(s-), \bar{X}_2(s-))ds + M(\tau^*)\\ &+ \int_0^{\tau^*} e^{-\beta s}[\psi_{x_1}(\bar{X}_1(s-), \bar{X}_2(s-)) - k\psi_{x_2}(\bar{X}_1(s-), \bar{X}_2(s-))]dC_{21}^C(s)\\ &+ \int_0^{\tau^*} e^{-\beta s}[\psi_{x_2}(\bar{X}_1(s-), \bar{X}_2(s-)) - k\psi_{x_1}(\bar{X}_1(s-), \bar{X}_2(s-))]dC_{12}^C(s)\\ &+ \sum_{\bar{X}_1(s-)\ne\bar{X}_1(s), \bar{X}_2(s-)\ne\bar{X}_2(s), s\le\tau^*} e^{-\beta s}[\psi(\bar{X}_1(s), \bar{X}_2(s)) - \psi(\bar{X}_1(s-), \bar{X}_2(s-))]\\ &- \int_0^{\tau^*} ae^{-\beta s}dL_1(s) - \int_0^{\tau^*}(1-a)e^{-\beta s}dL_2(s)\\ &+ \int_0^{\tau^*}(a - \psi_{x_1}(\bar{X}_1(s-), \bar{X}_2(s-)))e^{-\beta s}dL_1^C(s)\\ &+ \sum_{L_1(s+)\ne L_1(s), s<\tau^*}\int_0^{L_1(s+)-L_1(s)}(a - \psi_{x_1}(\bar{X}_1(s)-\alpha, \bar{X}_2(s)))e^{-\beta s}d\alpha\\ &+ \int_0^{\tau^*}((1-a) - \psi_{x_2}(\bar{X}_1(s-), \bar{X}_2(s-)))e^{-\beta s}dL_2^C(s)\\ &+ \sum_{L_2(s+)\ne L_2(s), s<\tau^*}\int_0^{L_2(s+)-L_2(s)}((1-a) - \psi_{x_2}(\bar{X}_1(s+), \bar{X}_2(s)-\alpha))e^{-\beta s}d\alpha,\end{aligned} \tag{19}$$

where M is a martingale and L_i^C, C_{ij}^C are the continuous parts of L_i, C_{ij}.
If $\bar{X}_1(s) > \bar{X}_1(s-)$,

$$\psi(\bar{X}_1(s), \bar{X}_2(s)) - \psi(\bar{X}_1(s-), \bar{X}_2(s-)) = (\psi_{x_1}(\xi) - k\psi_{x_2}(\xi))(\bar{X}_1(s) - \bar{X}_1(s-)),$$

where $\xi = (1-\iota)(\bar{X}_1(s-), \bar{X}_2(s-)) + \iota(\bar{X}_1(s), \bar{X}_2(s))$ and $\iota \in [0,1]$.
If $\bar{X}_1(s) < \bar{X}_1(s-)$,

$$\psi(\bar{X}_1(s), \bar{X}_2(s)) - \psi(\bar{X}_1(s-), \bar{X}_2(s-)) = (\psi_{x_2}(\zeta) - k\psi_{x_1}(\zeta))(\bar{X}_2(s) - \bar{X}_2(s-)),$$

where $\zeta = (1-\vartheta)(\bar{X}_1(s-), \bar{X}_2(s-)) + \vartheta(\bar{X}_1(s), \bar{X}_2(s))$ and $\vartheta \in [0,1]$.

We next describe some properties of the optimal value function.

Proposition 4. *The optimal value function V is locally bounded and continuous in $\mathcal{R}_+^2$.*

Proof. For $k=1$, we have proved that V is locally bounded in $\mathcal{R}_+^2$. Hence when $k \geq 1$, V is locally bounded trivially. Apparently, V is increasing both in x_1 and x_2.

For $h_1, h_2 \in [0, \min(x_1/2, x_2/2))$, consider the following strategy (L, C) and let $(\bar{X}_1, \bar{X}_2)$ be the controlled surplus process associated with (L, C). With initial surplus (x_1, x_2), pay no dividends until $\bar{\tau} := \inf\{t \geq 0 : \bar{X}_1(t) \geq x_1 + h_1 \text{ and } \bar{X}_2(t) \geq x_2 + h_2\}$. At $\bar{\tau}$, pay $(\bar{X}_1(t) - (x_1 + h_1))^+$ to Business One and $(\bar{X}_2(t) - (x_2 + h_2))^+$ to Business Two. Then

$$V(x_1, x_2) \geq E(1_{\{\bar{\tau} < \tau\}} e^{-\beta \bar{\tau} \wedge \tau}) V(x_1 + h_1, x_2 + h_2).$$

Hence,

$$0 \leq V(x_1 + h_1, x_2 + h_2) - V(x_1, x_2) \leq \left(\frac{1}{E(1_{\{\bar{\tau} < \tau\}} e^{-\beta \bar{\tau} \wedge \tau})} - 1 \right) V(x_1, x_2).$$

As $V(x_1, x_2)$ is locally bounded, when $h_1, h_2 \to 0$, $\bar{\tau} \wedge \tau \to 0$ and $1_{\{\bar{\tau} < \tau\}} \to 1$,

$$V(x_1 + h_1, x_2 + h_2) - V(x_1, x_2) \to 0.$$

For $h_1, h_2 \in (-\min(x_1/2, x_2/2), 0]$, consider the following strategy (L', C') and let $(\bar{X}_1', \bar{X}_2')$ be the controlled surplus process associated with (L', C'). With initial surplus $(x_1 + h_1, x_2 + h_2)$, pay no dividends until $\hat{\tau} := \inf\{t \geq 0 : \bar{X}_1'(t) \geq x_1 \text{ and } \bar{X}_2'(t) \geq x_2\}$. At $\hat{\tau}$, pay $(\bar{X}_1'(t) - x_1)^+$ to Business One and $(\bar{X}_2'(t) - x_2)^+$ to Business Two. Then

$$V(x_1 + h_1, x_2 + h_2) \geq E(1_{\{\hat{\tau} < \tau\}} e^{-\beta \hat{\tau} \wedge \tau}) V(x_1, x_2).$$

Hence,

$$\begin{aligned} 0 \leq V(x_1, x_2) - V(x_1 + h_1, x_2 + h_2) &\leq \left(\frac{1}{E(1_{\{\hat{\tau} < \tau\}} e^{-\beta \hat{\tau} \wedge \tau})} - 1 \right) V(x_1 + h_1, x_2 + h_2) \\ &\leq \left(\frac{1}{E(1_{\{\hat{\tau} < \tau\}} e^{-\beta \hat{\tau} \wedge \tau})} - 1 \right) V(x_1, x_2). \end{aligned}$$

As $V(x_1, x_2)$ is locally bounded, when $h_1, h_2 \to 0$, $\hat{\tau} \wedge \tau \to 0$ and $1_{\{\hat{\tau} < \tau\}} \to 1$,

$$V(x_1, x_2) - V(x_1 + h_1, x_2 + h_2) \to 0.$$

For $h_1 > 0, h_2 < 0$,

$$V(x_1, x_2 + h_2) \leq V(x_1 + h_1, x_2 + h_2) \leq V(x_1 + h_1, x_2).$$

When $h_1, h_2 \to 0$,

$$V(x_1 + h_1, x_2 + h_2) \to V(x_1, x_2).$$

A similar argument can be applied to the case $h_1 < 0, h_2 > 0$.

To conclude, V is continuous in $\mathcal{R}_+^2$. □

Definition 1. A continuous function $\bar{u}(\underline{u}):\mathcal{R}_+^2\to\mathcal{R}$ is a viscosity supersolution (subsolution) of (18) at $(x_1,x_2)\in\mathcal{R}_+^2$ if any twice continuously differentiable function $\varphi:\mathcal{R}_+^2\to\mathcal{R}$ with $\varphi(x_1,x_2)=\bar{u}(x_1,x_2)$ $(\underline{u}(x_1,x_2))$ such that $\bar{u}-\varphi$ $(\underline{u}-\varphi)$ reaches the minimum (maximum) at (x_1,x_2) satisfies

$$\max\Big(\mathcal{L}(\varphi)(x_1,x_2), a-\frac{\partial\varphi}{\partial x_1}(x_1,x_2),(1-a)-\frac{\partial\varphi}{\partial x_2}(x_1,x_2),\\ \frac{\partial\varphi}{\partial x_1}(x_1,x_2)-k\frac{\partial\varphi}{\partial x_2}(x_1,x_2),\frac{\partial\varphi}{\partial x_2}(x_1,x_2)-k\frac{\partial\varphi}{\partial x_1}(x_1,x_2)\Big)\le 0(\ge 0).$$

Proposition 5. *The optimal value function V is a viscosity supersolution of (18).*

Proof. Let φ be a test function. Assume (x_1,x_2) is the minimum point of $V-\varphi$ satisfying $V(x_1,x_2)=\varphi(x_1,x_2)$. For any fixed $l_1,l_2,\Delta x_1,\Delta x_2>0$ and $B_1>x_1,B_2>x_2$, consider the following strategy (L,C) and let $(\bar{X}_1,\bar{X}_2)$ be the controlled surplus process associated with (L,C). With initial surplus level (x_1,x_2), line One keeps paying dividends at rate l_1 and transferring money to line Two at rate Δx_1 until time $\bar{\tau}$, where $\bar{\tau}:=\inf\{t\ge 0:\bar{X}_1(t)=0$ or $\bar{X}_2(t)=0$ or $\bar{X}_1(t)\ge B_1$ or $\bar{X}_1(t)\ge B_2\}$. Similarly, line Two keeps paying dividends at rate l_2 and transferring money to line One at rate Δx_2 until time $\bar{\tau}$. Then,

$$\begin{aligned}\varphi(x_1,x_2)=V(x_1,x_2)\ge\ & E_{x_1,x_2}\left(a\int_0^{\bar{\tau}\wedge t}e^{-\beta s}l_1ds+(1-a)\int_0^{\bar{\tau}\wedge t}e^{-\beta s}l_2ds\right)\\ &+E_{x_1,x_2}\left(e^{-\beta\bar{\tau}\wedge t}V(\bar{X}_1(\bar{\tau}\wedge t),\bar{X}_2(\bar{\tau}\wedge t))\right)\\ \ge\ & E_{x_1,x_2}\left(a\int_0^{\bar{\tau}\wedge t}e^{-\beta s}l_1ds+(1-a)\int_0^{\bar{\tau}\wedge t}e^{-\beta s}l_2ds\right)\\ &+E_{x_1,x_2}\left(e^{-\beta\bar{\tau}\wedge t}\varphi(\bar{X}_1(\bar{\tau}\wedge t),\bar{X}_2(\bar{\tau}\wedge t))\right).\end{aligned}\tag{20}$$

By Itô's formula and the optional sampling theorem and temporarily denote by $\varphi(s)=\varphi(\bar{X}_1(s),\bar{X}_2(s))$, we have

$$\begin{aligned}&E_{x_1,x_2}\left(e^{-\beta\bar{\tau}\wedge t}\varphi(\bar{X}_1(\bar{\tau}\wedge t),\bar{X}_2(\bar{\tau}\wedge t))\right)-\varphi(x_1,x_2)\\ =\ &E_{x_1,x_2}\left(\int_0^{\bar{\tau}\wedge t}e^{-\beta s}\mathcal{L}(\varphi)(s)ds\right)\\ &+E_{x_1,x_2}\left(\int_0^{\bar{\tau}\wedge t}e^{-\beta s}[\varphi_{x_1}(s)(-l_1+\Delta x_2-k\Delta x_1)+\varphi_{x_2}(s)(-l_2+\Delta x_1-k\Delta x_2)]ds\right).\end{aligned}\tag{21}$$

By substituting (21) into (20), cancelling out the term $\varphi(x_1,x_2)$ and collecting the other terms, we have

$$\begin{aligned}0\ge&\lim_{t\downarrow 0}\frac{E_{x_1,x_2}(\int_0^{\bar{\tau}\wedge t}e^{-\beta s}\mathcal{L}(\varphi)(s)ds)}{t}\\ &+\lim_{t\downarrow 0}\frac{E_{x_1,x_2}(\int_0^{\bar{\tau}\wedge t}e^{-\beta s}[(a-\varphi_{x_1}(s))l_1+(\varphi_{x_1}(s)-k\varphi_{x_2}(s))\Delta x_2]ds)}{t}\\ &+\lim_{t\downarrow 0}\frac{E_{x_1,x_2}(\int_0^{\bar{\tau}\wedge t}e^{-\beta s}[(1-a-\varphi_{x_2}(s))l_2+(\varphi_{x_2}(s)-k\varphi_{x_1}(s))\Delta x_1]ds)}{t}\end{aligned}$$

or

$$\begin{aligned}0\ge&\ \mathcal{L}(\varphi)(x_1,x_2)+l_1(a-\varphi_{x_1})(x_1,x_2)+l_2(1-a-\varphi_{x_2}(x_1,x_2))\\ &+\Delta x_2(\varphi_{x_1}-k\varphi_{x_2})(x_1,x_2)+\Delta x_1(\varphi_{x_2}-k\varphi_{x_1})(x_1,x_2)\end{aligned}$$

By letting $l_1=l_2=\Delta x_1=\Delta x_2=0$, we have $0\ge\mathcal{L}(\varphi)(x_1,x_2)$. Similarly, by letting $l_1\to\infty,l_2=\Delta x_1=\Delta x_2=0$ and $\Delta x_1\to\infty,l_2=l_1=\Delta x_2=0$ and $l_2\to\infty,l_1=\Delta x_1=\Delta x_2=0$ and $\Delta x_2\to\infty,l_2=l_1=\Delta x_1=0$, the result follows. □

To prove V is a viscosity subsolution, we first prove a lemma that contains the main ingredients in the proof of Proposition 6. The proof of Lemma 1 is deferred to Appendix.

Lemma 1. *If the optimal value function V is not a viscosity subsolution of (18) at $(x_1,x_2)\in\mathcal{R}_+^2$, then there exist $\epsilon>0$, $h\in(0,\min(x_1/2,x_2/2))$ and twice continuously differentiable functions $\psi,\phi:\mathcal{R}_+^2\to\mathcal{R}$ such that*

1. $\psi_{x_1} \geq a, \psi_{x_2} \geq 1-a$ *for* $(x,y) \in [0, x_1+h] \times [0, x_2+h]$,
2. $\psi_{x_1} - k\psi_{x_2} \leq 0, \psi_{x_2} - k\psi_{x_1} \leq 0, \mathcal{L}(\psi) \leq -\epsilon\beta$ *for* $(x,y) \in [x_1-h, x_1+h] \times [x_2-h, x_2+h]$,
3. $V \leq \psi - \epsilon$ *for* $(x,y) \in [0, x_1+x_2+2h] \times [0, x_1+x_2+2h] \setminus (x_1-h, x_1+h) \times (x_2-h, x_2+h)$,
4. $V(x_1,x_2) = \psi(x_1,x_2)$, *and*
5. $\phi_{x_1} - k\phi_{x_2} \leq 0$ *for* $(x,y) \in (x_1-h, x_1+x_2+2h] \times [0, x_2+h)$, $\phi_{x_2} - k\phi_{x_1} \leq 0$ *for* $(x,y) \in [0, x_1+h] \times (x_2-h, x_1+x_2+2h)$.
6. $\phi_{x_1} \geq a, \phi_{x_2} \geq 1-a, \mathcal{L}(\phi) \leq -\epsilon\beta$ *for* $(x,y) \in [x_1-h, x_1+h] \times [x_2-h, x_2+h]$,
7. $V \leq \phi - \epsilon$ *for* $(x,y) \in [0, x_1+x_2+2h] \times [0, x_1+x_2+2h] \setminus (x_1-h, x_1+h) \times (x_2-h, x_2+h)$,
8. $V(x_1,x_2) = \phi(x_1,x_2)$.

PROPOSITION 6. *The optimal value function V is a viscosity subsolution of (18).*

Proof. We prove by contradiction. Assume that V is not a viscosity subsolution at $(x_1,x_2) \in \mathcal{R}_+^2$. let $(\bar{X}_1, \bar{X}_2)$ be the controlled surplus process associated with any control (L,C). Define $\bar{\tau} := \inf\{t > 0 : (\bar{X}_1(t), \bar{X}_2(t)) \notin (x_1-h, x_1+h) \times (x_2-h, x_2+h)\}$. Since $(0,0) \notin (x_1-h, x_1+h) \times (x_2-h, x_2+h)$ and $\tau = \inf\{t \geq 0 : \bar{X}_1(t) = 0$ and $\bar{X}_2(t) = 0\}$, we have $\tau \geq \bar{\tau}$.

• If $(\bar{X}_1(\bar{\tau}), \bar{X}_2(\bar{\tau})) \notin (x_1-h, x_1+h) \times (x_2-h, x_2+h)$, then $(\bar{X}_1(\bar{\tau}), \bar{X}_2(\bar{\tau})) \in (x_1-h, x_1+x_2+2h] \times [0, x_2+h) \setminus (x_1-h, x_1+h) \times (x_2-h, x_2+h)$ if $\bar{X}_1(\bar{\tau}) - \bar{X}_1(\bar{\tau}-) > 0$ and $(\bar{X}_1(\bar{\tau}), \bar{X}_2(\bar{\tau})) \in [0, x_1+h) \times (x_2-h, x_1+x_2+2h] \setminus (x_1-h, x_1+h) \times (x_2-h, x_2+h)$ if $\bar{X}_2(\bar{\tau}) - \bar{X}_2(\bar{\tau}-) > 0$ and $(\bar{X}_1(t), \bar{X}_2(t)) \in (x_1-h, x_1+h) \times (x_2-h, x_2+h)$ for $t \in [0, \bar{\tau})$. By Proposition 3 and Lemma 1,

$$\begin{aligned} &e^{-\beta\bar{\tau}}\phi(\bar{X}_1(\bar{\tau}), \bar{X}_2(\bar{\tau}) - \phi(x_1,x_2) \\ &\leq \int_0^{\bar{\tau}} e^{-\beta s}\mathcal{L}(\phi)ds - a\int_0^{\bar{\tau}} e^{-\beta s}dL_1(s) - (1-a)\int_0^{\bar{\tau}} e^{-\beta s}dL_2(s) + M(\bar{\tau}) \\ &\leq \int_0^{\bar{\tau}} e^{-\beta s}(-\epsilon\beta)ds - a\int_0^{\bar{\tau}} e^{-\beta s}dL_1(s) - (1-a)\int_0^{\bar{\tau}} e^{-\beta s}dL_2(s) + M(\bar{\tau}), \end{aligned} \tag{22}$$

where M is a zero-expectation martingale.

• If $(\bar{X}_1(\bar{\tau}), \bar{X}_2(\bar{\tau})) \in (x_1-h, x_1+h) \times (x_2-h, x_2+h)$, then $(\bar{X}_1(\bar{\tau}+), \bar{X}_2(\bar{\tau}+)) \in [0, x_1+h] \times [0, x_2+h] \setminus (x_1-h, x_1+h) \times (x_2-h, x_2+h)$ and $(\bar{X}_1(t), \bar{X}_2(t)) \in (x_1-h, x_1+h) \times (x_2-h, x_2+h)$ for $t \in [0, \bar{\tau}]$. By Proposition 3 and Lemma 1,

$$\begin{aligned} &e^{-\beta\bar{\tau}}\psi(\bar{X}_1(\bar{\tau}), \bar{X}_2(\bar{\tau})) - \psi(x_1,x_2) \\ &\leq \int_0^{\bar{\tau}} e^{-\beta s}\mathcal{L}(\psi)ds - a\int_0^{\bar{\tau}} e^{-\beta s}dL_1(s) - (1-a)\int_0^{\bar{\tau}} e^{-\beta s}dL_2(s) + M(\bar{\tau}), \end{aligned}$$

and

$$\begin{aligned} &e^{-\beta\bar{\tau}}\psi(\bar{X}_1(\bar{\tau}+), \bar{X}_2(\bar{\tau}+)) - \psi(x_1,x_2) \\ &\leq \int_0^{\bar{\tau}} e^{-\beta s}\mathcal{L}(\psi)ds - a\int_0^{\bar{\tau}+} e^{-\beta s}dL_1(s) - (1-a)\int_0^{\bar{\tau}+} e^{-\beta s}dL_2(s) + M(\bar{\tau}) \\ &\leq \int_0^{\bar{\tau}} e^{-\beta s}(-\epsilon\beta)ds - a\int_0^{\bar{\tau}+} e^{-\beta s}dL_1(s) - (1-a)\int_0^{\bar{\tau}+} e^{-\beta s}dL_2(s) + M(\bar{\tau}). \end{aligned} \tag{23}$$

Further, $\phi(x_1,x_2) = \psi(x_1,x_2) = V(x_1,x_2)$.

$$\begin{aligned} &V(x_1,x_2) \\ &= \sup_{L,C} E_{x_1,x_2}\left(a\int_0^{\tau} e^{-\beta s}dL_1(s) + (1-a)\int_0^{\tau} e^{-\beta s}dL_2(s)\right) \\ &= \sup_{L,C} E_{x_1,x_2}\begin{pmatrix} 1_{(\bar{X}_1(\bar{\tau}),\bar{X}_2(\bar{\tau}))\notin(x_1-h,x_1+h)\times(x_2-h,x_2+h)} \\ \times(e^{-\beta\bar{\tau}}V(\bar{X}_1(\bar{\tau}), \bar{X}_2(\bar{\tau})) + a\int_0^{\bar{\tau}} e^{-\beta s}dL_1(s) + (1-a)\int_0^{\bar{\tau}} e^{-\beta s}dL_2(s)) \\ +1_{(\bar{X}_1(\bar{\tau}),\bar{X}_2(\bar{\tau}))\in(x_1-h,x_1+h)\times(x_2-h,x_2+h)} \\ \times(e^{-\beta\bar{\tau}}V(\bar{X}_1(\bar{\tau}+), \bar{X}_2(\bar{\tau}+)) + a\int_0^{\bar{\tau}+} e^{-\beta s}dL_1(s) + (1-a)\int_0^{\bar{\tau}+} e^{-\beta s}dL_2(s)) \end{pmatrix} \\ &\leq \sup_{L,C} E_{x_1,x_2}\begin{pmatrix} 1_{(\bar{X}_1(\bar{\tau}),\bar{X}_2(\bar{\tau}))\notin(x_1-h,x_1+h)\times(x_2-h,x_2+h)} \\ \times(e^{-\beta\bar{\tau}}\phi(\bar{X}_1(\bar{\tau}), \bar{X}_2(\bar{\tau})) - \epsilon) + a\int_0^{\bar{\tau}} e^{-\beta s}dL_1(s) + (1-a)\int_0^{\bar{\tau}} e^{-\beta s}dL_2(s)) \\ +1_{(\bar{X}_1(\bar{\tau}),\bar{X}_2(\bar{\tau}))\in(x_1-h,x_1+h)\times(x_2-h,x_2+h)} \\ \times(e^{-\beta\bar{\tau}}\psi(\bar{X}_1(\bar{\tau}+), \bar{X}_2(\bar{\tau}+)) - \epsilon) + a\int_0^{\bar{\tau}+} e^{-\beta s}dL_1(s) + (1-a)\int_0^{\bar{\tau}+} e^{-\beta s}dL_2(s)) \end{pmatrix} \\ &\leq V(x_1,x_2) + E_{x_1,x_2}(-\epsilon e^{-\beta\bar{\tau}} - \epsilon\beta\int_0^{\bar{\tau}} e^{-\beta s}ds) \\ &= V(x_1,x_2) - \epsilon. \end{aligned}$$

The second equality above is due to dynamic programming, the first inequality is due to conditions 3 and 7 in Lemma 1. and the second inequality is due to inequalities (22) and (23). We come to a contradiction. □

We summarize the results in Propositions 4, 5 and 6 into the following theorem.

THEOREM 2. *The optimal value function V is a continuous viscosity solution of (18).*

To understand uniqueness of the viscosity solution, we adopt the technique used in Song and Zhu (2014) by constructing an auxiliary viscosity super- and subsolution to a variational inequality. The following lemma gives the construction method, where its proof is similar to that in Song and Zhu (2014).

LEMMA 2. *Let*

$$\bar{u}'(x_1,x_2)=e^{-\lambda(x_1+x_2)}\bar{u}(x_1,x_2),\ \underline{u}'(x_1,x_2)=e^{-\lambda(x_1+x_2)}\underline{u}(x_1,x_2), \tag{24}$$

for any $(x_1,x_2)\in\mathcal{R}_+^2$, where $\lambda>0$. Then $\bar{u}$ ($\underline{u}$) is a viscosity supersolution (subsolution) of HJB equation (18) if and only if $\bar{u}'$ ($\underline{u}'$) is a viscosity supersolution (subsolution) of

$$0=\max\Big\{\ \mathcal{H}_\lambda(\mathbf{x},V,D_V,D_V^2), a-e^{\lambda(x_1+x_2)}(\lambda V+\tfrac{\partial V}{\partial x_1}), 1-a-e^{\lambda(x_1+x_2)}(\lambda V+\tfrac{\partial V}{\partial x_2}), \\ \lambda(1-k)V+\tfrac{\partial V}{\partial x_1}-k\tfrac{\partial V}{\partial x_2}, \lambda(1-k)V+\tfrac{\partial V}{\partial x_2}-k\tfrac{\partial V}{\partial x_1}\ \Big\} \tag{25}$$

where $\mathbf{x}=(x_1,x_2)^T$ and

$$\begin{aligned}\mathcal{H}_\lambda(\mathbf{x},p,q,A)=&\ \tfrac{1}{2}tr((\sigma_1^2,\sigma_2^2)A)+(\mu_1+\lambda\sigma_1^2,\mu_2+\lambda\sigma_2^2)\cdot q\\ &+(\mu_1\lambda+\mu_2\lambda+\tfrac{1}{2}\lambda^2\sigma_1^2+\tfrac{1}{2}\lambda^2\sigma_2^2-\beta)p,\end{aligned} \tag{26}$$

for $(\mathbf{x},p,q,A)\in\mathcal{R}_+^2\times\mathcal{R}\times\mathcal{R}^2\times\mathcal{S}_2$, where $\mathcal{S}_n$ denotes the set of $n\times n$ symmetric matrices with real entries.

PROPOSITION 7. *Let $\bar{u}$, $\underline{u}$ be respectively viscosity super- and subsolution to the HJB equation (18) and satisfy*

$$|\bar{u}(x_1,x_2)|+|\underline{u}(x_1,x_2)|\le\kappa(1+|\mathbf{x}|^p), \tag{27}$$

for any $\mathbf{x}=(x_1,x_2)^T\in\mathcal{R}_+^2$, where κ and p be positive constants. If subsolution $\underline{u}$ is less than or equal to supersolution $\bar{u}$ at the boundary of $\mathcal{R}_+^2$, then we have

$$\bar{u}(x_1,x_2)\ge\underline{u}(x_1,x_2)$$

for any $\mathbf{x}\in\mathcal{R}_+^2$.

We call a function $u(x_1,x_2),\mathbf{x}=(x_1,x_2)^T\in\mathcal{R}_+^2$ satisfies the natural growth condition if

$$|u(x_1,x_2)|\le\kappa(1+|\mathbf{x}|^p), \tag{28}$$

where κ and p be positive constants. To prove Proposition 7, one need to make use of the concept superjets and subjets and verify the comparison principle, we refer the readers to Grandall, Ishii and Lions(1992) and Pham(2000) for technical details.

Proof. Suppose by contrary that

$$L:=\sup_{\mathbf{x}\in\mathcal{R}_+^2}[\underline{u}(x_1,x_2)-\bar{u}(x_1,x_2)]>0.$$

Define $\bar{u}'$, $\underline{u}'$ as in (24), where λ is a positive constant to be determined later. According to (28), we have

$$\lim_{x_1,x_2\to+\infty}|\bar{u}'(x_1,x_2)|+|\underline{u}'(x_1,x_2)|=0$$

Due to continuity of $\bar{u}'$ and $\underline{u}'$, there exists a $b>0$, such that

$$\tilde{L} := \sup_{\mathbf{x}\in\mathcal{R}_+^2}[\underline{u}'(x_1,x_2)-\bar{u}'(x_1,x_2)] = \max_{\mathbf{x}\in[0,b]^2}[\underline{u}'(x_1,x_2)-\bar{u}'(x_1,x_2)]>0.$$

Fix a $\varepsilon\in(0,1)$. Set

$$\omega_\varepsilon(\mathbf{x},\mathbf{y}) = \underline{u}'(\mathbf{x})-\bar{u}'(\mathbf{y})-\frac{1}{\varepsilon}|\mathbf{x}-\mathbf{y}|^2,$$

where $\mathbf{x},\mathbf{y}\in[0,b]^2$. Assume that ω_ε achieves its maximum at $(\mathbf{x}_\varepsilon,\mathbf{y}_\varepsilon)$. As $\varepsilon\to 0$, there exists a $\hat{\mathbf{x}}\in[0,b]^2$ such that $\mathbf{x}_\varepsilon,\mathbf{y}_\varepsilon\to\hat{\mathbf{x}}$. For any $\mathbf{x}\in[\mathbf{0},\mathbf{b}]$,

$$\underline{u}'(\mathbf{x}_\varepsilon)-\bar{u}'(\mathbf{y}_\varepsilon)\geq\omega_\varepsilon(\mathbf{x}_\varepsilon,\mathbf{y}_\varepsilon)\geq\omega_\varepsilon(\mathbf{x},\mathbf{x}),$$

hence $\underline{u}'(\hat{\mathbf{x}})-\bar{u}'(\hat{\mathbf{x}})=\tilde{L}$. Note that $\tilde{L}>0$ and $\underline{u}'\leq\bar{u}'$ at the boundary of $\mathcal{R}_+^2$, we have $\hat{x}$ is in the interior of $\mathcal{R}_+^2$ and therefore $(\mathbf{x}_\varepsilon,\mathbf{y}_\varepsilon)$ is in the interior of $\mathcal{R}_+^2$ if ε is sufficiently small. Also, since

$$\omega_\varepsilon(\mathbf{x}_\varepsilon,\mathbf{y}_\varepsilon)\geq\omega_\varepsilon(\hat{\mathbf{x}},\hat{\mathbf{x}}),$$

we have

$$\underline{u}'(\mathbf{x}_\varepsilon)-\bar{u}'(\mathbf{y}_\varepsilon)-\underline{u}'(\hat{\mathbf{x}})+\bar{u}'(\hat{\mathbf{x}})\geq\frac{1}{\varepsilon}|\mathbf{x}_\varepsilon-\mathbf{y}_\varepsilon|^2.$$

It follows that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}|\mathbf{x}_\varepsilon-\mathbf{y}_\varepsilon|^2=0,$$

and therefore as $\varepsilon\to 0$

$$\mathbf{y}_{i\varepsilon}=\mathbf{x}_{i\varepsilon}+o(\sqrt{\varepsilon}), i=1,2. \tag{29}$$

The function $\mathbf{x}\mapsto\underline{u}'(\mathbf{x})-\phi_1(\mathbf{x})$, $\mathbf{x}\in[0,b]^2$ achieves its maximum at $\mathbf{x}_\varepsilon$, where $\phi_1(\mathbf{x})=\bar{u}'(\mathbf{y}_\epsilon)+\frac{1}{\varepsilon}|\mathbf{x}-\mathbf{y}_\varepsilon|^2$. As $\underline{u}'$ is a viscosity subsolution to (25) and by Ishii's Lemma, we know that for some $M\in\mathcal{S}_2$, $(D_{\phi_1}(\mathbf{x}_\varepsilon),M)\in\bar{\mathcal{P}}^{2,+}\underline{u}'(\mathbf{x}_\varepsilon)$(the second-order "superjet" of $\underline{u}'$ at $\mathbf{x}_\varepsilon$), where

$$D_{\phi_1}(\mathbf{x})=-\frac{2}{\varepsilon}(\mathbf{y}_\varepsilon-\mathbf{x}),$$

and

$$\begin{aligned}0\leq\max\{\ &\mathcal{H}_\lambda(\mathbf{x}_\varepsilon,\underline{u}'(\mathbf{x}_\varepsilon),D_{\phi_1}(\mathbf{x}_\varepsilon),M), a-e^{\lambda(x_{1\varepsilon}+x_{2\varepsilon})}(\lambda\underline{u}'(\mathbf{x}_\varepsilon)-\tfrac{2}{\varepsilon}(y_{1\varepsilon}-x_{1\varepsilon}),\\ &1-a-e^{\lambda(x_{1\varepsilon}+x_{2\varepsilon})}(\lambda\underline{u}'(\mathbf{x}_\varepsilon)-\tfrac{2}{\varepsilon}(y_{2\varepsilon}-x_{2\varepsilon}),\lambda(1-k)\underline{u}'(\mathbf{x}_\varepsilon)-\tfrac{2}{\varepsilon}((y_{1\varepsilon}-ky_{2\varepsilon})-(x_{1\varepsilon}-kx_{2\varepsilon})),\\ &\lambda(1-k)\underline{u}'(\mathbf{x}_\varepsilon)-\tfrac{2}{\varepsilon}((y_{2\varepsilon}-ky_{1\varepsilon})-(x_{2\varepsilon}-kx_{1\varepsilon}))\ \}.\end{aligned}$$

Therefore, either

$$\mathcal{H}_\lambda(\mathbf{x}_\varepsilon,\underline{u}'(\mathbf{x}_\varepsilon),D_{\phi_1}(\mathbf{x}_\varepsilon),M)\geq 0, \tag{30}$$

or

$$\begin{aligned}&\max\{\ a-e^{\lambda(x_{1\varepsilon}+x_{2\varepsilon})}(\lambda\underline{u}'(\mathbf{x}_\varepsilon)-\tfrac{2}{\varepsilon}(y_{1\varepsilon}-x_{1\varepsilon}),1-a-e^{\lambda(x_{1\varepsilon}+x_{2\varepsilon})}(\lambda\underline{u}'(\mathbf{x}_\varepsilon)-\tfrac{2}{\varepsilon}(y_{2\varepsilon}-x_{2\varepsilon}),\\ &\lambda(1-k)\underline{u}'(\mathbf{x}_\varepsilon)-\tfrac{2}{\varepsilon}((y_{1\varepsilon}-ky_{2\varepsilon})-(x_{1\varepsilon}-kx_{2\varepsilon})),\lambda(1-k)\underline{u}'(\mathbf{x}_\varepsilon)-\tfrac{2}{\varepsilon}((y_{2\varepsilon}-ky_{1\varepsilon})-(x_{2\varepsilon}-kx_{1\varepsilon}))\ \}\geq 0,\end{aligned} \tag{31}$$

holds.

On the other hand, the function $\mathbf{y}\mapsto\bar{u}'(\mathbf{y})-\phi_2(\mathbf{y})$, $\mathbf{y}\in[0,b]^2$ achieves its minimum at $\mathbf{y}_\varepsilon$, where $\phi_2(\mathbf{y})=\underline{u}'(\mathbf{x}_\epsilon)-\frac{1}{\varepsilon}|\mathbf{y}-\mathbf{x}_\varepsilon|^2$. As $\bar{u}'$ is a viscosity supersolution to (25) and by Ishii's Lemma, we know that for some $N\in\mathcal{S}_2$, such that $(D_{\phi_2}(\mathbf{y}_\varepsilon),N)\in\bar{\mathcal{P}}^{2,-}\bar{u}'(\mathbf{y}_\varepsilon)$(the second-order "subjet" of $\bar{u}'$ at $\mathbf{x}_\varepsilon$), where

$$D_{\phi_2}(\mathbf{y})=-\frac{2}{\varepsilon}(\mathbf{y}-\mathbf{x}_\varepsilon),$$

and

$$0 \geq \max\{\ \mathcal{H}_\lambda(\mathbf{y}_\varepsilon, \bar{u}'(\mathbf{y}_\varepsilon), D_{\phi_2}(\mathbf{y}_\varepsilon), N), a - e^{\lambda(y_{1\varepsilon}+y_{2\varepsilon})}(\lambda\bar{u}'(\mathbf{y}_\varepsilon) - \tfrac{2}{\varepsilon}(y_{1\varepsilon} - x_{1\varepsilon}), \\ 1 - a - e^{\lambda(y_{1\varepsilon}+y_{2\varepsilon})}(\lambda\bar{u}'(\mathbf{x}_\varepsilon) - \tfrac{2}{\varepsilon}(y_{2\varepsilon} - x_{2\varepsilon}), \lambda(1-k)\bar{u}'(\mathbf{y}_\varepsilon) - \tfrac{2}{\varepsilon}((y_{1\varepsilon} - ky_{2\varepsilon}) - (x_{1\varepsilon} - kx_{2\varepsilon})), \\ \lambda(1-k)\bar{u}'(\mathbf{y}_\varepsilon) - \tfrac{2}{\varepsilon}((y_{2\varepsilon} - ky_{1\varepsilon}) - (x_{2\varepsilon} - kx_{1\varepsilon}))\ \}. \tag{32}$$

Case I: if (31) is true, combining (32), we have

$$0 \leq \max\{\ e^{\lambda(y_{1\varepsilon}+y_{2\varepsilon})}(\lambda\bar{u}'(\mathbf{y}_\varepsilon) - \tfrac{2}{\varepsilon}(y_{1\varepsilon} - x_{1\varepsilon})) - e^{\lambda(x_{1\varepsilon}+x_{2\varepsilon})}(\lambda\underline{u}'(\mathbf{x}_\varepsilon) - \tfrac{2}{\varepsilon}(y_{1\varepsilon} - x_{1\varepsilon})), \\ e^{\lambda(y_{1\varepsilon}+y_{2\varepsilon})}(\lambda\bar{u}'(\mathbf{y}_\varepsilon) - \tfrac{2}{\varepsilon}(y_{2\varepsilon} - x_{2\varepsilon})) - e^{\lambda(x_{1\varepsilon}+x_{2\varepsilon})}(\lambda\underline{u}'(\mathbf{x}_\varepsilon) - \tfrac{2}{\varepsilon}(y_{2\varepsilon} - x_{2\varepsilon})), \\ \lambda(1-k)(\underline{u}'(\mathbf{x}_\varepsilon) - \bar{u}'(\mathbf{y}_\varepsilon))\ \}.$$

As $k > 1$, we have either

$$\underline{u}'(\mathbf{x}_\varepsilon) - \bar{u}'(\mathbf{y}_\varepsilon) \leq 0, \tag{33}$$

or

$$\max\{\ e^{\lambda(y_{1\varepsilon}+y_{2\varepsilon})}(\lambda\bar{u}'(\mathbf{y}_\varepsilon) - \tfrac{2}{\varepsilon}(y_{1\varepsilon} - x_{1\varepsilon})) - e^{\lambda(x_{1\varepsilon}+x_{2\varepsilon})}(\lambda\underline{u}'(\mathbf{x}_\varepsilon) - \tfrac{2}{\varepsilon}(y_{1\varepsilon} - x_{1\varepsilon})), \\ e^{\lambda(y_{1\varepsilon}+y_{2\varepsilon})}(\lambda\bar{u}'(\mathbf{y}_\varepsilon) - \tfrac{2}{\varepsilon}(y_{2\varepsilon} - x_{2\varepsilon})) - e^{\lambda(x_{1\varepsilon}+x_{2\varepsilon})}(\lambda\underline{u}'(\mathbf{x}_\varepsilon) - \tfrac{2}{\varepsilon}(y_{2\varepsilon} - x_{2\varepsilon}))\ \} \geq 0. \tag{34}$$

(34) is equivalent to

$$\lambda(\underline{u}(\mathbf{x}_\varepsilon) - \bar{u}(\mathbf{y}_\varepsilon)) \\ \leq \max\{\ \tfrac{2}{\varepsilon}(e^{\lambda(x_{1\varepsilon}+x_{2\varepsilon})} - e^{\lambda(y_{1\varepsilon}+y_{2\varepsilon})})(y_{1\varepsilon} - x_{1\varepsilon}), \tfrac{2}{\varepsilon}(e^{\lambda(x_{1\varepsilon}+x_{2\varepsilon})} - e^{\lambda(y_{1\varepsilon}+y_{2\varepsilon})})(y_{2\varepsilon} - x_{2\varepsilon})\ \} \tag{35}$$

Let $\varepsilon \to 0$, due to (29), (35) becomes

$$\lambda(\underline{u}(\hat{\mathbf{x}}) - \bar{u}(\hat{\mathbf{x}})) \leq 0,$$

and (33) becomes

$$\underline{u}'(\hat{\mathbf{x}}) - \bar{u}'(\hat{\mathbf{x}}) \leq 0.$$

Therefore, in both cases (33) and (34) we have

$$\tilde{L} = \underline{u}'(\hat{\mathbf{x}}) - \bar{u}'(\hat{\mathbf{x}}) \leq 0.$$

Case I is impossible.
Case II: if (30) is true, combining (32), we have

$$\mathcal{H}_\lambda(\mathbf{x}_\varepsilon, \underline{u}'(\mathbf{x}_\varepsilon), D_{\phi_1}(\mathbf{x}_\varepsilon), M) - \mathcal{H}_\lambda(\mathbf{y}_\varepsilon, \underline{u}'(\mathbf{y}_\varepsilon), D_{\phi_2}(\mathbf{y}_\varepsilon), N) \geq 0,$$

which is equivalent to

$$\frac{1}{2}tr((\sigma_1^2, \sigma_2^2)(M - N)) + (\mu_1\lambda + \mu_2\lambda + \frac{1}{2}\lambda^2\sigma_1^2 + \frac{1}{2}\lambda^2\sigma_2^2 - \beta)(\underline{u}'(\mathbf{x}_\varepsilon) - \bar{u}'(\mathbf{y}_\varepsilon)) \geq 0.$$

We choose λ to be sufficiently close to 0 such that

$$\mu_1\lambda + \mu_2\lambda + \frac{1}{2}\lambda^2\sigma_1^2 + \frac{1}{2}\lambda^2\sigma_2^2 - \beta < 0.$$

By virtue of Ishii's lemma,

$$\lim_{\varepsilon\to 0}\frac{1}{2}tr((\sigma_1^2, \sigma_2^2)(M - N)) = 0.$$

Therefore, let $\varepsilon \to 0$, we have

$$\tilde{L} = \underline{u}'(\hat{\mathbf{x}}) - \bar{u}'(\mathbf{y}) \leq 0.$$

Case II is impossible. This contradicts to the assumption that $L = \sup_{\mathbf{x}\in\mathcal{R}_+^2}[\underline{u}(x_1, x_2) - \bar{u}(x_1, x_2)] > 0$. To summarize, for any $\mathbf{x} \in \mathcal{R}_+^2$, $\underline{u}(x_1, x_2) - \bar{u}(x_1, x_2) \leq 0$. □

REMARK 5. There is no uniqueness of the viscosity solution of HJB equation (18) if the boundary condition is not given. We can illustrate this with functions f in (7) and f_{sym} in (36). It is easy to verify that f satisfies (18) and is twice continuously differentiable, and therefore is a viscosity solution. f_{sym} is the optimal value function for a symmetric case (see Proposition 11) and therefore is a viscosity solution of (18) (see Theorem 2). It is obvious that f and f_{sym} are different even though they are both viscosity solutions of (18) and satisfy some polynomial growth conditions. The reason for this non-uniqueness is that f and f_{sym} have different values at the boundary of $\mathcal{R}_+^2$. In fact, even for no-transaction cost case ($k=1$), we have infinitely many smooth solutions of HJB equation (18) by replacing coefficient a in f with any c greater than or equal to a. Since there is no priori boundary condition for the optimal value function, we cannot imply a feasible viscosity solution of (18) is the optimal value function using Proposition 7.

The following proposition characterizes the viscosity solution to the HJB equation (18) and the optimal control. Define a family of continuous functions $\{u:\mathcal{R}_+^2 \to \mathcal{R}\}$ as $\tilde{C}^2_{a.e}$, if each u satisfies

- $u(0,0)=0$,
- u has bounded and continuous first derivatives a.e.,
- $\frac{\partial u}{\partial x_1}$, $\frac{\partial u}{\partial x_2}$ are non-increasing in x_1 and x_2 respectively a.e.,
- u has bounded and continuous second derivatives a.e.

PROPOSITION 8. *For any control (L,C) and initial value $(x_1,x_2)\in\mathcal{R}_+^2$, the corresponding value function $V_{L,C}(x_1,x_2)$ is bounded above by any viscosity supersolution $\bar{u}\in\tilde{C}^2_{a.e}$.*

Proof. Let $\bar{u}\in\tilde{C}^2_{a.e}$ be a viscosity supersolution of (18). Suppose $\bar{u}$ is twice differentiable at $(x_1,x_2)\in\mathcal{R}_+^2$. Then $\bar{u}$ can be approximated at (x,y) near (x_1,x_2) as follows:

$$\begin{aligned}\bar{u}(x,y) = {} & \bar{u}(x_1,x_2)+\bar{u}_{x_1}(x_1,x_2)(x-x_1)+\bar{u}_{x_2}(x_1,x_2)(y-x_2)\\ & +\tfrac{1}{2}\bar{u}_{x_1x_1}(x_1,x_2)(x-x_1)^2+\tfrac{1}{2}\bar{u}_{x_2x_2}(x_1,x_2)(y-x_2)^2\\ & +\tfrac{1}{2}\bar{u}_{x_1x_2}(x_1,x_2)(x-x_1)(y-x_2)+\tfrac{1}{2}\bar{u}_{x_2x_1}(x_1,x_2)(x-x_1)(y-x_2)\\ & +o((x-x_1)^2+(y-x_2)^2).\end{aligned}$$

Define, when $(x-x_1)^2+(y-x_2)^2$ is sufficiently small,

$$\begin{aligned}\varphi_n(x,y) = {} & \bar{u}(x_1,x_2)+\bar{u}_{x_1}(x_1,x_2)(x-x_1)+\bar{u}_{x_2}(x_1,x_2)(y-x_2)\\ & +\tfrac{1}{2}(\bar{u}_{x_1x_1}(x_1,x_2)-\tfrac{1}{n})(x-x_1)^2+\tfrac{1}{2}(\bar{u}_{x_2x_2}(x_1,x_2)-\tfrac{1}{n})(y-x_2)^2\\ & +\tfrac{1}{2}\bar{u}_{x_1x_2}(x_1,x_2)(x-x_1)(y-x_2)+\tfrac{1}{2}\bar{u}_{x_2x_1}(x_1,x_2)(x-x_1)(y-x_2).\end{aligned}$$

Hence $\bar{u}(x_1,x_2)=\varphi_n(x_1,x_2)$ and $\bar{u}-\varphi_n\geq 0$ in a neighborhood of (x_1,x_2). As $\bar{u}$ is continuous in $\mathcal{R}_+^2$, we can extend φ_n to $\mathcal{R}_+^2$ such that φ_n is twice continuously differentiable and $\bar{u}-\varphi_n$ reaches its minimum at (x_1,x_2) in $\mathcal{R}_+^2$. Therefore, φ_n is a test function of $\bar{u}$ in (x_1,x_2). We have

$$\begin{aligned}&\mathcal{L}(\varphi_n)(x_1,x_2)\leq 0,\\ &a-\frac{\partial\varphi_n}{\partial x_1}(x_1,x_2)\leq 0, (1-a)-\frac{\partial\varphi_n}{\partial x_2}(x_1,x_2)\leq 0,\\ &\frac{\partial\varphi_n}{\partial x_1}(x_1,x_2)-k\frac{\partial\varphi_n}{\partial x_2}(x_1,x_2)\leq 0, \frac{\partial\varphi_n}{\partial x_2}(x_1,x_2)-k\frac{\partial\varphi_n}{\partial x_1}(x_1,x_2)\leq 0,\end{aligned}$$

Let $n\to\infty$,

$$\begin{aligned}&\mathcal{L}(\bar{u})(x_1,x_2)\leq 0,\\ &a-\frac{\partial\bar{u}}{\partial x_1}(x_1,x_2)\leq 0, (1-a)-\frac{\partial\bar{u}}{\partial x_2}(x_1,x_2)\leq 0,\\ &\frac{\partial\bar{u}}{\partial x_1}(x_1,x_2)-k\frac{\partial\bar{u}}{\partial x_2}(x_1,x_2)\leq 0, \frac{\partial\bar{u}}{\partial x_2}(x_1,x_2)-k\frac{\partial\bar{u}}{\partial x_1}(x_1,x_2)\leq 0.\end{aligned}$$

Since $\bar{u}$ is twice differentiable a.e. in $\mathcal{R}_+^2$, we have

$$\begin{aligned}&\mathcal{L}(\bar{u})(x,y)\leq 0,\\ &a-\frac{\partial\bar{u}}{\partial x_1}(x,y)\leq 0, (1-a)-\frac{\partial\bar{u}}{\partial x_2}(x,y)\leq 0,\\ &\frac{\partial\bar{u}}{\partial x_1}(x,y)-k\frac{\partial\bar{u}}{\partial x_2}(x,y)\leq 0, \frac{\partial\bar{u}}{\partial x_2}(x,y)-k\frac{\partial\bar{u}}{\partial x_1}(x,y)\leq 0,\end{aligned}$$

a.e. in $\mathcal{R}_+^2$. Let $\phi(x,y)$ be a non-negative continuously differentiable function with support in $(0,1)\times(0,1)$ such that $\int_0^1\int_0^1 \phi(x,y)dxdy=1$. We define $\bar{u}_n:\mathcal{R}_+^2\to\mathcal{R}$ as the convolution:

$$\bar{u}_n(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{u}(x+s,y+t)n^2\phi(ns,nt)dsdt.$$

Then by standard technique,

- $\bar{u}_n$ is twice continuously differentiable in $\mathcal{R}_+^2$.
- $\bar{u}_n\to\bar{u}$ in a compact set in $\mathcal{R}_+^2$ and

$$\frac{\partial\bar{u}_n}{\partial x_1}\to\frac{\partial\bar{u}}{\partial x_1},\ \frac{\partial\bar{u}_n}{\partial x_2}\to\frac{\partial\bar{u}}{\partial x_2},\frac{\partial\bar{u}_n^2}{\partial^2 x_1}\to\frac{\partial\bar{u}^2}{\partial^2 x_1},\ \frac{\partial\bar{u}_n^2}{\partial^2 x_2}\to\frac{\partial\bar{u}^2}{\partial^2 x_2},$$

a.e. in $\mathcal{R}_+^2$.
We also claim the following inequalities.

- $$\begin{aligned}&\mathcal{L}(\bar{u}_n)(x,y)\le 0,\\ &a-\tfrac{\partial\bar{u}_n}{\partial x_1}(x,y)\le 0,\ (1-a)-\tfrac{\partial\bar{u}_n}{\partial x_2}(x,y)\le 0,\\ &\tfrac{\partial\bar{u}_n}{\partial x_1}(x,y)-k\tfrac{\partial\bar{u}_n}{\partial x_2}(x,y)\le 0,\ \tfrac{\partial\bar{u}_n}{\partial x_2}(x,y)-k\tfrac{\partial\bar{u}_n}{\partial x_1}(x,y)\le 0,\end{aligned}$$

in $\mathcal{R}_+^2$.
For any Δx, due to the absolute continuity of $\bar{u}$, we have

$$\begin{aligned}&\bar{u}_n(x+\Delta x,y)-\bar{u}_n(x,y)\\ =&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_0^{\Delta x}\tfrac{\partial\bar{u}}{\partial x_1}(x+\xi+s,y+t)d\xi\right)n^2\phi(ns,nt)dsdt.\end{aligned}$$

Then by the almost everywhere continuity of $\frac{\partial\bar{u}}{\partial x_1}$,

$$\frac{\partial\bar{u}_n}{\partial x_1}(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial\bar{u}}{\partial x_1}(x+s,y+t)n^2\phi(ns,nt)dsdt.$$

By a similar argument,

$$\frac{\partial\bar{u}_n}{\partial x_2}(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial\bar{u}}{\partial x_2}(x+s,y+t)n^2\phi(ns,nt)dsdt.$$

Therefore, we can have

$$\begin{aligned}&a-\tfrac{\partial\bar{u}_n}{\partial x_1}(x,y)\le 0,\ (1-a)-\tfrac{\partial\bar{u}_n}{\partial x_2}(x,y)\le 0,\\ &\tfrac{\partial\bar{u}_n}{\partial x_1}(x,y)-k\tfrac{\partial\bar{u}_n}{\partial x_2}(x,y)\le 0,\ \tfrac{\partial\bar{u}_n}{\partial x_2}(x,y)-k\tfrac{\partial\bar{u}_n}{\partial x_1}(x,y)\le 0,\end{aligned}$$

in $\mathcal{R}_+^2$. For any $\Delta x>0$, as $\frac{\partial\bar{u}}{\partial x_1}$ is decreasing,

$$\begin{aligned}&\tfrac{\partial\bar{u}_n}{\partial x_1}(x+\Delta x,y)-\tfrac{\partial\bar{u}_n}{\partial x_1}(x,y)\\ \le&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_0^{\Delta x}\tfrac{\partial^2\bar{u}}{\partial x_1^2}(x+\xi+s,y+t)d\xi\right)n^2\phi(ns,nt)dsdt.\end{aligned}$$

By the almost everywhere continuity of $\frac{\partial^2\bar{u}}{\partial x_1^2}$,

$$\frac{\partial^2\bar{u}_n}{\partial x_1^2}(x,y)\le\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial^2\bar{u}}{\partial x_1^2}(x+s,y+t)n^2\phi(ns,nt)dsdt.$$

Similarly,

$$\frac{\partial^2\bar{u}_n}{\partial x_2^2}(x,y)\le\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial^2\bar{u}}{\partial x_2^2}(x+s,y+t)n^2\phi(ns,nt)dsdt.$$

Therefore, we can have

$$\mathcal{L}(\bar{u}_n)(x,y) \le \mathcal{L}(\bar{u})(x,y) \le 0$$

in $\mathcal{R}_+^2$. For any control (L,C), let $(\bar{X}_1,\bar{X}_2)$ denote the corresponding controlled process. By Proposition 3,

$$\begin{aligned}&\bar{u}_n(\bar{X}_1(t\wedge\tau),\bar{X}_2(t\wedge\tau))e^{-\beta(t\wedge\tau)} - \bar{u}_n(x_1,x_2)\\ \le\ & M(t\wedge\tau) - a\int_0^{t\wedge\tau} e^{-\beta s}dL_1(s) - (1-a)\int_0^{t\wedge\tau} e^{-\beta s}dL_2(s),\end{aligned}$$

where M is a zero-expectation martingale. Therefore,

$$\bar{u}_n(x_1,x_2) \ge E_{x_1,x_2}\left(a\int_0^{t\wedge\tau} e^{-\beta s}dL_1(s) + (1-a)\int_0^{t\wedge\tau} e^{-\beta s}dL_2(s)\right).$$

By the Monotone Convergence Theorem, when $t\to\infty$,

$$\bar{u}(x_1,x_2) = \lim_{n\to\infty}\bar{u}_n(x_1,x_2) \ge V_{L,C}(x_1,x_2).$$

□

THEOREM 3. *Consider a family of admissible strategies $\{(L,C)\in\pi_{x_1,x_2} : (x_1,x_2)\in\mathcal{R}_+^2\}$. If the function $V_{L,C}$ is a viscosity supersolution of (18) and $V_{L,C}\in\tilde{C}^2_{a.e}$, then $V_{L,C}$ is the optimal value function.*

5. The Symmetric Case

In this section, we consider the symmetric surplus processes with transaction costs, i.e., $\mu_1=\mu_2=\mu, \sigma_1=\sigma_2=\sigma, a=1-a=1/2$. Let

$$\begin{aligned}&f_{sym}(x_1,x_2)\\ =&\begin{cases} aC(e^{\theta_1(x_1+kx_2)} - e^{-\theta_2(x_1+kx_2)}), & (x_1,x_2)\in A_1,\\ aC(e^{\theta_1(x_2+kx_1)} - e^{-\theta_2(x_2+kx_1)}), & (x_1,x_2)\in B_1,\\ a[C(e^{\theta_1 m} - e^{-\theta_2 m}) + x_1 + kx_2 - m], & (x_1,x_2)\in A_2,\\ a[C(e^{\theta_1 m} - e^{-\theta_2 m}) + x_2 + kx_1 - m], & (x_1,x_2)\in B_2,\\ a[C(e^{\theta_1 m} - e^{-\theta_2 m}) + x_2 + x_1 - \frac{2m}{1+k}], & (x_1,x_2)\in C,\end{cases}\end{aligned} \tag{36}$$

where area $A_i, B_i, i=1,2, C$ are given in Figure 5, $\theta_1, -\theta_2$ are roots of the equation

$$\left(\frac{1}{2}(1+k^2)\sigma^2\right)\theta^2 + (1+k)\mu\theta - \beta = 0, \tag{37}$$

with $\theta_2>\theta_1>0$ and

$$m = \frac{2(\ln\theta_2 - \ln\theta_1)}{\theta_1+\theta_2}, C = \frac{1}{\theta_1 e^{\theta_1 m} + \theta_2 e^{-\theta_2 m}}. \tag{38}$$

A similar argument as in Proposition 2 yields the following result.

PROPOSITION 9. *Function f_{sym} is continuous in $\mathcal{R}_+^2$ and differentiable and solves the HJB equation (18) in each of the areas $A_1\bigcup A_2$, $B_1\bigcup B_2$ and C.*

In what follows, we present a stationary strategy (L^*,C^*) with initial surpluses $(x_1,x_2)\in\mathcal{R}_2^+$.

1. If $(x_1,x_2)\in C$, line Two pays dividend $x_2-\frac{m}{k+1}$ and line One pays dividend $x_1-\frac{m}{k+1}$ and we go to 4.
2. If $(x_1,x_2)\in A_2$, line One pays directly an amount x_1+kx_2-m as dividend and we go to 4.
3. If $(x_1,x_2)\in B_2$, line Two pays directly an amount x_2+kx_1-m as dividend and we go to 5.
4. If $(x_1,x_2)\in A_1$, line One pays the accumulated amount $\max_{s\le t}\{\bar{X}_1(s)+k\bar{X}_2(s)-m\}$ up to time t until the surplus process escapes from A_1 , money is transferred from line One to line Two automatically when $\bar{X}_2$ hits zero, and we go to 5 or the process hits $(0,0)$.

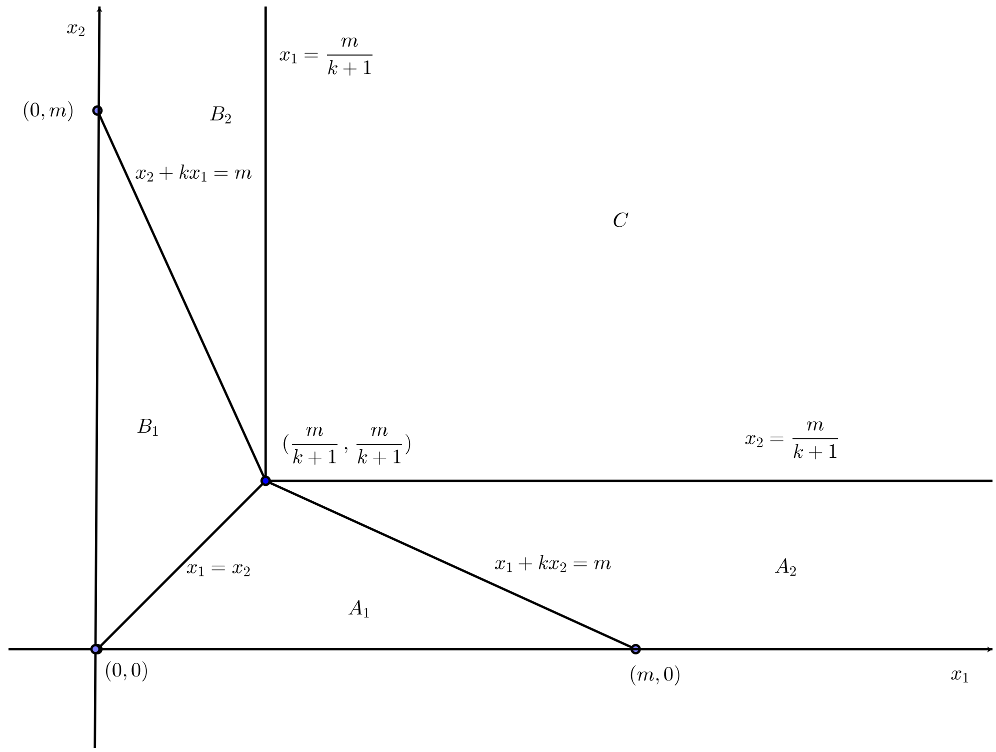

FIGURE 5. Symmetric Case

5. If $(x_1,x_2) \in B_1$, line Two pays the accumulated amount $\max_{s\le t}\{\bar{X}_2(s) + k\bar{X}_1(s) - m\}$ up to time t until the surplus process escapes from B_1 , money is transferred from line Two to line One automatically when $\bar{X}_1$ hits zero, and we go to 4 or the process hits $(0,0)$.

PROPOSITION 10. *The strategy (L^*, C^*) gives the value function f_{sym}.*

Proof. Let $\bar{X}_1, \bar{X}_2$ be the controlled surplus process with L^* and C^*. For $(x_1,x_2) \in A_1$, then $(\bar{X}_1(t), \bar{X}_2(t)) \in A_1 \bigcup B_1$ for any $t \ge 0$. Define $\tau_0 = 0$ and

$$\tau_i := \inf\{\tau \ge t > \tau_{i-1} \mid (\bar{X}_1(t), \bar{X}_2(t)) \in \{x_1 = x_2\}\}$$

Note that $\{\tau_i\}$ is an increasing sequence, $\tau_i \le \tau$ for any i and $\tau \in \{\tau_i\}$. Hence, $\tau_i \to \tau$.

Since $\bar{X}_1, \bar{X}_2$ are continuous stochastic processes, $\tau_i, i = 1,2,\dots$ are stopping times. By Proposition 9 and a similar argument as in Proposition 3, for $i = 1,2,\dots,$ we have

$$\begin{aligned} & e^{-\beta\tau_{i+1}\wedge\tau\wedge t} f_{sym}(\bar{X}_1(\tau_{i+1}\wedge\tau\wedge t), \bar{X}_2(\tau_{i+1}\wedge\tau\wedge t)) \\ & - e^{-\beta\tau_i\wedge\tau\wedge t} f_{sym}(\bar{X}_1(\tau_i\wedge\tau\wedge t), \bar{X}_2(\tau_i\wedge\tau\wedge t)) \\ = & -\int_{\tau_i\wedge\tau\wedge t}^{\tau_{i+1}\wedge\tau\wedge t} a e^{-\beta s} dL_1^*(s) - \int_{\tau_i\wedge\tau\wedge t}^{\tau_{i+1}\wedge\tau\wedge t} (1-a) e^{-\beta s} dL_2^*(s) + M(\tau_{i+1}\wedge\tau\wedge t) - M(\tau_i\wedge\tau\wedge t) \end{aligned}$$

where M is zero-expectation martingale. Therefore, for $i = 1,2,\dots,$

$$\begin{aligned} & e^{-\beta\tau_i\wedge\tau\wedge t} f_{sym}(\bar{X}_1(\tau_i\wedge\tau\wedge t), \bar{X}_2(\tau_i\wedge\tau\wedge t)) - f_{sym}(x_1,x_2) \\ = & -\int_0^{\tau_i\wedge\tau\wedge t} a e^{-\beta s} dL_1^*(s) - \int_0^{\tau_i\wedge\tau\wedge t} (1-a) e^{-\beta s} dL_2^*(s) + M(\tau_i\wedge\tau\wedge t). \end{aligned}$$

and

$$\begin{aligned} & f_{sym}(x_1,x_2) \\ = & E(\int_0^{\tau_i\wedge\tau\wedge t} a e^{-\beta s} dL_1^*(s) + \int_0^{\tau_i\wedge\tau\wedge t} (1-a) e^{-\beta s} dL_2^*(s)) \\ & + E(e^{-\beta\tau_i\wedge\tau\wedge t} f_{sym}(\bar{X}_1(\tau_i\wedge\tau\wedge t), \bar{X}_2(\tau_i\wedge\tau\wedge t))). \end{aligned}$$

Letting $i \to \infty$ and $t \to \infty$, $\tau_i \to \tau$,

$$f_{sym}(x_1,x_2) = V_{L^*,C^*}(x_1,x_2).$$

□

We verify that f_{sym} gives the optimal value function in the following proposition.

PROPOSITION 11. *The strategy (L^*,C^*) is the optimal strategy and the function f_{sym} gives the optimal value function in the symmetric case.*

Proof. One can easily check that $f_{sym} \in \tilde{C}^2_{a.e}$. From Proposition 9, f_{sym} is a viscosity supersolution of the HJB equation (18) in each of the areas $A_1 \cup A_2$, $B_1 \cup B_2$ and C. Note that at any (x_1,x_2) belongs to the segment $\overline{A_1 \cup A_2} \bigcap \overline{B_1 \cup B_2}$ or $\overline{A_1 \cup A_2} \bigcap C$ or $\overline{B_1 \cup B_2} \bigcap C$, f_{sym} is continuous but not differentiable and the following two inequalities hold.

$$\lim_{\delta x \nearrow 0} \frac{f_{sym}(x_1+\delta x, x_2) - f_{sym}(x_1,x_2)}{\delta x} > \lim_{\delta x \searrow 0} \frac{f_{sym}(x_1+\delta x, x_2) - f_{sym}(x_1,x_2)}{\delta x} \tag{39}$$

$$\lim_{\delta x \nearrow 0} \frac{f_{sym}(x_1, x_2+\delta x) - f_{sym}(x_1,x_2)}{\delta x} > \lim_{\delta x \searrow 0} \frac{f_{sym}(x_1, x_2+\delta x) - f_{sym}(x_1,x_2)}{\delta x} \tag{40}$$

Assume that φ is a test function at (x_1,x_2). We have $\varphi(x_1,x_2) = f_{sym}(x_1,x_2)$ and $\varphi(x,y) < f_{sym}(x,y)$ for $(x,y) \neq (x_1,x_2)$.

$$\begin{aligned} &\lim_{\delta x \nearrow 0} \tfrac{\varphi(x_1,x_2+\delta x)-\varphi(x_1,x_2)}{\delta x} \geq \lim_{\delta x \nearrow 0} \tfrac{f_{sym}(x_1,x_2+\delta x)-f_{sym}(x_1,x_2)}{\delta x} \\ &> \lim_{\delta x \searrow 0} \tfrac{f_{sym}(x_1,x_2+\delta x)-f_{sym}(x_1,x_2)}{\delta x} \geq \lim_{\delta x \searrow 0} \tfrac{\varphi(x_1,x_2+\delta x)-\varphi(x_1,x_2)}{\delta x} \end{aligned}$$

This means φ is not differentiable at (x_1,x_2), which contradicts φ is a test function. Hence such a test function does not exists. Function f_{sym} is a viscosity supersolution at points in the segment $\overline{A_1 \cup A_2} \bigcap \overline{B_1 \cup B_2}$ and $\overline{A_1 \cup A_2} \bigcap C$ and $\overline{B_1 \cup B_2} \bigcap C$. To summarize, f_{sym} is a viscosity supersolution in $\mathcal{R}^2_+$. This completes the proof. □

REMARK 6. In the symmetric case, when $(x_1,x_2) \in A_2(B_2)$, the optimal strategy (L^*,C^*) is Line One(Two) pays dividend directly until the surplus level hitting the line $\overline{A_1} \cap \overline{A_2}(\overline{B_1} \cap \overline{B_2})$. Note that at the boundary $x_2 = 0(x_1 = 0)$, Line One(Two) transfers money to Line Two(One) to support its surplus being non-negative with proportional transaction cost. The contribution to the dividends depends only on the level $x_1 + kx_2(x_2 + kx_1)$. Hence the optimal dividend strategy depends on the lines $x_1 + kx_2 = m(x_2 + kx_1 = m)$.

To find the optimal value function and the optimal control can be rather difficult due to the complication of the HJB equation (18) for general parameters even with $a = 1 - a = 1/2$. We leave further discussion to next section.

6. Conclusions and Further Discussion

In this paper, we consider the optimal dividend payment strategy for an insurance company, having two collaborating business lines, where their surplus processes are modeled by diffusion processes and the dividends paid by different business lines are weighted differently. We find the optimal dividend strategy when money is transferred between two business lines without transaction costs. We show the optimal value function is a continuous viscosity solution to the corresponding HJB equation when money is transferred with transaction costs. We also prove a verification theorem. Finally, we find the optimal solution to the problem with transaction costs in the symmetric case.

In the case of proportional transaction costs with general parameters, we cannot get the explicit solution. Motivated by the symmetric case, we propose a family of stationary curve strategies depending on $(x_1^o, x_2^o, g(x_2), h(x_1))$, with initial surpluses $(x_1,x_2) \in \mathcal{R}_2^+$, where g,h are functions of x_2 and x_1 respectively with $0 \leq x_2 \leq x_2^o, 0 \leq x_1 \leq x_1^o$. The curve strategies are presented as follows (see Figure 6).

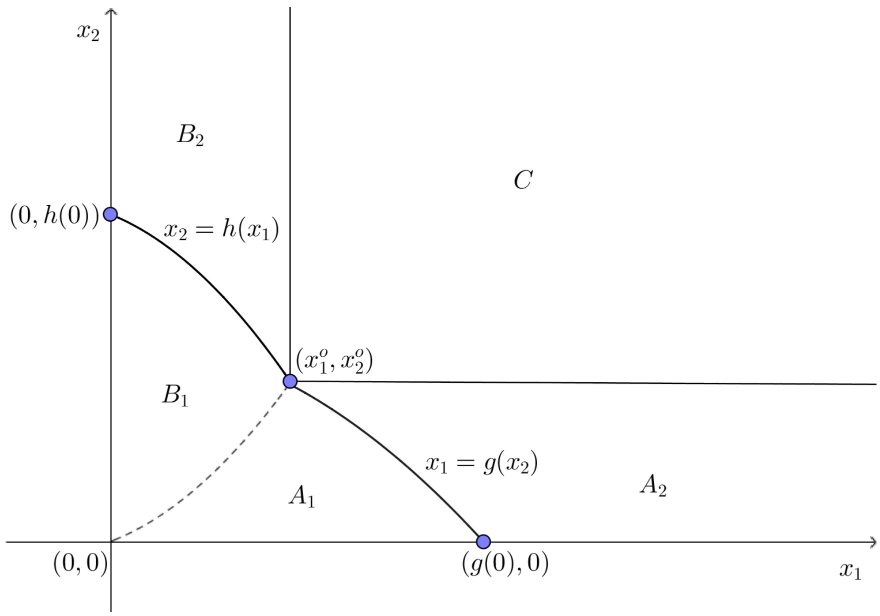

FIGURE 6. Curve Strategies

1. If $(x_1,x_2)\in C$, line Two pays dividend $x_2 - x_2^o$ and line One pays dividend $x_1 - x_2^o$ and we go to 4.
2. If $(x_1,x_2)\in A_2$, line One pays directly an amount $x_1 - g(x_2)$ as dividend and we go to 4.
3. If $(x_1,x_2)\in B_2$, line Two pays directly an amount $x_2 - h(x_1)$ as dividend and we go to 5.
4. If $(x_1,x_2)\in A_1$, line One pays the accumulated amount $\max_{s\le t}\{\bar{X}_1(s) - g(\bar{X}_2(s))\}$ up to time t until the surplus process escapes from A_1 , money is transferred from line One to line Two automatically when $\bar{X}_2$ hits zero, and we go to 5 or the process hits $(0,0)$.
5. If $(x_1,x_2)\in B_1$, line Two pays the accumulated amount $\max_{s\le t}\{\bar{X}_2(s) - h(\bar{X}_1(s))\}$ up to time t until the surplus process escapes from B_1 , money is transferred from line Two to line One automatically when $\bar{X}_1$ hits zero, and we go to 4 or the process hits $(0,0)$.

As a natural analogue of barrier strategies in two-dimensional context, we suggest for future research to search the optimal control among the family of curve strategies.

For further research, it is also interesting to study the case with the non-homogeneous proportional transaction costs. In this case, the controlled surplus processes are given by

$$\begin{aligned}\bar{X}_1(t) &= X_1(t) + C_{21}(t) - k_1 C_{12}(t) - L_1(t),\\ \bar{X}_2(t) &= X_2(t) + C_{12}(t) - k_2 C_{21}(t) - L_2(t),\end{aligned}$$

where $k_1, k_2 > 1$. The corresponding HJB equation becomes

$$\begin{aligned}0 &= \max\Big(\mathcal{L}(V)(x_1,x_2), a - \tfrac{\partial V}{\partial x_1}(x_1,x_2), (1-a) - \tfrac{\partial V}{\partial x_2}(x_1,x_2),\\ &\quad \tfrac{\partial V}{\partial x_1}(x_1,x_2) - k_1\tfrac{\partial V}{\partial x_2}(x_1,x_2), \tfrac{\partial V}{\partial x_2}(x_1,x_2) - k_2\tfrac{\partial V}{\partial x_1}(x_1,x_2)\Big),\\ 0 &= V(0,0).\end{aligned} \tag{41}$$

A similar argument gives

PROPOSITION 12. *The optimal value function function V with non-homogeneous proportional transaction cost is a continuous viscosity solution of the HJB equation (41). If we can find an admissible strategy $(L,C)\in\pi_{x_1,x_2}$, $(x_1,x_2)\in\mathcal{R}_+^2$ and $V_{L,C}$ is a viscosity solution of (41) and satisfies the natural growth condition (28), then $V_{L,C}$ is the optimal value function.*

Appendix. Proof of Lemma 1

If the optimal value function V is not a viscosity subsolution at $(x_1,x_2)\in\mathcal{R}_+^2$, there exist $\eta>0$ and a continuously differentiable function $\varphi:\mathcal{R}_+^2\to\mathcal{R}$ such that

- $\varphi(x_1,x_2)=V(x_1,x_2)$,
- $V(x,y)\le\varphi(x,y)$ in $\mathcal{R}_+^2$,
- $\max\{\mathcal{L}(\varphi),\varphi_{x_1}-k\varphi_{x_2},\varphi_{x_2}-k\varphi_{x_1},a-\varphi_{x_1},1-a-\varphi_{x_2}\}<-2\eta\beta$ at point (x_1,x_2).

Define $\bar\varphi(x,y)=\varphi(x,y)+(x-x_1)^4+(y-x_2)^4$. $\bar\varphi$ is continuously differentiable in $\mathcal{R}_+^2$ and

- $\bar\varphi(x_1,x_2)=V(x_1,x_2)$,
- $\bar\varphi(x,y)\ge V(x,y)+(x-x_1)^4+(y-x_2)^4$ in $\mathcal{R}_+^2$,
- $\max\{\mathcal{L}(\bar\varphi),\bar\varphi_{x_1}-k\bar\varphi_{x_2},\bar\varphi_{x_2}-k\bar\varphi_{x_1},a-\bar\varphi_{x_1},1-a-\bar\varphi_{x_2}\}<-2\eta\beta$ at point (x_1,x_2).

We can find $h\in(0,\min(x_1/2,x_2/2))$ such that

$$\max\{\mathcal{L}(\bar\varphi),\bar\varphi_{x_1}-k\bar\varphi_{x_2},\bar\varphi_{x_2}-k\bar\varphi_{x_1},a-\bar\varphi_{x_1},1-a-\bar\varphi_{x_2}\}<-\eta\beta, \tag{42}$$

for $(x,y)\in[x_1-2h,x_1+2h]\times[x_2-2h,x_2+2h]$.

Let $G(s,t)=\frac{1}{2\pi\sigma^2}e^{-\frac{s^2+t^2}{2\sigma^2}}$ be the Gaussian kernel. We define the convolution $v_n:\mathcal{R}_+^2\to\mathcal{R}$:

$$v_n=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\bar V(x-s,y-t)+h^4)n^2G(ns,nt)dsdt,$$

where $\bar V$ is the extension of V to $\mathcal{R}^2$:

$$\bar V(x,y)=\begin{cases}V(x,y), & (x,y)\in\mathcal{R}_+^2,\\ V(x,0)+ay, & x>0,y<0,\\ V(0,y)+ax, & x<0,y>0,\\ ax+ay, & \text{otherwise.}\end{cases}$$

By standard techniques, v_n is a smooth function and converges to $V+h^4$ uniformly in a compact set. Thus we can find n_0 large enough such that

$$V+2h^4\ge v_{n_0}\ge V+h^4/2,$$

for $(x,y)\in[0,x_1+x_2+2h]\times[0,x_1+x_2+2h]$. By Remark 4,

$$\begin{aligned}&\frac{v_n(x+\delta x,y)-v_n(x,y)}{\delta x}\\ &=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\bar V(x+\delta x-s,y-t)-\bar V(x-s,y-t)}{\delta x}n^2G(ns,nt)dsdt\\ &\ge a,\end{aligned}$$

$(v_{n_0})_{x_1}\ge a$ for $(x,y)\in\mathcal{R}^2$. Similarly, $(v_{n_0})_{x_2}\ge 1-a$ for $(x,y)\in\mathcal{R}^2$. Let χ be a continuously differentiable function satisfying $0\le\chi\le1$ and

- $\chi(x,y)=1$ for $(x,y)\in[x_1-h,x_1+h]\times[x_2-h,x_2+h]$,
- $\chi(x,y)=0$ for $(x,y)\notin(x_1-2h,x_1+2h)\times(x_2-2h,x_2+2h)$,
- $\chi_{x_1},\chi_{x_2}\ge0$ for $(x,y)\in[x_1-2h,x_1+h]\times[x_2-2h,x_2+h]$.

Define

$$\psi(x,y)=\chi(x,y)\bar\varphi(x,y)+(1-\chi(x,y))v_{n_0}(x,y)$$

and take $\epsilon\in(0,\min(\eta,h^4/2))$. We now verify that ψ satisfies condition 1-4 in Lemma 1.

Apparently, $\psi(x_1,x_2)=\bar\varphi(x_1,x_2)=V(x_1,x_2)$ and condition (4) holds. For $(x,y)\in[x_1-h,x_1+h]\times[x_2-h,x_2+h]$, $\psi(x,y)=\bar\varphi(x,y)$, hence condition (2) holds by (42). For $(x,y)\in[0,x_1+x_2+2h]\times[0,x_1+x_2+2h]\setminus(x_1-h,x_1+h)\times(x_2-h,x_2+h)$,

$$\bar\varphi(x,y)\ge V(x,y)+(x-x_1)^4+(y-x_2)^4\ge V(x,y)+2h^4\ge V(x,y)+\epsilon$$

and

$$v_{n_0}(x,y) \geq V(x,y) + h^4/2 \geq V(x,y) + \epsilon,$$

hence, $\psi(x,y) \geq V(x,y) + \epsilon$ and condition (3) holds. For $(x,y) \in (x_1 - h, x_1 + h) \times (x_2 - h, x_2 + h)$,

$$\psi = \bar{\varphi} \text{ and } \psi_{x_1} = \bar{\varphi}_{x_1} \geq a, \psi_{x_2} = \bar{\varphi}_{x_2} \geq 1 - a.$$

For $(x,y) \in [0, x_1 + h] \times [0, x_2 + h] \setminus (x_1 - h, x_1 + h) \times (x_2 - h, x_2 + h)$,

$$\bar{\varphi} \geq V + 2h^4 \geq v_{n_0},$$

and

$$\psi_{x_1} = \chi_{x_1}(\bar{\varphi} - v_{n_0}) + \chi\bar{\varphi}_{x_1} + (1-\chi)(v_{n_0})_{x_1} \geq a.$$

Combining above, for $(x,y) \in [0, x_1 + h] \times [0, x_2 + h]$, $\psi_{x_1} \geq a$. Similarly, for $(x,y) \in [0, x_1 + h] \times [0, x_2 + h]$, $\psi_{x_2} \geq 1 - a$. Therefore, condition (1) holds.

If we replace $\bar{V}$ by $\tilde{V}$:

$$\tilde{V}(x,y) = \begin{cases} V(x,y), & (x,y) \in \mathcal{R}_+^2, \\ V(x+y,0), & x > 0, y < 0, \\ V(0,x+y), & x < 0, y > 0, \\ ax + ay, & \text{otherwise}. \end{cases}$$

We can also find n_0 large enough such that

$$V + 2h^4 \geq v_{n_0} \geq V + h^4/2,$$

for $(x,y) \in [0, x_1 + x_2 + 2h] \times [0, x_1 + x_2 + 2h]$. By Remark 4,

$$\begin{aligned} & v_n(x + \delta x, y - k\delta x) - v_n(x,y) \\ = & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\bar{V}(x + \delta x - s, y - k\delta x - t) - \bar{V}(x - s, y - t)) n^2 G(ns, nt) ds dt \\ \leq & \ 0, \end{aligned}$$

$(v_{n_0})_{x_1} - k(v_{n_0})_{x_2} \leq 0$ for $(x,y) \in \mathcal{R}^2$. Similarly, $(v_{n_0})_{x_2} - k(v_{n_0})_{x_1} \leq 0$ for $(x,y) \in \mathcal{R}^2$. Replace χ by θ with θ being a continuously differentiable function and satisfying $0 \leq \theta \leq 1$ and

- $\theta(x,y) = 1$ for $(x,y) \in [x_1 - h, x_1 + h] \times [x_2 - h, x_2 + h]$,
- $\theta(x,y) = 0$ for $(x,y) \notin (x_1 - 2h, x_1 + 2h) \times (x_2 - 2h, x_2 + 2h)$,
- $\theta_{x_1} - k\theta_{x_2} \leq 0$ for $(x,y) \in [x_1 - h, x_1 + 2h] \times [x_2 - 2h, x_2 + h]$, $\theta_{x_2} - k\theta_{x_1} \leq 0$ for $(x,y) \in [x_1 - 2h, x_1 + h] \times [x_2 - h, x_2 + 2h]$.

Define $\phi(x,y) = \theta(x,y)\bar{\varphi}(x,y) + (1 - \theta(x,y))v_{n_0}(x,y)$. With $\epsilon \in (0, \min(\eta, h^4/2))$, the result for ϕ follows by a similar argument as with ψ.

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