

The Duration Derby: A Comparison of Duration Based Strategies in Asset Liability Management

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Abstract

Macaulay duration matched strategy is a key tool in bond portfolio immunization. It is well known that if term structures are not flat or changes are not parallel, then Macaulay duration matched portfolio can not guarantee adequate immunization. In this paper the approximate duration is proposed to measure the bond price sensitivity to changes of interest rates of non-flat term structures. Its performance in immunization is compared with those of Macaulay and key rate durations using the US Treasury strips and bond data. Approximate duration turns out to be a possible contender in asset liability management: it does not assume any particular structures or patterns of changes of interest rates, it does not need short selling of bonds, and it is easy to set up and rebalance the optimal portfolio with linear programming.

1 Introduction

Duration is a useful way of making a rough assessment of the effect of interest rate changes on single bonds and portfolios of bonds. (See Bierwag (1987), Bierwag, Corrado, and Kaufman (1990).) If one only could use two numbers to describe the characteristics of a bond the obvious ones are its price and its duration. Duration has also proved effective in matching asset portfolios and liability portfolios by matching their durations, though recent developments in decomposition and sampling aspects of stochastic programming means that this more precise approach becomes more viable for realistic problems. (See Birge and Louveaux (1998).)

However there are difficulties with the original Macaulay duration approach. It requires that the yield curve for the bond is flat even though the gilt market is usually suggesting something different and it does not deal with default risk explicitly. This paper reviews the first of these issues. An extension of the Macaulay duration, partial duration (Cooper (1977)) has been suggested as a way of dealing with non-flat yield curves. In this paper the idea of an approximate duration is introduced which is closer to the Macaulay duration idea of a second number to describe the relationship of an asset, liability or portfolio of such to interest rates. Unlike the Macaulay duration though this can be thought of as the median of the cash flow of the bond rather than the mean and hence cannot be obtained for a portfolio of bonds directly from the durations of the individual bonds. However a linear programming method of calculating this duration measure is described in the paper in the case of asset liability management.

The effectiveness of these duration measures is investigated by describing a simulation experiment using US Treasury strips and bond data to see how well these duration measures choose a portfolio of assets to match a given cash flow of liabilities. Three duration measures are compared in this experiment. They are Macaulay, approximate, and key rate durations.

Section 2 reviews the Macaulay duration and discusses the partial and key rate durations for non-flat term structures. Section 3 introduces the approximate duration approach. Section 4 describes how duration matching strategies can be applied to asset liability management problems. Section 5 deals with the “horse

race”—the derby—between the three asset management strategies based on the different definitions of duration. It describes the way the experiment is performed and discusses the results.

2 Macaulay and Partial Durations

The Macaulay duration of a bond can be identified with the maturity of a zero-coupon risk free bond which has the same value and the same response to a small change in interest rates as the original bond. Thus if a bond has an income stream c_t , $t = 1, \dots, T$, over separate periods until its maturity at T , and r is the implied interest rate or yield to maturity of the bond, the value of the bond V satisfies

$$V = \sum_t \frac{c_t}{(1+r)^t}.$$

If one matches this by a zero-coupon bond which pays out R at time D so that its value is $V_0 = R/(1+r)^D$ and both bonds have the same response to small changes of interest rates, then one would require

$$V = V_0 \text{ and } \frac{dV}{dr} = \frac{dV_0}{dr}.$$

This leads to the standard definition of Macaulay duration for a risk-free bond, namely,

$$D = \frac{1}{V} \sum_t \frac{tc_t}{(1+r)^t}.$$

The definition of Macaulay duration is based on the idea that the term structure is flat and that the only changes are parallel shifts. This is not what the market assumes and this has led to other definitions of duration. Suppose the term structure is not flat and the risk free spot rates are given by a vector $\mathbf{r} = (r_1, \dots, r_T)$ then the value of a bond with income stream c_t is

$$V(\mathbf{r}) = \sum_t \frac{c_t}{(1+r_t)^t}. \quad (1)$$

If, on the other hand, the risk-free forward rates are given by a vector $\mathbf{f} = (f_1, \dots, f_T)$ then the value of a bond with income stream c_t is

$$V(\mathbf{f}) = \sum_t \frac{c_t}{\prod_{s \leq t} (1+f_s)}. \quad (2)$$

Whichever formulation is used, one has to model the term structure or equivalently the discount factor b_t such that $V(b) = \sum_t b_t c_t$ and $b_t = (1+r_t)^{-t}$ in (1) and $b_t = 1/\prod_{s \leq t} (1+f_s)$ in (2). There are two main approaches to modeling the term structure. The first is to choose a specific form of the yield curve and use the market data to estimate its parameters. Thus Haugen (1997) suggested a spot rate curve of the form

$$r(t) = (a + bt)e^{-dt} + c. \quad (3)$$

The parameters can be easily estimated using nonlinear regression methods and the model has the advantage that the parameters have an obvious interpretation: $a = r(0) - r(\infty)$ is the difference of the short rate and the long rate, $b = r'(0) + d(r(0) - r(\infty))$ is related to the short rate slope and the overall structure of interest rates, $c = r(\infty)$ is the long rate, and $d = -r''(\infty)/r'(\infty)$ is the ratio of curvature to slope in the long run but is also the rate of convergence to the long rate.

A second approach is to describe the movements in the term structure by a set of factors. In this case it is assumed that

$$r(t) = \sum_i a_i F_i(t) + w(t)$$

where $w(t)$ is a stochastic process with zero mean. The factors $F_i(t)$ are determined empirically (see Dahl (1993)) using factor analysis on the historical returns of pure discount bonds or the historical estimated term structures. Ho (1992) suggested that changes of spot rate curves are determined by changes of some key rates. Suppose for example the first, fifth and twenty-fifth year spot rates are taken as key rates, and changes of them are a_1, a_2, a_3 , respectively. Then changes of spot rates are determined by linear interpolation of changes of three key rates, that is,

$$\begin{aligned} r(t) - r_0(t) &= \\ &= \begin{cases} \frac{1}{4}((5-t)a_1 + (t-1)a_2) & \text{if } t \leq 5 \\ \frac{1}{20}((25-t)a_2 + (t-5)a_3) & \text{if } 5 \leq t \leq 25 \\ a_3 & \text{if } t \geq 25 \end{cases} \end{aligned} \quad (4)$$

where r_0 is the initial spot rate curve. In both cases one ends up with a discount function $b_t(\mathbf{a})$ which is a function of a few critical parameters, i.e. $b_t(\mathbf{a}) = b(t, a_1, \dots, a_n)$. Thus whichever model of spot rate (zero coupon bond yield) curve one chooses, one arrives at a model for the value of a bond which depends on a vector of parameters $\mathbf{a} = (a_1, \dots, a_n)$ which describe the spot rates or forward rates, so that

$$V(\mathbf{a}) = \sum_t b_t(\mathbf{a})c_t. \quad (5)$$

Following the analogy with the derivation of the Macaulay duration, one would ask what is the maturity of a zero coupon risk-free bond paying out R at time D (so its value is $V_0(\mathbf{a}) = Rb_D(\mathbf{a})$) that has the same value as the previous bond and the same response to small changes in risk-free rates.

The problem is that there are now a number of ways the risk free rate can change, not just the parallel shifts in the term structure that is implicit in the Macaulay duration. What is normally suggested in the literature is to calculate the duration for each of the ways that

this rate can change and seek to match asset and liability portfolios in each of these durations. One assumes that each change in the risk free rate corresponds to a change in one of the parameters that make up the risk free interest rate term structure and hence the discount factors $b_t(\mathbf{a})$. Cooper (1977) first suggested this approach and subsequently these durations became called partial durations. Given the bond price model of (5), then the i th partial duration is

$$D_i = -\frac{1}{V(\mathbf{a})} \frac{\partial V(\mathbf{a})}{\partial a_i}.$$

As examples, consider using the spot rate curve formulation of (3) and assume the short rate, the short rate slope, and the long rate are independent factors. This leads to partial durations of the form

$$\begin{aligned} D_1 &= \sum_t t(1+dt)e^{-dt}C_t \\ D_2 &= \sum_t t^2e^{-dt}C_t \\ D_3 &= \sum_t t(1-(1+dt)e^{-dt})C_t \end{aligned}$$

where $C_t = c_t(1+r(t))^{-(t+1)}/V(\mathbf{a})$. Here D_1 is the duration to the short rate, D_2 to the short rate slope, and D_3 to the long rate.

If key rates are used to describe term structure model, then their partial durations, or key rate durations, can be computed in the same way. For example, consider using the first, fifth, and twenty-fifth year rates as key rates as in (4), then partial durations of bonds to these key rates are

$$\begin{aligned} D_1 &= \sum_{t \leq 5} \frac{1}{4}(5-t)tC_t \\ D_2 &= \sum_{t \leq 5} \frac{1}{4}(t-1)tC_t + \sum_{5 < t \leq 25} \frac{1}{20}(25-t)tC_t \\ D_3 &= \sum_{5 < t \leq 25} \frac{1}{20}(t-5)tC_t + \sum_{t > 25} tC_t \end{aligned}$$

where $C_t = c_t(1+r_0(t))^{-(t+1)}/V(\mathbf{a})$. Here D_1 is the duration to the first year key rate, D_2 to the fifth year key rate, and D_3 to the twenty-fifth year key rate.

Given one is seeking to allow for all the possible changes in the term structure that one has identified one would expect fitting portfolios by matching all their partial durations would be much more successful than just matching on the one Macaulay duration. This is what Chambers, Carleton, and McEnally (1988) investigated and they did find an improvement in immunizing the terminal values of the portfolio, when transaction costs are ignored.

3 Approximate Durations

There is an alternative duration measure that may be more robust than the Macaulay duration and which has the advantage that the user can specify which types of change to the interest rate term structure are of most concern to him. This approach minimizes the weighted sum of the errors of the sensitivity of the bond to changes in each of the parameters in the yield curve. This duration, called the approximate duration, is obtained as follows for a bond whose price $V(\mathbf{a})$ is given by (5). As in the case with the Macaulay duration, one wishes to find the maturity, D , of the zero-coupon risk free bond, paying R which most closely matches the weighted sum of the individual changes in the yield curve. The price of such a bond is $V_0(\mathbf{a}) = Rb_D(\mathbf{a})$. The weighting put on the individual changes is given by the weights vector $w = (w_i)$ which is chosen subjectively to reflect one's perceptions about importance of individual factors. The approximate duration D_w is defined as the optimal solution to the following problem:

$$\begin{aligned} \text{Minimize} \quad & \sum_i w_i \left| \frac{\partial V(\mathbf{a})}{\partial a_i} - \frac{\partial V_0(\mathbf{a})}{\partial a_i} \right| \\ \text{subject to} \quad & V(\mathbf{a}) = V_0(\mathbf{a}) \end{aligned} \quad (6)$$

The approximate duration of a bond is a generalization of Macaulay duration to non-flat term structures of interest rates, since if the term structure is flat with interest rate r as the only factor, then $V(r) = \sum_t c(t)/(1+r)^t$ and $V_0(r) = R/(1+r)^D$, and (6) reduces to $V(r) = V_0(r)$ and $w_1|dV/dr - dV_0/dr|$ is minimized. The minimum is achieved if and only if $D_w = \sum_t tc_t(1+r)^{-t}/V(r)$ which is independent of the weight w and equals the Macaulay duration. One can define the approximate duration for any types of interest rate models described above. As an example its calculation is described in the case where the parameters are all the forward interest rates, $\mathbf{f} = (f_1, \dots, f_T)$. In this case D_w is the optimal solution to the problem:

$$\begin{aligned} \text{Minimize} \quad & \sum_s w_s \left| \frac{\partial V(\mathbf{f})}{\partial f_s} - \frac{\partial V_0(\mathbf{f})}{\partial f_s} \right| \\ \text{subject to} \quad & V(\mathbf{f}) = V_0(\mathbf{f}) \end{aligned} \quad (7)$$

Since bonds usually have an integer valued maturity one should first look at the duration D_w which is the integer that minimizes this value. The difference between the s th partial derivatives in (7), if $s \leq D$, is

$$\frac{\partial V}{\partial f_s} - \frac{\partial V_0}{\partial f_s} = -\sum_{t \geq s} \frac{b_t(\mathbf{f})c_t}{(1+f_s)} + \frac{V_0(\mathbf{f})}{(1+f_s)} = \sum_{t < s} \frac{b_t(\mathbf{f})c_t}{(1+f_s)}$$

where $b_t(\mathbf{f}) = 1/\prod_{u \leq t} (1+f_u)$ is a discount factor. The last equality comes from the fact that $V_0(\mathbf{f}) = V(\mathbf{f})$. If $s > D$ then

$$\frac{\partial V}{\partial f_s} - \frac{\partial V_0}{\partial f_s} = -\sum_{t \geq s} \frac{b_t(\mathbf{f})c_t}{(1+f_s)}.$$

If cash flows c_t are non-negative for all t , then the function to be minimized in (7) can be written as

$$E(D) = \sum_{s \leq D} \tilde{w}_s \sum_{t < s} b_t(\mathbf{f})c_t + \sum_{s > D} \tilde{w}_s \sum_{t \geq s} b_t(\mathbf{f})c_t \quad (8)$$

where $\tilde{w}_s = w_s/(1 + f_s)$. Therefore the approximate duration D_w is the minimum solution of $E(D)$ over $D = 1, 2, \dots, T$, i.e.,

$$E(D_w) = \min_{1 \leq D \leq T} E(D).$$

If some cash flows c_t are negative (such as in the case where bonds might be short sold), then a linear integer programming can be used to find the approximate duration as follows:

$$\text{Minimize } \sum_s \tilde{w}_s(\alpha_s + \beta_s)$$

subject to the following constraints:

$$\begin{aligned} \sum_{t \geq s} b_t(\mathbf{f})c_t + \alpha_s - \beta_s &= V(\mathbf{f}) \left(\sum_{u=1}^s y_u \right) \\ \sum_{u=1}^T y_u &= 1 \\ \alpha_s, \beta_s, y_s &\geq 0 \end{aligned}$$

for all $s = 1, \dots, T$, and y_s are integers.

The approximate duration D_w is a different measure of stability than the Macaulay duration family of measures which concentrate on one dimensional changes in the forward rates. To see this compute the difference $E(D+1)$ and $E(D)$ in the positive cash flow case to get

$$\begin{aligned} &E(D+1) - E(D) \\ &= \tilde{w}_{D+1} \left(\sum_{t < D+1} b_t(\mathbf{f})c_t - \sum_{t \geq D+1} b_t(\mathbf{f})c_t \right) \end{aligned}$$

and it follows that the minimum occurs at

$$D_w = \max \left\{ D : \sum_{t \leq D} b_t(\mathbf{f})c_t < \sum_{t \geq D+1} b_t(\mathbf{f})c_t \right\}.$$

This is the median time of the discounted cash flows whereas the Macaulay duration is the mean time. Note that the optimal solution D_w is independent of the weighting w of the importance of the different periods sensitivities, provided they are non-zero. So one can drop the subscript w from the D_w .

Unlike the mean, the median of a linear combination of measures need not be the linear combination of the individual medians and hence the approximate duration D of a portfolio of bonds has to be calculated by considering the total cash flows in the portfolio rather than by combining the approximate durations of the individual bonds.

4 Asset Liability Management with Transaction Costs

Asset liability management is concerned with selecting a bond portfolio such that value of asset is the same as that of liability no matter how interest rates change. Suppose there are n bonds in an asset portfolio. Then the value of the asset is

$$V_A = \sum_j V_j x_j$$

where V_j is the present value of bond j and x_j is the number of bond j in the portfolio. If the term structure is flat and has only parallel shifts, then the duration of the asset portfolio is derived from those of individual bonds by

$$D_A = \sum_j D_j y_j \quad (9)$$

where D_j is the Macaulay duration of bond j and $y_j = x_j V_j / V_A$ is the current price weighted proportion of bond j in the portfolio. An immunized asset liability portfolio can be set up with two equality constraints:

$$V_A = V_L \text{ and } D_A = D_L$$

where V_L is the present value of the liability and D_L the Macaulay duration of the liability. There may be several solutions to the two constraints. An objective function is then used to select an "optimal" solution that minimizes the cost, or maximizes the yield, etc.

Suppose there are transaction costs of buying or selling bonds, which are proportional to number of bonds bought or sold then minimizing the transaction cost of each period is a natural choice of objective function. If the portfolio is periodically rebalanced with the policy that all old bonds are sold or bought at rebalancing time and a new portfolio is set up to hedge against the new liability, the transaction cost is $\sum_j |x_j|$, the total number of bonds bought and sold. (Assume the proportional costs are the same for all bonds.)

In Sections 3 and 4, several variants of duration are introduced. Each of them can be used as a way of managing an asset portfolio which is meant to cover a liability by matching the asset durations with the liability durations. The following strategies are considered, where initially assume no short selling of bonds in the asset portfolio is allowed, i.e., $x_j \geq 0$ for all j .

Macaulay duration matched strategy. The first approach is to use the Macaulay durations of bond portfolios for non-flat term structures. Suppose D_j is Macaulay duration for bond j , then the Macaulay duration D_A of the portfolio is defined by (9). This approach is using the yield of each bond to define its duration, rather than the yield of the portfolio, see Bierwag, Corrado, and Kaufman (1990) for a discussion of

this point. The latter approach is more difficult here because the durations are themselves needed to define what is the optimal portfolio. The optimal portfolio is selected by solving a linear programming problem that minimizes the total number of bonds $\sum_j x_j$ in the portfolio subject to equality constraints $V_A = V_L$ and $D_A = D_L$, and non-negativity constraints $x_j \geq 0$ for all j .

Approximate duration matched strategy. The second approach is to use approximate duration. Since no short selling of bonds is allowed, all cash flows are non-negative. The approximate duration of the bond portfolio D_A is computed from

$$E_A(D_A) = \min_D E_A(D) \quad (10)$$

where $E_A(D)$ is defined by (8) with cash flow $\sum c_t^j x_j$ at time t . Exchange order of summation to get

$$E_A(D) = \sum_j E_j(D) x_j$$

where $E_j(D)$ is defined by (8) with cash flow c_t replaced by c_t^j at time t . To have approximate duration matched portfolio one requires $D_A = D_L$ which implies D_L is the minimum solution to (10), which is equivalent to the following inequality constraints:

$$\sum_j E_j(D_L) x_j \leq \sum_j E_j(D) x_j \quad (11)$$

for $D = 1, \dots, T$. In general, there will be a number of portfolios which satisfy all the inequalities. The Macaulay duration matched strategy chooses the objective with the smallest total number of bonds. It might be more appropriate instead to try and get the difference between the asset and the liability portfolio durations closer, i.e. instead of getting just the durations to match $D_A = D_L$, have the errors in the durations to agree as well, so $D_A = D_L$ and $E_A(D_A) = E_L(D_L)$. If the liability occurs at only one time point then $E_L(D_L)$ is always zero and so $E_A(D_A)$ should be as close to zero as possible. Then instead of taking the objective function to be to minimize $\sum_j x_j$, one seeks to minimize $\sum_j x_j + \sum_j E_j(D_L) x_j$. The optimal portfolio is selected by solving an LP subject to equality constraint $V_A = V_L$, inequality constraints (11), and non-negativity constraints $x_j \geq 0$ for all j .

Key rate duration matched strategy. The third approach is to match key rate durations of the bond portfolio with those of the liability. Suppose a set of key rates determines changes of the term structure and D_j^i is the key rate i duration for bond j . Then the key rate i duration D_A^i of the portfolio is defined by (9) with D_j replaced by D_j^i . The optimal portfolio is selected by minimizing $\sum_j x_j$ subject to equality constraints

$V_A = V_L$ and $D_A^i = D_L^i$ for all key rates i , and non-negativity constraints $x_j \geq 0$ for all j . Unfortunately, quite often there are no feasible solutions satisfying all constraints above. One has to relax the restriction of no short selling of bonds to ensure the existence of feasible solutions. The optimal portfolio is then selected by minimizing $\sum_j |x_j|$ subject to the same set of the equality constraints. Note that the above optimization problem is an LP by writing $x_j = x_j^+ - x_j^-$ with $x_j^+, x_j^- \geq 0$ and $|x_j| = x_j^+ + x_j^-$.

5 Horse Race Data and Result

In this section performances of asset liability management with different duration strategies are compared using US Treasury strips and Bonds. The risk free spot rate curves can be derived from US Treasury bonds with bootstrapping technique or linear programming method. Since derived spot rates are very close to Treasury strips rates (see Allen, Thomas, and Zheng (2000)), the latter are used to construct spot rate curves for simplicity. Six option free Treasury bonds with maturity one, two, three, five, ten, and twenty-five years are chosen each year to form a selection universe of the bond portfolio. The data source is the Wall Street Journal (New York Edition) around the 15th of February each year from 1994 to 2001.

A simple example is used to illustrate how the portfolios are set up, optimized, and rebalanced, for the different duration matching strategies. Suppose there is a two year liability from February 1999 to February 2001 with face value \$1,000,000. A bond portfolio is set up in 1999 from six Treasury bonds. The values of these bonds and the liability are computed by (1) where spot rates are the 1999 Treasury strips rates. Three portfolios are set up with different duration matched strategies. No short selling of bonds is allowed for Macaulay and approximate duration matched portfolios but this restriction is relaxed for key rate duration matched portfolios (with 1, 5, and 25 year spot rates as key rates). For each strategy the corresponding durations for six bonds and the liability are first computed and then a linear programming problem is solved as discussed in detail in Section 4.

The portfolio is rebalanced once a year. After a year the values of these bonds and the liability are computed with the 2000 Treasury strips rates. The values of the bond portfolios are then compared with the value of the liability to see their performances. The results are that Macaulay duration gains \$602, approximate duration gains \$1392, and key rate duration gains \$13. All bonds are sold or bought and gains or losses are brought forward to the maturity of the liability (February 2001) with the corresponding values \$640, \$1480, and \$13,

respectively, after compounded with the 1 year strips rate.

The same process is now repeated for the second of the two years, where there is a one year liability (from February 2000). All duration matched strategies produce the same optimal solution in the second year of buying one year bonds which hedge the liability perfectly, i.e., the gains/losses at the maturity of the liability (February 2001) are zero.

A set of horse races are carried out: The liabilities all have face value \$1,000,000 but with different length of durations in the range of two to seven years. All portfolios are set up and analyzed with the observed US Treasury strips and bond data. The gains/losses of portfolios with different duration strategies are listed in Table 1.

The approximate duration matched strategy does well in all tests: it has the largest average gain of \$3430, it has the smallest maximum loss of \$1250, and it has the largest maximum gain of \$10892. If short selling of bonds is not allowed, it is also the safest method (with the smallest standard deviation of \$4345). Key rate duration matched strategy is a suitable choice for immunization since it has the smallest standard deviation, small maximum loss, and near-zero average gain/loss. The main disadvantage of the key rate duration matched strategy is that short selling of bonds must be allowed to ensure feasible solutions and in practice this may not be possible. Macaulay duration matched strategy has reasonable performance in immunization, but is slightly inferior to the approximate duration matched strategy.

6 Conclusions

In this paper the approximate duration is proposed to measure the sensitivity of bond prices to changes of interest rates and to use it in bond portfolio immunization. The approximate duration matched strategy is compared with Macaulay and other commonly used duration matched strategies using the US Treasury strips and bond data. The horse race result shows that approximate duration matched strategy is a possible contender for an asset liability management strategy: it does not assume any particular structures or patterns of changes of interest rates (in contrast to Macaulay duration), it does not need short selling of bonds (in contrast to key rate duration), it is easy to find the optimal portfolio with linear programming, and it is robust to changes of interest rates.

Table 1. Gains/Losses of Duration Matched Strategies in Immunization

Liability/Strategy	Macaulay	Approx	Key Rate
2 Year: 1999-2001	640	1480	13
3 Year: 1998-2001	799	248	37
4 Year: 1997-2001	933	-1250	1169
5 Year: 1996-2001	5297	4875	617
6 Year: 1995-2001	-8039	4335	-1352
7 Year: 1994-2001	3439	10892	-2640
Average	512	3430	-359
Std Deviation	4580	4345	1398
Maximum Loss	-8039	-1250	-2640
Maximum Gain	5297	10892	1169

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