Mathematics is painting without the brush;

painting is mathematics without the chalk.

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Keynote lecture give at:
INTERNATIONAL CONFERENCE "EXCELLENCE: EDUCATION & HUMAN DEVELOPMENT", ATHENS, GREECE, 1 - 4 JULY 2009

Abstract:
Although at a superficial level mathematics and painting may be perceived as of very different nature, they are profoundly similar at a deep conceptual and functional level. The similarity goes far beyond immediate relationships such as the fact that geometry plays a significant role in both disciplines. The deep and significant kinship between math and painting becomes evident, when one considers that both disciplines are concerned with a symbolic description of aspects of the surrounding world. Both painting and mathematics struggle to express, by abstraction, the general behind the specific and to establish the essential and relevant. Both disciplines try to digest and analyse notions such as open versus closed, or figurative versus non-figurative, or finite versus infinite. Both activities make use of conjectures and explorations.

It is important, not least for the teaching of mathematics, to realise that mathematics is fundamentally a discipline that is profoundly similar to the arts, music and humanities and in particular to painting. Mathematics will then not be considered a unique or alien discipline, but can be approached with playfulness and experimentation along the traditions used in the teaching of art, where rigor and exploration goes hand in hand.

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INTRODUCTION

Mathematics is the science of pattern; painting is articulation through pattern.

The idea that there should be some barrier between different human activities is, of course, new. We all know that Leonardo da Vinci was as much a painter as he was a scientist. Many people may also know that Isaac Newton considered his theological interests and investigations as being of equal value and importance as his scientific work. These few remarks just to indicate that simultaneous interest and gift for art and for mathematics are far from contradictory, but rather on the contrary can be wonderfully complementary.

Mathematics and painting are related in many ways and at many levels. One relation is the technical. One may think of the geometrical methods behind linear perspective drawing as for instance developed by the Italian architect and engineer Filippo Brunelleschi around 1400. In contrast one can mention the thematic relation represented by Maurits Escher. He was fascinated by mathematical concepts and ideas, which inspiring him to develop pictures depicting what, in a strict reality, is impossible, such as periodically connected water falls perpetually falling downwards.

Here I'll argue that at a deep level mathematics and painting are similar. Fundamentally, both mathematics and painting strive to develop a representation of the surrounding reality. And in both cases one is aiming at a representation possessing a large degree of universality. We see that this endeavour has been successful indeed in cases such as Euclid's geometry or the wall paintings of Knossos. Despite the dramatic change in worldview during the centuries that separates us from Euclid or from Knossos' artist, we understand and appreciate the mathematics or the art. I will emphasise the following fact. Namely, that painting and mathematics share a programme aimed at developing symbolic, concise, and often very abstract, representations of reality. This is in both cases done by a two-stage process. First the essential, the most general aspects of reality are mentally digested. Then follows a process of experimentation by which one tries to establish the most adequate way to represent in symbolic form the identified structures or themes. This shared agenda is the reason for some very close mental and conceptual parallels between mathematics and painting. Below I will discuss, how notions such as, say, open and closed, or finite and infinite are themes considered, manipulated and investigated in both mathematics and painting. As part of my argument I will relate some examples of mathematical formalism and concepts to a selection of paintings.

Finally I'll examine some teaching strategies for mathematics inspired by the close relation between painting and mathematics.

THE TECHNICAL RELATION

That mathematics and painting can be related is easily seen when one thinks of the use of mathematics to analyse paintings. Perspective has been approached from a mathematical angle for centuries and in more recent time fractals have been realised to be the best way to geometrically describe many natural occurring objects such as mountains or clouds. A few years ago Taylor, Micolich and Jonas (1991) attracted considerable interest by presenting a quantitative analysis of Pollock's drip paintings. The mathematical analysis consisted in measuring how the paint is distributed across the canvas. The authors concluded that Pollock's drops of paint forms fractal geometrical structures. From this technical investigation Taylor et al. then pointed out that in a way Pollock's paintings are figurative since also many objects in nature have fractal shapes. This kind of relation between painting and mathematics is an external purely analytic one (Jensen 2002). It demonstrates the usefulness of mathematics applied to the investigation of paintings. The relationship established this way between mathematics and painting is similar to the relationship between mathematics and any subject that can be studied by the use of mathematics. We can for instance with great success make use of mathematics when we design a bridge but the actual construction of the bridge may be an enterprise entirely different from doing mathematics. The purpose of bridge building and of mathematics remains very different although one (mathematics) may be a useful *component* in the other (bridge building). The relationship between mathematics and painting is different. They both share a common aim and purpose; moreover, the mental activities involved in the two have many similarities.

THE CONCEPTUAL PARALLELS

One objective, mathematics and paintings obviously share, consists in the striving to represent aspects of reality that cannot adequately be captured by words alone. A painting of a person may at one level represent, say, a woman but at another level may seek to reach far beyond the concrete object. Perhaps most clearly this is seen in the aspiration of the icon painters of Orthodox Christianity to create imagery that reaches into a different dimension of reality. When the religiously minded person meditate on an icon of Saint Mary, it is not the portrait of a women that develops in the mind of the beholder, it is rather a set of transcendental concepts related to the *Theotokos*, the *God-bearer* or *the one who gave birth to God*. The icon is used to generate, or communicate, abstractions far beyond the immediate physical world. The icon attempts to handle concepts that cannot be reduced to the low "bandwidth" of our daily language. Obviously it is not only religious art that finds its proper level of expression beyond the concrete. Any painting, whether figurative or abstract, is, to one degree or another, an endeavour to create in the mind of the observer perceptions and feelings that are larger than what can be capture by a bare listing of the shapes on the canvas.

Holy Icon of Theotokos Threnody (ca. 1860-1880) by Ioasaf Athonites

The situation is similar in mathematics. A conceptual universe is constructed with some anchoring points in the physical observable reality, from which explorations into the realm of the abstract are launched. An example can be the route from natural whole numbers through irrational number to the complex numbers. In this example we begin with concepts that can be represented by material objects, say number of beads on an abacus, and we arrive at concepts like imaginary numbers, for which we can't find a simple tangible representation[1]. However, the imaginary numbers are certainly able to reach into a realm of conceptual reality that is very real. We would for example not be able to represent the quantum mechanical world of atoms, if we didn't make use of complex numbers – or some equivalent algebraic structure.

Here it is also natural to mention the fact that both painting and mathematics make use of representations that, when observed by a person with the appropriate background, create mental associations not directly referred to. Recall how the above Icon of Theotokos, when viewed by, say, an orthodox Christian, can stimulate associations to entities not visually present in the picture. Now compare this transcendental content of the icon to the effect on a mathematically trained mind of the diffusion equation

$$\frac{\partial \varphi}{\partial t} = D\nabla^2\varphi .$$

For the uninitiated the equation probably doesn't make much sense, whereas for one trained in mathematics concepts such as heat, random walks, cream in coffee and many more may spring to mind. And this despite the equation in no direct way refers to any of these entities.

The conceptual similarity between mathematics and painting indicated here is, in my opinion, the underlying reason for a number of communalities in how painting and mathematics can be structured. Both disciplines make conscious and fundamental use of emergence or, what may be denoted, non-linear construction. I will first illustrate the idea by an example and then point to the similarity between Euclid's geometry and Kandinsky's theory of the spiritual significance of geometrical objects and shapes.

Let us consider the following simplistic artistic creation to illustrate the fact that both painting and mathematics will combine components together to obtain a sum that is greater and typically of an entirely different nature than the components. E.g. a smiley will use " •", "•", "|" and " ◡ " and "|" to represent a mental mood of a person . The smiley  generates emotion in the receiver, whereas the unassembled individual components: " •", "•", "|" and " ◡ " don't. Let us point out that the generation of an emotional state by the smiley in this example really involves an infinite number of interacting components. These components consist of the entire cognitive machinery of the observer together with the "cultural" heritage, which enables the observer to register and to interpret the emotional content of the four geometrical shapes organised in a smiley configuration.

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[1] Identification of complex numbers with points in the plane may appear to be a simple representation equivalent to representing natural numbers by beads or real numbers by length of intervals. However it should be kept in mind that the geometrical representation of complex numbers only came late. The reason is probably that the relation between points and complex numbers is a fairly complicate one and that points in the plane only represent complex numbers, if we assume the appropriate algebra to make adding and multiplication of the points equivalent to adding and multiplying complex numbers.

Similar, in non-linear mathematics new emergent properties – which are “bigger than the sum of the parts” – are obtained in various ways. In all cases some kind of interaction between the parts plays an essential role. A simple example is the study of non-linear functions of few variables, say f(x,y) = (x+y)^2. The behaviour of the square sum is qualitatively different from the linear behaviour of the individual independent variables “x” and “y”. A more interesting case (and more similar to the smiley example above) is where the new emergent entity is obtained in the limit of infinitely many interacting variables. The mathematically description of phase transitions is an interesting example. Think of a collection of water molecules, the rigidity of the solid ice that appears as the temperature is lowered, is mathematically understood by analysing the interaction amongst virtually an infinite number of molecules. The emergence of rigidity out of collective interaction amongst the molecules can be seen as an analogy to, the way interaction between the components “ •”, ”•”, “|” and “”,  and the brain together transforms the little painting  into an emotional state.

Let us now briefly mention the remarkable similarity between Euclid’s geometry and the attempt by Kandinsky to establish an analytic and axiomatic foundation for the art of painting. A few quotes will illustrate the parallels.

From the first Book of Eucild we have (Euclid’s Elements web reference):

Definition 1. A point is that which has no part.
Definition 2. A line is breadthless length.
Definition 3. The ends of a line are points.
Definition 4. A straight line is a line which lies evenly with the points on itself.
Definition 5. A surface is that which has length and breadth only.
Definition 6. The edges of a surface are lines.
…

As we all know Eucild establishes his geometry by deducing relationships between the objects introduced in his set of the definitions.

Now compare this with Kandinsky’s book *Point and Line to Plane* from 1914 (Kandinsky 1979). He writes:

<u>Elements:</u> The first unavoidable question is, naturally, the question of the art elements, which are the building materials of works of art and which, as such, must be different in every art.

Kandinsky then moves on to introduce definitions:

<u>The Geometric Point:</u> The geometrical point is an invisible thing. Therefore, it must be defined as an incorporeal thing. Considered in terms of substance, it equals zero.

Hidden in this zero, however, are various attributes which are “human” in nature. ….

<u>The geometric line</u> is an invisible thing. It is the track made by the moving point; that is, its product. It is created by movement – specifically through the destruction of the intense self-contained response of the point…

The term “Basic Plane” is understood to mean the material plane which is called upon to receive the content of the work of art.

It will be designated by BP.
The schematic BP is bounded by 2 horizontal and 2 vertical lines, and is thereby set off as an individual thing in the realm of its surroundings.

Kandinsky, like Euclid, derives from the definitions properties of his building blocks. He notes, for example:

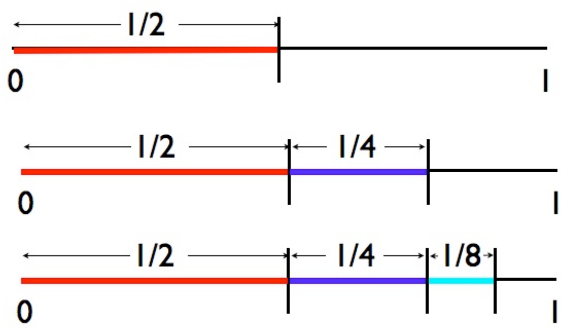
The line, is therefore, the greatest antithesis to the pictorial proto-element – the point. Viewed in the strictest sense, it can be designated as a secondary element.

These two small extracts should suffice to illustrate that both mathematics and painting can, at least sometimes, be treated as deductive axiomatic endeavours. Some would perhaps argue that the close relation between Euclid and Kandinsky is a consequence of both being concerned with geometry in one form or another. One might therefore want to argue that the example of Euclid and Kandinsky doesn't demonstrate a general kinship between mathematics and painting. This point can be addressed by considering concepts, which are not of simple geometrical nature. I have elsewhere discussed how the mathematical concepts *open* and *closed*, used in set theory, sometimes can be related to the overall ambience of paintings (Jensen 2002). All I want to repeat here is that a painting like van Gogh's "Room in Arles" (1889) (Musee d'Orsay, Paris) conveys the impression of a corner of the world closed off and isolated from the rest of the universe. The doors are closed and a bed blocks one and a chair the other. The room has a window, but the view through the window is somehow obstructed, perhaps by shutters on the outside. J.M.W. Turner expresses the opposite in his painting "Snowstorm" (1842) (Tate Britain, London). In Turner's painting one senses how the boundary of the painting is a formality. The picture is open and expresses clearly that the entire world is engulfed in a tremendous and all consuming storm.

To expand upon how the same notion is investigated in our two disciplines let us now consider a concept that goes beyond any possible direct physical representation. The idea of infinity is central in mathematics. Zeno's paradox – a very early example of the tortoise and Achilles – is probably known to most and involves the ability to add infinitely many numbers together. The reason Achilles will be able to overtake the slow tortoise is, of course, that Achilles only needs a finite time to cover the distance between him and the tortoise. The tortoise managed to confuse Achilles by breaking the distance up in to a ½ plus a ¼ plus an ⅛ etc. The tortoise did this to make it appear that Achilles would need an infinite long time to move through the infinity of fractional distances. A simple drawing makes clear that the sum

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$$

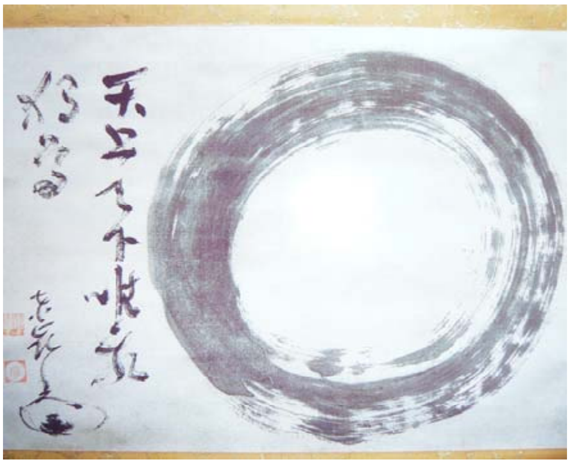
simply consists of first moving halfway from 0 to 1, followed by moving halfway from ½ to 1 and so on. Thus we keep adding half of the distance between our present position and 1:



From this it becomes clear that

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots=1.$$

I.e. nice and finite. The mind and mathematics can handle the infinite in one flash and end up with a finite result. It is irrelevant that the actual computational process, if carried out, of adding the infinitely many small fractions together will last forever.



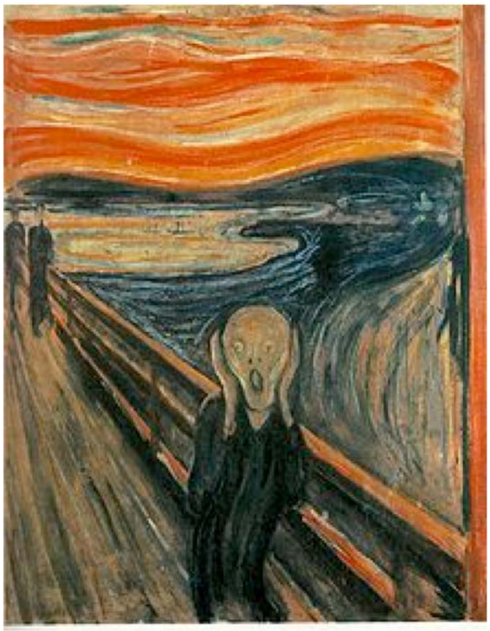
Zen Circle, or Enso, by Torei Enji (1721-1792)

The painter can, equally well as the mathematician, represent the infinite. This is superbly demonstrated in the following Enso, or Zen Circle, by Torei Enji (1721-1792); drawn with ink on paper. The Enso is in Zen Buddhism considered to symbolise enlightenment as well as strength and elegance. It also represents the universe and the void, simultaneously. I.e. one may, I suggest, think of the Enso as a representation of the infinite, the never ending and inexhaustible. It is illuminating to contrast the sense of perpetual never ceasing motion conveyed by the circular shape, ○, of the Enso to the associations induced in our mind by a triangle △. The vertices of the triangle interrupt motion and therefore the metal impression of the triangle remains in the realm of the finite.

Parallel strategies

It is interesting and illuminating to review how the same approach can be identified both in mathematics and in painting. I believe there are numerous cases of such parallels, however here I will only discuss a few brief examples. Let us first consider how figurative versus non-figurative analysis is used. In painting, Kandinsky is a towering exponent of the attempt to communicate using non-figurative codes. Much mathematics is certainly rooted in geometry and the analysis of spatial structures, but there are fields such as number theory or group algebra in which direct figurative analogy is hardly used. In both cases after a while working in the field a mathematician will develop a mental intuitive representation of the abstract mathematical structures living in a realm beyond the figurative. This is similar to Kandinsky's programme mentioned above, in which he tried to identify the non-figurative spiritual value of abstract painted structures.

As another example of parallel approaches in painting and mathematics I want to mention the effort to extract the essential and leave out irrelevant details. Think of Edward Munch's *The Scream* or, as it was originally named, *The Scream of Nature*. The bare minimum is included to allow the overall composition to render a sense of ineffable angst. The heart penetrating effect of the painting is very much an effect of particular details *not* being allowed to distract. Had the faces or the human bodies been elaborated in more detail, our mind would immediately start to generate associations to particular people known to us. Similarly the broad strokes of the background don't allow us to see this as a particular geographical location. We are therefore forced to experience the painting as a prototype of general relevance to our own emotional life. If the screaming figure in the front had reminded me of my aunt, or the two dark figures in the background could be thought of as the two bad guys always making trouble Saturday nights in my little home town; I might not have realised that the figure in front is *myself* on my way across one of the multitude of real, or virtual bridges, I have to cross on my way through life.



The Scream by Edward Munch, Oil, tempera and pastel on cardboard.
National Gallery, Oslo

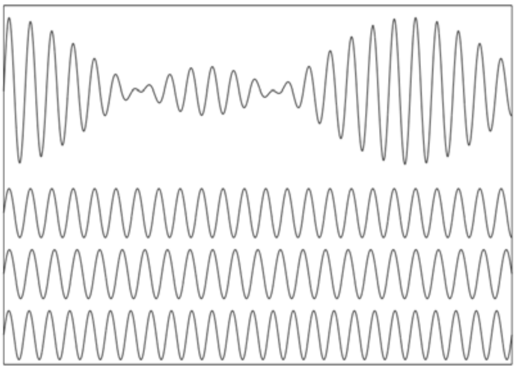
The situation is very similar in mathematics. Let us think of Newton's equation of motion

$$m\frac{d^2x}{dt^2} = F.$$

The equation describes the trajectory, i.e. the position $x(t)$ for any time *t*, of any object of mass *m* subject to the force *F*. The equation makes clear that only the mass and the force matter. We do not need to worry about the colour or the shape of the object. Both Munch's painting and Newton's law are able to study the issue in question, world angst or particle motion, respectively, neglecting an infinity of aspects. And in fact it is a crucial part of the understanding to realise that most details are irrelevant

In more concrete ways the painter and the mathematician also often use corresponding techniques. I want to mention just two such examples. The first is superposition. That is adding components together to obtain a whole that is different, and sometimes in various ways also bigger, than the collection of the parts. Superposition in mathematics can for instance consist of adding oscillating waves together to get a new temporal behaviour. The following equation adds together three sine waves each with frequencies, θ_1, θ_2 and θ_3, a little different from the others, to produce a new time signal $f(t)$:

$$f(t) = \sin(\theta_1 t) + \sin(\theta_2 t) + \sin(\theta_3 t).$$



The result of the three added waves together with the three original waves is shown in the figure to the right. Interference between the three components leads to a behaviour of the sum that is markedly different in nature from the behaviour of the individual components. This effect is of course what string musicians make use of when they tune their instrument by listening for the sound "beat" between nearly tuned stings.

The emergence of a new quality due to the combined effect of components can also be seen in the painting below of landmarks from London.






The painting was painted from memory after moving around in London and tries to capture the effect of the myriad of contrasts one encounters in the great metropolis. The photos are included in retrospect and collected from the Internet. Although some of these structures are at quite separate locations in London, the net impression, left on the mind of a person moving through the city, is one crowded with buildings and people. In the memory the individual landmark has a tendency to loose its integrity and is recollected as part of a blend. The non-realistic juxtaposition in the painting attempts to generate a sensation of the overload of the mind experienced after a trip through London.

Our final example of conceptual parallels between mathematics and painting is concerned with the use of hierarchies. In mathematics one encounters many different kinds of hierarchical structures. Think of how the natural numbers, $\mathbb{N}$, sit inside the integers, $\mathbb{Z}$, which are located inside the rational numbers, $\mathbb{Q}$, which are part of the real numbers, $\mathbb{R}$, which in turn are inside the complex numbers, $\mathbb{C}$. Or more simply stated by use of mathematical notation

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

Modern mathematics is built around the theory of abstract sets, which is a formalised way of dealing with things that are to be found within each other. Set theory is used persistently across all fields of mathematics and has allowed a much greater precision than was possible before the development of modern set theory. Russell's paradox is a famous example of complications that can arise when hierarchies and self-reference are combined. Russell suggested to "consider a set containing exactly the sets that are not members of themselves". These considerations lead Russell to a paradox similar to the one introduced by the mythical *Cretan* Epimenides, when he claimed: "That all Cretans are liars". So surely, hierarchies are powerful mathematical structures.

In fact hierarchical structures are so powerful and so natural that paintings often make use of them to evoke a global and delocalised impression. Think e.g. of Picasso's paintings such as "Portrait of Ambroise Vollard" or "The Clarinet Player".

The painting to the right is a semi-figurative example of a hierarchical composition. The same theme repeats itself in a nested fashion throughout the painting whereby it becomes difficult to identify regions more important than others. Perhaps this helps to establish an impression of the theme's all encompassing transcendental character.

CONSEQUENCES

Let us now turn to the teaching of mathematics. The above outlined parallels between painting and mathematics suggest it may be useful to introduce children to mathematics by a broad open-minded experimental strategy. The similarities between mathematics and painting also imply that essentially everyone can become acquainted with mathematics. It is not unusual to hear people, and even teachers, presenting the viewpoint that a certain type of brain is need to do mathematics. Perhaps this may be true for doing mathematics at the highest research level; in the same way that Picasso had a certain aptitude for painting, Mozart clearly had unusual musical abilities and Einstein a better feeling for physics than most others. However, all this doesn't mean that either you have Picasso's talent or you don't paint. Or you either have Mozart's genius or you can't sing a song, or you either have Einstein's intellect or you simply cannot understand a thing about physics. I believe that the similarities between mathematics and painting, which I have discussed above, indicate that doing mathematics is natural to the human way of thinking. Namely, in the same way as it is in the nature of most children and adults to make drawings once in a while. Small children make paintings and drawings, whenever they have a pencil and a piece of paper (or a wall with a nice piece of wall paper) within reach. Presumably children do this as a way to digest the surrounding world. Adults make scribbles when they sit and think. Scribbles are a symbolic representation of thoughts. Scribbling is an aid to thinking no matter whether one scribbles subconsciously or as a conscious component of an analytic thinking process. My point is that basic aspects of mathematical thinking is so close to the processes involved in painting that everyone that has the ability to make simple drawings or scribble is able to do mathematics at least at some basic level.

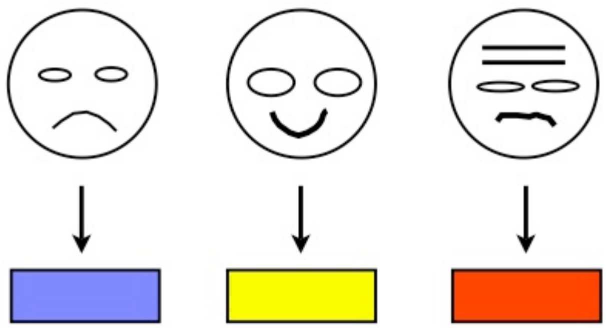
So why don't we see children and adults sit and do mathematics spontaneously the same way they playfully make drawings. The reason is the same as why we don't all impulsively make exclamations in Aramaic. It is not because of some inability in our brain or because we don't posses a special talent for Aramaic. It is because we need some training in the specifics of the Aramaic language. In the same way we need to provide training in mathematics to enable people develop their natural potential. If we consider the teaching of mathematics from the perspective that mathematics and painting share essential similarities we are likely to change our pedagogical attitudes. I suggest it may be useful to approach some of the conceptual ideas of mathematics by asking pupils or students to make some paintings or drawings aimed at directing the students' imagination to the most important overall aspects of the mathematical topic to be discussed. Perhaps students may this way realise that before learning the technical aspects of mathematics we need to develop our ability to

1 – do analysis: identification of the essential.
2 – learn to combine and compose.
3 – get acquainted with symbolic language

The following three concrete examples should help to clarify how I imagine this can be done.

(A) Mathematics topic: Set theory – open and closed sets.
Painting exercise: Draw a structure that reaches beyond the paper, draw another one that is entirely confined to the paper with no reference to anything exterior to the drawing.

(B) Mathematics topic: Functional, i.e. $y = y(x) \mapsto J[y]$
Painting exercise: make a series of smileys of different mood: sad, happy, angry, sick etc. Associate to each mood a single colour that most closely represent the mood.



(C) Mathematics topic: Be able to see the content behind the formalism.
Painting exercise: Discuss the emotional content of Munch's Scream.

CONCLUSION

We are at the moment faced with a paradox and a serious problem. Our world is increasingly relying on mathematics. Every aspect of modern life is permeated by mathematics: commerce, technology, engineering, science, medicine, biology, sociology, psychology, linguistics etc. all make heavily use of mathematics. However, at the same time we experience throughout the world – at least in the West – a significant decline of the level of mathematics taught in primary education as well as a fall at secondary and higher education in the number of students choosing subjects involving mathematics. I believe a change in attitude is needed. We must make it evident to pupils and students that mathematics involves much more than the learning by heart of mechanical algorithms from a bygone era. By relating the conceptual foundation of mathematics to other human forms of expression, in particular to the art of painting, we may be able to convey the understanding that mathematics is not a dusty irrelevant enterprise. We will be able to demonstrate that mathematics is one of the most exciting intellectual activities humans can participate in. We want to make our pupils and students realise that it is as rewarding to learn from spending time together with Euler's thoughts as it is to be exposed to Picasso's paintings and Mozart's music.

ACKNOWLEDGEMENT

I am indebted to Vibeke Nordmark Hansen, Barbara Nordmark Jeldtoft Jensen and Paolo Sibani for helpful comments on the manuscript.

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http://aleph0.clarku.edu/~djoyce/java/elements/bookI/bookI.html

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