Induced replication and the assessment of models

Heather Battey* and Nancy Reid†

June 2, 2026

Abstract

The paper is a first attempt at a foundation of inference for the assessment of semiparametric and other highly-parametrised models, seeking reconciliation with the low-dimensional Fisherian foundations. We highlight the possibility, in some contexts, of avoiding the usual semiparametric considerations, which typically require estimation of nuisance components through kernel smoothing or basis expansion, with the associated difficulties of tuning-parameter choice that blur the distinction between estimation and model assessment. A key aspect of the present work is the inducement of replication under the postulated model. This can be cast in terms of some non-standard inferential separations, in the vein of ancillarity/co-ancillarity and sufficiency/co-sufficiency separations, allowing the replacement of out-of-sample prediction error as a criterion for semiparametric model assessment by a type of within-sample prediction error. Framed in this light are new formulations in example settings, including model assessment for the proportional hazards model, for a time-dependent Poisson process with semiparametric intensity function, and for matched-pair and two-group problems. Also subsumed within the framework is a post-reduction inference approach to the construction of confidence sets of sparse regression models. Perturbative cases are probed, both analytically and numerically. We conclude by emphasising open challenges and further unifying perspectives.

Some key words: co-sufficiency; exchangeability; foundations of model inference; inferential separations; model adequacy; post-reduction inference.

1 Introduction

Two widely used approaches to assessing regression models, semiparametric or otherwise, are to substitute unknown regression parameters by estimates and check residuals for any anomalous behaviour, or to assess the predictive performance of the fitted model. The visually compelling but sometimes informal approaches based on residuals, through their connection to co-sufficiency in parametric models, can be viewed as a way of evaluating predictive accuracy within sample.

If a statistic $S = s(Y)$ constructed from observations $Y = (Y_1, \ldots, Y_n)$ is sufficient for a parameter θ, then it contains all the information in the data relevant for inference on θ,

*Department of Mathematics, Imperial College London, UK, h.battey@imperial.ac.uk.

†Department of Statistical Sciences, University of Toronto, Canada.

potentially leaving information remaining in Y for assessment of the model, an observation due to Bartlett (1937). Specifically if the observed value y^o is extreme when calibrated against the conditional distribution of Y given the observed value $S = s^o$, then this provides evidence against the model (Barndorff-Nielsen and Cox, 1994, p. 29). We call this residual information after conditioning *co-sufficient* in line with terminology used in earlier work. The idea, although conceptually appealing, typically does not translate to a convenient statistical procedure, as the conditional distribution is multidimensional, often constrained to a manifold of inexplicit form, and usually does not have a convenient analytic expression. Simulation to calculate extremal regions is not straightforward either, as simple bootstrap draws from the data violate the constraint $S = s^o$. Early progress was made by Engen and Lillegård (1997), Lindqvist and Taraldsen (2005), and Lockhart et al. (2007). More recently, Barber and Janson (2022) substantially advanced the methodology and associated theory. See also Battey (2024) for some geometric perspectives and Battey, Rasines & Tang (2025) for an analytic approach for a particular setting, outlined in §3.3. The purpose of the present paper is to explore other inferential structures, some being generalisations of the sufficiency/co-sufficiency separation, others involving a form of ancillarity/co-ancillarity separation.

If a model contains a genuinely infinite-dimensional component, there is no hope for either inference or model assessment from n observations. In practice, however, semiparametric models are not genuinely infinite dimensional. For example, smoothness assumptions imply an approximating basis of order smaller than n, implicitly constraining the effective parameter space. The smoothness might be imposed using a truncated basis expansion, by regularisation, or by kernel smoothing. The associated tuning-parameter selection for such estimators is typically based on cross-validation, blurring the distinction between estimation and model assessment.

The present work is a first attempt to reconcile inference for semiparametric and other highly-parametrised models with the Fisherian parametric foundations, by illustrating the feasibility and merits of an approach to model assessment that evades estimation of the high- or infinite-dimensional nuisance component. The conceptual points are conveyed via deliberately idealised examples that isolate relevant structures in an incisive form. A general theory in this vein probably entails a means of inducing approximate replication, in a sense that will become clear.

The following example exposes two important points: that smoothness or other assumptions on infinite-dimensional nuisance parameters can sometimes be removed, and that model assessment can be performed without estimating nuisance parameters, and without an additional sample to assess out of sample prediction error.

Example 1.1. (Battey and Cox, 2020). Matched-comparison studies are an effective method of experimentation in many applied fields in which the units within a pair share the same outcome distribution in the absence of treatment differences. Let (Y_{j1}, Y_{j0}) be outcomes on treated and untreated units in the jth of m matched pairs of units. Consider a model in which (Y_{j1}, Y_{j0}) are independent and exponentially distributed with rates $\gamma_j\psi$, γ_j respectively, ψ being a parameter representing treatment effect and γ_j capturing arbitrary within-pair dependencies and between-pair differences. By treating γ_j as a fixed arbitary constant, the model is semiparametric, most easily seen on noting that, had covariates $x \in$

$\mathbb{R}^p$ been available, the same model would have been valid with $\gamma_j = h(x_j)$ for $h : \mathbb{R}^p \to \Gamma$ of arbitrary and unspecified form. The formulation in terms of $m+1$ unknown parameters thus makes fewer assumptions than a standard semiparametric formulation postulating a smoothness class for h.

Battey and Cox (2020, §5) proposed the following approach to model assessment in the present example, implicitly broadening Bartlett's (1937) sufficiency separation to allow interest-parameter-dependent sufficient statistics.

Suppose, for an analogue of a proof by contradiction, that the true distribution belongs to the postulated model and has true parameter value ψ^*, then $Y_{j0} + Y_{j1}\psi^* =: S_j(\psi^*)$ is sufficient for γ_j and has density function

$$f_{S_j(\psi^*)}(s) = \gamma_j^2 s \exp(-\gamma_j s), \tag{1}$$

i.e., $S_j(\psi^*)$ is gamma distributed with shape parameter 2 and rate parameter γ_j. The conditional density of Y_{j1} at y_{j1}, given $S_j(\psi^*) = s_j(\psi^*)$, is $\psi^*/s_j(\psi^*)$ uniformly in y_{j1}, showing that Y_{j1} is conditionally uniformly distributed between 0 and $s_j(\psi^*)/\psi^*$. Equivalently $U_j(\psi^*) := Y_{j1}\psi^*/s_j(\psi^*)$ has a standard uniform distribution for all $j = 1, \ldots, m$. The above conclusions are invalidated if ψ^* is replaced by any other value ψ_0, or if the model is wrong, suggesting a route to assessment of model adequacy via the empirical behaviour of $U_1(\psi_0), \ldots, U_m(\psi_0)$ for a postulated value ψ_0, which can be viewed as analogues of residuals.

Since some postulated values ψ_0 may produce a distribution for $U_1(\psi_0), \ldots, U_m(\psi_0)$ that is hard to distinguish from a standard uniform sample even when the model is violated, it is sensible, unless m is small, to replace ψ_0 by the realisation of a consistent estimator $\hat{\psi}$. In this example, the transformed random variables $Z_j = Y_{j1}/Y_{j0}$ are, under the model, identically distributed with density function

$$f_Z(z; \psi^*) = \frac{\psi^*}{(1 + \psi^* z)^2}. \tag{2}$$

Since (2) is free of all the pair-specific nuisance parameters, inference on ψ^* can be constructed from a likelihood function based on (2), the resulting maximum-likelihood estimator $\hat{\psi}$ being consistent as $m \to \infty$. An approach replacing ψ^* by $\hat{\psi}$ in $U_1(\psi^*), \ldots, U_m(\psi^*)$ is formally justified through an application of Proposition 2.1 in §2.3. □

Example 1.1 replaces the usual semiparametric approach of assessing prediction error after regularised estimation by an in-sample assessment based on the structure of the postulated model, avoiding, without sample splitting, any post-selection model inference issues that would arise from using the same sample to choose tuning parameters and to assess the model.

Section 2 extracts the most important structure from Example 1.1, aiming to probe the foundations of model inference for semiparametric and other highly parametrised formulations. A requirement of the framework is an inducement of a replication of known form if and only if the model is correctly specified to an adequate approximation. The underlying logic is that of proof by contradiction, familiar from classical statistical testing and illustrated in Example 1.1.

Exact replication may not always be achievable, and it remains an open question whether a mechanism can always be found to approximate it. The present paper makes a more modest contribution, showing through carefully-constructed examples the diversity of ways in which internal replication can be induced.

2 Broad formulation

2.1 Joint assessment of a model and its interest parameters

Suppose that there is structure in the postulated model ensuring the existence of new independent random variables $U_j(\psi_0)$, $j = 1, \ldots, m$, for every candidate value of the interest parameter ψ_0, such that $U_j(\psi_0)$ follows a standard uniform distribution if and only if the true distribution with parameter value ψ^* belongs to the postulated model and $\psi_0 = \psi^*$. Standard uniformity is a convenient convention and is equivalent to the existence of any set of m ψ_0-dependent random variables whose distribution is known under the postulated model at $\psi_0 = \psi^*$. The special case where $m = 1$ and $U_1(\psi_0) = U_1$ does not depend on ψ_0 can be viewed as a framing of Bartlett's (1937) approach to parametric model assessment based on co-sufficiency. Operationalisation of his proposal has received little attention, but in cases where this has been attempted, the approach appears to have little or no power without modification (Barber and Janson, 2022; Battey, Rasines & Tang, 2025). When an extension to $m > 1$ is available, which is achieved by inducing internal replication, power to detect an erroneous model at a given $\psi_0 = \psi^*$ is in principle achievable provided that $U_1(\psi_0), \ldots, U_m(\psi_0)$ are not statistically indistinguishable from a sample of standard uniform random variables when the postulated model is wrong.

The following initial discussion considers model adequacy in terms of a confidence set for the parameter ψ, before introducing in §2.3 a more powerful approach mirroring that in Example 1.1.

A simple way to assess joint compatibility of the model and a parameter value ψ_0 is by Fisher's method for combining p-values (Fisher, 1932), whose traditional usage is in meta analysis. Specifically, with G_{2m} the distribution function of a χ^2 random variable with $2m$ degrees of freedom, the two-tailed p-value is $2\min\{p(\psi_0), 1 - p(\psi_0)\}$, where $p(\psi_0) := G_{2m}(-2\sum \log U_j(\psi_0))$. The resulting α-level confidence set for the interest parameter ψ under correct specification of the model is

$$\mathcal{C}(\alpha) := \Big\{\psi_0 \in \Psi : 2\min\{p(\psi_0), 1 - p(\psi_0)\} > \alpha\Big\}. \tag{3}$$

If the confidence set is empty at a chosen level α, that casts doubt on the adequacy of the model. The use of Fisher's statistic in (3) is convenient for analytic calculations of power because the expectation and variance of $R_j = -\log U_j$ simplify, by integration by parts, to

$$\mathbb{E}(R_j) = \int_0^1 \frac{F_U(u)}{u} du, \quad \mathbb{V}(R_j) = -2\int_0^1 \frac{\log(u) F_U(u)}{u} du - \Big(\int_0^1 \frac{F_U(u)}{u} du\Big)^2, \tag{4}$$

where F_U is the distribution function of U_j.

Both moments increase as the density function f_U of U concentrates near zero, and the mean exceeds half the variance, and hence departs from the null χ^2_2 behaviour, when

f_U departs from uniformity asymmetrically towards zero. When departures instead occur towards 1, the mean and variance are typically both too small to be compatible with the null distribution, although in this case sensitivity can be increased by replacing $U_j(\psi_0)$ by $1 - U_j(\psi_0)$ in $p(\psi_0)$ from (3). Thus, if the majority of departures from uniformity are in the same direction, $\mathcal{C}(\alpha)$ is empty with high probability for sufficiently large m when the postulated model is misspecified. More irregular departures from uniformity may be better detected by alternative combination rules (see e.g. Birnbaum, 1954; Heard and Rubin-Delanchy, 2018).

2.2 Two parallel analyses

In subsequent sections, we present the main conceptual development of the paper via examples. These illustrate different ways through which internal replication can be induced under the postulated model, and through which contradictions are forced when the model is violated. Before turning to those constructions, we analyse the behaviour of the resulting confidence set $\mathcal{C}(\alpha)$, assuming the existence of random variables $U_j(\psi_0)$. The aim is not to advocate Fisher's method but to supply qualitative insight through a reasonably general analysis, showing how systematic departures from uniformity translate to eventual emptiness of $\mathcal{C}(\alpha)$ as $m \to \infty$. Calculations at this level of generality are inevitably idealised.

When the model is erroneous or if it contains the true distribution but $\psi_0 \neq \psi^*$, then $U_j(\psi_0)$, $j = 1, \ldots, m$, are by definition not standard uniform, and for most of the example cases we have in mind, are not identically distributed either. Since $U_j(\psi_0)$ is necessarily supported on $[0, 1]$, insight can be obtained by approximating its density function using the parametric family

$$(1 - \vartheta_j)u^{-\vartheta_j}, \quad 0 < u < 1, \quad 0 < \vartheta_j < 1, \tag{5}$$

so that the null distribution is recovered in the limit as $\vartheta_j \to 0$. The model (5) is deliberately oversimplified in that it specifies the directions of departure from uniformity to be in the same direction for all j. Specifically, the distribution (5) concentrates towards zero for all ϑ_j, the opposite behaviour achieved only by replacing $U_j(\psi_0)$ by $1 - U_j(\psi_0)$.

Under (5), the kth moment of $R_j := -\log U_j(\psi_0)$ has the convenient form

$$\mathbb{E}(R_j^k) = \frac{\Gamma(k+1)}{(1-\vartheta_j)^k}, \quad k \in \{1, 2, \ldots\}.$$

The mean and variance of $R := 2\sum_j R_j$ are

$$\mathbb{E}(R) = 2\sum_{j=1}^{m}(1-\vartheta_j)^{-1}, \quad \mathbb{V}(R) = 4\sum_{j=1}^{m}(1-\vartheta_j)^{-2}, \tag{6}$$

recovering the χ^2_{2m} mean and variance as $\vartheta_j \to 0$ for all j and showing that R has a larger mean and variance than a χ^2_{2m} random variable, with the discrepancies increasing as $m \to \infty$.

On letting k_α be the $1-\alpha$ quantile of the χ^2_{2m} distribution, Markov's inequality in the form presented in Appendix B shows that

$$\mathrm{pr}(R \geq k_\alpha) \geq 1 - \inf_{s>0}\left(e^{sk_\alpha} \prod_{j=1}^{m} \frac{1-\vartheta_j}{1-\vartheta_j+2s} \right). \tag{7}$$

The constituent terms are in the interval $(0,1)$ for all s in the permissible range, so the product tends to zero exponentially fast in $m \to \infty$.

For a parallel analysis that does not involve specifying a parametric family for $U_j(\psi_0)$, let $\mu_m(\psi_0) = \sum_{j=1}^m \mathbb{E}(R_j)$ and $\tau_m(\psi_0) = (\sum_{j=1}^m \mathrm{var}(R_j))^{1/2}$. Then by the central limit theorem $\tau_m(\psi_0)(\sum_{j=1}^m R_j - \mu_m(\psi_0))$ is asymptotically standard normal for large m, thus

$$\mathrm{pr}(R \geq k_\alpha) \simeq 1 - \Phi\left(\frac{k_\alpha - 2\mu_m(\psi_0)}{2\tau_m(\psi_0)} \right) \simeq 1 - \Phi\left(\frac{2m + 2\sqrt{m} z_\alpha - 2\mu_m(\psi_0)}{2\tau_m(\psi_0)} \right), \quad (m \to \infty),$$

where z_α is the $1-\alpha$ quantile of the standard normal distribution and where we have used the normal approximation to the χ^2_{2m} distribution for large m. Under the null, where the distribution of $U_j(\psi_0)$ is uniformly distributed for all j, R_j has a unit exponential distribution, implying that $\mu_m(\psi_0) = m$ and $\tau_m(\psi_0) = \sqrt{m}$. In this case, the previous display reduces to $\mathrm{pr}(R \geq k_\alpha) \simeq \alpha$, which is the exact answer for any m. When the null distribution is violated, $\mathrm{pr}(R \geq k_\alpha) \to 1$ if and only if $(k_\alpha - 2\mu_m(\psi_0))/2\tau_m(\psi_0) \to -\infty$, which requires that the mean grows faster than the standard deviation as $m \to \infty$. There is a parallel calculation for the left tail.

2.3 Model assessment using a null-consistent estimator of ψ^*

Example 1.1 illustrates a situation in which a consistent estimator of the interest parameter is available when the postulated model contains the true distribution, without any need to estimate the high-dimensional nuisance component. When a null-consistent estimator is available, and when the effective sample size is large enough that sampling variability of $\hat{\psi}$ around ψ^* does not materially affect the null distribution, there is no need to assess $U_j(\psi_0)$ for all plausible values ψ_0; instead $\hat{\psi}$ can be used in place of ψ_0, leading to the statistic

$$\hat{R}_m = -2 \sum_{j=1}^{m} \log U_j(\hat{\psi}), \tag{8}$$

and its complementary version with $1 - U_j$ in place of U_j. The discussion following (5) illustrates that the two need not be interchangeable for the purpose of refuting a false postulated model, one version tending to have higher power than the other depending on the form of departure. If $\hat{R}_m$ is extreme when calibrated against the χ^2_{2m} distribution, that casts doubt on the adequacy of the model. This statement is formally justified under a bounded derivative condition on the function defining U_j as outlined in Proposition 2.1.

Proposition 2.1. Let Z_j be any continuous random variable for which $U_j(\psi) = g(Z_j, \psi)$ has a standard uniform distribution for some unknown value of ψ under the postulated model. Let $\hat{\psi}$ be an estimator satisfying $\hat{\psi} \to_p \psi_0^*$ as some $n \to \infty$, where ψ_0^* is a constant.

(i) if g has bounded derivative $\nabla_\psi g(z,\psi)$, uniformly in $\psi \in (\hat{\psi}, \psi_0^*)$, then $U_j(\hat{\psi}) \rightsquigarrow U_j(\psi_0^*)$ as $n \to \infty$; (ii) if the postulated model is true, so that $\psi_0^* = \psi^*$, and if the condition of (i) is satisfied, or if $\log g(z,\psi)$ has bounded derivative uniformly in $\psi \in (\hat{\psi}, \psi^*)$, then $\hat{R}_m \rightsquigarrow R_m^*$ as $m \to \infty$, where $R_m^* = -2\sum_{j=1}^m \log U_j(\psi^*)$.

In Example 1.1, $\hat{\psi}$ was constructed from the same $Z_1, \ldots, Z_m$ used to construct $U_1, \ldots, U_m$. Since this need not be the case, n in Proposition 2.1 is not necessarily equal to m. Sections 3.3 and 3.4 provide examples where the values differ.

Proposition 2.1 establishes weak convergence of $\hat{R}_m$ to a χ^2_{2m} random variable. Quantification of the approximation error is more difficult as the constituent terms in the sum are dependent. There may be a route to it via Chen (1975) on noting that $\text{pr}(R_m^* > t) = \text{pr}(N_t < m)$, where N_t is the number of events in the interval $(0, t)$ in a Poisson process of rate 1.

When the model is violated, $\hat{\psi}$ either converges in probability to a a fixed value $\psi_0^* \neq \psi^*$, violating asymptotic standard uniformity of each $U_j(\hat{\psi})$, or it has a non-degenerate limit distribution. This second scenario not only violates asymptotic uniformity of $U_j(\hat{\psi})$ for every j but also induces a complicated dependence between the $U_j(\hat{\psi})$, $j = 1, \ldots, m$ because of the shared dependence on $\hat{\psi}$, whose distribution depends on nuisance parameters when the postulated model is erroneous. In general, any such dependence is expected to exacerbate the distributional violations in the aggregate (8), so may well be beneficial for detecting departures from the model. Another possibility when the model is violated is that $U_j(\hat{\psi}) = 0$ with non-negligible probability, refuting the χ^2_{2m} approximation to the distribution of (8).

2.4 Falsifiable induced replication

A perspective of this paper is that different model structures can provide other inferential separations useful for model assessment, complementing Bartlett's (1937) sufficiency separation. In the most direct of these, there are statistics $Z_1, \ldots, Z_m$, each of which has the same conditional distribution $F_{Z|S}$, say, given the sufficient statistic $S = s$ for the nuisance parameters if and only if the postulated model is true. Example 1.1 is a version in which $S = S(\psi_0)$ is constructed using the postulated ψ_0 and is sufficient for the nuisance parameters only when the postulated model is correct with $\psi_0 = \psi^*$. In this case, under mild regularity conditions, $U_j(\psi_0) = F_{Z|S}(Z_j \mid s(\psi_0))$ will follow a standard uniform distribution under the same conditions, and otherwise will violate standard uniformity unless the postulated model-ψ_0 pair belongs to an equivalence class of the true model within which distributions induce identical conditional distributions.

In standard low-dimensional parametric contexts, an additional route to assessing model adequacy is to calibrate the observed value of an ancillary statistic against its marginal distribution. The analogue in the semiparametric examples motivating the present work is based on ancillarity for the nuisance parameter, a construction not typically used in parametric inference. Additional flexibility in the construction of replicates $U_j(\psi_0)$ uses a notion of conditional co-sufficiency, introduced in §3.4 for assessing the semiparametric proportional hazards model (Cox, 1972).

3 Non-matched examples

3.1 Introduction

The most obvious settings in which inducible replication is present under the postulated model are those in which there is a degree of replication for the nuisance parameter by design, as in Example 1.1. As illustrated in the example, there is no exchangeability across the outcomes, either within or between pair, but each nuisance parameter appears twice among the distributions of the outcomes, allowing replication to be easily induced under postulated models with an amenable structure. We start, however, with some different settings to provide the sharpest contrast to Example 1.1. Section 3.2 illustrates a more abstract form of internal replication after conditioning, §3.3 illustrates how internal replication can be achieved artificially through a carefully-designed randomisation scheme, and §3.4 illustrates the possibility of extending the notion of co-sufficiency to conditional co-sufficiency. We then return in §4 to the matched pair setting of Example 1.1, where we formalise some other constructions for achieving internal replication that do not directly correspond to co-sufficiency but follow the same broad principles.

3.2 Semiparametric time-dependent Poisson process

The following is a semi-parametric generalisation of a model proposed by Cox (1955) and best approached via Cox and Lewis (1966, pp. 45–46). Consider a time-dependent Poisson process with intensity function $\lambda_i(t) = e^{\gamma_i+\beta t}$ for the ith of n individuals. The model could be suitable for sequences of health events, in which γ_i captures individual-specific effects and β captures the general effect of ageing. It is semiparametric in the sense that γ_i encapsulates arbitrary dependence on perhaps unmeasured covariates, and might be modelled in a more conventional semiparametric formulation as $\gamma_i = h(x_i)$, where h is an unknown function of the covariate vector x_i. We avoid fitting h and the associated semiparametric considerations by eliminating all the nuisance parameters γ_i from the analysis. The calculations leading to elimination of nuisance parameters are more involved in this model than in Example 1.1, but follow the same principles.

Suppose that in a fixed interval $(0, t_0)$, events are recorded at times $(t_{i1}, \ldots, t_{im_i})$ for individual i. The likelihood contribution for the ith individual is

$$L(\lambda_i; t_{i1}, \ldots, t_{im_i}) = \exp\Big\{-\int_0^{t_0} \lambda_i(u)du\Big\} \prod_{j=1}^{m_i} \lambda_i(t_{ij}),$$

which, for the special form $\lambda_i(t) = e^{\gamma_i+\beta t}$ becomes

$$L(\gamma_i, \beta; t_{i1}, \ldots, t_{im_i}) = \exp\Big\{m_i\gamma_i + \beta \sum_{j=1}^{m_i} t_{ij} - e^{\gamma_i}(e^{\beta t_0} - 1)/\beta\Big\}. \tag{9}$$

Thus, the jointly sufficient statistics for (γ_i, β) based on the ith individual are $(m_i, \sum_{j=1}^{m_i} t_{ij})$, and the conditional distribution of $\sum_{j=1}^{m_i} t_{ij}$ given m_i eliminates γ_i. There is, therefore, an induced replication in the conditional distributions across the n individuals. The marginal

probability mass function for the number of events for the ith individual is Poisson with rate

$$\int_0^{t_0} \lambda_i(u)du = e^{\gamma_i}(e^{\beta t_0} - 1)/\beta.$$

Thus, on using the right hand side of (9), viewed as a function of $t_{i1}, \ldots, t_{im_i}$ for fixed parameter values, the conditional density is

$$f(t_{i1}, \ldots, t_{im_i} \mid m_i) = \frac{m_i! \beta^{m_i} \exp(\beta \sum_{j=1}^{m_i} t_{ij})}{(e^{\beta t_0} - 1)^{m_i}}, \tag{10}$$

which is the probability density function of an ordered sample of m_i random variables each with density function

$$f_T(t; \beta) = \frac{\beta e^{\beta t}}{(e^{\beta t_0} - 1)}, \quad 0 \le t \le t_0, \tag{11}$$

(Cox and Lewis, 1966, pp. 45–46). Equation (11) reduces to $1/t_0$ in the limit as $\beta \to 0$. It follows that the conditional distribution of $S_i = \sum_{j=1}^{m_i} T_{ij}$ given m_i is that of the sum of m_i independent random variables with density function (11). Cox and Lewis (1966) do not proceed further and suggest that a conditional likelihood based on (10) be used for inference on β. To assess the model via the approach of §2.3, explicit calculation of the conditional distribution of S_i given m_i is needed. Appendix C shows by way of a Laplace transform that the conditional distribution function with support $s < m_i t_0$ is

$$F_{S_i|M_i}(s \mid m_i; \beta) = \frac{\beta^{m_i}}{(e^{\beta t_0} - 1)^{m_i} \Gamma(m_i)} \sum_{v=0}^{\lfloor s/t_0 \rfloor} \binom{m_i}{v} (-1)^v e^{\beta v t_0} \int_0^{s - v t_0} e^{\beta w} w^{(m_i - 1)} dw, \tag{12}$$

where the integral is a gamma integral. Although this alternating sum can in principle be evaluated at the observed values of S_i, in our numerical work summarised in §5 we found that estimation of the conditional distribution of S_i was numerically more stable when based on Monte-Carlo replicates of sums of m_i random variables drawn from the density function (11). For large m_i, a normal approximation to the distribution can be used. Up to Monte Carlo sampling error, the resulting statistics follow a standard uniform distribution at the true value of β under correct specification of the model. The version based on the analogue of (8) uses the maximum likelihood estimate $\hat{\beta}$ in place of β, where $\hat{\beta}$ maximises the conditional log likelihood function based on the product of joint conditional density functions (10) over the n individuals.

Two adaptations of the model with intensity function $\lambda_i(t)$ as above have a quadratic trend of the form $\lambda_i(t) = \exp(\gamma_i + \beta_1 t + \beta_2 t^2)$ or a power-law trend of the form $\lambda_i(t) = e^{\gamma_i} t^{\beta}$. These are semiparametric analogues of models discussed by Cox (1955, p. 138), where the essential elements of the above argument apply on noting that, in the first case, m_i, $S_{i1} = \sum_{j=1}^{m_i} T_i$ and $S_{i2} = \sum_{j=1}^{m_i} T_i^2$ are jointly sufficient in the ith sample for γ_i, β_1 and β_2, and in the second case, m_i and $S_i = \sum_{j=1}^{m_i} \log T_i$ are jointly sufficient for γ_i and β.

3.3 Post-reduction confidence sets of models

A qualitatively related approach to assessing model adequacy was presented by Battey, Rasines & Tang (2025) from a different perspective. Their focus was on constructing

confidence sets of sparse models in a high-dimensional regression framework, where the set of all the variables that might be contemplated is large. Battey and Cox (2018) used sample splitting to ensure appropriate coverage properties of the confidence set of models after preliminary reduction. The purpose of Battey, Rasines & Tang (2025) was to avoid discarding information at the model assessment phase by using the sufficiency/co-sufficiency separation.

Let Y be the outcome vector in the normal-theory linear model. For any postulated model with a low-dimensional set of covariates, let $\mathcal{X} \subset \mathbb{R}^n$ be their column space, and let $\mathcal{X}^\perp$ be its orthogonal complement, so that $\mathcal{X} \oplus \mathcal{X}^\perp = \mathbb{R}^n$. It is shown in Battey, Rasines & Tang (2025) that the co-sufficient information for assessment of a given postulated model is contained in the statistic $Q = V^{\mathrm{T}}Y/\|V^{\mathrm{T}}Y\|_2$, where V is an orthogonal basis for $\mathcal{X}^\perp$. Under the postulated model Q is uniformly distributed on the surface of the unit hypersphere embedded in $\mathcal{X}^\perp$.

There is no power from a single observation Q to detect departures from uniformity on the sphere. Using a generalisation of the randomisation scheme introduced by Rasines & Young (2023), Battey, Rasines & Tang (2025) showed how to construct k statistics $\tilde{Q}_1, \ldots, \tilde{Q}_k$ that are independent and identically distributed on the surface of the unit sphere if and only if the postulated model is correct, assessed in that work using the Rayleigh test for uniformity on the sphere. Here we reframe the resulting assessment of model adequacy under the unifying framework of §2 and suggest an alternate test statistic based on the $m = k(k-2)/2$ inner products between all possible pairs $\tilde{Q}_i$, $\tilde{Q}_j$ of synthetic replicates after projection onto $\mathcal{X}^\perp$. Under the model hypothesis with d_0 explanatory variables, the m inner products (cosines of the angles between projected replicates), $Z_1, \ldots, Z_m$ say, each have density function (Fisher, 1915)

$$f_Z(z) = \frac{\Gamma((n-d_0)/2)}{\sqrt{\pi}\Gamma\{(n-d_0-1)/2\}}(1-z^2)^{(n-d_0-3)/2}, \quad -1 < z < 1. \tag{13}$$

The framework of §2 in this paper is an alternative to the Rayleigh test. The distribution function corresponding to (13) is

$$F_Z(z) = \frac{1}{2} + \frac{2\Gamma((n-d_0)/2)\mathrm{sign}(z)}{\sqrt{\pi}\Gamma\{(n-d_0-1)/2\}} \int_0^{z^2} u^{1/2-1}(1-u)^{(n-d_0-1)/2-1}du, \tag{14}$$

where the definite integral is the incomplete beta integral of arguments $1/2$, $(n-d_0-1)/2$ and r^2. The m inner products are converted to standard uniform random variables in the usual way $U_j = F_Z(Z_j)$, $j = 1, \ldots m$. Strictly speaking, only the angles between disjoint pairs of replicates are independent; in numerical work, however, it appears that if m is small relative to $n - d_0$, the dependence across all m of them is sufficiently weak that the χ^2_{2m} null distribution of (8) holds to close approximation; see column 1 of Table 5 in the supplementary file, where the empirical coverage probabilities of the model confidence sets and the number of false models in the set are reported.

Superficially, this example appears totally different from those introduced in previous sections, but the logic is the same: for every model postulated for inclusion in the confidence set, replication under this model is induced via the sequence of steps leading to $U_1, \ldots, U_m$. If their realised values are collectively extreme when calibrated against the distribution

expected under the postulated model, this casts doubt on the model, leading to its exclusion from the confidence set.

3.4 Proportional hazards and conditional co-sufficiency

The proportional hazards model $h(y;x) = h_0(y)g(x_i^{\mathrm{T}}\beta)$ is unusual in that the infinite-dimensional nuisance parameter, the baseline hazard function h_0, can be eliminated in the partial likelihood analysis for the regression coefficient β (Cox, 1972, 1975), circumventing estimation of h_0. This section frames partial likelihood in terms of a notion of conditional sufficiency for h_0, pointing to a corresponding notion of conditional co-sufficiency for assessment of the model.

On each of a number of independent individuals there is a survival time T_i, and individuals are subject to non-informative censoring, so that the observable outcome is $Y_i = \min\{T_i, c_i\}$ in addition to an indicator $d_i = 1$ if $Y_i = T_i$, and $d_i = 0$ otherwise. We initially consider the uncensored setting $d_i = 1$ for all i. Let $y_{(1)} < y_{(2)} < \cdots < y_{(n)}$ denote the ordered failure times and let i_j be the index of the individual who fails at time $y_{(j)}$. Thus $i = i_j$ if and only if $y_i = y_{(j)}$. Together, $y_{(1)}, \ldots, y_{(n)}$ and $i_1, \ldots, i_n$ are equivalent to the unordered failure times $y_1, \ldots, y_n$, in the sense that the latter can be recovered from the former, and vice versa. Let $\mathscr{R}_j := \{i : y_i \geq y_{(j)}\}$ be the set of individuals still at risk of failure at time $y_{(j)}$.

The following arguments implicitly underpin the development of Cox (1972). By ignoring the indices $i_1, \ldots, i_n$, the ordered survival times $y_{(1)}, \ldots, y_{(n)}$ are detached from any covariate dependence and therefore must, in the absence of the information supplied by $i_1, \ldots, i_n$, tell us nothing about how covariates influence the distribution of survival times, a version of ancillarity for β in the presence of nuisance parameters. The distribution of these ordered times depends primarily on h_0, the common instantaneous probability of failure at baseline.

Let $I_1, \ldots, I_n$ denote the random variables whose realisations are $i_1, \ldots, i_n$, and consider the conditional distribution of $I_1, \ldots, I_n$ given $y_{(1)}, \ldots, y_{(n)}$. Write

$$\mathscr{H}_j = \{y_{(1)}, \ldots, y_{(j)}, i_1, \ldots, i_{j-1}\}$$

for the history up to time $y_{(j)}$. The conditional probability that $i_j = i$, i.e. the probability that a failure occurring at time $y_{(j)}$ is contributed by individual i, conditional on $\mathscr{H}_j$, is the familiar expression appearing in the partial likelihood for β:

$$p_{j,i}(\beta^*) := \mathrm{pr}(I_j = i \mid \mathscr{H}_j) = \frac{h(y_{(j)}; x_i)}{\sum_{k \in \mathscr{R}_j} h(y_{(j)}; x_k)} = \frac{g(x_i^{\mathrm{T}}\beta^*)}{\sum_{k \in \mathscr{R}_j} g(x_k^{\mathrm{T}}\beta^*)}. \tag{15}$$

Although the conditioning set is the entire history $\mathscr{H}_j$, the expression (15) is free of $y_{(1)}, \ldots, y_{(j-1)}$, a type of conditional sufficiency, for the nuisance parameter h_0, of the increasing sets of order statistics $\{y_{(1)}, \ldots, y_{(j-1)}\}$, conditional on a failure having occurred at time $y_{(j)}$.

The above discussion suggests an assessment of the proportional hazards assumption based on the conditional distributions of the indices I_j given the relevant history $\mathscr{H}_j$. This is a version of co-sufficiency relative to the nuisance parameter, and leads to assessment

of the model based on the same inferential partition that enabled inference on β^*. This is defensible provided that the sample size is large enough to reliably estimate β^*; it is similar to the approach based on Structure 3 in §4 to be presented, which is simpler. We explore the possibility of using the conditional distributions of the indices, first highlighting the difficulties before presenting a resolution.

The partial likelihood is based on

$$\begin{aligned} \mathrm{pr}(I_1 = i_1, \ldots, I_n = i_n) &= \prod_{j=1}^{n} \mathrm{pr}(I_j = i_j \mid i_1, i_2, \ldots, i_{j-1}) \\ &= \prod_{j=1}^{n} \frac{g(x_{i_j}^{\mathrm{T}}\beta^*)}{\sum_{k\in\mathscr{R}_j} g(x_k^{\mathrm{T}}\beta^*)}, \end{aligned} \tag{16}$$

leading to a consistent estimator of β^* with familiar properties (Cox, 1972; Wong, 1982). Construction of uniformly distributed random variables under the proportional hazards structure is, however, difficult, primarily because of the changing risk set. To see this, consider two candidate analogues for the probability integral transform based on the indices:

$$\sum_{\ell\leq I_j} p_{j,\ell}(\beta^*), \qquad \sum_{\ell\in\mathscr{R}_j, \ell\leq I_j} \frac{g(x_{i_\ell}^{\mathrm{T}}\beta^*)}{\sum_{k\in\mathscr{R}_j} g(x_k^{\mathrm{T}}\beta^*)}.$$

These both violate standard uniformity: in the first case the risk set changes for every index ℓ in the sum, so that the probabilities do not sum to one; in the second case the risk set is fixed, but the intersection of $\mathscr{R}_j$ and $\{\ell < I_j\}$ is empty, so that only the last element in the sum contributes.

The simplest way to achieve the required replication, although almost certainly not the most efficient way to proceed, is to split the sample at random into m blocks $j = 1, \ldots, m$ of equal size and to form the partial log-likelihood $\ell_j(\beta)$ based on (16) in each block. The partial likelihood scores $S_j(\beta) = \nabla_\beta \ell_j(\beta)$ are then approximately normally distributed at $\beta = \beta^*$ as $n/m \to \infty$. Independent approximately standard uniform replicates $U_1(\beta^*), \ldots, U_m(\beta^*)$ can be obtained from the probability integral transforms for the score statistics when β^* is known. Since β^* is unknown, it is replaced by the full-sample partial likelihood estimate $\hat{\beta}$ based on (16), as discussed in §2.3.

Since the partial likelihood score on the full sample is asymptotically normally distributed, it may be possible to use a form of randomised inference in the vein of Rasines & Young (2023) or Dharamshi et al. (2026), but it is unclear to what extent the theoretical guarantees are affected by violation of exact normality; this investigation requires a paper of its own.

4 Matched-pair and two-group examples

4.1 Inferential structures for matched-pair examples

The most obvious settings in which inducible replication is present under the postulated model are those in which principles of experimental design have been applied, most simply,

the matched-comparison designs of Example 1.1 leading to outcomes $(Y_{j1}, Y_{j0})_{j=1}^m$ on the treated and untreated units for the m pairs.

Example 4.1 (Continuation of Example 1.1)**.** The statistic $U_j(\psi_0)$ constructed in Example 1.1 arises also through a valuable complementary perspective. On replacing the observed value $s_j(\psi_0)$ by the corresponding random variable, we obtain

$$U_j(\psi_0) = \frac{Y_{j1}\psi_0}{S_j(\psi_0)} = \frac{\psi_0 Z_j}{1+\psi_0 Z_j}, \tag{17}$$

where $Z_j = Y_{j1}/Y_{j0}$. The distribution function of Z_j under the proportional rates model of Example 1.1 is

$$F_Z(z;\psi^*) = \frac{\psi^* z}{1+\psi^* z}. \tag{18}$$

Thus (17) is $F_Z(Z_j;\psi_0)$, showing through a different route that uniformity is achieved if and only if the postulated model is an adequate approximation with $\psi_0 = \psi^*$. The transformation to $Z_j = Y_{j1}/Y_{j0}$ is the most immediate way to eliminate the nuisance parameters in this scale model. It was not a priori obvious, however, that this simple route to internal replication preserves the co-sufficient information, in the generalised sense used in Example 1.1. From (18) it is easy to demonstrate that the conditions of Proposition 2.1 are satisfied. □

The set of random variables $Z_1, \ldots, Z_m$ from Example 4.1 might be viewed from a different perspective as ancillary for the nuisance parameters, in the sense that, from observation of these statistics alone, estimation of $\gamma_1, \ldots, \gamma_m$ is impossible. There is some ambiguity of terminology in these highly parametrised models where generalised inferential separations are needed. In particular, in a conventional parametric context, an ancillary statistic must be part of the minimal sufficient statistic, and the co-sufficient information is that left over after conditioning on the latter. When there is no exact sufficiency reduction, such as here, the separations are less clearly defined.

Example 4.2. Suppose that the true distribution of (Y_{j1}, Y_{j0}) belongs to an additive exponential model with rates $\gamma_j + \Delta$ for the treated individual in pair j and γ_j for the untreated individual, with true value Δ^* for the treatment parameter. This model would arise, for instance, for the first event time in a Poisson process, where an additional source of events is present in the treatment group. In the additive model, the statistic $S_j = Y_{j1} + Y_{j0}$ is sufficient for the nuisance parameter γ_j and the conditional distribution function of Y_{j1} given $S_j = s_j$ is

$$F_{Y_{j1}|S_j}(y \,|\, s_j\,; \Delta^*) = \frac{1-e^{-\Delta^* y}}{1-e^{-\Delta^* s_j}}. \tag{19}$$

Provided that the model is correctly specified, the random variables $U_j(\Delta^*) := F_{Y_{j1}|S_j}(Y_{j1} \,|\, s_j\,; \Delta^*)$ are uniformly distributed conditional on $S_j = s_j$, and therefore also unconditionally. The above conclusion is invalidated if Δ^* is replaced by another value Δ_0. □

Replication-inducing constructions for matched pairs fall into two broad classes. The first eliminates pair-specific nuisance parameters by conditioning on a co-sufficient statistic,

as illustrated in Examples 1.1 and 4.2; the second eliminates them by transformation to a random variable whose distribution is free of nuisance parameters, as illustrated by Example 4.1. Each of these may or may not depend on the parameter of interest, leading to four generic inferential structures.

Structure 1. *The joint density function $f(y_1, y_0; \gamma_j, \psi)$ of (Y_{j1}, Y_{j0}) is such that there is a statistic S_j, not depending on ψ, such that $S_j := s(Y_{j1}, Y_{j0})$ is sufficient for γ_j and the conditional distribution function of $F_{Y_{j1}|S_j}(y \mid s_j; \psi)$ is continuous and bijective in y for any ψ, and injective in ψ for any fixed y.*

To exploit Structure 1, it is immaterial whether Y_{j1} or Y_{j0} is used in the conditional distribution.

Structure 2. *The second co-sufficiency structure parallels Structure 1, except that the sufficient statistic depends on the interest parameter ψ, as in Example 1.1.*

The two co-sufficiency structures contain all the residual information having eliminated the nuisance parameters through conditioning. Structures 1 and 2 lead therefore to natural analogues of Bartlett's (1937) argument. In view of (17), however, there is also interest in replication induced via a version of ancillarity. In an idealised parametric analysis, an ancillary statistic can be calibrated against its marginal distribution for assessment of the model. The appropriate analogue in highly parametrised settings is ancillarity for the nuisance parameters in the relaxed sense that, from observation of ancillary random variables alone, the nuisance parameters cannot be estimated. The above relaxed definition of ancillarity was implicit in constructions used by R. A. Fisher and D. R. Cox (e.g. Fisher, 1935; Cox, 1958). Many formalised versions of approximate ancillarity for an interest parameter in the presence of nuisance parameters have been put forward for conditional inference on a parameter of interest (e.g. Barndorff-Nielsen and Cox, 1994, p.38). Ancillary for a nuisance parameter is not, however, a standard concept, and its relevance for model assessment is an observation new to this work.

Structure 3. *The joint density function $f(y_1, y_0; \gamma_j, \psi)$ of (Y_{j1}, Y_{j0}) is such that there is a transformation t, not depending on ψ, such that $Z_j := t(Y_{j1}, Y_{j0})$ has a distribution function $F_Z(z; \psi)$ that is continuous and bijective in z for any fixed ψ, injective in ψ for any fixed z, and functionally independent of the nuisance parameter γ_j. While Structure 3 only applies exactly in the case of transformation models, Structure 4 extends beyond these classes, as illustrated below in Example 4.5.*

Structure 4. *Structure 4 parallels Structure 3, except that the transformation depends on the interest parameter ψ.*

The following proposition formalises the sense in which each structure induces internal replication. Here we drop pair-specific indices for notational conciseness. The propositions refer, for each of Structures 1–4, to equivalence classes of density functions within which the approach has no power to reject the postulated model. If the true density function violates the postulated model but belongs to the stated equivalence class, then the model violation will not be detected at any sample size. An analogy is to Markov equivalence classes in Gaussian graphical models, where multiple causal models give rise to the same distribution over the outcomes and therefore are statistically indistinguishable.

Proposition 4.1. Let (Y_1, Y_0) have a joint density function, provisionally assumed to belong to the model $\mathcal{M} = \{f(y_1, y_0; \gamma, \psi) : \gamma \in \Gamma, \psi \in \Psi\}$, where f satisfies one of Structures 1–4. When any such structure holds, define, respectively

$$\begin{aligned} U^{(1)}(\psi_0) &:= F_{Y_1|S}(Y_1 \,|\, s; \psi_0), & S &= s(Y_1, Y_0), \\ U^{(2)}(\psi_0) &:= F_{Y_1|S(\psi_0)}(Y_1 \,|\, s(\psi_0)), & S(\psi_0) &= s(Y_1, Y_0; \psi_0), \\ U^{(3)}(\psi_0) &:= F_Z(Z; \psi_0), & Z &= t(Y_1, Y_0), \\ U^{(4)}(\psi_0) &:= F_{Z(\psi_0)}(Z(\psi_0)), & Z(\psi_0) &= t(Y_1, Y_0; \psi_0). \end{aligned}$$

Then $U^{(k)}(\psi_0) \sim U(0,1)$ if and only if $\mathcal{M}$ holds with true parameter value $\psi^* = \psi_0$, or if the true distribution of (Y_0, Y_1) with density function g^* belongs to the equivalence class $\mathcal{E}^{(k)}(\psi_0)$, as specified in Appendix E.

Proof. A proof is given in the appendix. □

The four equivalence classes presented in the appendix are best explained with reference to the previous examples. Thus, in the context of Examples 1.1 and 4.1, the relevant equivalence class is $\mathcal{E}^{(3)}(\psi_0)$ and consists of all distributions over the original random variables (Y_{j1}, Y_{j0}) for which the induced distribution over the ratios $Z_j = Y_{j1}/Y_{j0}$ is identical to that postulated under $\mathcal{M}$ with the postulated value of ψ_0. It is clear from this example and others that members of the equivalence class, if any such member exists, must be such that any pair-specific heterogeneity is eliminated through the same operation as under the postulated model, as otherwise no replication would be induced and standard uniformity for every pair could not be achieved. Elimination of pair dependence is a necessary but not a sufficient condition. Consider, for instance, a true distribution belonging to the Weibull family with multiplicative treatment effect on the rate scale and a postulated model that is exponential, a special case, then ratios $Z_j = Y_{j1}/Y_{j0}$ eliminate the pair parameters in both cases, but no members of the Weibull model with non-unit shape parameter belong to the equivalence class $\mathcal{E}^{(3)}(\psi_0)$ for any value of ψ_0, as the induced distribution over Z_j is different.

For a different perspective, consider fitting ψ by maximum likelihood in the family of densities for the induced random variable corresponding to Structure 1 or 3, i.e. the conditional and marginal density functions. If the true density function g^* belongs to $\{\mathcal{E}^{(k)}(\psi_0) : \psi_0 \in \Psi\}$ for $k = 1, 3$ respectively, then the limiting maximum likelihood solution ψ_0^* under the postulated model gives zero Kullback-Leibler divergence within the induced families. In other words, there is a point in the parameter space Ψ for which the induced model on $f_{Y_1|S}$ or f_Z respectively intersects with a "true model" on the induced random variables, this being any family of induced models that contains the true distribution over the induced random variables.

While Proposition 4.1 ensures appropriate coverage when the postulated model is correct, it does not automatically translate to a powerful method for rejecting false models, even when they do not intersect with the equivalence classes $\{\mathcal{E}^{(k)}(\psi_0) : \psi_0 \in \Psi\}$, since when the postulated model is violated at ψ_0, the notional replicates $U_j^{(k)}(\psi_0)$ typically depend on the nuisance parameters γ_j, and the summation in (3) does not necessarily produce an aggregate that is sufficiently far in the tail of a χ^2_{2m} distribution. It is partly

for this reason that the version based on $\hat{\psi}$ from §2.3 is preferred. We explore the above examples in §4.2.

4.2 Some local power analyses

Here we consider examples in which the true distribution belongs to a model that departs only slightly from that postulated. This is done partly with a view to assessing the effectiveness of (3) for joint assessment of the model and its parameters relative to the approach in §2.3, where a consistent estimator of the interest parameter is used. In the first of the examples below, the same preliminary reductions eliminate the nuisance parameters under both models, so that there is no excess variability under the true model, relative to that postulated, induced through the pair-specific nuisance parameters.

Example 4.3. Let the postulated model be the proportional rates model of Example 1.1 and suppose that true distribution of the paired outcomes is Weibull with rate parameters γ_j, and $\gamma_j\psi^*$, respectively for untreated and treated individuals in the jth pair. Suppose further that the shape parameter of these Weibull distributions is $\varsigma \neq 1$, so that the postulated model is misspecified. In spite of this, the same pairwise operation eliminates γ_j under both models. Specifically, the distribution and density functions of the ratios $Z_j = Y_{j1}/Y_{j0}$ are

$$F_Z(z) = \frac{\psi^* z^\varsigma}{1+\psi^* z^\varsigma}, \qquad f_Z(z) = \frac{\psi^* \varsigma z^{\varsigma-1}}{(1+\psi^* z^\varsigma)^2}, \tag{20}$$

and, by monotonicity, the distribution function of $U_j(\psi_0)$ in (17) is

$$\mathrm{pr}(U_j(\psi_0) \leq u) = \mathrm{pr}\Big(Z_j \leq \frac{u}{\psi_0(1-u)}\Big) = \frac{\psi^* u^\varsigma}{\psi_0^\varsigma(1-u)^\varsigma + \psi^* u^\varsigma}, \qquad 0 \leq u \leq 1, \tag{21}$$

which is non-uniform for all $\varsigma \neq 1$ and all ψ_0.

Although no choice of ψ_0 gives uniformity, the expectation of $R_j = -\log U_j(\psi_0)$ can be shown via (4) to be monotonically decreasing in ψ_0, implying the existence of a unique value of ψ_0 at which $\mathbb{E}(R_j) = 1$. Consequently, departures from exponentiality in the direction of the Weibull model are difficult to detect using the aggregation criterion (3).

For comparison, consider maximum likelihood estimation based on the induced density based on (18). When the true model is Weibull with ς close to 1, the limiting maximiser ψ_0^* of the expected log-likelihood satisfies $\psi_0^* = \psi^* + O(\varsigma - 1)$ (see Appendix F.1). Thus, the estimator remains locally consistent despite model misspecification, and by Proposition 2.1 and equation (21),

$$\mathrm{pr}(U_j(\hat{\psi}) \leq u) \to \frac{\psi^{*(1-\varsigma)} u^\varsigma}{(1-u)^\varsigma + \psi^{*(1-\varsigma)} u^\varsigma} \neq u, \quad j = 1, \ldots, m. \tag{22}$$

This shows that the criterion (8) has power to detect departures from the model even though the maximum likelihood estimator of ψ^*, based on the transformed random variables Z_j, is consistent in spite of the model misspecification. In particular, from (4), $\mathbb{E}(R_j) = 1$ only when $\varsigma = 1$, the point at which the Weibull model collapses to the exponential model. Figure 1 plots the logarithm of both $\mathbb{E}(R_j)$ and $\mathbb{V}(R_j)$, obtained by

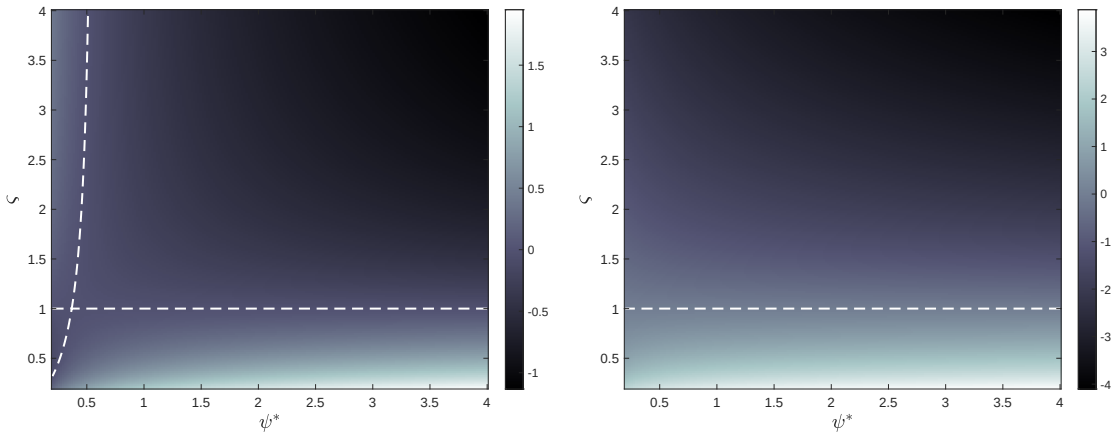

Figure 1: Logarithm of $\mathbb{E}(R_j)$ and $\mathbb{V}(R_j)$ calculated from (22) and (4), showing how the sensitivity to detect model misspecification varies with ς and ψ^*. Dashed lines indicate regions of the parameter space for the true Weibull model where $\mathbb{E}(R_j)$ and $\mathbb{V}(R_j)$ coincide with their putative values under the hypothesised exponential model.

numerical integration as a function of ς and ψ^*. This shows that (8) is most sensitive to departures from the postulated model in the direction of $\varsigma < 1$, with neighbourhoods of $\varsigma = 1$ and ψ^* being difficult to detect, uniformly over the values of the other variable. There is power to detect departures from the model when (ς, ψ^*) take values in the upper right quadrant of the reported range. □

Example 4.4. Another type of departure from the proportional exponential model of Example 1.1 is in the direction of the additive-rates exponential model of Example 4.2, with treatment effect Δ^*. In the additive model, the nuisance parameters are eliminated by conditioning on the pairwise sums, while for the multiplicative model, conditioning is on $Y_{j0} + Y_{j1}\psi_0$ for postulated values of ψ. Since there is no true value ψ^* when the model is erroneous, this second conditioning argument is ineffective at eliminating the nuisance parameters except at $\psi_0 = 1$.

Since Structure 3 leads to an identical inferential procedure as Structure 2 under the multiplicative model, interest lies in the behaviour of $Z_j = Y_{j1}/Y_{j0}$ under the additive model, which has distribution function

$$F_{Z_j}(z) = \text{pr}(Z_j \leq z) = \frac{(\gamma_j + \Delta^*)z}{\gamma_j + (\gamma_j + \Delta^*)z}. \tag{23}$$

Since (18) and (23) agree at the null effect value $\Delta^* = 0$, which corresponds to $\psi = 1$, we expect greater difficulty in detecting inadequacy of the multiplicative model for small values of Δ^*. A more exact analysis of power echoing the general discussion of §2.2 requires calculation of the density function of $U_j(\psi_0) = \psi_0 Z_j/(1 + \psi_0 Z_j)$ when the distribution of Z_j is specified by (23). By monotonicity,

$$\text{pr}(U_j(\psi_0) \leq u) = \text{pr}\Big(Z_j \leq \frac{u}{\psi_0(1-u)}\Big) = \frac{(\gamma_j + \Delta^*)u}{\gamma_j\psi_0(1-u) + (\gamma_j + \Delta^*)u}, \quad 0 \leq u \leq 1, \tag{24}$$

with corresponding density function

$$f_U(u) = \frac{\gamma_j(\gamma_j + \Delta^*)\psi_0}{(\gamma_j\psi_0(1-u) + (\gamma_j + \Delta^*)u)^2}, \quad 0 \leq u \leq 1.$$

This is not of the form (5), even approximately for Δ^* close to 0, but exact calculation via (4) shows that the expectation of $R_j = -\log U_j(\psi_0)$ is

$$\mathbb{E}(R_j) = \frac{\gamma_j + \Delta^*}{\gamma_j + \Delta^* - \gamma_j\psi_0} \log\Big(\frac{\gamma_j + \Delta^*}{\gamma_j\psi_0}\Big) = \frac{\eta_j \log \eta_j}{\eta_j - 1}, \quad \gamma_j + \Delta^* > 0,$$

where $\eta_j = (\gamma_j + \Delta^*)/\gamma_j\psi_0$. This is monotonically increasing in $\eta_j > 0$ and crosses $\mathbb{E}(R_j) = 1$ at $\eta_j = 1$, which corresponds to $\Delta^* > \gamma_j(\psi_0 - 1)$. Thus, sensitivity in the right tail tends to increase with Δ^*, and values $\psi_0 \leq 1$ are more easily detected as erroneous than $\psi_0 > 1$ when $\Delta^* > 0$, confirming intuition. The criterion $\Delta^* > \gamma_j(\psi_0 - 1)$ does, however, suggest that there are larger values of postulated ψ_0 that make the erroneous postulated model difficult to refute on the basis of the aggregation criterion (3), in spite of violation of standard uniformity; this is supported by numerical checks. It seems necessary in this example, therefore, to use a statistic based on $\hat{\psi}$, e.g. (8), echoing the conclusion from the previous example.

Appendix F.2 presents a local asymptotic expansion of the maximum likelihood solution in a neighbourhood of the point of intersection $\Delta^* = 0$ of the two models showing, as expected, that there is little sensitivity to departures from the postulated multiplicative model in the direction of the additive model at small Δ^*. □

The purpose of Examples 4.3 and 4.4 is to provide insight into the behaviour by way of some special cases where intuition can be recovered from direct calculation. These perturbative cases are ones in which the postulated model is so close to the true one that there is little harm in using it as a basis for inference. More substantial violations of modelling assumptions are probed by simulation in §G.1 of the supplement.

4.3 Extension to unbalanced strata

The experimental setting of §4 is a special case of a more general two-group problem arising in observational settings. Specifically, observations on treated and untreated individuals are stratified into groups that are as similar as possible. This typically leads to unbalanced strata, having a different number of individuals in the treated and untreated groups. Inference is based on the sufficient statistics S_{j1} and S_{j0} within treatment groups and strata, which can be treated in an analogous way to the paired observations of §4, as the strata sizes r_{j1} and r_{j0} are known.

Let $(Y_{ij1})_{i=1}^{r_{j1}}$ and $(Y_{ij0})_{i=1}^{r_{j0}}$ be observations within the jth stratum for treated and untreated individuals respectively. As a first example, if the observations $(Y_{ij1})_{i=1}^{r_{j1}}$ and $(Y_{ij0})_{i=1}^{r_{j0}}$ are normally distributed with means $\gamma_j + \psi^*$ and γ_j and variance τ, the likelihood contribution to the jth stratum depends on the data only through $S_{j1} = \sum_{i=1}^{r_{j1}} Y_{ij1}/r_{j1}$ and $S_{j0} = \sum_{i=1}^{r_{j0}} Y_{ij0}/r_{j0}$. The difference $Z_j = \sum_{i=1}^{r_{j1}} Y_{ij1}/r_{j1} - \sum_{i=1}^{r_{j0}} Y_{ij0}/r_{j0}$ is normally distributed of mean ψ^* and variance $\tau r_{j0}r_{j1}/(r_{j0} + r_{j1})$, which can be handled as discussed

in §2.3 with τ also estimated. This is an example of Structure 3, but the same answer is achieved via a conditioning argument based on Structure 1.

If the individual observations are Poisson distributed counts with rates $\gamma_j\psi^*$ and γ_j (Cox and Wong, 2010), the sufficient statistics S_{j1} and S_{j0} are sums of these counts, Poisson distributed with rates $r_{j1}\gamma_j\psi^*$ and $r_{j0}\gamma_j$. The distribution of S_{j1} or S_{j0} conditional on $S_{j1}+S_{j0}$ eliminates γ_j in the usual way. This is an example of Structure 1.

Example 4.5 is more interesting as it relies on Structure 4 to eliminate the nuisance parameters.

Example 4.5. If the originating variables are exponentially distributed with rates $\gamma_j\psi^*$ and γ_j, the sufficient statistics S_{j1} and S_{j0} are gamma-distributed sums, with shape and rate parameters $(r_{j1}, \gamma_j\psi^*)$ and (r_{j0}, γ_j) respectively. Then

$$Z_j(\psi^*) := \frac{r_{j0}\psi^* S_{j1}}{r_{j1}S_{j0}}$$

has the F distribution with parameters $2r_{j1}$ and $2r_{j0}$. The strategy of §2.3 based on Proposition 4.1 then applies directly. □

5 Brief summary of numerical work

The supplementary file confirms the expected behaviour for several of the examples above, showing recovery of nominal error rates when the postulated model is true and high sensitivity to detect departures from an erroneous postulated model when a different semiparametric model generated the data. For brevity, we summarise one example that illustrates these properties.

The postulated model is the inhomogeneous Poisson process discussed in §3.2 with semiparametric intensity function $\lambda_i(t) = e^{\gamma_i+\beta t}$ for the ith individual. Simulation is not straightforward and details are available in the supplement. To assess cases where the postulated model is erroneous, we also simulated from an inhomogeneous Poisson process with semiparametric intensity function specified differently as $\lambda_i(t) = e^{\gamma_i}t^\rho$ with $\rho \in \{-0.5, 0, 0.5, 1\}$.

By inducing an internal replication as discussed in §3.2, the postulated model is assessed for its compatibility with the data and the results are reported in Table 1, revealing high power to detect departures from the postulated model even at values of ρ very close to the point of intersection $\rho = 0$. Left and right sensitivity refers to whether $U_j(\hat{\psi})$ or $1 - U_j(\hat{\psi})$ was used in the construction of Fisher's statistic (8).

6 Discussion and open problems

The work provides new perspectives on the assessment of semiparametric and highly parametrised models, illustrating the possibility and value, in settings where the model admits this, of circumventing estimation of the infinite- or high-dimensional component. The broad principle is to use the postulated model to induce replication of known form if and only if the postulated model holds to an adequate approximation. We presented

Direction of sensitivity	ρ	Square root of number of individuals n					
		3	4	5	6	7	8
left	**0**	**0.03**	**0.04**	**0**	**0.03**	**0.02**	**0.05**
right	**0**	**0.03**	**0.03**	**0.02**	**0.03**	**0.01**	**0.03**
left	0.1	0.18	0.34	0.42	0.61	0.73	0.92
right	0.1	0.11	0.17	0.23	0.33	0.47	0.51
left	0.2	0.72	0.88	0.98	1	1	1
right	0.2	0.66	0.81	0.95	0.99	1	1

Table 1: The generating process is the inhomogeneous Poisson process with intensity function $\lambda_i(t) = e^{\gamma_i + \rho \log t}$; the postulated model has intensity function $\lambda_i(t) = e^{\gamma_i + \beta t}$. Proportion of 200 simulation runs in which the 5% test based on $F_{S_i|M_i}$ and (8) rejects the postulated model. The point of model intersection $\rho = 0$ is highlighted.

deliberately idealised examples illustrating diverse routes to such inducement, in the hope of suggesting, to others with different perspectives, new directions for research on semi-parametric model assessment.

Assessment of a statistical model for its compatibility with the data is inevitably a discrete problem unless competing models are nested, or can be artificially nested, allowing model space to be continuously traversed through variation of a parameter. By contrast, the simplest approaches to inference on the parameters of a given statistical model often involve maximisation of a log-likelihood or other objective function, and therefore typically invoke a notion of continuity on the parameter space. Perhaps for this reason, valuable inferential structures, such as interest-parameter-dependent co-sufficiency, and ancillarity for a nuisance parameter, have been overlooked in the literature, yet emerge naturally for model assessment.

There are broad principles underpinning all of the examples presented, extracted in §2, however the exact manner in which the replication is induced appears to be problem-specific. This raises the question of whether any general routes to the exact or approximate inducement of internal replication might be formulated.

In this vein, an important open problem concerns the possibility of eliminating nuisance parameters approximately. A notion of approximate exchangeability was formulated by Barber and Janson (2022) for a particular context, and this may well be useful across the range of settings considered here. An alternative might seek to collapse nuisance parameters into a low-dimensional summary that is both estimable and relatively insensitive to local perturbations under the postulated model.

Battey et al. (2024) attempted a constructive approach to finding transformations of observable random variables whose distributions are free or approximately free of nuisance parameters. They framed known examples in which nuisance parameters are straightforwardly eliminated in terms of integro-differential equations; these could be converted to standard types of partial differential equations and solved by established methods. The purpose of that work was not to solve easy examples via an unnecessarily complicated method, but to show how the results could be obtained through an application of general theory, in the hope that this might generalise. Such generalisation is a difficult open chal-

lenge. While Battey et al. (2024) sought nuisance-eliminating transformations by solving differential equations analytically, an alternative might seek those transformations numerically, in the vein of Box & Cox (1964).

Less directly applicable, although related, are notions of approximate sufficiency and approximate ancillarity from work in the 80s and 90s on approximate conditional inference (e.g. Cox, 1985; Barndorff-Nielsen, 1980; McCullagh, 1984; Fraser and Reid, 1988). In those papers, the objective is different, and the relevant inferential structures are in a sense the opposite of those required for model assessment. The conditional inference literature does not directly address the problem of approximate inducement of internal replication in the presence of high-dimensional nuisance components.

Inducement of internal replication under a postulated model can also be viewed as a form of inducement of population-level sparsity. Battey (2023) framed four examples from this perspective that sought to achieve, through traversal of data-transformation space or of parametrisation space, a population-level sparsity that was not present in the initial formulation. The work cited in the previous paragraph is one example, another is parameter orthogonalisation (Cox and Reid, 1987). It is possible in view of this, although not obvious, that reparametrisation-based assessments of model adequacy might be available: if the model is violated, the reparametrisation fails to induce the population-level sparsity.

A reader has indicated a further example (Diggle et al., 2007). In this semiparametric model for longitudinal data, possibly subject to drop-out, an internal replication is induced through transformation to a cumulative scale. In the induced model, the nonparametric component is reduced to a martingale random effect, whose distribution is arbitrary and evaded in the analysis.

The same reader has asked about the trade-off between avoiding estimation of a high-dimensional nuisance parameter, as favoured in the present paper, and having available an estimate of the nuisance parameter, even if the estimate is poor. In the context of most of our examples, this might entail a reformulation $\gamma_i = h(x_i)$ where h is now the nuisance parameter to be estimated semiparametrically. Standard approaches to model assessment would then assess prediction performance out of sample. The development of a general framework for fair comparison seems challenging.

Acknowledgements

We are grateful to Daniel Farewell for comments on a first draft. HB was supported by a UK Engineering and Physical Sciences Research Council research fellowship (EP/T01864X/1).

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A Proof of Proposition 2.1

Proof. The proof of the first part is essentially that of Slutsky's theorem. Convergence in distribution of $U_j(\hat{\psi})$ to $U_j(\psi_0^*)$ is metricised by both the Prohorov and bounded Lipschitz metrics, the latter defined for any two probability measures P and Q as

$$\beta(P,Q) := \sup\Big\{\Big|\int f d(P-Q)\Big| : \|f\|_{\mathrm{BL}} \leq M\Big\},$$

for any finite M, where $\|f\|_{\mathrm{BL}} := \|f\|_{\mathrm{L}} + f_\infty$ and $\|f\|_{\mathrm{L}} := \sup(x \neq y)|f(x) - f(y)|/d(x,y)$, where d is a metric. By Dudley (2002, Theorem 11.3.3), $\beta(P,Q) \to 0$ if and only if $P \to Q$. Proposition 2.1 is thus established by showing that $\mathbb{E}f(U_j(\hat{\psi})) \to \mathbb{E}f(U_j(\psi_0^*))$ for all bounded functions f with bounded Lipschitz norm. Let $\mathcal{A}$ be the event $\{|\hat{\psi} - \psi_0^*| \leq \varepsilon\}$. For an arbitrary $\eta > 0$, let $\varepsilon = \eta/2\|f\|_{\mathrm{L}}$ and let n_0 be a sample size such that $\mathrm{pr}(|\hat{\psi} - \psi_0^*| > \varepsilon) < \eta/4\|f\|_\infty$ for $n > n_0$. Thus, for n exceeding n_0,

$$\begin{aligned}
|\mathbb{E}f(U_j(\hat{\psi})) - \mathbb{E}f(U_j(\psi_0^*))| &\leq \big|\mathbb{E}\big[\big\{f(U_j(\hat{\psi})) - f(U_j(\psi_0^*))\big\}\mathbb{1}(\mathcal{A})\big]\big| \\
&\quad + \big|\mathbb{E}\big[\big\{f(U_j(\hat{\psi})) - f(U_j(\psi_0^*))\big\}\mathbb{1}(\mathcal{A}^c)\big]\big| \\
&\leq \|f\|_{\mathrm{L}}\,\mathbb{E}\big\{|\hat{\psi} - \psi_0^*|\,\mathbb{1}(\{|\hat{\psi} - \psi_0^*| \leq \varepsilon\})\big\} + 2\|f\|_\infty \mathrm{pr}(|\hat{\psi} - \psi_0^*| > \varepsilon) \\
&\leq \eta/2 + \eta/2 = \eta.
\end{aligned}$$

The first conclusion (i) follows by the arbitrariness of η.

The same strategy does not work for the sum $\hat{R}_m$ as there is accumulation of estimation error accross the m terms. Let $\hat{R}_j := -2\log U_j(\hat{\psi})$, $R_j^* := -2\log U_j(\psi^*)$ and

$$\hat{\mathbb{P}}_m(A) := \mathrm{pr}\Big(\sum_{j=1}^m \hat{R}_j \in A\Big), \quad \mathbb{P}_m^*(A) := \mathrm{pr}\Big(\sum_{j=1}^m R_j^* \in A\Big)$$

The Prohorov metric is

$$\rho(\hat{\mathbb{P}}_m, \mathbb{P}_m^*) := \inf\{\varepsilon > 0 : \hat{\mathbb{P}}_m(A) \leq \mathbb{P}_m^*(A^\varepsilon) + \varepsilon, \quad A \in \mathscr{B}(\mathbb{R}^+)\}$$

where $\mathscr{B}(\mathbb{R}^+)$ is the Borel sigma algebra on $\mathbb{R}^+$, and $A^\varepsilon := \{a \in \mathbb{R}^+ : d(a,A) < \varepsilon\}$. Following Dudley (2002, Ch. 11.3), introduce the function

$$h(a) := 0 \vee \big(1 - \tfrac{1}{2\varepsilon} d(x,A)\big),$$

so that $h(x) = 1$ if and only if $x \in A$ and $h(x) = 0$ for x more than 2ε away from the boundary of A. Then

$$\begin{aligned}
\hat{\mathbb{P}}_m(A) &\leq \int_{\mathbb{R}^+} h(x)\mathbb{P}_m^*(dx) + \int_{\mathbb{R}^+} h(x)|\hat{\mathbb{P}}_m(dx) - \mathbb{P}_m^*(dx)| \\
&\leq \mathbb{P}_m^*(A^{2\varepsilon}) + \int_{\mathbb{R}^+} h(x)|\hat{\mathbb{P}}_m(dx) - \mathbb{P}_m^*(dx)|. \qquad (25)
\end{aligned}$$

In the second term

$$\hat{\mathbb{P}}_m(dx) = \mathrm{pr}\Big(\sum_{j=1}^m \hat{R}_j \in [x, x+dx)\Big) = \mathrm{pr}\Big(\frac{1}{m}\sum_{j=1}^m \hat{R}_j \in \Big[\frac{x}{m}, \frac{x+dx}{m}\Big)\Big)$$

and similarly for $\mathbb{P}^*_m(dx)$.

Under a true postulated model, $U_j(\psi^*) \neq 0$ almost surely, $U_j(\hat{\psi}) \neq 0$ almost surely for sufficiently large n, and part (i) shows that $U_j(\hat{\psi}) \rightsquigarrow U_j(\psi^*)$. It follows that the average $\bar{R}$ of the $\hat{R}_j$ converges in probability to $\mathbb{E}(R^*_j) = 2$. Let $B^{\varepsilon/2}$ be a ball of radius $\varepsilon/2$ around 2 and let m_0 be such that for all $m > m_0$, $\mathrm{pr}(\bar{R} \in \big(B^{\varepsilon/2}\big)^c) \leq \varepsilon/2$, $\mathrm{pr}(\bar{R}^* \in \big(B^{\varepsilon/2}\big)^c) \leq \varepsilon/2$ and

$$|\,\mathrm{pr}(\bar{R}) \in B^{\varepsilon/2}) - \mathrm{pr}(\bar{R}^*) \in B^{\varepsilon/2})| \leq \varepsilon.$$

Then since, for any interval I

$$\int_{\mathbb{R}^+} h(x)\Big|\mathrm{pr}\big(\bar{R}) \in I \cap B^{\varepsilon/2}\big) - \mathrm{pr}(\bar{R}^* \in I \cap B^{\varepsilon/2})\Big| = |\,\mathrm{pr}(\bar{R}) \in B^{\varepsilon/2}) - \mathrm{pr}(\bar{R}^* \in B^{\varepsilon/2})|,$$

it follows by (25) that $\hat{\mathbb{P}}_m(A) \leq \mathbb{P}^*_m(A^{2\varepsilon}) + 2\varepsilon$, showing by Dudley (2002, Theorem 11.3.3) that $\hat{R}_m \rightsquigarrow R^*_m$.

□

B Derivation of equation (7)

This is a standard argument. Let k_α be the $1-\alpha$ quantile of the χ^2_{2m} distribution, Markov's inequality implies

$$\mathrm{pr}(R \geq k_\alpha) = 1 - \mathrm{pr}(-R \geq -k_\alpha) \geq 1 - e^{sk_\alpha}\mathbb{E}(e^{-sR}), \quad s > 0.$$

Direct calculation shows that

$$\mathbb{E}(e^{-sR}) = \prod_{j=1}^{m} \frac{1-\vartheta_j}{1-\vartheta_j+2s}, \quad s > 0.$$

C Derivation of equation (12)

The Laplace transform of (11) is

$$f^*_T(z) = \frac{\beta(e^{(\beta-z)t_0}-1)}{(e^{\beta t_0}-1)(\beta-z)},$$

thus the conditional density function of S_i is, for any $\tau > \beta$,

$$\begin{aligned} f_{S_i}(s \mid m_i;\beta) &= \frac{\beta^{m_i}}{(e^{\beta t_0}-1)^{m_i}} \frac{1}{2\pi i} \int_{\tau-i\infty}^{\tau+i\infty} e^{zs} \frac{(e^{(\beta-z)t_0}-1)^{m_i}}{(\beta-z)^{m_i}} dz \\ &= \frac{\beta^{m_i}}{(e^{\beta t_0}-1)^{m_i}} \sum_{v=0}^{m_i} \binom{m_i}{v} (-1)^v e^{v\beta t_0} \frac{1}{2\pi i} \int_{\tau-i\infty}^{\tau+i\infty} \frac{e^{sz}e^{-vzt_0}}{(z-\beta)^{m_i}} dz \end{aligned}$$

by the binomial formula. Let

$$k^*(z) = \frac{e^{z(s-vt_0)}}{(z-\beta)^{m_i}}.$$

Its contour integral is the residue at $z = \beta$ which, for a pole of order m_i, is

$$\begin{aligned}\mathrm{Res}(k^*(z), \beta) &= \frac{1}{(m_i-1)!} \lim_{z\to\beta} \frac{d^{(m_i-1)}}{dz^{(m_i-1)}} \Big\{(z-\beta)^{m_i} k^*(z)\Big\} \\ &= \frac{1}{(m_i-1)!} \lim_{z\to\beta} \frac{d^{(m_i-1)}}{dz^{(m_i-1)}} e^{z(s-vt_0)} \\ &= \frac{(s-vt_0)^{(m_i-1)}}{(m_i-1)!} e^{\beta(s-vt_0)}.\end{aligned}$$

Thus,

$$\begin{aligned}f_{S_i}(s \mid m_i; \beta) &= \frac{\beta^{m_i} e^{\beta s}}{(e^{\beta t_0}-1)^{m_i}\Gamma(m_i)} \sum_{v=0}^{m_i} \binom{m_i}{v} (-1)^v (s-vt_0)^{(m_i-1)} \mathbb{1}\{s > vt_0\}, \\ &= \frac{\beta^{m_i} e^{\beta s}}{(e^{\beta t_0}-1)^{m_i}\Gamma(m_i)} \sum_{v=0}^{\lfloor s/t_0 \rfloor} \binom{m_i}{v} (-1)^v (s-vt_0)^{(m_i-1)}, \quad s < m_i t_0 \end{aligned} \tag{26}$$

and zero otherwise. Integration gives

$$F_{S_i|M_i}(s \mid m_i; \beta) = \frac{\beta^{m_i}}{(e^{\beta t_0}-1)^{m_i}\Gamma(m_i)} \sum_{v=0}^{\lfloor s/t_0 \rfloor} \binom{m_i}{v} (-1)^v e^{\beta v t_0} \int_0^{s-vt_0} e^{\beta w} w^{(m_i-1)} dw.$$

This is equation (12).

D Proof of Proposition 4.1

Proof. For notational convenience, introduce $V^{(k)}$ for the, perhaps notional, random variable relevant for exploiting a postulated model with Structure k. This is Z and $Z(\psi_0)$ for $k = 3$ and $k = 4$ respectively and is the notional random variable $Y_\bullet(S)$ and $Y_\bullet(S(\psi_0))$ for $k = 1$ and $k = 2$. These are not functions of S in the conventional sense, and are notional in the sense that they typically cannot be expressed in terms of the original random variables, but once $S = s$ or $S(\psi_0) = s(\psi_0)$ is observed, $Y_\bullet(S)$ and $Y_\bullet(S(\psi_0))$ have the conditional distribution of $Y_\bullet$ given $S = s$ or $S(\psi_0) = s(\psi_0)$.

Temporarily dropping superscripts, let $F_V(v; \psi_0)$ be the distribution function of V under the assumption that the postulated model is true at ψ_0. Let $G_V^*(v)$ be the true distribution of V. Thus, if G_V^* belongs to the family $\mathcal{F}_V := \{F_V(\cdot; \psi),\ \psi \in \Psi\}$, then there exists a ψ^* such that $G_V^*(v) = F_V(v; \psi^*)$.

One direction is by direct calculation: if $\psi_0 = \psi^*$, then the distribution of $F_V(V; \psi_0)$ is uniform. For the converse direction, suppose for a contradiction that $F_V(V; \psi_0)$ is uniformly distributed but that $G_V^*(v) \neq F_V(v; \psi_0)$. By uniformity,

$$u = \mathrm{pr}\big(F_V(V; \psi_0) \leq u\big) \tag{27}$$

for all $u \in [0, 1]$. First suppose that $G_V^* \in \mathcal{F}_V$. Since $F_V(v, \psi)$ is bijective in v for any fixed ψ, there exists a $u \in [0, 1]$ such that $F_V^{-1}(u; \psi_0) \neq F_V^{-1}(u; \psi_0')$ for any $\psi_0' \neq \psi_0$. Thus, for

any such value of u, $F_V\big(F_V^{-1}(u;\psi_0);\psi^*\big) \neq F_V\big(F_V^{-1}(u;\psi^*);\psi^*\big)$, where the right hand side is equal to u by definition. It follows that $F_V\big(F_V^{-1}(u;\psi_0);\psi^*\big) \neq u$, which contradicts (27).

Suppose now that uniformity is achieved and $G_V^* \notin \mathcal{F}_V$. Then $G_V^*(F_V^{-1}(u;\psi_0))$ replaces $F_V\big(F_V^{-1}(u;\psi_0);\psi^*\big)$ in the argument of the previous paragraph. Equality for all u can only be achieved at the points ψ_0 where $G_V^*(v) = F_V(v;\psi_0)$. But at such points G_V^* belongs to $\mathcal{F}_V$, a contradiction.

The remaining question is whether there are distributions over (Y_0, Y_1) that induce the same distribution $G_V^*(v) = F_V(v;\psi_0)$ over V that would hold if the postulated model for (Y_0, Y_1) were true at ψ_0. The corresponding equivalence classes are most easily expressed in terms of density functions $g^* = g^*_{Y_0,Y_1}$, inducing a density function g_V^* on V. These are the sets $\mathcal{E}^{(k)}(\psi_0)$ from Proposition 4.1, made explicit in Appendix E. □

E Equivalence classes for Proposition 4.1

Proposition 4.1 refers to equivalence classes of density functions g over pairs (Y_0, Y_1), within which the approach to achieving internal replication based on Structures 1–4 has no power to reject the postulated model. In other words, if the true density function violates the postulated model in the directions of any model belonging to the corresponding equivalence class, then this will not be detected at any sample size. As in the statement of Proposition 4.1, we drop the pair-index subscripts. With $|J|$ the absolute Jacobian determinant corresponding to the transformation from (Y_0, Y_1) to (S, Y_1) for $k = 1$, to $(S(\psi_0), Y_1)$ for $k = 2$, to (Z, Y_1) for $k = 3$ and to $(Z(\psi_0), Y_1)$ for $k = 4$, the equivalence classes are:

$$\mathcal{E}^{(1)}(\psi_0) = \Big\{ g : \frac{g\big(y_0(s,y_1),y_1\big)\,|J|}{\int_{\mathcal{Y}_1(s)} g\big(y_0(s,y_1),y_1\big)\,|J|\,dy_1} = f_{Y_1|S}(y_1 \mid s;\psi_0) \Big\},$$

$$\mathcal{E}^{(2)}(\psi_0) = \Big\{ g : \frac{g\big(y_0(s(\psi_0),y_1),y_1\big)\,|J|}{\int_{\mathcal{Y}_1(s(\psi_0))} g\big(y_0(s(\psi_0),y_1),y_1\big)\,|J|\,dy_1} = f_{Y_1|S(\psi_0)}\big(y_1 \mid s(\psi_0)\big) \Big\},$$

$$\mathcal{E}^{(3)}(\psi_0) = \Big\{ g : \int_{\mathcal{Y}_1} g\big(y_0(z,y_1),y_1\big)\,|J|\,dy_1 = f_Z(z;\psi_0) \Big\},$$

$$\mathcal{E}^{(4)}(\psi_0) = \Big\{ g : \int_{\mathcal{Y}_1} g\big(y_0(z(\psi_0),y_1),y_1\big)\,|J|\,dy_1 = f_{Z(\psi_0)}\big(z(\psi_0)\big) \Big\}.$$

The above equivalence relations are in terms of generic expressions and usually do not specify the simplest route to calculation in particular cases. For instance, the distribution of Z_j from Example 4.1 can typically be calculated directly without explicit preliminary transformation from (Y_0, Y_1) to (Z, Y_1).

F Derivations for §4.2

F.1 Local consistency under the Weibull model

Maximisation of the log-likelihood function based on (18) when the true distribution is given by (20) produces a maximum likelihood estimator $\hat{\psi} \to_p \psi_0^*$, where ψ_0^* is the value of ψ that maximises the expected log-likelihood function

$$\psi_0^* = \underset{\psi \in \mathbb{R}^+}{\operatorname{argmax}} \left(\log \psi - 2 \int_0^\infty \log(1 + \psi z) \frac{\psi^* \varsigma z^{\varsigma - 1} dz}{(1 + \psi^* z^\varsigma)^2} \right). \tag{28}$$

This solution satisfies

$$\frac{1}{\psi_0^*} = 2\psi^* \varsigma \int_0^\infty \frac{z^\varsigma dz}{(1 + \psi^* z^\varsigma)^2 (1 + \psi_0^* z)}. \tag{29}$$

Since we are interested in the behaviour near the point of intersection of the two models $\varsigma = 1$, write $\varsigma = 1 + \varepsilon$ and expand the integral for small ε, giving

$$\frac{1}{\psi_0^*} = 2\psi^*(1 + \varepsilon) \left(\int_0^\infty \frac{z dz}{(1 + \psi^* z)^2 (1 + \psi_0^* z)} + O(\varepsilon) \right)$$

On expanding the integral in partial fractions it follows that, to first order in ε, ψ_0^* solves

$$\frac{\psi^*}{\psi_0^*} = 2\psi^*(1 + \varepsilon) \left(\frac{\psi^* \log(\psi^*/\psi_0^*) + \psi_0^* - \psi^*}{(\psi^* - \psi_0^*)^2} \right).$$

Equivalently, $x = \psi^*/\psi_0^*$ solves

$$(x - 1)^2 = 2(1 + \varepsilon) x (\log x + 1/x - 1). \tag{30}$$

The left and right hands sides are both convex with a unique minimum at $x = 1$ for any ε. Thus, to first order in ε, the maximum likelihood solution ψ_0^* under the postulated model is equal to the true value ψ^*.

F.2 Local asymptotic analysis for the additive exponential model

The limiting maximum likelihood solution is in this case more complicated in view of the nuisance parameters, but by the law of large numbers the average of the log-likelihood contributions converges to the average of the expected log-likelihood contributions, thus

$$\psi_0^* = \lim_{m \to \infty} \underset{\psi \in \mathbb{R}^+}{\operatorname{argmax}} \left(\log \psi - \frac{2}{m} \sum_{j=1}^m \gamma_j (\gamma_j + \Delta^*) \int_0^\infty \frac{\log(1 + \psi z) dz}{(\gamma_j + (\gamma_j + \Delta^*) z)^2} \right).$$

Differentiation shows that ψ_0^* solves

$$\frac{1}{\psi_0^*} = \lim_{m \to \infty} \frac{2}{m} \sum_{j=1}^m \gamma_j (\gamma_j + \Delta^*) \int_0^\infty \frac{z dz}{(1 + \psi_0^* z)(\gamma_j + (\gamma_j + \Delta^*) z)^2}.$$

Analogously to Example 4.3 we can expand the integrand around $\Delta^* = 0$, which corresponds to the point at which the two models intersect, giving, to first order

$$\frac{1}{\psi_0^*} = \left(\frac{\psi_0^* - 1 - \log \psi_0^*}{(\psi_0^* - 1)^2} + O(\Delta^*) \right) \lim_{m \to \infty} \frac{2}{m} \sum_{j=1}^{m} \frac{(\gamma_j + \Delta^*)}{\gamma_j}.$$

To the same order, this is of almost identical form to (30) with the term $1 + \varepsilon$ replaced by the limiting average a of $(\gamma_j + \Delta^*)/\gamma_j$. The first-order solution under the notional asymptotic regime $\Delta^* \to 0$ is thus the logical one $\psi_0^* = 1$ for all a, $\psi^* = 1$ and $\Delta^* = 0$ corresponding to the null treatment effect and the point at which the two models intersect. Thus, intuitively, there is little sensitivity to departures from the postulated multiplicative model in the direction of the additive model at small Δ^*.

G Numerical checks

G.1 Matched pairs

Data were generated from a joint model for matched pairs $(Y_{j0}, Y_{j1})_{j=1}^m$ in which an additive treatment effect Δ^* operates on the rate scale for Weibull distributed outcomes with baseline rate parameters $(\gamma_j)_{j=1}^m$ and shape parameter ς, where $(\gamma_j)_{j=1}^m$ were generated from a standard uniform distribution. The assumed model is exponential with a multiplicative rate parameter as in Example 1.1. This notional parameter was estimated by maximum likelihood based on the density function of the ratios $Z_j = Y_{j1}/Y_{j0}$ which, from (18) is $\psi/(1 + \psi z)^2$. Fisher's statistic $2 \sum_{j=1}^m \log U_j(\hat{\psi})$ was then constructed based on (18) with ψ^* replaced by $\hat{\psi}$ and z replaced by Z_j; Table 2 reports the results. At $(\Delta^*, \varsigma) = (0, 1)$, the true model intersects with the postulated model with null treatment effect, captured by value $\psi = 1$. Thus, the proportion of rejected tests of size α would be α by construction if ψ was perfectly estimated. Table 2 reports a beneficial under-rejection rate in that case. Elsewhere in the parameter space, power to detect departures from the model increases to 1 with increasing m except when $\varsigma = 1$, where the true Weibull distribution collapses to the exponential and therefore is closer to the point at which the true and postulated models intersect. Both the true distribution and those in the postulated model are in this example constructed from a parameter space whose dimension diverges at the same rate as the sample size m, so both models are semiparametric.

We also assessed the approach based on (3) without estimation of the notional parameter ψ, computing the proportion of confidence sets for ψ that were empty. If the parameter space of permissible values is constrained to some reasonable interval such as $[0, 4]$, then such an approach tends to be more powerful for the sample sizes reported, but a larger range, say $[0, 10]$ requires a much larger sample size in order for the confidence set to be empty with high probability; numerical results are not reported.

Table 3 reverses the roles of the two types of models, taking the baseline distribution of Y_{j0} to be the same as before but obtaining the distribution of Y_{j1} through multiplication of the rate parameter γ_j by a value ψ^*. The postulated model, by contrast, is the exponential additive rates model of Example 4.2. There is high power to detect the erroneous model,

Direction of sensitivity	(Δ^*, ς)	Square root of number of pairs m					
		5	8	11	14	17	20
left	(0, 0.5)	0.657	0.977	1	1	1	1
right	(0, 0.5)	0.640	0.983	1	1	1	1
left	**(0, 1)**	**0.002**	**0.001**	**0**	**0**	**0**	**0**
right	**(0, 1)**	**0.001**	**0.001**	**0**	**0.001**	**0**	**0**
left	(0, 2)	0	0.221	0.994	1	1	1
right	(0, 2)	0	0.232	0.997	1	1	1
left	(1, 0.5)	0.717	0.982	1	1	1	1
right	(1, 0.5)	0.685	0.990	1	1	1	1
left	(1, 1)	0.008	0.015	0.020	0.042	0.075	0.134
right	(1, 1)	0.002	0.003	0	0.005	0.007	0.010
left	(1, 2)	0	0	0.105	0.420	0.774	0.951
right	(1, 2)	0	0.005	0.647	0.992	1	1
left	(2, 0.5)	0.731	0.983	1	1	1	1
right	(2, 0.5)	0.696	0.992	1	1	1	1
left	(2, 1)	0.010	0.025	0.030	0.073	0.134	0.236
right	(2, 1)	0.002	0.003	0.002	0.006	0.012	0.023
left	(2, 2)	0	0	0.041	0.215	0.555	0.825
right	(2, 2)	0	0.004	0.482	0.979	1	1

Table 2: Weibull generating distribution with additive treatment parameter Δ^* and shape ς. The assumed model is exponential with multiplicative treatment parameter ψ. Proportion of 1000 simulation runs in which the 5% test based on (8) rejects the postulated model. The point of model intersection $(\Delta^*, \varsigma) = (0, 1)$ is highlighted.

and at the point $(\psi^*, \varsigma) = (1, 1)$ at which the true and postulated models coincide, the nominal level is recovered asymptotically.

G.2 Time-dependent Poisson process

Simulation of an inhomogeneous Poisson process can be performed most simply using the argument of Cox and Lewis (1966, pp. 27–28) that on the transformed time scale $\tau_i(t) = \int_0^t \lambda_i(u)du$ the process is Poisson of unit constant rate. It follows that the sequence $T_{i1}, \ldots, T_{im_i}$ of ordered event times for individual i satisfy $\tau_i(T_{ik_i}) = \sum_{j=1}^{k_i} E_{ij}$, where E_{ij} are independent unit exponential random variables. For $\lambda_i(t) = e^{\gamma_i + \beta t}$, we have $\tau_i(t) = e^{\gamma_i}(e^{\beta t} - 1)/\beta$, and the event times are distributed as

$$T_{ik_i} \stackrel{d}{=} \tau_i^{-1}\Big(\sum_{j=1}^{k_i} E_{ij}\Big) = \log\Big(1 + (\beta/e^{\gamma_i})\sum_{j=1}^{k_i} E_{ij}\Big)/\beta, \tag{31}$$

the limit as $\beta \to 0$ being $T_{ik_i} = e^{-\gamma_i}\sum E_{ij}$ as expected. To check that the nominal error rates are recovered under the postulated model, we used this generating process with γ_i drawn from a standard normal distribution, generating unit exponential random variables

Direction of sensitivity	(ψ^*, ς)	Square root of number of pairs m					
		5	8	11	14	17	20
left	(0.5, 0.5)	0.442	0.589	0.763	0.859	0.926	0.964
right	(0.5, 0.5)	0.846	0.995	1	1	1	1
left	(0.5, 1)	0.231	0.623	0.907	0.991	0.998	1
right	(0.5, 1)	0.239	0.669	0.942	0.998	1	1
left	(0.5, 2)	0.875	1	1	1	1	1
right	(0.5, 2)	0.002	0.035	0.182	0.536	0.817	0.939
left	(1, 0.5)	0.708	0.925	0.990	1	1	1
right	(1, 0.5)	0.565	0.881	0.984	1	1	1
left	**(1, 1)**	**0.079**	**0.068**	**0.082**	**0.059**	**0.070**	**0.058**
right	**(1, 1)**	**0.061**	**0.075**	**0.073**	**0.060**	**0.065**	**0.054**
left	(1, 2)	0.046	0.399	0.826	0.985	0.999	1
right	(1, 2)	0.073	0.450	0.894	0.991	1	1
left	(2, 0.5)	0.896	0.999	1	1	1	1
right	(2, 0.5)	0.329	0.466	0.606	0.740	0.873	0.936
left	(2, 1)	0.253	0.626	0.925	0.993	1	1
right	(2, 1)	0.252	0.620	0.922	0.991	1	1
left	(2, 2)	0.011	0.042	0.192	0.515	0.763	0.901
right	(2, 2)	0.889	1	1	1	1	1

Table 3: Weibull generating distribution with multiplicative treatment prameter ψ^* and shape ς. The assumed model is exponential with additive treatment parameter Δ. Proportion of 1000 simulation runs in which the 5% test based on (8) rejects the postulated model. The point of model intersection $(\psi^*, \xi) = (1, 1)$ is highlighted.

to use in (31) until their sum exceeded the endpoint of the transformed timescale $\tau_i(t_0)$ with $t_0 = 5$. Since we wish to assess cases where the model is misspecified, we also simulated using the power-law intensity function $\lambda_i(t) = e^{\gamma_i} t^\rho$ with $\rho \in \{-0.5, 0, 0.5, 1\}$. We generated

$$T_{ik_i} \stackrel{d}{=} \tau_i^{-1}\Big(\sum_{j=1}^{k_i} E_{ij}\Big) = \left(\frac{(\rho+1)\sum_{j=1}^{k_i} E_{ij}}{e^{\gamma_i}}\right)^{1/(\rho+1)} \tag{32}$$

until the corresponding sum of unit exponentials exceeded $\tau_i(t_0) = e^{\gamma_i} t_0^{\rho+1}/(\rho+1)$. At $\rho = \beta = 0$ the true and postulated models intersect, but since the likelihood function (10) is indeterminate there due to division by zero, we do not necessarily expect to recover the nominal level.

From n individuals we estimated β in the postulated model by maximising the conditional log-likelihood function

$$\ell(\beta) = \sum_{i=1}^{n} m_i \log\Big(\frac{\beta}{e^{\beta t_0} - 1}\Big) + \beta \sum_{i=1}^{n}\sum_{j=1}^{m_i} t_{ij} \tag{33}$$

based on (10). The model was subsequently assessed using the statistic $-2\sum_{i=1}^{n} \log U_i(\hat\beta)$ where $U_i(\hat\beta) = F_{S_i|M_i}(S_i \,|\, m_i; \hat\beta)$ and the conditional distribution was approximated either

through Monte Carlo sampling using 1000 replicates under the postulated model if m_i was below 40, or by a normal approximation to the distribution of the sum under the postulated model if m_i exceeded 40. An analogous statistic was computed with $\hat{\beta}$ replaced by its true value. The results are reported in Table 4 for data generated under the postulated model and in Table 1 for data generated under the power-law model.

Table 4 shows that the procedure rarely rejects the true model when the conditional maximum likelihood estimate is used in the construction of each $U_i(\hat{\beta})$. When the true value of β is used, it attains a rejection rate close to the nominal level, the discrepancy being primarily due to Monte Carlo sampling error in the approximation to $F_{S_i|M_i}(S_i \,|\, m_i; \hat{\beta})$. Table 1 of the main paper reports the power to detect departures from the postulated model.

Direction of sensitivity	β	estimated/true	Square root of number of individuals n					
			3	4	5	6	7	8
left	0	true	0.05	0.08	0.02	0.07	0.05	0.06
right	0	true	0.05	0.07	0.04	0.05	0.02	0.05
left	0	estimated	0.03	0.04	0	0.03	0.02	0.05
right	0	estimated	0.03	0.03	0.02	0.03	0.01	0.03
left	1	true	0.02	0.07	0.06	0.04	0.07	0.09
right	1	true	0.03	0.04	0.03	0.03	0.05	0.05
left	1	estimated	0.01	0.01	0	0	0	0
right	1	estimated	0	0	0.01	0	0	0
left	2	true	0.06	0.03	0.06	0.05	0.06	0.05
right	2	true	0.06	0.04	0.04	0.07	0.03	0.04
left	2	estimated	0	0	0	0	0	0
right	2	estimated	0.01	0	0	0.01	0	0

Table 4: The generating process is the inhomogeneous Poisson process with intensity function $\lambda_i(t) = e^{\gamma_i + \beta t}$ as postulated. Proportion of 200 simulation runs in which the 5% test based on $F_{S_i|M_i}$ and (8) rejects the true model.

G.3 Confidence sets of regression models

This section presents numerical performance of the approach discussed in §3.3. The original procedure of Battey, Rasines & Tang (2025) was intended for the situation in which the full set of variables contemplated is inordinately large, necessitating some preliminary reduction by variable screening prior to assessment of low-dimensional subsets of variables. Since, in the present paper, we are illustrating a particular point, we simplify the problem by assuming that the number of starting variables is 15, so that no preliminary reduction is needed. If that were actually the case, there would be no advantage to using anything other than a likelihood-ratio test of every low-dimensional model against the comprehensive model, but it is reassuring to see coverage probabilities for the confidence sets based on $U_j = F_Z(Z_j)$ from (14). These are reported as a function of the number of synthetic replicates k and the sample size n. We also report the simulation-average size of the

resulting confidence set of models, i.e. the number of false models that are included in the confidence set on average over simulation runs.

The experiment was conducted as follows. In each of 500 simulations, the n rows of the $n \times d$ covariate matrix X were drawn from a mean-zero normal distribution with correlation 0.9 between $s + a$ of the d variables, and correlation zero elsewhere, where $d = 15$, $s = 5$ is the number of signal variables and $a = 3$ is the number of noise variables that are correlated with signal variables. Associated with X is a vector θ of regression coefficients with entries 1 in the positions corresponding to the signal variables, and zeros elsewhere. The outcome vector was constructed as $Y = X\theta + \varepsilon$, where the entries of ε were taken as standard normally distributed.

Direction of sensitivity	$(n,\ k)$	Coverage	$\hat{\mathbb{E}}\lvert\mathcal{M}\backslash\mathcal{S}\rvert$	$\frac{\hat{\mathbb{E}}(\#\text{ false models})}{\#\text{ models tested}}$
left	(100, 4)	0.944	613	0.124
right	(100, 4)	0.954	603	0.122
left	(100, 8)	0.954	508	0.103
right	(100, 8)	0.962	504	0.102
left	(100, 12)	0.964	477	0.097
right	(100, 12)	0.956	476	0.096
left	(100, 16)	0.956	462	0.093
right	(100, 16)	0.968	460	0.093
left	(200, 4)	0.950	270	0.055
right	(200, 4)	0.954	250	0.051
left	(200, 8)	0.956	213	0.043
right	(200, 8)	0.958	206	0.041
left	(200, 12)	0.958	197	0.040
right	(200, 12)	0.958	192	0.039
left	(200, 16)	0.956	190	0.038
right	(200, 16)	0.950	186	0.038

Table 5: Simulated coverage probability and expected size of the confidence sets of models from 500 simulations. These were constructed from (8) based on the distribution (14) of the cosine angles between projected synthetic replicates under each postulated sparse model.

For every possible model of size $d_0 \leq 5$, the corresponding columns X_0 of X were extracted. For this postulated model we constructed a basis for the orthogonal projection onto the null space of X_0 by taking the $n - d_0$ eigenvectors of $I - X_0(X_0^{\mathrm{T}}X_0)^{-1}X_0$ corresponding to the unit eigenvalues. Let V_0 denote this $n \times (n - d_0)$ matrix of eigenvectors. From the vector of outcomes Y, $k \in \{4, 8, 12, 16\}$ synthetic replicates $\tilde{Y}_1, \ldots, \tilde{Y}_k$ were generated according to equation (4.3) of Battey, Rasines & Tang (2025) and the corresponding co-sufficient replicates were constructed as $\tilde{Q}_j = V_0^{\mathrm{T}}\tilde{Y}_j / \|V_0^{\mathrm{T}}\tilde{Y}_j\|_2$. The statistics $Z_1, \ldots, Z_m$ were then computed as the $m = k(k-1)/2$ inner products $\langle \tilde{Q}_j, \tilde{Q}_i \rangle$, $j \neq i$. When the postulated model is true, these follow the distribution (14) and $U_j = F_Z(Z_j)$ has a standard uniform distribution. The rejection regions for the model were thus defined

in the usual way via (8). Table 5 reports the resulting coverage probabilities of the 0.95 nominal-coverage confidence sets and the average number of false models in the set from 500 simulations.

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