HIGHER HIDA THEORY FOR SIEGEL MODULAR FORMS

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Abstract. We develop a version of Hida theory for higher coherent cohomology on Siegel Shimura varieties.

Contents

1. Introduction

1.1. **The coherent cohomology of Siegel Shimura varieties.** Let us first introduce the Siegel Shimura varieties and the coherent cohomology of automorphic vector bundles. A standard reference is [Har90]. Let $(G = \mathrm{GSp}_{2g}, \mathcal{H}_g)$ be the Siegel Shimura datum. Let $\{S_K\}_{K \subseteq G(\mathbb{A}_f)}$ be the tower of Shimura varieties, defined over the reflex field $\mathbb{Q}$. It parametrizes polarized abelian schemes of dimension g with level structure. It carries an action of $G(\mathbb{A}_f)$. We let $S^{tor}_{K,\Sigma}$ be a toroidal compactification of S_K and we denote by $D_{K,\Sigma}$ the boundary divisor. We let P_μ be the Siegel parabolic of G associated to μ. We let M_μ be the unipotent radical of P_μ.

Let us fix a maximal torus and a Borel $T \subseteq B \subseteq P_\mu \subseteq G$. Let $\mathrm{Rep}(M_\mu)$ be the category of finite dimensional algebraic representations of M_μ. Irreducible representations are labeled by M_μ-dominant weights $\kappa \in X^\star(T)^{M_\mu,+}$. The tower of Shimura varieties carries a natural M_μ-torsor, equivariant for the $G(\mathbb{A}_f)$-action, and this torsor provides a functor for each level K and choice of cone decomposition Σ to the category of locally free sheaves over $S^{tor}_{K,\Sigma}$ (see [Har90]):

$$\begin{array}{rcl} \mathrm{Rep}(M_\mu) & \to & VB(S^{tor}_{K,\Sigma}) \\ \kappa \in X^\star(T)^{M_\mu,+} & \mapsto & \omega^\kappa \end{array}$$

We consider $\mathrm{colim}_K \mathrm{R}\Gamma(S^{tor}_{K,\Sigma}, \omega^\kappa)$ as well as the cuspidal counterpart $\mathrm{colim}_K \mathrm{R}\Gamma(S^{tor}_{K,\Sigma}, \omega^\kappa(-D_{K,\Sigma}))$. These are independent of the choice of Σ. They are complexes of smooth admissible $G(\mathbb{A}_f)$-representations, and if we tensor with $\mathbb{C}$, these cohomologies can be computed by automorphic forms by [Su18].

1.2. **Hida theory.** Let p be a prime number. In classical Hida theory one defines p-adic modular forms as the space of functions over the ordinary Igusa variety. Here are some of the important properties of this space (see the book [Hid04] and the references given there):

1. This is a flat and p-adically complete $\mathbb{Z}_p$-algebra. It carries an action of Hecke operators and of the weight torus $T(\mathbb{Z}_p)$.
2. There is an injective map from classical modular forms $\mathrm{H}^0(S^{tor}_{K,\Sigma}, \omega^\kappa)$ to p-adic modular forms which is a restriction map from the Shimura variety to its ordinary locus.
3. The control theorem shows that in regular and cohomological weight, the ordinary part of p-adic modular forms identifies with ordinary classical modular forms.
4. The interpolation theorem shows that the ordinary part of cuspidal p-adic modular forms can be organized into a finite projective module over the Iwasawa algebra $\mathbb{Z}_p[[T(\mathbb{Z}_p)]]$.

The goal of the current paper is to set up a theory (that we call higher Hida theory) that works for higher cohomological degrees (not only for modular forms). Results in this direction have already appeared in [Pil20], [BCGP21], [LPSZ19], [BP20] for $g = 1$ and $g = 2$. The rational theory (that we called higher Coleman theory) has been developed in [BP21].

The various Hida theories are parametrized by a certain set of Kostant representatives in the Weyl group. Each such element of the Weyl group determines certain support conditions on the ordinary Igusa variety and we construct higher Hida theory as the ordinary part of cohomology with support condition on the Igusa variety.

Remark 1.2.1. We found the definition of the support conditions rather subtle. In particular, unless we are in the case of the degree 0 or top degree Hida theories, there doesn't seem to be a canonical choice of support condition. We found that there exists lots of suitable support conditions, but that they all give the same ordinary cohomology. For this reason we don't define a space of higher p-adic modular forms, but only its ordinary part.

1.3. **The set ^MW.** In order to describe the theory, we need to introduce the set ^MW. Let W be the Weyl group of G and W_M be the Weyl group of M

Definition 1.3.1. *We let ^MW be the set of minimal length representatives of the quotient $W_M\backslash W$.*

We give several interpretations and constructions involving this set.

1.3.2. *Combinatorial description.* We have $X^\star(T)^{M_\mu,+} = \cup_{w \in {}^MW} wX^\star(T)^+$. Let $\kappa \in X^\star(T)^{M_\mu,+}$. We define $C(\kappa) = \{w \in {}^MW, -w^{-1}w_{0,M}(\kappa+\rho) \in X^\star(T)^+\}$. We let $\nu + \rho = -w^{-1}w_{0,M}(\kappa+\rho)$ for any $w \in C(\kappa)$. This is the dominant representative of the Hodge-Tate cocharacter of automorphic forms contributing to the coherent cohomology in weight κ. If $\nu+\rho$ is regular, then the set $C(\kappa)$ contains a unique element.

1.3.3. *The ordinary part of classical cohomology.* Let us recall how the ordinary part of coherent cohomology is defined from a representation theoretic perspective. If π is a smooth admissible representation of $G(\mathbb{Q}_p)$ defined over $\mathbb{Q}_p$, we let $J_B(\pi)$ be its Jacquet module (see [Cas]). This is a smooth admissible $T(\mathbb{Q}_p)$-representation. We now introduce the ordinary part of the Jacquet module. Given a character $\chi : T(\mathbb{Q}_p) \to \bar{\mathbb{Q}}_p^\times$, composing with the valuation $v : \bar{\mathbb{Q}}_p^\times \to \mathbb{R}$, we obtain a map $v(\chi) : T(\mathbb{Q}_p)/T(\mathbb{Z}_p) \to \mathbb{R}$. We can think of $v(\chi)$ as an element of $X^\star(T)_\mathbb{R}$.

Definition 1.3.4. *We say that a character $\chi : T(\mathbb{Q}_p) \to \bar{\mathbb{Q}}_p^\times$ is ordinary in weight $\kappa \in X^\star(T)^{M,+}$ if $v(\chi) = w^{-1}w_{0,M}(\kappa+\rho)+\rho = -\nu$.*

We let $J_B^{ord}(\mathrm{colim}_K \mathrm{R}\Gamma(S^{tor}_{K,\Sigma}, \omega^\kappa))$ be the sum of the generalized eigenspaces for ordinary characters in weight κ in the Jacquet module $J_B(\mathrm{colim}_K \mathrm{R}\Gamma(S^{tor}_{K,\Sigma}, \omega^\kappa))$.

Remark 1.3.5. There is a partial order relation on $X^\star(T)_\mathbb{R}$, where $\lambda \geq \lambda'$ if $\lambda - \lambda'$ is a linear combination of positive roots with non-negative coefficients. The Katz-Mazur inequality postulates that $v(\chi) \geq -\nu$ (see [BP21], conjecture 5.10.7, thm. 5.10.12 and thm. 6.10.1). Our ordinarity condition is therefore precisely the matching of the Newton and Hodge polygon.

There is another definition of the ordinary part, using the Hecke algebra action. If $K_p \subseteq G(\mathbb{Q}_p)$ is a compact open subgroup admitting an Iwahori decomposition: $K_p = U_{K_p} \times T_{K_p} \times \bar{U}_{K_p}$, there is an algebra morphism $\mathbb{Z}[T^+(\mathbb{Q}_p)/T_{K_p}] \to \mathcal{C}^0_c(K_p\backslash G(\mathbb{Q}_p)/K_p, \mathbb{Z})$, $t \mapsto [K_ptK_p]$.

We let $\mathrm{R}\Gamma(S^{tor}_{K^pK_p}, \omega^\kappa)^{ord}$ be the direct factor of $\mathrm{R}\Gamma(S^{tor}_{K^pK_p}, \omega^\kappa)$ where $T^+(\mathbb{Q}_p)$ acts via invertible operators and the associated characters of $T(\mathbb{Q}_p)$ are ordinary in weight κ. There is a natural quasi-isomorphism

$$\mathrm{R}\Gamma(S^{tor}_{K^pK_p}, \omega^\kappa)^{ord} = J_B^{ord}(\mathrm{colim}_K \mathrm{R}\Gamma(S^{tor}_{K,\Sigma}, \omega^\kappa))^{T_{K_p}}$$

of complexes of smooth $T(\mathbb{Q}_p)$-representations.

1.3.6. *Archimedean representation theory and limits of discrete series.* We recall how the set ${}^MW$ occurs in the parametrization of limits of discrete series.

Theorem 1.3.7 ([Har90], thm. 3.4)**.** *Let $\kappa \in X^\star(T)^{M_\mu,+}$ and $w \in C(\kappa)$. There exists a unique non degenerate limit of discrete series representation $\pi_\infty(\kappa, w)$ of $G(\mathbb{R})$, with the property that $\pi_\infty(\kappa,w) \otimes V_\kappa$ has $(\mathfrak{p}, K_\infty)$-cohomology in degree $\ell(w)$.*

Let $\pi = \pi_\infty(\kappa, w) \otimes \pi_f$ be an automorphic representation. Then $\pi_f \hookrightarrow \mathrm{colim}_K \mathrm{H}^{\ell(w)}(S^{tor}_{K,\Sigma}, \omega^\kappa) \otimes \mathbb{C}$. When $w = Id$, $\pi_\infty(\kappa, Id)$ is a holomorphic limit of discrete series and the representation π is generated by holomorphic modular forms.

1.3.8. *p-adic geometry.* We explain how the set ${}^MW$ appears naturally in the p-adic geometry of the Shimura variety. We consider the perfectoid Shimura variety and its Hodge-Tate period map ([Sch15]):

$$\pi_{HT} : \mathcal{S}_{K^p} \to \mathcal{FL} = P_\mu\backslash G$$

The set ${}^MW$ is the fixed point set for the action of $T(\mathbb{Q}_p)$ on $\mathcal{FL}$. Each fiber $\pi_{HT}^{-1}(\{w\})$ is a perfectoid Igusa variety ([CS17]) which admits a very nice integral structure. There is a perfectoid p-adic formal scheme $\mathfrak{IG}_{K^p}$ which parametrizes ordinary abelian varieties A together with an isomorphism $A[p^\infty] \simeq \mu^g_{p^\infty} \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^g$, a polarization, and a K^p level structure. The fiber $\pi_{HT}^{-1}(\{w\})$ identifies with $\mathfrak{IG}_{K^p} \times_{\mathrm{Spa}(\mathbb{Z}_p,\mathbb{Z}_p)} \mathrm{Spa}(\mathbb{Q}_p^{cycl}, \mathbb{Z}_p^{cycl})$. Let J_{ord} be the group of polarized self quasi-isogenies of $\mu^g_{p^\infty} \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^g$ over $\mathbb{Z}_p$-algebras where p is nilpotent. It acts naturally on $\mathfrak{IG}_{K^p}$. Let $P'(\mathbb{Q}_p)$ be the subgroup of J_{ord} of self quasi-isogenies which lift to characteristic 0. We see that $P'(\mathbb{Q}_p)$ is a locally profinite group scheme over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ which is a form of the constant group scheme $P(\mathbb{Q}_p)$. For any compact open subgroup $K_{p,P} \subseteq P'(\mathbb{Q}_p)$, we have a smooth formal scheme $\mathfrak{IG}_{K^pK_{p,P}}$ which is the quotient of $\mathfrak{IG}_{K^p}$ by $K_{p,P}$, and we get a tower:

$$\{\mathfrak{IG}_{K^pK_{p,P}}\}_{K_{p,P}\subseteq P'(\mathbb{Q}_p)}$$

where the transition maps are finite flat. The action of $T(\mathbb{Q}_p)$ on $\pi_{HT}^{-1}(w)$ results in a map $T(\mathbb{Q}_p) \to P'(\mathbb{Q}_p)$. In summary, we can attach to w the data:

$$\big\{\{\mathfrak{IG}_{K^pK_{p,P}}\}_{K_{p,P}\subseteq P'(\mathbb{Q}_p)},\ T(\mathbb{Q}_p) \xrightarrow{w} P'(\mathbb{Q}_p)\big\}.$$

In particular, we get some privileged Hecke correspondences for $t \in T^+(\mathbb{Q}_p)$:

$$\begin{array}{ccc} & \mathfrak{IG}_{K^pK_{p,P}\cap tK_{p,P}t^{-1}} & \\ \swarrow & & \searrow \\ \mathfrak{IG}_{K^pK_{p,P}} & & \mathfrak{IG}_{K^pK_{p,P}} \end{array}$$

1.4. **Higher Hida complexes.** We can attach to each w a cohomology theory $\mathrm{R}\Gamma_w(K^p, cusp/\emptyset) \in \mathrm{D}(\mathbb{Z}_p[T(\mathbb{Q}_p)] \otimes \mathbb{T}^{sph})$ by considering the ordinary part of the cohomology with certain support condition depending on w of toroidal compactifications of these Igusa varieties. Here, we assume that $K^p = \prod_{\ell\neq p} K_\ell$ and $\mathbb{T}^{sph}$ is the abstract Hecke algebra equal to the restricted tensor product of the spherical Hecke algebras at all primes ℓ for which $K_\ell = G(\mathbb{Z}_\ell)$ is hyperspecial. The support conditions are chosen according to the Hecke correspondences parametrized by $t \in T^+(\mathbb{Q}_p) \xrightarrow{w} P'(\mathbb{Q}_p)$. We now describe the main properties of these cohomology.

Let $T'(\mathbb{Z}_p)$ be the pro-p subgroup of $T(\mathbb{Z}_p)$. We let $\mathcal{C}^0(T'(\mathbb{Z}_p), \mathbb{Z}_p)$ be the $T'(\mathbb{Z}_p)$-module of continuous functions on $T'(\mathbb{Z}_p)$, with value in $\mathbb{Z}_p$. Its $\mathbb{Z}_p$-dual is the algebra of measure on $T'(\mathbb{Z}_p)$, equal to the completed group algebra $\Lambda' = \mathbb{Z}_p[[T'(\mathbb{Z}_p)]]$.

Definition 1.4.1. *We say that a complex in $\mathrm{D}(\mathbb{Z}_p[T'(\mathbb{Z}_p)])$ is admissible if it can be represented by a bounded complex of $T'(\mathbb{Z}_p)$-modules $M^\bullet$ where M^i is isomorphic to $\mathcal{C}^0(T'(\mathbb{Z}_p), \mathbb{Z}_p)^{n_i}$. We say that an admissible complex has amplitude $[a, b]$ if it can be represented by a complex $M^\bullet$ as before concentrated in the range $[a, b]$.*

Remark 1.4.2. If a complex is admissible, then its dual is a perfect complex over the Iwasawa algebra Λ'. If it has amplitude $[a, b]$, then its dual has amplitude $[-b, -a]$ over Λ'.

Theorem 1.4.3. *The cohomologies $\mathrm{R}\Gamma_w(K^p, cusp/\emptyset)$ enjoy the following properties:*

(1) *(admissibility) $\mathrm{R}\Gamma_w(K^p, cusp/\emptyset)$ is admissible*
(2) *(Integral cohomological vanishing) If $g = 1$ or $g = 2$, $\mathrm{R}\Gamma_w(K^p, cusp)$ has amplitude $[0, \ell(w)]$, and $\mathrm{R}\Gamma_w(K^p)$ has amplitude $[\ell(w), \frac{g(g+1)}{2}]$.*
(3) *(Rational cohomological vanishing) $\mathrm{R}\Gamma_w(K^p, cusp)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ has amplitude $[0, \ell(w)]$, $\mathrm{R}\Gamma_w(K^p)\otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ has amplitude $[\ell(w), \frac{g(g+1)}{2}]$*
(4) *(Classicality in regular weight) Let $\nu \in X^\star(T)^+$. Let $\kappa = -w_{0,M}w(\nu+\rho)-\rho \in X^\star(T)^{M,+}$. Let $K_p = U_{K_p} \times T_{K_p} \times \bar{U}_{K_p}$ be a compact open subgroup of $G(\mathbb{Q}_p)$ admitting an Iwahori decomposition. Let $n \in \mathbb{Z}_{\geq 0}$ be such that $\{t \in T(\mathbb{Z}_p), t = 1 \mod p^n\} \subseteq T_{K_p}$. Then there is a $T(\mathbb{Q}_p)$-equivariant quasi-isomorphism*

$$\mathrm{RHom}_{T_{K_p}}(\nu, \mathrm{R}\Gamma_w(K^p)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(\zeta_{p^n})(-\nu) = \mathrm{R}\Gamma(S^{tor}_{K^pK_p}, \omega^\kappa)^{ord} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\zeta_{p^n})$$

and similarly for cuspidal cohomology.
(5) *(Cousin spectral sequence) Let $\kappa \in X^\star(T)^{M,+}$. Let $\nu = -w^{-1}w_{0,M}(\kappa+\rho)-\rho$ for any $w \in C(\kappa)$. Let $K_p = U_{K_p} \times T_{K_p} \times \bar{U}_{K_p}$ be a compact open subgroup of $G(\mathbb{Q}_p)$ admitting an Iwahori decomposition. Let $n \in \mathbb{Z}_{\geq 0}$ be such that $\{t \in T(\mathbb{Z}_p), t = 1 \mod p^n\} \subseteq T_{K_p}$.*
There is a $T(\mathbb{Q}_p)$-equivariant spectral sequence:

$$\oplus_{w\in C(\kappa),\ell(w)=p}\mathrm{RHom}_{T_{K_p}}(\nu, \mathrm{R}\Gamma_w(K^p)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(\zeta_{p^n})(-\nu) \Rightarrow \mathrm{H}^{p+q}(S^{tor}_{K^pK_p}, \omega^\kappa)^{ord} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\zeta_{p^n}).$$

Remark 1.4.4. We conjecture that the integral vanishing theorem 2) above holds for any g and any w. It holds if $w = Id$ or $w = w_0^M$ is the longest element of ${}^MW$.

Remark 1.4.5. The point 4) is a special case of 5). Indeed, when the weight is regular the spectral sequence trivially degenerates as $\#C(\kappa) = 1$.

Remark 1.4.6. In point 4) and 5) we need to extend scalars to $\mathbb{Q}_p(\zeta_{p^n})$ because the definition of the level structure on the Shimura variety and the Igusa variety differ slightly.

There is another way to organize the information contained in the cohomology $\mathrm{R}\Gamma_w(K^p, cusp/\emptyset)$ by using the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[T(\mathbb{Z}_p)]]$. We can construct perfect complexes of Λ-modules:

$$\begin{aligned} M_w^\bullet &= \mathrm{RHom}_{T(\mathbb{Z}_p)}(1, \mathrm{R}\Gamma_w(K^p) \otimes_{\mathbb{Z}_p} \Lambda) \\ M_{w,cusp}^\bullet &= \mathrm{RHom}_{T(\mathbb{Z}_p)}(1, \mathrm{R}\Gamma_w(K^p, cusp) \otimes_{\mathbb{Z}_p} \Lambda). \end{aligned}$$

The ordinary Hecke algebra $\mathbb{T}_w$ and $\mathbb{T}_{w,cusp}$ are the finite Λ-algebras equal to the image of $\mathbb{T}^{sph}$ in $\mathrm{End}(M_w^\bullet)$ and $\mathrm{End}(M_{w,cusp}^\bullet)$.

The following interpolation property follows easily from theorem 1.4.3:

Proposition 1.4.7. *For any $\nu = \nu_{alg}\chi : T(\mathbb{Z}_p) \to \mathbb{Z}_p(\zeta_{p^n})^\times$, with $\nu_{alg} \in X^\star(T)^+$ and χ a finite order character, and for $\kappa = -w_{0,M}w(\nu + \rho) - \rho$, we have:*

$$M_w^\bullet \otimes_{\Lambda, -\nu} \mathbb{Q}_p(\zeta_{p^n}) = \mathrm{R}\Gamma(S^{tor}_{K^pK_p}, \omega^\kappa)^{ord} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\zeta_p^n)[\chi]$$

where $K_p = U_{K_p} \times T_{K_p} \times \bar{U}_{K_p}$ is a compact open subgroup of $G(\mathbb{Q}_p)$ admitting an Iwahori decomposition, $T_{K_p} \subseteq \mathrm{Ker}\chi$ and $[\chi]$ means the χ-isotypic part for the action of diamond operators in $T(\mathbb{Z}_p)/T_{K_p}$. A similar statement holds for cuspidal cohomology.

We can now state the duality theorem, using the modules $M_w^\bullet$. There is an involution of ${}^MW$, $w \mapsto w_{0,M}ww_0$.

Theorem 1.4.8 (Serre duality)**.** *There is a perfect pairing $M_w^\bullet \otimes_\Lambda M^\bullet_{w_{0,M}ww_0,cusp} \to \Lambda[-d]$, for which the adjoint of $t \in T(\mathbb{Q}_p)$ is w_0t^{-1} and the adjoint of $[K_\ell g K_\ell]$ is $[K_\ell g^{-1} K_\ell]$ for some prime $\ell \neq p$ such that $K_\ell = G(\mathbb{Z}_\ell)$. This pairing is compatible with the classical Serre duality when specializing to locally algebraic dominant weights.*

In [BP21], we developed higher Coleman theory which is the theory of overconvergent modular forms in higher cohomological degree. We prove the natural compatibility between higher Coleman and higher Hida theory, which we state informally as follows:

Theorem 1.4.9. *The slope 0 part of higher Coleman theory canonically identifies with the rational part of higher Hida theory.*

We can also use the present paper to produce lattices in overconvergent cohomology and deduce lower bounds on slopes of overconvergent cohomology.

Theorem 1.4.10. *The lower bounds on slopes conjectures 5.9.2 and 6.8.1 in* [BP21] *hold true.*

As a consequence, one can replace the strongly small slope condition in [BP21] (in the Siegel case) by the small slope condition.

1.5. **An application to the cohomology of arithmetic groups.** Let N be an integer divisible by p. Let us consider the arithmetic group $\Gamma = \{\gamma \in \mathrm{Sp}_{2g}(\mathbb{Z}), \gamma = 1 \mod N\}$. Let $\mathrm{H}^\star(\Gamma, \mathbb{Z})$ be the cohomology of Γ acting on $\mathbb{Z}$. This is a finite $\mathbb{Z}$-module, acted on by the Hecke algebra $\mathbb{T}_N$ generated by the double classes $\Gamma t \Gamma$ for $t \in \mathrm{Sp}_{2g}(\mathbb{Q})$ in the usual way:

$$\Gamma t \Gamma : \mathrm{H}^\star(\Gamma, \mathbb{Z}) \stackrel{\mathrm{res}}{\to} \mathrm{H}^\star(t^{-1}\Gamma t \cap \Gamma, \mathbb{Z}) \stackrel{t}{\to} \mathrm{H}^\star(t\Gamma t^{-1} \cap \Gamma, \mathbb{Z}) \stackrel{\mathrm{cores}}{\to} \mathrm{H}^\star(\Gamma, \mathbb{Z}).$$

For $0 \leq i \leq g$, let $t_i = \mathrm{diag}(p, \cdots, p, 1, \cdots, 1, p^{-1}, \cdots, p^{-1})$ be the diagonal matrix with $2i$ many 1's. The double classes $\{\Gamma t_i \Gamma\}_{0 \leq i \leq g}$ generate a commutative subalgebra of $\mathbb{T}_N$.

We now consider $\mathrm{H}^\star(\Gamma, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathrm{H}^\star(\Gamma, \mathbb{Z}_p)$ and its rational part $\mathrm{H}^\star(\Gamma, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}_p = \mathrm{H}^\star(\Gamma, \mathbb{Q}_p)$. Let us define $\mathrm{H}^\star(\Gamma, \mathbb{Z}_p)^{ord} \subseteq \mathrm{H}^\star(\Gamma, \mathbb{Z}_p)$ to be the direct factors where the operators $\Gamma t_i \Gamma$ act invertibly and $\mathrm{H}^\star(\Gamma, \mathbb{Q}_p)^{ord} = \mathrm{H}^\star(\Gamma, \mathbb{Z}_p)^{ord} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Theorem 1.5.1. *We have $\mathrm{H}^i(\Gamma, \mathbb{Q}_p)^{ord} = 0$ for $0 \leq i < \frac{g(g+1)}{2}$.*

Remark 1.5.2. This theorem is to be compared with the general belief that the tempered part of $\mathrm{H}^\star(\Gamma,\mathbb{Q})$ should be concentrated in degrees $\geq \frac{g(g+1)}{2}$ (see [Ven17], section 4.2 for example). Temperedness is a condition on the archimedean size of Hecke eigenvalues, while ordinarity is a condition on the p-adic size.

We also remark that in theorem 5.12.11 of [BP21] we proved a variant of theorem 1.5.1 with coefficients in an algebraic representation of slightly regular weight. However the proof in the case of trivial coefficients depends on the methods of this paper.

1.6. **Organization of the paper.** In the section 2 we develop abstractly the formalism of cohomology with partial support and of correspondences acting on these cohomology. We show that there is at most one reasonable way to attach to a scheme and a correspondence over this scheme an ordinary cohomology theory (the ordinary part of a cohomology with partial support). In the section 3, we review the theory of Siegel modular varieties and their integral models. At a deep level, there are plenty of (very singular) integral models for the Shimura variety. However, there is a very good integral theory for the ordinary Igusa varieties. We show that the various finite level ordinary Igusa varieties can be organized into a tower of smooth formal schemes very similarly to the tower of Shimura varieties over $\mathbb{Q}$. In section 4, we focus on the Shimura variety of deep Iwahori level and we work in fixed weight. We manage to produce integral models for the classical cohomology of the Shimura variety which carry and action of the Hecke operators at p. We construct a Cousin spectral sequence which computes the ordinary part of the cohomology of these integral models in terms of local cohomologies which turn out to be the various higher Hida theories in a given weight. In section 5, we prove that the higher Hida theories interpolate over the weight space, just like in classical Hida theory. In section 6, we compare higher Hida theory and higher Coleman theory by studying overconvergent versions of higher Hida theory.

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2. Cohomology with support

2.1. **Compactification.** Let S be a noetherian affine scheme. We work in the category of S-schemes.

2.1.1. *Compactification of a scheme.* Let X be a separated, finite type S-scheme.

Definition 2.1.2. *A compactification of X (over S) is a proper S-scheme $\bar{X}$ together with an open immersion $j_X : X \hookrightarrow \bar{X}$.*

Compactifications of X form in a natural way a category, with maps $\xi : (j_X : X \hookrightarrow \bar{X}) \to (j'_X : X \hookrightarrow \bar{X}')$ being given by maps of S-scheme $\xi : \bar{X} \to \bar{X}'$, extending the identity of X. Moreover, the category is cofiltered.

Remark 2.1.3. By a theorem of Nagata, compactifications exist. See [Sta22], thm TAG 0F41.

Lemma 2.1.4. *Let $\xi : (j_X : X \hookrightarrow \bar{X}) \to (j'_X : X \hookrightarrow \bar{X}')$ be a map of compactifications. Then $\xi^\star X = X$.*

Proof. The map $\xi : \xi^\star X \to X$ is a separated map. It has a section $s : X \to \xi^\star X$. The section is a closed immersion. Since $s(X)$ is open dense in $\xi^\star X$ we deduce that $\xi^\star X = s(X)$. □

We recall that a closed subscheme D of $\bar{X}$ is called an effective Cartier divisor if $D = V(\mathscr{I})$ where $\mathscr{I}$ is an invertible ideal in $\mathscr{O}_{\bar{X}}$.

Lemma 2.1.5. *Compactifications $j_X : X \hookrightarrow \bar{X}$ with the property that $\bar{X} \setminus X$ admits the structure of an effective Cartier divisor in $\bar{X}$ are cofinal among compactifications.*

Proof. Let D be the closed complement of $\bar{X}$ endowed with the reduced scheme structure. We have $D = V(\mathscr{I})$. We let $\bar{X}'$ be the blow-up of X at the ideal $\mathscr{I}$. Then by [Sta22], Lemma Tag 02OS, $X \hookrightarrow \bar{X}'$ is a compactification mapping to $X \hookrightarrow \bar{X}$. □

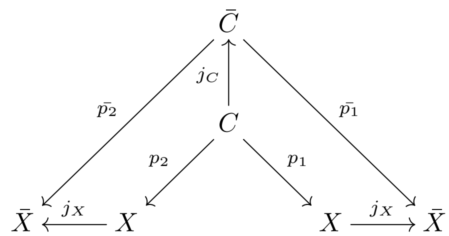
Lemma 2.1.6. *Let $\mathscr{F}$ be a locally free sheaf of finite rank over X. The set of compactifications $(j_X : X \hookrightarrow \bar{X})$ with the property that $\mathscr{F}$ extends to a locally free sheaf over $\bar{X}$ is cofinal among compactifications.*

Proof. Let $\bar{X}$ be a compactification. We can take an arbitrary extension of $\mathscr{G}$ to $\bar{X}$ by [Sta22], TAG 01 PE, then perform a suitable blow-up $\bar{X}'$ and take the strict transform to find an extension $\bar{\mathscr{G}}'$ of $\mathscr{G}$ to $\bar{X}'$ which is locally free ([RG71], thm. 5.2.2). □

2.1.7. *Compactification of a correspondence.* We adopt the following definition of correspondence:

Definition 2.1.8. *A correspondence is a quadruple (C, X, p_1, p_2), where X and C are separated, finite type schemes over S, coming with two projections $p_1, p_2 : C \rightrightarrows X$ and such that $p_1 \times p_2 : C \to X \times X$ is a proper map.*

Definition 2.1.9. *A compactification (denoted $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2)$) of a correspondence (C, X, p_1, p_2) is a diagram:*



where j_C and j_X are open immersions, $\bar{X}$ and $\bar{C}$ are proper, and $C = \bar{C} \times_{\bar{X} \times \bar{X}} X \times X$.

Remark 2.1.10. In particular $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2)$ is a correspondence.

Lemma 2.1.11. *Compactifications of a correspondence exist.*

Proof. Let (C, X, p_1, p_2) be a correspondence. Let $\bar{X}$ and $\bar{C}'$ be compactifications of X and C. Let $\bar{C}$ be the schematic closure of C in $\bar{C}' \times \bar{X} \times \bar{X}$. We have a map $C \to \bar{C} \times_{\bar{X} \times \bar{X}} X \times X$ which is a dense open immersion of proper schemes over $X \times X$. Therefore this is also a proper map, hence an isomorphism. □

A map between compactifications

$$(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2) \to (\bar{C}', \bar{X}', \bar{p}_1', \bar{p}_2')$$

is the data of maps $\bar{C} \to \bar{C}'$ and $\bar{X} \to \bar{X}'$ inducing the identity on C and X and making all the obvious diagrams commute.

Lemma 2.1.12. *Let $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2)$ be a compactification of (C, X, p_1, p_2). Let $(X \hookrightarrow \bar{X}') \to (X \hookrightarrow \bar{X})$ be a map of compactifications of X. There exists a map $(\bar{C}', \bar{X}', \bar{p}_1', \bar{p}_2') \to (\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2)$ of compactifications of (C, X, p_1, p_2), compatible with the given map $(X \hookrightarrow \bar{X}') \to (X \hookrightarrow \bar{X})$.*

Proof. We take $\bar{C}' = \bar{C} \times_{\bar{X} \times \bar{X}} \bar{X}' \times \bar{X}'$. □

Lemma 2.1.13. *The category of compactifications of (C, X, p_1, p_2) is cofiltered: if $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2)$ and $(\bar{C}', \bar{X}', \bar{p}_1', \bar{p}_2')$ are two compactifications, there exists a compactification $(\bar{C}'', \bar{X}'', \bar{p}_1'', \bar{p}_2'')$ with maps $(\bar{C}'', \bar{X}'', \bar{p}_1'', \bar{p}_2'') \to (\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2)$ and $(\bar{C}'', \bar{X}'', \bar{p}_1'', \bar{p}_2'') \to (\bar{C}', \bar{X}', \bar{p}_1', \bar{p}_2')$.*

Proof. We can take $\bar{X}'' = \bar{X} \times \bar{X}'$ and $\bar{C}'' = \bar{C} \times \bar{C}'$. □

2.1.14. *Closed subschemes.* Let $\bar{X}$ be a scheme. Let D be a closed subscheme of $\bar{X}$. We recall that D is called locally principal if $D = V(\mathscr{I})$ and there exists an invertible sheaf $\mathscr{L}$ over $\bar{X}$ and a map $\mathscr{L} \to \mathscr{O}_{\bar{X}}$ whose image is $\mathscr{I}$. We will often denote $\mathscr{L}$ by $\mathscr{O}_{\bar{X}}(-D)$, even if the choice of $\mathscr{L}$ is not unique. In the case that $\mathscr{I}$ itself is an invertible sheaf, D is called Cartier. In this case $\mathscr{I}$ is also denoted by $\mathscr{O}_{\bar{X}}(-D)$ in the literature.

Definition 2.1.15. *Let $D = V(\mathscr{I})$ and $D' = V(\mathscr{I}')$ be closed subschemes of $\bar{X}$.*

(1) *We write $D \subseteq D'$ when D is a subset of D'. In other words, $\mathscr{I}' \subseteq \sqrt{\mathscr{I}}$.*

(2) *Let* $s \in \mathbb{Q}_{>0}$. *We write* $D \leq sD'$ *if we can write* $s = \frac{p}{q}$ *with* $p, q \in \mathbb{Z}_{\geq 0}$ *and* $(\mathscr{I}')^p \subseteq (\mathscr{I})^q$.

Remark 2.1.16. We warn the reader that $D \leq D'$ implies that $D \subseteq D'$ as subsets of $\bar{X}$ but not that we have a map $D \to D'$ as subschemes of $\bar{X}$.

Lemma 2.1.17. (1) *If* D *and* D' *are Cartier divisors, then* $D \leq D'$ *implies that there exists* $n_0 \in \mathbb{Z}_{\geq 0}$ *such that we have a canonical injective map:* $\mathscr{O}_{\bar{X}}(-n_0 D') \to \mathscr{O}_{\bar{X}}(-n_0 D)$.

(2) *If* $\bar{X}$ *is normal and* D *and* D' *are Cartier divisors, then* $D \leq D'$ *implies that we have a canonical injective map:* $\mathscr{O}_{\bar{X}}(-D') \to \mathscr{O}_{\bar{X}}(-D)$.

(3) *If* $\bar{X}$ *is quasi-compact,* $D = V(\mathscr{I})$ *and* $D' = V(\mathscr{I}')$ *are locally principal closed subschemes and* $D \leq sD'$ *for* $0 < s < 1$, *then for any* $s < s' < 1$, *for any choice of maps* $\mathscr{O}_{\bar{X}}(-D) \to \mathscr{I}$ *and* $\mathscr{O}_{\bar{X}}(-D') \to \mathscr{I}'$, *there exists* p' *and* q' *such that* $s' = \frac{p'}{q'}$ *and such that we have a canonical diagram:*

$$\begin{array}{ccc} \mathscr{O}_{\bar{X}}(-p'D') & \longrightarrow & \mathscr{O}_{\bar{X}}(-q'D) \\ \downarrow & & \downarrow \\ (\mathscr{I}')^{p'} & \longrightarrow & (\mathscr{I})^{q'} \end{array}$$

Proof. We consider a local situation first. Let $\bar{X} = \operatorname{Spec} A$ and $D = V(f)$, $D' = V(f')$. In the first case, by definition, $(f')^{n_0}A \hookrightarrow (f)^{n_0}A$. In the second case, we use the normality assumption to deduce that $(\frac{f}{f'})^{n_0} \in A$ implies $\frac{f}{f'} \in A$. Therefore $(f')A \hookrightarrow (f)A$. In the last case we suppose $s = \frac{p}{q}$ and $(f')^p = af^q$ where a is unique up to an element of f^q-torsion. For any k we deduce that $(f')^{kp} = a^k f^q f^{(k-1)q}$ where $a^k f^q$ depends only of f' and f. We therefore get a canonical diagram (which only depends on choices of generators of $(f)A$ and $(f')A$):

$$\begin{array}{ccc} A & \xrightarrow{a^k f^q} & A \\ \downarrow{\scriptstyle (f')^{kp}} & & \downarrow{\scriptstyle f^{(k-1)q}} \\ (f')^{kp}A & \longrightarrow & (f)^{(k-1)q}A \end{array}$$

We put $s' = \frac{p'}{q'}$. We have $kp = p'k\frac{p}{p'}$ and $(k-1)q = q'k\frac{p}{p'}\frac{p'q}{pq'}\frac{k-1}{k}$. We find that $\frac{p}{p'}\frac{p'q}{pq'}\frac{k-1}{k} \geq 1$ for k large enough. We therefore get a canonical map for a large enough k:

$$\begin{array}{ccc} A & \longrightarrow & A \\ \downarrow{\scriptstyle (f')^{kp'}} & & \downarrow{\scriptstyle f^{kq'}} \\ (f')^{kp'}A & \longrightarrow & (f)^{kq'}A \end{array}$$

We now change p' to kp' and q to kq'. Doing this on a finite affine open covering of $\bar{X}$ (and possibly increasing p' and q' by a multiple), gives a canonical map: $\mathscr{O}_{\bar{X}}(-p'D') \to \mathscr{O}_{\bar{X}}(-q'D)$. □

2.1.18. *Expanded and contracted subschemes.* Let $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2)$ be a correspondence.

Definition 2.1.19. (1) *We say that a closed subscheme* $D \subseteq \bar{X}$ *is expanded by the correspondence if* $\bar{p}_2^\star D \subseteq \bar{p}_1^\star D$.

(2) *We say that a closed subscheme* D *is strictly expanded if* $\bar{p}_2^\star D \leq s\bar{p}_1^\star D$ *for some* $0 < s < 1$.

(3) *We say that a closed subscheme* D *is compactly expanded if* $\bar{p}_2^\star D \subseteq \overset{\circ}{\bar{p}_1^\star D}$ [1].

(4) *We say that a closed subscheme* $D \subseteq \bar{X}$ *is contracted by the correspondence if* $\bar{p}_1^\star D \subseteq \bar{p}_2^\star D$.

(5) *We say that a closed subscheme* D *is strictly contracted if* $\bar{p}_1^\star D \leq s\bar{p}_2^\star D$ *for some* $0 < s < 1$.

(6) *We say that a closed subscheme* D *is compactly contracted if* $\bar{p}_1^\star D \subseteq \overset{\circ}{\bar{p}_2^\star D}$.

Remark 2.1.20. The notion of expanded/contracted is set theoretical. The notion of compactly expanded/contracted is topological. The notion of strictly expanded/contracted is schematic.

[1] For a subset T of a topological space V we let $\overset{\circ}{T}$ be the interior of T in V

Example 2.1.21. Let $S = \mathrm{Spec}\ \mathbb{F}_p$. We consider the Frobenius correspondence. Namely, $\bar{C} = \bar{X}$, and we let $p_2 = F : \bar{X} \to \bar{X}$ be the Frobenius map and $p_1 : \bar{X} \to \bar{X}$ be the identity. Then for any closed subscheme D, we have $\bar{p}_2^\star D = p\bar{p}_1^\star D$, so D is strictly contracted.

Remark 2.1.22. If D is compactly expanded, we see that the closure of $\bar{C} \setminus \bar{p}_1^\star D$ in $\bar{C}$ is contained in $\bar{C} \setminus \bar{p}_2^\star D$. We chose the adjective compactly to indicate that there is a connection with the theory of compact operators. See section 2.10.

2.1.23. *Dynamic correspondences.* We introduce certain key definitions.

Definition 2.1.24.
(1) *A dynamic correspondence is a data $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D_+, D_-)$, where $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2)$ is a correspondence, $\bar{p}_1$ and $\bar{p}_2$ are proper maps, D_- and D_+ are closed subschemes of $\bar{X}$, such that D_+ is expanded by $\bar{C}$ and D_- is contracted by $\bar{C}$.*
(2) *A strict dynamic correspondence $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D_+, D_-)$ is a dynamic correspondence where D_+ is strictly expanded and D_- is strictly contracted.*
(3) *A compact dynamic correspondence $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D_+, D_-)$ is a dynamic correspondence with the property that D_+ is compactly expanded, and D_- is compactly contracted.*

2.1.25. *Dynamic compactifications of a correspondence.* Let (C, X, p_1, p_2) be a correspondence.

Definition 2.1.26. *A dynamic compactification of (C, X, p_1, p_2) is a dynamic correspondence $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D_+, D_-)$, where $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2)$ is a compactification of the correspondence (C, X, p_1, p_2) and $\bar{X} \setminus X = D_- \cup D_+$. A strict dynamic compactification is a dynamic compactification which is a strict dynamic correspondence. A compact dynamic compactification is a dynamic compactification which is a compact dynamic correspondence.*

Remark 2.1.27.
(1) We are asking that the support of $D_- \cup D_+$ is $\bar{X} \setminus X$.
(2) We are not imposing that D_- and D_+ are disjoint.
(3) A given compactification $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2)$ can often be enriched in different ways into a dynamic compactification.
(4) We don't know if a correspondence always admits a dynamic compactification.

A (non-strict) map between dynamic compactifications

$$(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D_+, D_-) \to (\bar{C}', \bar{X}', \bar{p}'_1, \bar{p}'_2, D'_+, D'_-)$$

is a map of compactifications with the property that for the map $\xi : \bar{C} \to \bar{C}'$, we have $D_+ \supseteq \xi^\star D'_+$ and $D_- \supseteq \xi^\star D'_-$. We say that the map is strict if $D_+ = \xi^\star D'_+$ and $D'_- = \xi^\star D'_-$.

Lemma 2.1.28. *Let $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D_+, D_-)$ be a dynamic (resp. strict dynamic, resp. compact dynamic) compactification of (C, X, p_1, p_2). Let $(\bar{C}', \bar{X}', \bar{p}'_1, \bar{p}'_2)$ be a compactification of (C, X, p_1, p_2) and let $\xi : (\bar{C}', \bar{X}', \bar{p}'_1, \bar{p}'_2) \to (\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2)$ be a map of compactifications. Then $(\bar{C}', \bar{X}', \bar{p}'_1, \bar{p}'_2, \xi^\star D_+, \xi^\star D_-)$ is a dynamic (resp. strict dynamic, resp. compact dynamic) compactification.*

Proof. By lemma 2.1.4, $\xi^\star(D_+ \cup D_-) = \bar{X}' \setminus X$. The rest follows easily. □

Lemma 2.1.29. *Let $\mathcal{C} = (\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D_+, D_-)$ be a dynamic (resp. strict dynamic, resp. compact dynamic) compactification. There exists a dynamic (resp. strict dynamic, resp. compact dynamic) compactification $\mathcal{C}' = (\bar{C}', \bar{X}', \bar{p}'_1, \bar{p}'_2, D'_+, D'_-)$ and a strict map $\mathcal{C}' \to \mathcal{C}$ with the property that D'_+ and D'_- are Cartier divisors on $\bar{X}'$ and $\bar{p}_i^\star D'_+$, $\bar{p}_i^\star D'_-$ for $i = 1, 2$ are Cartier Divisors in $\mathcal{C}'$.*

Proof. We let $D_+ = V(\mathscr{I})$ and $D_- = V(\mathscr{J})$. We let $\bar{X}'$ be the blow-up of $\bar{X}$ at $D_+ \cup D' = V(\mathscr{I}\mathscr{J})$. Let D'_+ and D'_- be the pull backs of D_+ and D_-. They are Cartier divisors by [Sta22], TAG 02ON. We let $\bar{C}''$ be $\bar{C} \times_{\bar{X} \times \bar{X}} \bar{X}' \times \bar{X}'$. We now let $\bar{C}'$ be the blow-up of $\bar{C}''$ at $\bar{p}_1^\star D'_+ \cup \bar{p}_2^\star D'_+ \cup \bar{p}_1^\star D'_- \cup \bar{p}_2^\star D'_-$. □

Lemma 2.1.30. *Let $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2)$ be a compactification and let*

$$(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D_+ = V(\mathscr{I}), D_- = V(\mathscr{J})) \text{ and } (\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D'_+ = V(\mathscr{I}'), D'_- = V(\mathscr{J}'))$$

be two enrichments as a dynamic (resp. strict dynamic, resp. compact dynamic) compactification. Then $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D'_+ \cup D_+ = V(\mathscr{I}\mathscr{I}'), D'_- \cup D_- = V(\mathscr{J}\mathscr{J}'))$ is a dynamic (resp. strict dynamic, resp. compact dynamic) compactification.

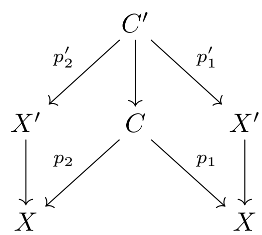
Proof. Obvious. □

Lemma 2.1.31. *The category of dynamic (resp. strict dynamic, resp. compact dynamic) compactifications of* $(C, X, p_1, p_2,)$ *where maps are non strict maps is cofiltered: if* $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D_+, D_-)$ *and* $(\bar{C}', \bar{X}', \bar{p}'_1, \bar{p}'_2, D'_+, D'_-)$ *are two dynamic (resp. strict dynamic, resp. compact dynamic) compactifications, there exists a dynamic (resp. strict dynamic, resp. compact dynamic) compactification* $(\bar{C}'', \bar{X}'', \bar{p}''_1, \bar{p}''_2, D''_+, D''_-)$ *with maps*

$$\begin{array}{rcl}(\bar{C}'', \bar{X}'', \bar{p}''_1, \bar{p}''_2, D''_+, D''_-) & \to & (\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D_+, D_-) \\ (\bar{C}'', \bar{X}'', \bar{p}''_1, \bar{p}''_2, D''_+, D''_-) & \to & (\bar{C}', \bar{X}', \bar{p}'_1, \bar{p}'_2, D'_+, D'_-).\end{array}$$

Proof. It follows from lemma 2.1.13 that we can find a compactification: $(\bar{C}'', \bar{X}'', \bar{p}''_1, \bar{p}''_2)$ with maps of compactification $\xi : (\bar{C}'', \bar{X}'', \bar{p}''_1, \bar{p}''_2) \to (\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2)$ and $\xi' : (\bar{C}'', \bar{X}'', \bar{p}''_1, \bar{p}''_2) \to (\bar{C}', \bar{X}', \bar{p}'_1, \bar{p}'_2)$. Now we take $D''_+ = \xi^\star D_+ \cup (\xi')^\star D'_+$ and $D''_- = \xi^\star D_- \cup (\xi')^\star D'_-$. This works by lemma 2.1.30. □

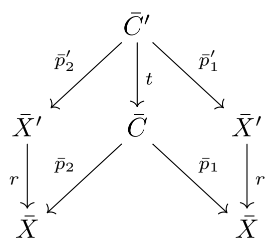
2.1.32. *Permanence property.* We consider a commutative diagram of correspondences (C, X, p_1, p_2) and (C', X', p'_1, p'_2):



Lemma 2.1.33. *We assume that in the above diagram the maps* $C' \to C$ *and* $X' \to X$ *are proper and that* (C, X, p_1, p_2) *admits a dynamic (resp. strict dynamic, resp. compact dynamic) compactification* $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D_+, D_-)$. *Then* (C', X', p'_1, p'_2) *admits a dynamic (resp. strict dynamic, resp. compact dynamic) compactification.*

Proof. We let $\bar{X}''$ and $\bar{C}''$ be compactifications of X' and C'. We let $\bar{X}'$ be the closure of X' in $\bar{X}'' \times \bar{X}$ and $\bar{C}'$ be the closure of C' in $\bar{C}'' \times \bar{C}$. We claim that $\bar{X}' \times_{\bar{X}} X = X'$. Indeed, the natural map $X' \to \bar{X}' \times_{\bar{X}} X$ is both an open immersion and a proper map (since $X' \to X$ is assumed to be proper). Similarly, we see that $\bar{C}' \times_{\bar{C}} C = C'$. We thus deduce that $C' = \bar{C}' \times_{\bar{X}' \times \bar{X}'} X' \times X'$.

We have a diagram:



We now let $D'_+ = r^\star D_+$ and $D'_- = r^\star D_-$. It is easy to check that this compactification has the desired properties. For example, let us assume that $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D_+, D_-)$ is strict dynamic. The identity $\bar{p}_2^\star D_+ \leq s\bar{p}_1^\star D_+$ implies $t^\star \bar{p}_2^\star D_+ \leq st^\star \bar{p}_1^\star D_+$ and thus $(\bar{p}'_2)^\star D'_+ \leq s(\bar{p}'_1)^\star D'_+$. One proceeds similarly with D'_-. We thus see that $(\bar{C}', \bar{X}', \bar{p}'_1, \bar{p}'_2, D'_+, D'_-)$ is a strict dynamic compactification of (C', X', p'_1, p'_2).

□

Let (C, X, p_1, p_2) be a correspondence and $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2)$ be a compactification.

Lemma 2.1.34. *Assume that* $(\bar{C}^{red}, \bar{X}^{red}, \bar{p}_1^{red}, \bar{p}_2^{red})$ *which is a compactification of* $(C^{red}, X^{red}, p_1^{red}, p_2^{red})$ *admits a dynamic (resp. strict dynamic, resp. compact dynamic) compactification* $(\bar{C}^{red}, \bar{X}^{red}, \bar{p}_1, \bar{p}_2, D_+^{red}, D_-^{red})$. *Then* $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D_+^{red}, D_-^{red})$ *is a dynamic (resp. strict dynamic, resp. compact dynamic) compactification.*

Proof. The cases of a dynamic or compact dynamic are obvious as these notions are set theoretical or topological. We do the case of a strict dynamic compactifications. We assume that there exists $0 < s < 1$ such that $(\bar{p}_2^{red})^\star D_+^{red} \leq s(\bar{p}_1^{red})^\star D_+^{red}$. We prove that a similar inequality holds over $\bar{C}$. We can check it locally. Let U_1, U_2 be open affines of $\bar{X}$ and let $V \subseteq \bar{C}$ be an open affine of $\bar{C}$ such that $\bar{p}_i(V) \subseteq U_i$. Let $s_1, \cdots, s_d$ be generators the ideal of D_+^{red} over U_1^{red} and let $\hat{s}_1, \cdots, \hat{s}_d$ be lifts of them to $\mathscr{O}_{\bar{X}}(U_1)$. The ideal of D_+^{red} over U_1 is generated by $\{\hat{s}_1, \cdots, \hat{s}_d, \sqrt{0}_{U_1}\}$ where $\sqrt{0}_{U_1}$ is the nilradical of $\mathscr{O}_{U_1}$.

Similarly, we find generators $\{\hat{t}_1, \cdots, \hat{t}_{d'}, \sqrt{0}_{U_2}\}$ of D_+^{red} over U_2. By assumption, for $s = \frac{p}{q}$ we have

$$(\bar{p}_1^\star(\hat{s}_1, \cdots, \hat{s}_d, \sqrt{0}_{U_1}))^p \subseteq (\bar{p}_2^\star(\hat{t}_1, \cdots, \hat{t}_{d'}))^q + \sqrt{0}_V.$$

Let r be such that $(\sqrt{0}_V)^r = (0)$. We see that for all $l \geq 0$,

$$(\bar{p}_2^\star(\hat{t}_1, \cdots, \hat{t}_{d'}))^q + \sqrt{0}_V)^l \subseteq (\bar{p}_2^\star(\hat{t}_1, \cdots, \hat{t}_{d'}))^{ql-r}.$$

We deduce that

$$(\bar{p}_1^\star(\hat{s}_1, \cdots, \hat{s}_d, \sqrt{0}_{U_1}))^{pl} \subseteq (\bar{p}_2^\star(\hat{t}_1, \cdots, \hat{t}_{d'}, \sqrt{0}_{U_2}))^{ql-r}.$$

For l large enough, $pl < ql - r$. The case of D_-^{red} is similar. We conclude as $\bar{C}$ is quasi-compact. $\square$

2.2. **Formalism of cohomology with support.** We use the material from [CSb]. We assume that all schemes are of finite type over $\mathbb{Z}$. For any such scheme X, we have a triangulated category of solid $\mathscr{O}_X$-modules, denoted $D(\mathscr{O}_{X,\blacksquare})$, and there is a six functor formalism. There is a fully faithful functor $D(\mathscr{O}_X) \to D(\mathscr{O}_{X,\blacksquare})$ where $D(\mathscr{O}_X)$ is the usual derived category of quasi-coherent $\mathscr{O}_X$-modules. The objects of $D(\mathscr{O}_X)$ are by definition the discrete objects of $D(\mathscr{O}_{X,\blacksquare})$.

2.2.1. *Base change maps.* Let X be an S-scheme. Consider a cartesian diagram where j_X and j_C are open immersions:

$$\begin{array}{ccc} C & \xrightarrow{j_C} & \bar{C} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \bar{p}} \\ X & \xrightarrow{j_X} & \bar{X} \end{array}$$

Lemma 2.2.2. *Let $\mathscr{F}$ be an object of $D(\mathscr{O}_{X,\blacksquare})$.*

(1) *We have a natural map $\bar{p}^\star(j_X)_\star\mathscr{F} \to (j_C)_\star p^\star\mathscr{F}$,*
(2) *We have a natural isomorphism $\bar{p}^\star(j_X)_!\mathscr{F} \to (j_C)_! p^\star\mathscr{F}$,*
(3) *We have a natural map $(j_C)_! p^! \mathscr{F} \to \bar{p}^!(j_X)_!\mathscr{F}$.*
(4) *We have a natural isomorphism: $(j_C)_\star p^! \mathscr{F} \to \bar{p}^!(j_X)_\star\mathscr{F}$.*

Proof. For the first point we just follow the adjunctions. We have:

$$\begin{aligned} \mathrm{Hom}(\bar{p}^\star(j_X)_\star\mathscr{F}, (j_C)_\star p^\star\mathscr{F}) &= \mathrm{Hom}((j_C)^\star\bar{p}^\star(j_X)_\star\mathscr{F}, p^\star\mathscr{F}) \\ &= \mathrm{Hom}(p^\star j_X^\star(j_X)_\star\mathscr{F}, p^\star\mathscr{F}) \\ &= \mathrm{Hom}(p^\star\mathscr{F}, p^\star\mathscr{F}) \end{aligned}$$

The map is given by the identity of $p^\star\mathscr{F}$. The second point is the proper base change theorem (note that $\bar{C}$ and X are tor-independent !), see [CSb], Lecture XI. The other points are similar. $\square$

2.2.3. *Cohomology with partial support.* Let $\bar{X}$ be an S scheme. Let D_+, D_- be two closed subschemes of $\bar{X}$. We let $D = D_+ \cup D_-$ and we let $X = \bar{X} \setminus D$.

Consider $X \overset{j_-}{\hookrightarrow} \bar{X} \setminus D_+ \overset{j_+}{\hookrightarrow} \bar{X}$ as well as $X \overset{i_+}{\hookrightarrow} \bar{X} \setminus D_- \overset{i_-}{\hookrightarrow} \bar{X}$. Let $\mathscr{F}$ be an object of $D(\mathscr{O}_{X,\blacksquare})$.

Definition 2.2.4. *We define* $\mathrm{R}\Gamma_{D_+,D_-}(\bar{X}, \mathscr{F}) := \mathrm{R}\Gamma(\bar{X}, (j_+)_\star(j_-)_!\mathscr{F})$ *and* $\mathrm{R}\Gamma_{D_-,D_+}(\bar{X}, \mathscr{F}) := \mathrm{R}\Gamma(\bar{X}, (i_-)_!(i_+)_\star\mathscr{F})$.

Remark 2.2.5. If we have an intermediate open subscheme $X \overset{j}{\hookrightarrow} X' \hookrightarrow \bar{X}$ and $\mathscr{F}$ is an object of $D(\mathscr{O}_{X',\blacksquare})$, then we abuse notation and write $\mathrm{R}\Gamma_{D_-,D_+}(\bar{X}, \mathscr{F})$ (respectively $\mathrm{R}\Gamma_{D_+,D_-}(\bar{X}, \mathscr{F})$) for $\mathrm{R}\Gamma_{D_-,D_+}(\bar{X}, j^\star\mathscr{F})$ (respectively $\mathrm{R}\Gamma_{D_+,D_-}(\bar{X}, j^\star\mathscr{F})$).

Lemma 2.2.6. *Let $\bar{X}$ be an S-scheme. Let D_+, D_- and D'_+, D'_- be closed subschemes of $\bar{X}$. We assume that $D'_- \subseteq D_-$ and $D_+ \subseteq D'_+$. We also let $\mathscr{F} \in D(\mathscr{O}_{\bar{X}\setminus\{D'_-\cup D_+\},\blacksquare})$. We have natural maps*

$$\begin{array}{rcl} \mathrm{R}\Gamma_{D_+,D_-}(\bar{X},\mathscr{F}) & \to & \mathrm{R}\Gamma_{D'_+,D'_-}(\bar{X},\mathscr{F}) \\ \mathrm{R}\Gamma_{D_-,D_+}(\bar{X},\mathscr{F}) & \to & \mathrm{R}\Gamma_{D'_-,D'_+}(\bar{X},\mathscr{F}) \end{array}$$

Proof. We only treat the first case. We can also assume that either $D'_+ = D_+$ or $D'_- = D_-$. Let us first assume that $D'_- = D_-$ and $D'_+ \supseteq D_+$. Consider the following diagram:

$$\begin{array}{ccccc} \bar{X}\setminus\{D_+\cup D_-\} & \xrightarrow{j_-} & \bar{X}\setminus D_+ & \xrightarrow{j_+} & \bar{X} \\ \uparrow{\scriptstyle i'} & & \uparrow{\scriptstyle i} & \nearrow{\scriptstyle j'_+} & \\ \bar{X}\setminus\{D'_+\cup D_-\} & \xrightarrow{j'_-} & \bar{X}\setminus D'_+ & & \end{array}$$

Let $\mathscr{F}$ be a sheaf on $\bar{X}\setminus\{D_+\cup D_-\}$. We need to produce a canonical map

$$j_{+,\star}j_{-,!}\mathscr{F} \to j'_{+,\star}j'_{-,!}(i')^\star\mathscr{F}.$$

Using that $j'_+ = j_+\circ i$, it suffices to produce a canonical map

$$j_{-,!}\mathscr{F} \to i_\star j'_{-,!}(i')^\star\mathscr{F}.$$

By adjunction, this amounts to a map:

$$\mathscr{F} \to j_-^\star i_\star j'_{-,!}(i')^\star\mathscr{F}.$$

We can use the flat base change map $j_-^\star i_\star = i'_\star(j'_-)^\star$ and the adjunction $(j'_-)^\star j'_{-,!} = Id$ to deduce that this boils down to the natural map given by adjunction:

$$\mathscr{F} \to i'_\star(i')^\star\mathscr{F}.$$

We now treat the case that $D'_+ = D_+$ and $D'_- \subseteq D_-$. Consider the following diagram:

$$\begin{array}{ccccc} \bar{X}\setminus\{D_+\cup D'_-\} & \xrightarrow{j'_-} & \bar{X}\setminus D_+ & \xrightarrow{j_+=j'_+} & \bar{X} \\ \uparrow{\scriptstyle i} & \nearrow{\scriptstyle j_-} & & & \\ \bar{X}\setminus\{D_+\cup D_-\} & & & & \end{array}$$

Let $\mathscr{F}$ be a sheaf on $\bar{X}\setminus\{D_+\cup D'_-\}$. We need to produce a canonical map

$$j_{+,\star}j_{-,!}i^\star\mathscr{F} \to j'_{+,\star}j'_{-,!}\mathscr{F}.$$

This map is easily seen to be induced by the adjunction map $i_!i^\star \to Id$. $\square$

Lemma 2.2.7. *Let $\bar{p} : \bar{C} \to \bar{X}$ be a morphism. Let $D_+, D_- \subseteq \bar{X}$ be two closed subschemes. Let $X = \bar{X}\setminus\{D_+\cup D_-\}$. Let $\bar{p}^\star D_+, \bar{p}^\star D_-$ be the corresponding two closed subschemes of $\bar{C}$. Let $C = \bar{C}\setminus\bar{p}^\star\{D_+\cup D_-\}$ and let $p : C \to X$ be the induced projection. Let $\mathscr{F}$ be an object of $D(\mathscr{O}_{X,\blacksquare})$. Then we have canonical pull back maps:*

$$\begin{array}{rcl} \mathrm{R}\Gamma_{D_+,D_-}(\bar{X},\mathscr{F}) & \to & \mathrm{R}\Gamma_{\bar{p}^\star D_+,\bar{p}^\star D_-}(\bar{C},p^\star\mathscr{F}) \\ \mathrm{R}\Gamma_{D_-,D_+}(\bar{X},\mathscr{F}) & \to & \mathrm{R}\Gamma_{\bar{p}^\star D_-,\bar{p}^\star D_+}(\bar{C},p^\star\mathscr{F}) \end{array}$$

If $\bar{p}$ is proper, we also have canonical trace maps:

$$\begin{array}{rcl} \mathrm{R}\Gamma_{\bar{p}^\star D_+,\bar{p}^\star D_-}(\bar{C},p^!\mathscr{F}) & \to & \mathrm{R}\Gamma_{D_+,D_-}(\bar{X},\mathscr{F}) \\ \mathrm{R}\Gamma_{\bar{p}^\star D_-,\bar{p}^\star D_+}(\bar{C},p^!\mathscr{F}) & \to & \mathrm{R}\Gamma_{D_-,D_+}(\bar{X},\mathscr{F}) \end{array}$$

Proof. We give the proof only for the first and third maps. We consider the diagram:

$$\begin{array}{ccccc} C & \xrightarrow{j'_-} & \bar{C}\setminus\bar{p}^\star D_+ & \xrightarrow{j'_+} & \bar{C} \\ \downarrow{\scriptstyle p} & & \downarrow & & \downarrow{\scriptstyle \bar{p}} \\ X & \xrightarrow{j_-} & \bar{X}\setminus D_+ & \xrightarrow{j_+} & \bar{X} \end{array}$$

By lemma 2.2.2 we have the following maps:

$$\begin{array}{rcl} \bar{p}^\star j_{+,\star} j_{-,!}\mathscr{F} & \to & j'_{+,\star} j'_{-,!} p^\star \mathscr{F} \\ j'_{+,\star} j'_{-,!} p^! \mathscr{F} & \to & \bar{p}^! j_{+,\star} j_{-,!}\mathscr{F} \end{array}$$

and the maps of the lemma are easily deduced. □

Lemma 2.2.8. *In the situation of the last lemma, assume that p is an isomorphism and that $\bar{p}$ is proper. Then all the maps are quasi-isomorphism.*

Proof. We only prove that $\mathrm{R}\Gamma_{D_+,D_-}(\bar{X},\mathscr{F}) \to \mathrm{R}\Gamma_{\bar{p}^\star D_+,\bar{p}^\star D_-}(\bar{C}, p^\star\mathscr{F})$ is an isomorphism. The remaining points are left to the reader. We consider the diagram:

$$\begin{array}{ccccc} X & \xrightarrow{j'_-} & \bar{C}\setminus\bar{p}^\star D_+ & \xrightarrow{j'_+} & \bar{C} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle \bar{p}} \\ X & \xrightarrow{j_-} & \bar{X}\setminus D_+ & \xrightarrow{j_+} & \bar{X} \end{array}$$

Since p' is proper, $p'_\star = p'_!$. We deduce that

$$\bar{p}_\star j'_{+,\star} j'_{-,!}\mathscr{F} = j_{+,\star} p'_\star j'_{-,!}\mathscr{F} = j_{+,\star} j_{-,!}\mathscr{F}.$$

□

2.2.9. *Cohomology and dynamic correspondences.* Let $\mathcal{C} = (\bar{C},\bar{X},\bar{p}_1,\bar{p}_2,D_+,D_-)$ be a dynamic correspondence. Let $\mathscr{F} \in D(\mathscr{O}_{X,\blacksquare})$. Let $T : p_2^\star\mathscr{F} \to p_1^!\mathscr{F}$ be a map in $D(\mathscr{O}_{C,\blacksquare})$.

Proposition 2.2.10. *The map T induces endomorphisms $T_{\mathcal{C}}$ of $\mathrm{R}\Gamma_{D_-,D_+}(\bar{X},\mathscr{F})$ and $\mathrm{R}\Gamma_{D_+,D_-}(\bar{X},\mathscr{F})$ as follows:*

$$\begin{array}{ccccccc} \mathrm{R}\Gamma_{\bar{p}_2^\star D_+,\bar{p}_2^\star D_-}(\bar{C},p_2^\star\mathscr{F}) & \longrightarrow & \mathrm{R}\Gamma_{\bar{p}_1^\star D_+,\bar{p}_2^\star D_-}(\bar{C},p_2^\star\mathscr{F}) & \longrightarrow & \mathrm{R}\Gamma_{\bar{p}_1^\star D_+,\bar{p}_2^\star D_-}(\bar{C},p_1^!\mathscr{F}) & \longrightarrow & \mathrm{R}\Gamma_{\bar{p}_1^\star D_+,\bar{p}_1^\star D_-}(\bar{C},p_1^!\mathscr{F}) \\ \uparrow & & & & & & \downarrow \\ \mathrm{R}\Gamma_{D_+,D_-}(\bar{X},\mathscr{F}) & & & & & & \mathrm{R}\Gamma_{D_+,D_-}(\bar{X},\mathscr{F}) \end{array}$$

$$\begin{array}{ccccccc} \mathrm{R}\Gamma_{\bar{p}_2^\star D_-,\bar{p}_2^\star D_+}(C,p_2^\star\mathscr{F}) & \longrightarrow & \mathrm{R}\Gamma_{\bar{p}_1^\star D_-,\bar{p}_2^\star D_+}(C,p_2^\star\mathscr{F}) & \longrightarrow & \mathrm{R}\Gamma_{\bar{p}_1^\star D_-,\bar{p}_2^\star D_+}(C,p_1^!\mathscr{F}) & \longrightarrow & \mathrm{R}\Gamma_{\bar{p}_1^\star D_-,\bar{p}_1^\star D_+}(C,p_1^!\mathscr{F}) \\ \uparrow & & & & & & \downarrow \\ \mathrm{R}\Gamma_{D_-,D_+}(X,\mathscr{F}) & & & & & & \mathrm{R}\Gamma_{D_-,D_+}(X,\mathscr{F}) \end{array}$$

Proof. We only do the first case. The maps

$$\begin{array}{rcl} \mathrm{R}\Gamma_{D_+,D_-}(\bar{X},\mathscr{F}) & \to & \mathrm{R}\Gamma_{\bar{p}_2^\star D_+,\bar{p}_2^\star D_-}(\bar{C},p_2^\star\mathscr{F}) \\ \mathrm{R}\Gamma_{\bar{p}_1^\star D_+,\bar{p}_1^\star D_-}(\bar{C},p_1^!\mathscr{F}) & \to & \mathrm{R}\Gamma_{D_+,D_-}(\bar{X},\mathscr{F}) \end{array}$$

are given by lemma 2.2.7. The maps

$$\begin{array}{rcl} \mathrm{R}\Gamma_{\bar{p}_2^\star D_-,\bar{p}_2^\star D_+}(\bar{C},p_2^\star\mathscr{F}) & \to & \mathrm{R}\Gamma_{\bar{p}_1^\star D_-,\bar{p}_2^\star D_+}(\bar{C},p_2^\star\mathscr{F}) \\ \mathrm{R}\Gamma_{\bar{p}_1^\star D_-,\bar{p}_2^\star D_+}(\bar{C},p_1^!\mathscr{F}) & \to & \mathrm{R}\Gamma_{\bar{p}_1^\star D_-,\bar{p}_1^\star D_+}(\bar{C},p_1^!\mathscr{F}) \end{array}$$

are given by lemma 2.2.6. The middle map $\mathrm{R}\Gamma_{\bar{p}_1^\star D_-,\bar{p}_2^\star D_+}(C,p_2^\star\mathscr{F}) \to \mathrm{R}\Gamma_{\bar{p}_1^\star D_-,\bar{p}_2^\star D_+}(C,p_1^!\mathscr{F})$ is produced by T as we now explain. We consider the chain of open immersions:

$$C \overset{j_{C,-}}{\to} \bar{C}\setminus\bar{p}_1^\star D_+ \overset{j_{C,+}}{\to} \bar{C}$$

We deduce from T a map:

$$(j_{C,+})_\star (j_{C,-})_! p_2^\star \mathscr{F} \to (j_{C,+})_\star (j_{C,-})_! p_1^! \mathscr{F}$$

which induces the middle map $\mathrm{R}\Gamma_{\bar{p}_1^\star D_-, \bar{p}_2^\star D_+}(C, p_2^\star \mathscr{F}) \to \mathrm{R}\Gamma_{\bar{p}_1^\star D_-, \bar{p}_2^\star D_+}(C, p_1^! \mathscr{F})$.

□

Let (C, X, p_1, p_2) be a correspondence. Let $\mathcal{C} = (\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D_+, D_-)$ be a dynamic compactification. The proposition below states that the endomorphism $T_{\mathcal{C}}$ only depends on the $\mathcal{C}$ up to strict morphism.

Proposition 2.2.11. *Let $\mathcal{C} = (\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D_+, D_-) \to \mathcal{C}' = (\bar{C}', \bar{X}', \bar{p}'_1, \bar{p}'_2, D'_+, D'_-)$ be a strict map. The canonical quasi-isomorphisms given by lemma 2.2.8:*

$$\begin{array}{rcl} \mathrm{R}\Gamma_{D_+,D_-}(\bar{X}, \mathscr{F}) & = & \mathrm{R}\Gamma_{D'_+,D'_-}(\bar{X}', \mathscr{F}) \\ \mathrm{R}\Gamma_{D_-,D_+}(\bar{X}, \mathscr{F}) & = & \mathrm{R}\Gamma_{D'_-,D'_+}(\bar{X}', \mathscr{F}) \end{array}$$

are equivariant for the actions of $T_{\mathcal{C}}$ and $T_{\mathcal{C}'}$.

Proof. We only give the proof in the first case. This follows from the existence of the following commutative diagrams:

$$\begin{array}{ccccc} \mathrm{R}\Gamma_{D'_+,D'_-}(\bar{X}', \mathscr{F}) & \longrightarrow & \mathrm{R}\Gamma_{(\bar{p}'_2)^\star D'_+,(\bar{p}'_2)^\star D'_-}(\bar{C}', (\bar{p}'_2)^\star \mathscr{F}) & \longrightarrow & \mathrm{R}\Gamma_{(\bar{p}'_1)^\star D'_+,(\bar{p}'_2)^\star D'_-}(\bar{C}', (\bar{p}'_2)^\star \mathscr{F}) \longrightarrow \\ \| & & \uparrow & & \| \\ \mathrm{R}\Gamma_{D_+,D_-}(\bar{X}, \mathscr{F}) & \longrightarrow & \mathrm{R}\Gamma_{(\bar{p}_2)^\star D_+,(\bar{p}_2)^\star D_-}(\bar{C}, (\bar{p}_2)^\star \mathscr{F}) & \longrightarrow & \mathrm{R}\Gamma_{(\bar{p}_1)^\star D'_+,(\bar{p}_2)^\star D'_-}(\bar{C}, (\bar{p}_2)^\star \mathscr{F}) \longrightarrow \end{array}$$

$$\begin{array}{ccccc} \longrightarrow \mathrm{R}\Gamma_{(\bar{p}'_1)^\star D'_+,(\bar{p}'_2)^\star D'_-}(\bar{C}', (\bar{p}'_1)^! \mathscr{F}) & \longrightarrow & \mathrm{R}\Gamma_{(\bar{p}'_1)^\star D'_+,(\bar{p}'_1)^\star D'_-}(\bar{C}', (\bar{p}'_1)^! \mathscr{F}) & \longrightarrow & \mathrm{R}\Gamma_{D'_+,D'_-}(\bar{X}', \mathscr{F}) \\ \| & & \downarrow & & \| \\ \longrightarrow \mathrm{R}\Gamma_{(\bar{p}_1)^\star D_+,(\bar{p}_2)^\star D_-}(\bar{C}, (\bar{p}_1)^! \mathscr{F}) & \longrightarrow & \mathrm{R}\Gamma_{(\bar{p}_1)^\star D_+,(\bar{p}_1)^\star D_-}(\bar{C}, (\bar{p}_1)^! \mathscr{F}) & \longrightarrow & \mathrm{R}\Gamma_{D_+,D_-}(\bar{X}, \mathscr{F}) \end{array}$$

□

2.3. Computations of the cohomology.

In this section, we give explicit formulas that allows us to compute the cohomology with support as limits and colimits of classical cohomologies.

2.3.1. *Computing the cohomology with support.* We consider an open immersion $j : X \to \bar{X}$. Let $\bar{X} \setminus X = D$. Assume that there exists an invertible sheaf $\mathscr{O}_{\bar{X}}(-D)$ and a morphism $\mathscr{O}_{\bar{X}}(-D) \to \mathscr{O}_{\bar{X}}$ whose image is an ideal $\mathscr{I}$ which defines D (in other words, D is a locally principal subscheme). Let $\mathscr{F}$ be in $D(\mathscr{O}_{X,\blacksquare})$. We choose an object $\overline{\mathscr{F}} \in D(\mathscr{O}_{\bar{X},\blacksquare})$ such that $j^\star \overline{\mathscr{F}} = \mathscr{F}$. We let $\overline{\mathscr{F}}(-nD) = \overline{\mathscr{F}} \otimes_{\mathscr{O}_{\bar{X}}} \mathscr{O}_{\bar{X}}(-D)^{\otimes n}$. This notation may be slightly non standard when D is not a Cartier divisor. We have maps $\overline{\mathscr{F}}(-(n+1)D) \to \overline{\mathscr{F}}(-nD)$ forming an inverse system. We recall that

$$\lim_n \overline{\mathscr{F}}(-nD) = \mathrm{cone}(\prod_n \overline{\mathscr{F}}(-nD) \to \prod_n \overline{\mathscr{F}}(-nD))[1]$$

where the map sends $(m_n)_n$ to $(m_n - m_{n+1})_n$.

Proposition 2.3.2. *Assume that $\overline{\mathscr{F}}$ is pseudo-coherent. In $D(\mathscr{O}_{\bar{X},\blacksquare})$, we have a canonical isomorphism*

$$j_! \mathscr{F} = \lim_n \overline{\mathscr{F}}(-nD).$$

Proof. We first construct a canonical map: $j_! \mathscr{F} \to \lim_n \overline{\mathscr{F}}(-nD)$. By adjunction, this amounts to a map: $\mathscr{F} \to j^\star \lim_n \overline{\mathscr{F}}(-nD)$. Since $j^\star$ commutes with limits, and since $\lim_n j^\star \overline{\mathscr{F}}(-nD) = \mathscr{F}$, the canonical map is induced by the identity of $\mathscr{F}$.

We now reduce to some local computations. We may assume that $\bar{X} = \mathrm{Spec}\ A$ and that $X = \mathrm{Spec}\ A[1/f]$.

We have a map $\mathrm{Spec}\ A[1/f] = A[X]/(Xf-1) \hookrightarrow \mathrm{Spec}\ A[X] \to \mathrm{Spec}\ A$. Let M be a pseudo-coherent object of $D((A[1/f], A[1/f])_\blacksquare)$ and let $\bar{M}$ be a pseudo-coherent object in $D((A,A)_\blacksquare)$ such that $\bar{M} \otimes_{(A,A)_\blacksquare} (A[1/f], A[1/f])_\blacksquare = M$. By definition ([CSb], Lecture VIII) $j_! M = M \otimes_{(A[X],A)_\blacksquare} A((X^{-1}))/A[X]$. Moreover, it is true that

$$j_! M = \overline{M} \otimes_{(A,A)_\blacksquare} A[X]/(Xf-1) \otimes_{(A[X],A)_\blacksquare} A((X^{-1}))/A[X].$$

Since $\overline{M}$ is pseudo-coherent, we find that

$$\overline{M} \otimes_{(A,A)_\blacksquare} A((X^{-1}))/A[X] = \overline{M} \otimes_{(A,A)_\blacksquare} A[[X^{-1}]]X^{-1} = \prod_{n \geq 1} \overline{M} X^{-n} := \overline{M}[[X^{-1}]]X^{-1}.$$

Therefore,

$$\begin{aligned} j_! M &= \overline{M} \otimes_A A((X^{-1}))/A[X] \otimes_{A[X]} A[X]/(Xf-1) \\ &= \mathrm{cone}(\overline{M}[[X^{-1}]]X^{-1} \xrightarrow{1-Xf} \overline{M}[[X^{-1}]]X^{-1})[1] \end{aligned}$$

□

Remark 2.3.3. If $\mathscr{F}$ is a coherent sheaf over X, then it admits an extension $\overline{\mathscr{F}}$ to a coherent sheaf over $\bar{X}$, and $\overline{\mathscr{F}}$ is pseudo-coherent (see [Sta22], TAG 0G41). Thus, the proposition applies in this setting.

Corollary 2.3.4. *Under the assumptions of proposition 2.3.2, we have* $\mathrm{R}\Gamma(\bar{X}, j_! \mathscr{F}) = \lim_n \mathrm{R}\Gamma(\bar{X}, \overline{\mathscr{F}} \otimes_{\mathscr{O}_{\bar{X}}} \mathscr{O}_{\bar{X}}(-nD))$.

Proof. We have:

$$\begin{aligned} \mathrm{R}\Gamma(\bar{X}, j_! \mathscr{F}) &= \mathrm{R}\Gamma(\bar{X}, \lim_n \overline{\mathscr{F}} \otimes_{\mathscr{O}_{\bar{X}}} \mathscr{O}_{\bar{X}}(-nD)) \\ &= \lim_n \mathrm{R}\Gamma(\bar{X}, \overline{\mathscr{F}} \otimes_{\mathscr{O}_{\bar{X}}} \mathscr{O}_{\bar{X}}(-nD)) \end{aligned}$$

□

Proposition 2.3.5. *Assume that $S = \mathrm{Spec}\ A$ with A an artinian ring. Assume that $\mathscr{F}$ is a coherent sheaf and assume that $\bar{X}$ is proper. We let $\mathscr{I}$ be the ideal of D. Then* $\mathrm{H}^i(X, j_! \mathscr{F}) = \lim_n \mathrm{H}^i(\bar{X}, \mathscr{I}^n \mathscr{F})$.

Remark 2.3.6. This is Hartshorne's definition of cohomology with support ([Har72]).

Proof. We first claim that the natural map $\lim_n \overline{\mathscr{F}}(-nD) \to \lim_n \mathscr{I}^n \overline{\mathscr{F}}$ is a quasi-isomorphism. This is a local question so we can assume $\mathscr{I}$ is generated by an element $f \in \mathscr{O}_{\bar{X}} = A$ and that $\overline{\mathscr{F}}$ corresponds to a finite type A-module M. The map between inverse systems writes:

$$\begin{array}{ccccccc} \longrightarrow & M & \xrightarrow{f} & M & \xrightarrow{f} & M \\ & \downarrow{\scriptstyle f^2} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle 1} \\ \longrightarrow & f^2 M & \longrightarrow & fM & \longrightarrow & M \end{array}$$

The cone is given by $(\lim_n M[f^n])[1]$ where $M[f^n]$ is the submodule of M of elements annihilated by f^n and the transition maps are given by multiplication by f. As $\mathrm{colim}_n M[f^n]$ is a finite type A-module (being a submodule of M), we deduce that $M[f^n] = M[f^{n+1}]$ for all n big enough. This implies that $\{M[f^n]\}_n$ is an essentially zero inverse system and so its limit is zero. We deduce that $\mathrm{R}\Gamma(\bar{X}, j_! \mathscr{F}) = \mathrm{R}\Gamma(\bar{X}, \lim_n \mathscr{I}^n \overline{\mathscr{F}})$. Moreover, the system $\{\mathrm{H}^i(\bar{X}, \mathscr{I}^n \overline{\mathscr{F}})\}_n$ satisfies the Mittag-Leffler property (since it is a system of finite A-modules), hence $\mathrm{H}^i(\bar{X}, j_! \mathscr{F}) = \lim_n \mathrm{H}^i(\bar{X}, \mathscr{I}^n \overline{\mathscr{F}})$.

□

2.3.7. *Computing direct images.* We now proceed to compute direct images. We consider an open immersion $j : X \to \bar{X}$. Let $\bar{X} \setminus X = D$. We assume that D is a locally principal subscheme, whose ideal is the image of a map $\mathscr{O}_{\bar{X}}(-D) \to \mathscr{O}_{\bar{X}}$ for an invertible sheaf $\mathscr{O}_{\bar{X}}(-D)$. Let $\mathscr{F}$ be in $D(\mathscr{O}_{X,\blacksquare})$. We choose an object $\overline{\mathscr{F}} \in D(\mathscr{O}_{\bar{X},\blacksquare})$ such that $j^\star \overline{\mathscr{F}} = \mathscr{F}$.

Proposition 2.3.8. *Assume that $\overline{\mathscr{F}}$ is discrete. In $D(\mathscr{O}_{\bar{X},\blacksquare})$, we have a canonical isomorphism*

$$j_\star \mathscr{F} = \mathrm{colim}_n \overline{\mathscr{F}}(nD).$$

Proof. We first construct a map $\mathrm{colim}_n \overline{\mathscr{F}}(nD) \to j_\star \mathscr{F}$. By adjunction this boils down to a map $j^\star \mathrm{colim}_n \overline{\mathscr{F}}(nD) \to \mathscr{F}$. Since $j^\star$ commutes with colimits, and $j^\star \overline{\mathscr{F}}(nD) = \mathscr{F}$, this map is simply induced by the identity of $\mathscr{F}$. We reduce to compute that this map is an isomorphism in the local case. For a (discrete) module M over a ring A and $f \in A$, we have $M \otimes_A A[1/f] = \mathrm{colim}_{\times f} M$. We use that if M is discrete the (usual) tensor product $M \otimes_A A[1/f]$ is solid in $D(A[1/f]_\blacksquare)$. □

Corollary 2.3.9. *We have* $\mathrm{R}\Gamma(\bar{X}, j_\star \mathscr{F}) = \mathrm{colim}_n \mathrm{R}\Gamma(\bar{X}, \overline{\mathscr{F}}(nD))$.

Proof. As $\bar{X}$ is quasi-separated, this is a consequence of [Sta22], TAG 01FF. □

2.3.10. *Computing the cohomology with partial support.* We consider an open immersion $X \hookrightarrow \bar{X}$ of finite type schemes over Spec $\mathbb{Z}$. We assume that $\bar{X} \setminus X = D$ is the union $D = D_+ \cup D_-$ of two locally principal subschemes. Let $\overline{\mathscr{F}}$ be a coherent sheaf on $\bar{X}$ with restriction $\mathscr{F}$ to X.

Corollary 2.3.11. *We have:*

$$\begin{aligned} \mathrm{R}\Gamma_{D_-, D_+}(\bar{X}, \mathscr{F}) &= \mathrm{colim}_{n_+ \geq 0} \lim_{n_- \geq 0} \mathrm{R}\Gamma(\bar{X}, \overline{\mathscr{F}}(n_+ D_+ - n_- D_-)) \\ \mathrm{R}\Gamma_{D_+, D_-}(\bar{X}, \mathscr{F}) &= \lim_{n_- \geq 0} \mathrm{colim}_{n_+ \geq 0} \mathrm{R}\Gamma(\bar{X}, \overline{\mathscr{F}}(n_+ D_+ - n_- D_-)) \end{aligned}$$

Proof. Consider the chain of open immersions $X \overset{j_-}{\to} \bar{X} \backslash D_+ \overset{j_+}{\to} \bar{X}$. Then $j_{+,\star} j_+^\star \overline{\mathscr{F}} = \mathrm{colim}_{n_+} \overline{\mathscr{F}}(n_+ D_+)$ and therefore $j_{+,\star} j_{-,!} \mathscr{F} = \lim_{n_- \geq 0} \mathrm{colim}_{n_+ \geq 0} \overline{\mathscr{F}}(n_+ D_+ - n_- D_-)$. Consider the chain of open immersions $X \overset{i_+}{\to} \bar{X} \setminus D_- \overset{i_-}{\to} \bar{X}$. Then $(i_+)_\star \mathscr{F} = \mathrm{colim}_{n_+} \overline{\mathscr{F}}(n_+ D_+)$. Since $(i_-)_!$ commutes with colimit, we deduce that $i_{-,!} i_{+,\star} \mathscr{F} = \mathrm{colim}_{n_+ \geq 0} \lim_{n_- \geq 0} \overline{\mathscr{F}}(n_+ D_+ - n_- D_-)$. We then take cohomology and use that cohomology commutes with projective and inductive limits for quasi-compact and quasi-separated schemes. □

2.3.12. *Action of a correspondence.* We consider a dynamic compactification $\mathcal{C} = (\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D_+, D_-)$ of a correspondence (C, X, p_1, p_2). We continue to assume that $\bar{X} \backslash X = D$ is the union $D = D_+ \cup D_-$ of two locally principal subschemes. We also assume that we have a cohomological correspondence $T : p_2^\star \mathscr{F} \to p_1^! \mathscr{F}$ where $\mathscr{F}$ is a coherent sheaf over X. We let $\overline{\mathscr{F}}$ be an extension to a coherent sheaf on $\bar{X}$.

Proposition 2.3.13. *There exists $n \geq 0$ and a map in $D(\mathscr{O}_{\bar{C}, \blacksquare})$*

$$T : \bar{p}_2^\star \overline{\mathscr{F}}(-nD_-) \to \bar{p}_1^! \overline{\mathscr{F}}(nD_+)$$

which induces the map $T : p_2^\star \mathscr{F} \to p_1^! \mathscr{F}$ after restricting to C.

Proof. Let $j_C : C \to \bar{C}$ be the open immersion, with complement $D_C = \bar{p}_1^\star D_+ + \bar{p}_2^\star D_-$. Since $j_C^\star \bar{p}_2^\star = p_2^\star j_X^\star$ and $j_C^\star \bar{p}_1^! = p_1^! j_X^\star$ (recall that $j_X^\star = j_X^!$ and $j_C^\star = j_C^!$), the map T is equivalently a map:

$$T : j_C^\star \bar{p}_2^\star \overline{\mathscr{F}} \to j_C^\star \bar{p}_1^! \overline{\mathscr{F}}$$

or a map $\mathrm{colim}_{n \geq 0} \bar{p}_2^\star \overline{\mathscr{F}}(nD_C) \to \mathrm{colim}_{n \geq 0} \bar{p}_1^! \overline{\mathscr{F}}(nD_C)$ (using that $\bar{p}_2^\star \overline{\mathscr{F}}$ and $\bar{p}_1^! \overline{\mathscr{F}}$ are discrete). We deduce a map $\bar{p}_2^\star \overline{\mathscr{F}} \to \mathrm{colim}_{n \geq 0} \bar{p}_1^! \overline{\mathscr{F}}(nD_C)$. Since $\bar{p}_2^\star \overline{\mathscr{F}}$ is pseudo coherent, this map factors through a map $\bar{p}_2^\star \overline{\mathscr{F}} \to \bar{p}_1^! \overline{\mathscr{F}}(nD_C)$ for n large enough. Now, since $nD_C = n\bar{p}_1^\star D_+ + n\bar{p}_2^\star D_-$, if we tensor by $\mathscr{O}_{\bar{C}}(-n\bar{p}_2^\star D_-)$ and use the projection formula, we deduce the claim. □

The following corollary expresses a certain continuity property of the map $T_{\mathcal{C}}$ constructed in proposition 2.2.10.

Corollary 2.3.14. *Assume that we have maps $p_2^\star \mathscr{O}_{\bar{X}}(D_+) \to p_1^\star \mathscr{O}_{\bar{X}}(D_+)$ and $p_2^\star \mathscr{O}_{\bar{X}}(-D_-) \to p_1^\star \mathscr{O}_{\bar{X}}(-D_-)$. For all $k_+, k_- \geq 0$ and n as in proposition 2.3.13, the cohomological correspondence T induces maps:*

$$\mathrm{R}\Gamma(\bar{X}, \overline{\mathscr{F}}(-nD_- - k_- D_- + k_+ D_+)) \to \mathrm{R}\Gamma(\bar{X}, \overline{\mathscr{F}}(nD_+ - k_- D_- + k_+ D_+))$$

which induce, after passing to the limit and colimit as in corollary 2.3.11, the map $T_{\mathcal{C}}$ of proposition 2.2.10.

Proof. We have maps $p_2^\star \mathscr{O}_{\bar{X}}(k_+ D_+ - k_- D_-) \to p_1^\star \mathscr{O}_{\bar{X}}(k_+ D_+ - k_- D_-)$. We simply twist the cohomological correspondence of proposition 2.3.13 to obtain the desired map in cohomology. □

2.4. **Ordinary part.** We give an abstract definition of the ordinary part of a complex acted on by endomorphisms, and apply this to cohomology with partial support.

2.4.1. *Ordinary part of a complex.* Let $M \in D(A_\blacksquare)$. Let T be an operator acting on M so that we can view M as an object of $D((A[T], A)_\blacksquare)$.

Definition 2.4.2. *The ordinary part $M^{ord} \in D(A[T, T^{-1}]_\blacksquare)$ of M is defined by the formula $M^{ord} = M \otimes_{(A[T],A)} A[T, T^{-1}]_\blacksquare$.*

Let us explain that M^{ord} is the correct object in the cases of interest.

Lemma 2.4.3. *Assume that A is an artinian local ring and that M is a bounded complex of finite A-modules equipped with an action of an operator T. There is a unique direct sum decomposition $M = M^{ord} \oplus M^{nord}$ where M^{nord} is an $A[[T]]$-module.*

Proof. We consider the algebra B generated by T in $\mathrm{End}(M)$. This is a finite A-algebra. It splits into a product of local A-algebras, $B = \prod_i B_i$. We have idempotents e_i attached to each B_i and we let $M_i = e_i M$. We now distinguish two cases:

(1) If $T \in \mathfrak{m}_{B_i}$, then there exist n such that $T^n = 0$. It follows that M_i is in $D((A[[T]], A)_\blacksquare)$. Since $A[[T]] \otimes_{(A[T],A)_\blacksquare} A[T, T^{-1}]_\blacksquare = 0$, we deduce that $M_i^{ord} = 0$.
(2) If $T \notin \mathfrak{m}_{B_i}$, then M_i is an object of the classical derived category $D(A[T, T^{-1}])$ and we can view it as a discrete object of $D(A[T, T^{-1}]_\blacksquare)$, so $M_i^{ord} = M_i$.

□

We now discuss certain situations where M is a colimit or a limit of objects.

Lemma 2.4.4. *Assume that $M = \mathrm{colim}_{i\in I} \lim_{j\in J} M_{i,j}$ in $D(A_\blacksquare)$ where I is a filtered category and J is a cofiltered category. We assume that an endomorphism T acts on M and that the action is induced by actions on each $M_{i,j}$.*

(1) *We have $M^{ord} = \mathrm{colim}_{i\in I} \lim_{j\in J} M_{i,j}^{ord}$.*
(2) *Assume that I and J are the natural numbers $\mathbb{Z}_{\geq 0}$. Assume that there are maps $T' : M_{i,j} \to M_{i-1,j+1}$ so that the following diagrams commute:*

$$\begin{array}{ccc} M_{i-1,j+1} & \longrightarrow & M_{i,j+1} \\ \uparrow{\scriptstyle T'} & & \uparrow{\scriptstyle T'} \\ M_{i,j} & \longrightarrow & M_{i+1,j} \end{array} \qquad\qquad \begin{array}{ccc} M_{i,j+1} & \xleftarrow{T'} & M_{i+1,j} \\ \downarrow & & \downarrow \\ M_{i,j} & \xleftarrow{T'} & M_{i+1,j-1} \end{array}$$

and moreover such that T is the composition

$$M_{i,j} \to M_{i+1,j} \to M_{i,j+1} \to M_{i,j}.$$

Then $M^{ord} = M_{i,j}^{ord}$ for any i, j.
(3) *Assume that each $M_{i,j}$ is a bounded complex of finite A-modules and that A is artinian. Then $M = M^{ord} \oplus M^{nord}$ where M^{nord} is an $A[[T]]$-module.*

Proof. We have that $M = \mathrm{colim}_{i\in I} \lim_{j\in J} M_{i,j}$ in $D((A[T], A)_\blacksquare)$. Since $- \otimes_{(A[T],A)_\blacksquare} A[T, T^{-1}]_\blacksquare$ commutes with colim and lim, we deduce that $M^{ord} = \mathrm{colim}_{i\in I} \lim_{j\in J} M_{i,j}^{ord}$. Under the factorization assumption, we see that the maps in the formula $M^{ord} = \mathrm{colim}_{i\in I} \lim_{j\in J} M_{i,j}^{ord}$ are quasi-isomorphism. The last point follows from lemma 2.4.3. □

Definition 2.4.5. *Let $M, N \in \mathrm{D}(A_\blacksquare)$. A map $T : M \to N$ is said to have finite rank if it has a factorization $M \to C \to N$ where C is a complex whose cohomology groups are finite A-modules.*

Lemma 2.4.6. *Let $T : M \to M$ be a finite rank endomorphism. Assume that A is an Artinian local ring. Then $M^{ord} \in D(A_\blacksquare)$ is a complex whose cohomology groups are finite A-modules.*

Proof. Let $M \to C \to M$ be a factorization of T, where C is a complex with finite cohomology. We can also think of T as an endomorphism of C by composing the arrows of the factorization of T. It is clear that the induced map $M^{ord} \to C^{ord}$ is a quasi-isomorphism. But $\mathrm{H}^i(C^{ord})$ is a direct factor of $\mathrm{H}^i(C)$ by the above discussion. □

2.5. **Ordinary part of cohomology.** We let (C, X, p_1, p_2) be a correspondence. We recall that all schemes are over $S = \mathrm{Spec}\ A$. We let $\mathscr{F}$ be a coherent sheaf over X. We let $T : p_2^\star \mathscr{F} \to p_1^! \mathscr{F}$ be a cohomological correspondence. We assume that there exists a strict dynamic compactification $\mathcal{C} = (\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D_+, D_-)$.

Theorem 2.5.1. (1) *The cohomology groups* $\mathrm{R}\Gamma_{D_+,D_-}(\bar{X}, \mathscr{F})^{ord}$ *and* $\mathrm{R}\Gamma_{D_-,D_+}(\bar{X}, \mathscr{F})^{ord}$ *are canonically quasi-isomorphic.*

(2) *If A is artinian, they are represented by bounded complexes of finite A-modules.*

(3) *If* $\mathcal{C}' = (\bar{C}', \bar{X}', \bar{p}'_1, \bar{p}'_2, D'_+, D'_-) \to \mathcal{C}$ *is a map of strict dynamic compactifications, there are canonical quasi-isomorphisms:*

$$\mathrm{R}\Gamma_{D_+,D_-}(\bar{X}, \mathscr{F})^{ord} \to \mathrm{R}\Gamma_{D'_+,D'_-}(\bar{X}', \mathscr{F})^{ord}$$

and

$$\mathrm{R}\Gamma_{D_-,D_+}(\bar{X}, \mathscr{F})^{ord} \to \mathrm{R}\Gamma_{D'_-,D'_+}(\bar{X}', \mathscr{F})^{ord}.$$

The theorem implies that the ordinary cohomology depends only on the data (C, X, p_1, p_2) (recall that the category of strict dynamic compactifications is cofiltered by lemma 2.1.31), the sheaf $\mathscr{F}$ and the cohomological correspondence T. We simply denote it by $\mathrm{R}\Gamma(X, \mathscr{F})^{C-ord}$.

2.5.2. *Proof of the first two points.* We first improve on the results of section 2.3.12. By proposition 2.2.11 and lemma 2.1.29, we can assume that D_+ and D_- are locally principal subschemes.

Lemma 2.5.3. *After changing D_+ and D_- by a common multiple nD_+ and nD_- with $n \in \mathbb{Z}_{>0}$, there is an integer m such for all $k_+ \geq m$ and $k_- \geq m$, the map $T : p_2^\star \mathscr{F} \to p_1^! \mathscr{F}$ extends to a map:*

$$T : \bar{p}_2^\star \overline{\mathscr{F}}(k_+D_+ - k_-D_-) \to \bar{p}_1^! \overline{\mathscr{F}}(k_+D_+ - k_-D_-)$$

which moreover factors as:

$$\begin{array}{ccc}
\bar{p}_2^\star \overline{\mathscr{F}}((k_+ + 1)D_+ - k_-D_-) & \searrow & \\
\uparrow & & \\
\bar{p}_2^\star \overline{\mathscr{F}}(k_+D_+ - k_-D_-) & \longrightarrow & \bar{p}_1^! \overline{\mathscr{F}}(k_+D_+ - k_-D_-) \\
& & \uparrow \\
& & \bar{p}_1^! \overline{\mathscr{F}}(k_+D_+ - (k_- + 1)D_-)
\end{array}$$

Proof. By proposition 2.3.13, there exists $n \geq 0$ such that the cohomological correspondence T factors into a map

$$T : p_2^\star \overline{\mathscr{F}}(-nD_-) \to p_1^! \overline{\mathscr{F}}(nD_+).$$

By lemma 2.1.17, there exists $0 < p < q \in \mathbb{Z}$ and maps: $p_2^\star \mathscr{O}_{\bar{X}}(qD_+) \to p_1^\star \mathscr{O}_{\bar{X}}(pD_+)$, $p_2^\star \mathscr{O}_{\bar{X}}(-pD_-) \to p_1^\star \mathscr{O}_{\bar{X}}(qD_-)$ we can twist the cohomological correspondence to obtain maps

$$T : p_2^\star \overline{\mathscr{F}}(k_+qD_+ - (n + pk_-)D_-) \to p_1^! \overline{\mathscr{F}}((pk_+ + n)D_+ - qk_-D_-)$$

There exists $k_0 \in \mathbb{Z}_{\geq 0}$ such that $n + (k+1)p \leq kq$ for all $k \geq k_0$. For all $k_+, k_- \geq k_0$, we thus get maps:

$$T : p_2^\star \overline{\mathscr{F}}((k_+ + 1)qD_+ - k_-qD_-) \to p_1^! \overline{\mathscr{F}}(k_+qD_+ - (k_- + 1)qD_-)$$

We can now replace D_+ and D_- by qD_+ and qD_-. □

Proposition 2.5.4. *We have*

$$\mathrm{R}\Gamma_{D_+,D_-}(\bar{X}, \mathscr{F})^{ord} \simeq \mathrm{R}\Gamma_{D_-,D_+}(\bar{X}, \mathscr{F})^{ord} \simeq \mathrm{R}\Gamma(\bar{X}, \overline{\mathscr{F}}(k_+D_+ - k_-D_-))^{ord}$$

for any $k_+ \geq m$ and $k_- \geq m$. In particular, if A is artinian the ordinary cohomology is represented by a bounded complex of finite A-modules.

Remark 2.5.5. This proposition can be viewed as giving a control theorem, identifying the ordinary cohomology with the cohomology of a sheaf over $\bar{X}$ with bounded zeros and poles. See section 2.11 for more statements in this direction.

Proof. We have a commutative diagram:

$$\begin{array}{ccc}
\mathrm{R}\Gamma(\bar{X}, \overline{\mathscr{F}}(k_+D_+ - k_-D_-)) & \longrightarrow & \mathrm{R}\Gamma(\bar{X}, \overline{\mathscr{F}}((k_+ + 1)D_+ - k_-D_-)) \\
\uparrow & \overset{T}{\swarrow} & \uparrow \\
\mathrm{R}\Gamma(\bar{X}, \overline{\mathscr{F}}(k_+D_+ - (k_- + 1)D_-)) & \longrightarrow & \mathrm{R}\Gamma(\bar{X}, \overline{\mathscr{F}}((k_+ + 1)D_+ - (k_- + 1)D_-))
\end{array}$$

where horizontal and vertical maps are induced by allowing more poles along D_+ or imposing less zeros along D_-. The diagonal map (called by abuse of notation T) is the map given by the factorization of T as given by lemma 2.5.3. The endomorphism T of each of the complexes is given by going around the diagram once. On the ordinary part, each of the endomorphism T of the complex is a quasi-isomorphism, implying that all vertical and horizontal maps are also quasi-isomorphisms. The final statement follows from the property that $\mathrm{R}\Gamma(\bar{X}, \overline{\mathscr{F}}(k_+D_+ - k_-D_-))$ is a bounded complex of finite A-modules and that $\mathrm{R}\Gamma(\bar{X}, \overline{\mathscr{F}}(k_+D_+ - k_-D_-))^{ord}$ is a direct factor by lemma 2.4.4. □

2.5.6. *Independence on the choice of a dynamic compactification.* We assume that we have two strict dynamic compactifications $\mathcal{C} = (\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D_+, D_-)$ and $\mathcal{C}' = (\bar{C}', \bar{X}', \bar{p}_1', \bar{p}_2', D_+', D_-')$. By proposition 2.2.11, we are free to replace $\mathcal{C}$ by another strict dynamic correspondence $\mathcal{D}$, admitting a strict map $\mathcal{D} \to \mathcal{C}$, and similarly for $\mathcal{C}'$. By lemma 2.1.13, we can therefore assume that $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2) = (\bar{C}', \bar{X}', \bar{p}_1', \bar{p}_2')$. We can therefore consider three strict dynamic correspondences $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2)$ with divisors (D_+, D_-), (D_+', D_-'), $(D_+' \cup D_+, D_-' \cup D_-)$. This reduces us to consider the case that $D_+' \supseteq D_+$ and $D_-' \supseteq D_-$. We can first treat the case that $D_+' \supseteq D_+$ and $D_-' = D_-$ and then the case $D_+' = D_+$ and $D_-' \subseteq D_-$. We therefore concentrate on the case $D_+' \supseteq D_+$ and $D_-' = D_-$ as the other is similar. We can also assume all divisors are Cartier by lemma 2.1.29 and proposition 2.2.11. There exists $a, b > 0$ such that $D_+' \leq aD_+ + bD_-$. Proposition 2.5.4 applies for some m that works simultaneously for $\mathcal{C} = (\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D_+, D_-)$ and $\mathcal{C}' = (\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D_+', D_-)$ (we also allow ourselves to replace D_-, D_+ and D_+' be a multiple).

We have maps

$$\mathrm{R}\Gamma(\bar{X}, \overline{\mathscr{F}}(k_+D_+ - k_-D_-)) \to \mathrm{R}\Gamma(\bar{X}, \overline{\mathscr{F}}(k_+D_+' - k_-D_-))$$

$$\to \mathrm{R}\Gamma(\bar{X}, \overline{\mathscr{F}}(ak_+D_+ - (k_- - bk_+)D_-)) \to \mathrm{R}\Gamma(\bar{X}, \overline{\mathscr{F}}(ak_+D_+' - (k_- - bk_+)D_-))$$

Taking $k_+ \geq m$ and $k_- \geq m + bk_+$ and applying the ordinary projector, all the above map become quasi-isomorphisms.

2.6. **Base change and perfect complexes.** In this section, we consider base change formulas for the ordinary cohomology and use it to investigate its structure.

2.6.1. *First base change formula.* Let (C, X, p_1, p_2) be a correspondence admitting a strict dynamical compactification. Let $\mathscr{F}$ be a coherent sheaf over X. Let $T : p_2^\star\mathscr{F} \to p_1^!\mathscr{F}$ be a cohomological correspondence.

We suppose that $S = \mathrm{Spec}\ A$. Let M be an A-module of finite type. Let $p : X \to S$. We assume that $\mathscr{F}$ is A-flat. We get a map $T \otimes 1 : p_2^\star(\mathscr{F} \otimes_A M) \to p_1^!(\mathscr{F} \otimes_A M)$, using that $p_2^\star(\mathscr{F}) \otimes_A^L M = p_2^\star(\mathscr{F} \otimes_A M)$ and that we have a map $p_1^!(\mathscr{F}) \otimes_A^L M \to p_1^!(\mathscr{F} \otimes_A M)$ by the projection formula. Indeed, by adjunction such a map is equivalent to a map $(p_1)_!(p_1^!(\mathscr{F}) \otimes_A^L M) \to (\mathscr{F} \otimes_A M)$ and we can use the projection formula to see that $(p_1)_!(p_1^!(\mathscr{F}) \otimes_A^L M) = ((p_1)_! p_1^! \mathscr{F}) \otimes_A^L M$ and we have a map given by adjunction $(p_1)_! p_1^! \mathscr{F} \to \mathscr{F}$.

Proposition 2.6.2. *Under the assumptions above, we have the following base change formula:*

$$\mathrm{R}\Gamma(X, \mathscr{F})^{C-ord} \otimes_A^L M = \mathrm{R}\Gamma(X, \mathscr{F} \otimes_A M)^{C-ord}.$$

Remark 2.6.3. Our assumption that $\mathscr{F}$ is A-flat is only used to make sure that $\mathscr{F} \otimes_A M$ is again a coherent sheaf (and in particular is bounded below) so that $\mathrm{R}\Gamma(X, \mathscr{F} \otimes_A M)^{C-ord}$ is a well defined notion. The assumption that M is a discrete A-module is crucial.

Proof. We take a strict dynamic compactification $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D_+, D_-)$. We can even assume that D_+ and D_- are locally principal subschemes. We will first prove that

$$\mathrm{R}\Gamma_{D_+,D_-}(\bar{X}, \mathscr{F}) \otimes^L_A M = \mathrm{R}\Gamma_{D_+,D_-}(\bar{X}, \mathscr{F} \otimes_A M).$$

The proposition follows by applying ordinary projectors. We have a sequence of map $X \overset{j_-}{\hookrightarrow} \bar{X} \setminus D_+ \overset{j_+}{\hookrightarrow} \bar{X}$. We let $\bar{p} : \bar{X} \to S$ be the projection. By the projection formula: $\bar{p}_\star((j_+)_\star(j_-)_!\mathscr{F}) \otimes^L_A M = \bar{p}_\star((j_+)_\star(j_-)_!(\mathscr{F}) \otimes_{\mathscr{O}_{\bar{X}}} \bar{p}^\star M)$. We now need to see that

$$(j_+)_\star(j_-)_!(\mathscr{F}) \otimes_{\mathscr{O}_{\bar{X}}} \bar{p}^\star M = (j_+)_\star((j_-)_!(\mathscr{F}) \otimes_{\mathscr{O}_{\bar{X}\setminus D_+}} j_+^\star \bar{p}^\star M).$$

This is a consequence of lemma 2.6.4 below, noting that the map j_+ is affine. We then use one more time the projection formula to deduce that $(j_+)_\star((j_-)_!(\mathscr{F}) \otimes_{\mathscr{O}_{\bar{X}\setminus D_+}} j_+^\star \bar{p}^\star M) = (j_+)_\star((j_-)_!(\mathscr{F} \otimes_{\mathscr{O}_X} p^\star M)$. □

Lemma 2.6.4. *Let $A \to B$ be a morphism of finite type $\mathbb{Z}$-algebras. Let $M \in D(A)$ be a complex of discrete A-modules and let $N \in D(B_\blacksquare)$. Then $N \otimes^L_{A_\blacksquare} M = N \otimes^L_{B_\blacksquare} (B_\blacksquare \otimes^L_A M)$.*

Proof. We have to see that $N \otimes^L_{A_\blacksquare} M$ is already B-solid. In principle this is only a complex of B-modules which is A-solid. We can write M as a colimit of bounded above complexes. This reduces us to the case that M is bounded above. We can resolve M by a complex of free A-modules. We thus reduce to the case that N is a solid B-module and $M = A^{\oplus I}$ is a free A-module. The claim now follows from the fact that if N is a solid B-module then $N^{\oplus I}$ is a solid B-module (the category of solid B-modules is stable under colimits inside condensed B-modules). □

2.6.5. *Construction of perfect complexes.* We now show that under certain assumptions, the ordinary cohomology can actually be represented by a perfect complex.

Lemma 2.6.6. *Let A be a local ring. Let $M^0 \overset{d_0}{\to} M^1 \overset{d_1}{\to} M^2$ be a complex of finite free A-modules. Assume that $M^0 \otimes_A A/\mathfrak{m}_A \to M^1 \otimes_A A/\mathfrak{m}_A \to M^2 \otimes_A A/\mathfrak{m}_A$ is exact. Then $M^0 \to M^1 \to M^2$ is exact, moreover $\ker(d_0)$ is a direct factor of M^0, $\mathrm{Im}(d_0)$ is a direct factor of M_1 and $\mathrm{Im}(d_1)$ is a direct factor of M^2.*

Proof. We take elements $x_1, \cdots, x_n$ in M^0 with the property that $d_0(x_1), \cdots, d_0(x_n)$ reduce modulo $\mathfrak{m}_A$ to a basis of $\ker(d_1 \otimes A/\mathfrak{m}_A)$. We get a commutative diagram:

$$\begin{array}{ccccc} M^0 & \xrightarrow{d_0} & M^1 & \xrightarrow{d_1} & M^2 \\ \uparrow & \nearrow & & & \\ A^n & & & & \end{array}$$

where the i-th vector e_i of the canonical basis of A^n is mapped to x_i. Since we can find elements $y_1, \cdots, y_{n'}$ of M^1 with the property that $d_0(x_1), \cdots, d_0(x_n), y_1, \cdots, y_{n'}$ reduce to a basis of $M^1/\mathfrak{m}_A$. We consider the map $A^n \oplus A^{n'} \to M^1$, sending the i-the canonical basis vector of A^n to $d_0(x_i)$ and the j-th canonical basis vector of $A^{n'}$ to y_j. This map induces an isomorphism modulo $\mathfrak{m}_A$. By applying Nakayama's lemma, we deduce that the map is an isomorphism. It follows that A^n injects in M^1 and is a direct factor. The map $M^1/A^n \to M^2$ induces an injective map modulo $\mathfrak{m}_A$. By repeating the same argument as before, we deduce that M^1/A^n injects as a direct factor in M^2. If follows that $A^n = \ker(d_1)$. Since $A^n \subseteq \mathrm{Im}(d_0)$, we deduce that $\mathrm{Im}(d_0) = \ker(d_1)$. The map $M^0 \to \mathrm{Im}(d_0)$ has a splitting given by $A^n \simeq \mathrm{Im}(d_0) \to M^0$, so $\ker(d_0)$ is a direct factor of M^0. □

Lemma 2.6.7. *Let A be a local ring. Let $M^\bullet$ be a pseudo-coherent object of the usual derived category $D(A)$. We assume that $M^\bullet \otimes^L_A A/\mathfrak{m}_A$ is a perfect complex of amplitude $[a, b]$. Then $M^\bullet$ is a perfect complex of amplitude $[a, b]$.*

Proof. We can represent $M^\bullet$ by a complex in $C^-(A)$ of finite free A-modules. We consider the truncation $M^\bullet_{\tau \geq a, \tau \leq b}$. The lemma 2.6.6 shows that this is a perfect complex, quasi-isomorphic to $M^\bullet$. □

Let (C, X, p_1, p_2) be a correspondence admitting a strict dynamical compactification. Let $\mathscr{F}$ be a coherent sheaf over X. Let $T : p_2^\star\mathscr{F} \to p_1^!\mathscr{F}$ be a cohomological correspondence. We suppose that $S = \mathrm{Spec}\ A$ is an artinian local ring. We assume that $\mathscr{F}$ is A-flat.

Theorem 2.6.8. *Under the above assumptions, the ordinary cohomology* $\mathrm{R}\Gamma(X, \mathscr{F})^{C-ord}$ *is represented by a perfect complex of (discrete) A-modules.*

Proof. By lemma 2.6.7, it suffices to prove that $\mathrm{R}\Gamma(X, \mathscr{F})^{C-ord} \otimes^L_A A/\mathfrak{m}_A$ is a bounded complex. This follows from the base change formula proposition 2.6.2 and theorem 2.5.1. □

2.6.9. *Group action and base change.* Let (C, X, p_1, p_2) be a correspondence admitting a strict dynamic compactification. Let $\mathscr{F}$ be a coherent sheaf over X. Let $T : p_2^\star\mathscr{F} \to p_1^!\mathscr{F}$ be a cohomological correspondence.

Let G be a finite abelian group acting linearly on $\mathscr{F}$. We assume that the cohomological correspondence $T : p_2^\star\mathscr{F} \to p_1^!\mathscr{F}$ commutes with the G-action (in other words, this is a map in $D(\mathscr{O}_C[G]_\blacksquare)$. We suppose that $S = \mathrm{Spec}\ A$ and that $\mathscr{F}$ is flat as an $A[G]$-module. We let $I_G \subseteq A[G]$ be the augmentation ideal (the kernel of the augmentation map $A[G] \to A$). We let $\mathscr{F}/I_G = \mathscr{F} \otimes_{A[G]} A$. We deduce a map $T : p_2^\star\mathscr{F}/I_G \to p_1^!\mathscr{F}/I_G$.

Under all these assumptions, we prove:

Theorem 2.6.10. *The ordinary cohomology is an object of $D(A[G])$, and we have the specialization formula* $\mathrm{R}\Gamma(X, \mathscr{F})^{C-ord} \otimes^L_{A[G]} A = \mathrm{R}\Gamma(X, \mathscr{F}/I_G)^{C-ord}$*. If A is an artinian ring, then* $\mathrm{R}\Gamma(X, \mathscr{F})^{C-ord}$ *is a perfect complex of $A[G]$-modules.*

Proof. By proposition 2.6.2, we have that $\mathrm{R}\Gamma(X, \mathscr{F})^{C-ord} \otimes^L_{A[G]} A = \mathrm{R}\Gamma(X, \mathscr{F}/I_G)^{C-ord}$. The final claim follows from theorem 2.6.8, as then $A[G]$ is a product of local rings. □

We discuss briefly the projectivity assumption that the sheaf $\mathscr{F}$ is $A[G]$-flat.

Lemma 2.6.11. *Let $r : Y \to X$ be a finite étale cover with group G, then $r_\star\mathscr{O}_Y$ is a finite locally projective sheaf of $\mathscr{O}_X[G]$-module. If furthermore $X \to \mathrm{Spec}\ A$ is flat, then $r_\star\mathscr{O}_Y$ if $A[G]$-flat.*

Proof. We can suppose $X = \mathrm{Spec}\ B$ and $Y = \mathrm{Spec}\ C$ are affine. We have an isomorphism $C \otimes_B C = C[G]$, $c \otimes c' \mapsto (c'g(c))_{g \in G}$ (this isomorphism is G-equivariant for the action of G on the first factor). We note that $C[G]$ is a projective $B[G]$-module since C is a projective B-module. There is a G-equivariant inclusion $C \to C \otimes_B C$, $c \mapsto c \otimes 1$. Conversely, we can choose an element c_0 in C such that $\mathrm{Tr}(c_0) = 1$ since C is finite étale over B. The G-equivariant map $C \otimes_B C \to C$, $(c \otimes c') \mapsto c\mathrm{Tr}(c'c_0)$ is a section to $C \to C \otimes C$. □

2.6.12. *Second base change formula.* We assume that (C, X, p_1, p_2) is a correspondence admitting a strict dynamic compactification. We let $S' = \mathrm{Spec}\ A' \to S = \mathrm{Spec}\ A$ be a finite morphism. We let $X' = X \times_S S'$ and similarly for all spaces and morphisms. We deduce that (C', X', p'_1, p'_2) admits a strict dynamic compactification (take the base change via $S' \to S$ of a strict dynamic compactification of (C, X, p_1, p_2)). Let $\mathscr{F}$ be a coherent sheaf over X, and $T : p_2^\star\mathscr{F} \to p_1^!\mathscr{F}$ be a cohomological correspondence. We assume that X and S' are tor independent and that $\mathscr{F}$ is A-flat. We have a cohomological correspondence: $T' : (p'_2)^\star\mathscr{F}' \to (p'_1)^!\mathscr{F}'$ deduced by base change (we use here [CSb], prop. 11.4 and the tor-independence of X and S').

Proposition 2.6.13. *Under the above assumptions we have the base change formula:*

$$\mathrm{R}\Gamma(X, \mathscr{F})^{C-ord} \otimes^L_A A' = \mathrm{R}\Gamma(X', \mathscr{F}')^{C'-ord}$$

Proof. We prove that

$$\mathrm{R}\Gamma_{D_+, D_-}(\bar{X}, \mathscr{F}) \otimes^L_A A' = \mathrm{R}\Gamma_{D'_+, D'_-}(\bar{X}', \mathscr{F}').$$

The proposition follows by applying ordinary projectors. As in the proof of proposition 2.6.2, we have that $\mathrm{R}\Gamma_{D_+, D_-}(\bar{X}, \mathscr{F}) \otimes^L_A A' = \mathrm{R}\Gamma_{D_+, D_-}(\bar{X}, \mathscr{F} \otimes^L_A A')$. So we need to see that $\mathscr{F} \otimes^L_A A' = \mathscr{F}'$. Note that $\mathscr{F}' = \mathscr{F} \otimes^L_{\mathscr{O}_X} \mathscr{O}_{X'}$ and $\mathscr{O}_{X'} = \mathscr{O}_X \otimes_A A'$ (this last tensor product is not derived by definition of X' as the underived fiber product). By tor independence $\mathscr{O}_{X'} = \mathscr{O}_X \otimes^L_A A'$ and so we deduce that $\mathscr{F} \otimes^L_A A' = \mathscr{F}'$. □

2.7. **Duality.** Let $p : X \to S$ be a separated morphism of finite type schemes over $\mathbb{Z}$. We let $D_S = \underline{\mathrm{RHom}}(-, \mathscr{O}_S)$ and $D_X = \underline{\mathrm{RHom}}(-, p^! \mathscr{O}_S)$. The following is a restatement of the adjunction property between $p^!$ and $p_!$.

Theorem 2.7.1 ([CSb], Lecture XI)**.** *Let $\mathscr{F} \in D(\mathscr{O}_{X,\blacksquare})$. We have $p_\star D_X(\mathscr{F}) = D_S(p_! \mathscr{F})$.*

Consider an open immersion $X \to \bar{X}$. We let $\bar{X} \setminus X = D_1 \cup D_2$ where D_1, D_2 are closed subschemes.

Proposition 2.7.2. *Let $\mathscr{F}$ be a locally free sheaf over X. Assume that $X \to S$ is smooth and that $\mathscr{F}$ extends to a locally free sheaf $\overline{\mathscr{F}}$ over $\bar{X}$. Assume also that D_1 and D_2 are Cartier divisors. Finally, assume that $S = \mathrm{Spec}\ A$ and that $\bar{X}$ is proper. Then*

$$\mathrm{RHom}(\mathrm{R}\Gamma_{D_{1,-},D_{2,+}}(\bar{X}, \mathscr{F}), A) = \mathrm{R}\Gamma_{D_{1,+},D_{2,-}}(\bar{X}, D_X(\mathscr{F})).$$

Proof. We consider the inclusions: $X \stackrel{j_2}{\to} \bar{X} \setminus D_1 \stackrel{j_1}{\to} \bar{X}$. By theorem 2.7.1, we have to prove that $D_{\bar{X}}((j_1)_!(j_2)_\star \mathscr{F}) = (j_1)_\star (j_2)_! D_X(\mathscr{F})$. First of all, by theorem 2.7.1, we have $D_{\bar{X}}((j_1)_!(j_2)_\star \mathscr{F}) = (j_1)_\star D_{\bar{X} \setminus D_1}((j_2)_\star \mathscr{F})$. We therefore reduce to the case that $D_1 = \emptyset$. We let $j_2 = j : X \to \bar{X}$ and we need to prove that $D_{\bar{X}}(j_\star \mathscr{F}) = j_! D_X(\mathscr{F})$. We assume that $\bar{X} \setminus X$ is $V(\mathscr{I})$ for an invertible sheaf $\mathscr{I}$. We have $j_\star \mathscr{F} = \mathrm{colim} \mathscr{I}^{-n} \overline{\mathscr{F}}$. We deduce that

$$\begin{aligned} D_{\bar{X}}(j_\star \mathscr{F}) &= \underline{\mathrm{RHom}}(j_\star \mathscr{F}, \bar{p}^! \mathscr{O}_S) \\ &= \lim_n \mathscr{I}^n \underline{\mathrm{RHom}}(\overline{\mathscr{F}}, \bar{p}^! \mathscr{O}_S) \\ &= \lim_m \lim_n \mathscr{I}^n \underline{\mathrm{RHom}}(\overline{\mathscr{F}}, (\bar{p}^! \mathscr{O}_S)_{\sigma_{\leq m}}) \end{aligned}$$

In the last equation, $(\bar{p}^! \mathscr{O}_S)_{\sigma_{\leq m}}$ is the stupid truncation ([Sta22], TAG 0018). Since $\mathrm{RHom}(\overline{\mathscr{F}}, (\bar{p}^! \mathscr{O}_S)_{\sigma_{\leq m}})$ is pseudo-coherent, by proposition 2.3.2, we have $\lim_n \mathscr{I}^n \underline{\mathrm{RHom}}(\overline{\mathscr{F}}, (\bar{p}^! \mathscr{O}_S)_{\sigma_{\leq m}}) = j_! \mathrm{RHom}(\mathscr{F}, (p^! \mathscr{O}_S)_{\sigma_{\leq m}})$. Since p is smooth, $(p^! \mathscr{O}_S)_{\sigma_{\leq m}} = p^! \mathscr{O}_S$ for m large enough.

□

We now assume that we have a correspondence (C, X, p_1, p_2) where C and X are smooth over S and that $\mathscr{F}$ is a locally free sheaf of finite rank. We let $T : p_2^\star \mathscr{F} \to p_1^! \mathscr{F}$ be a cohomological correspondence.

Proposition 2.7.3. *Applying D_C to T we obtain a transpose of the cohomological correspondence:*

$$T^t : p_1^\star D_X(\mathscr{F}) \to p_2^! D_X(\mathscr{F}).$$

Proof. We first check that $D_C(p_2^\star \mathscr{F}) = p_2^! D_X(\mathscr{F})$. We have:

$$\begin{aligned} D_C(p_2^\star \mathscr{F}) &= \underline{\mathrm{RHom}}(p_2^\star \mathscr{F}, p_2^! p^! \mathscr{O}_S) \\ &= p_2^! p^! \mathscr{O}_S \otimes (p_2^\star \mathscr{F})^\vee \\ &= p_2^!(p^! \mathscr{O}_S \otimes \mathscr{F}^\vee) \\ &= p_2^! D_X(\mathscr{F}) \end{aligned}$$

We next check that $D_C(p_1^! \mathscr{F}) = p_1^\star D_X(\mathscr{F})$. It follows from the assumption that C and X are smooth over S that $p_1^! \mathscr{O}_X$ is an invertible object.

$$\begin{aligned} D_C(p_1^! \mathscr{F}) &= \underline{\mathrm{RHom}}(p_1^! \mathscr{F}, p_1^! p^! \mathscr{O}_S) \\ &= \underline{\mathrm{RHom}}(p_1^! \mathscr{O}_X \otimes p_1^\star \mathscr{F}, p_1^! \mathscr{O}_X \otimes p_1^\star p^! \mathscr{O}_S) \\ &= \underline{\mathrm{RHom}}(p_1^\star \mathscr{F}, p_1^\star p^! \mathscr{O}_S) \\ &= p_1^\star D_X(\mathscr{F}) \end{aligned}$$

□

We let C^t be the transpose of C.

Theorem 2.7.4. *Let (C, X, p_1, p_2) be a correspondence with C and X smooth over $S = \mathrm{Spec}\ A$. We assume that it admits a strict dynamical compactification and that A is artinian. Let $\mathscr{F}$ be a locally free sheaf of finite rank and $T : p_2^\star \mathscr{F} \to p_1^! \mathscr{F}$ be a cohomological correspondence. Then we have a perfect pairing:*

$$\mathrm{R}\Gamma(X, \mathscr{F})^{C-ord} \times \mathrm{R}\Gamma(X, D_X(\mathscr{F}))^{C^t-ord} \to A$$

This precisely means that

$$\mathrm{R}\Gamma(X, \mathscr{F})^{C-ord} = \mathrm{RHom}(\mathrm{R}\Gamma(X, D_X(\mathscr{F}))^{C^t-ord}, A)$$

and

$$\mathrm{R}\Gamma(X, D_X(\mathscr{F}))^{C^t-ord} = \mathrm{RHom}(\mathrm{R}\Gamma(X, \mathscr{F})^{C-ord}, A).$$

Proof. We can first assume that (C, X, p_1, p_2) admits a dynamical compactification $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D_+, D_-)$ with the property that $\mathscr{F}$ extends to a locally free sheaf $\overline{\mathscr{F}}$ and such that D_+ and D_- are Cartier divisors (by lemma 2.1.6, lemma 2.1.12 and lemma 2.1.29). We first apply proposition 2.7.2, to obtain $\mathrm{RHom}(\mathrm{R}\Gamma_{D_-,D_+}(\bar{X}, \mathscr{F}), A) = \mathrm{R}\Gamma_{D^t_+, D^t_-}(\bar{X}, D_X(\mathscr{F}))$ where $D^t_+ = D_-$ and $D^t_- = D_+$. By proposition 2.7.3, this isomorphism is equivariant for the action of C and C^t (we call T the corresponding endomorphism). By lemma 2.4.4, the complexes $\mathrm{R}\Gamma_{D_-,D_+}(\bar{X}, \mathscr{F})$ and $\mathrm{R}\Gamma_{D^t_+, D^t_-}(\bar{X}, D_X(\mathscr{F}))$ have a decomposition into an ordinary and a non-ordinary part. Moreover, the non-ordinary part is an $A[[T]]$-module. Since $A[T, T^{-1}]_{\blacksquare} \otimes_{A[T]} A[[T]] = 0$ it follows easily that passing to the ordinary part gives

$$\mathrm{R}\Gamma(X, \mathscr{F})^{C-ord} = \mathrm{RHom}(\mathrm{R}\Gamma(X, D_X(\mathscr{F}))^{C^t-ord}, A).$$

□

2.8. **Iterating the correspondence.** We let (C, X, p_1, p_2) be a correspondence. In this section we will consider iteration of the correspondence and prove that the ordinary part of cohomology can actually be realized in the cohomology of the iterated correspondence. We let $C^{(1)} = C$, $p_i^{(1)} = p_i$ and $C^{(0)} = X$. By induction, we define $(C^{(n)}, C^{(n-1)}, p_1^{(n)}, p_2^{(n)})$. We let $C^{(n)} = C^{(n-1)} \times_{p_1^{(n-1)}, C^{(n-2)}, p_2^{(n-1)}} C^{(n-1)}$. We let $p_2^{(n)}$ be the projection on the first factor and $p_1^{(n)}$ be the projection on the second factor.

Lemma 2.8.1. *Assume that (C, X, p_1, p_2) admits a dynamic (resp. strict dynamic, resp. compact dynamic) compactification. Then*

$$(C^{(n)}, C^{(n-1)}, p_1^{(n)}, p_2^{(n)})$$

admits a dynamic (resp. strict dynamic, resp. compact dynamic) compactification.

Proof. We let $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D_+, D_-)$ be a dynamic compactification. We define inductively

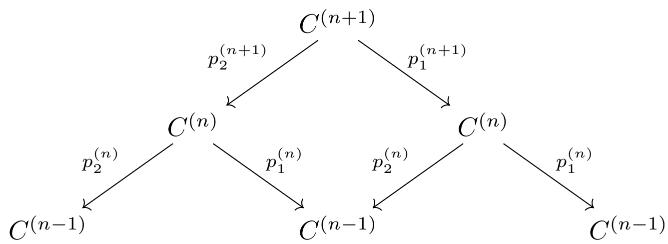
$$\bar{C}^{(n)} = \bar{C}^{(n-1)} \times_{\bar{p}_1^{(n-1)}, \bar{C}^{(n-2)}, \bar{p}_2^{(n-1)}} \bar{C}^{(n-1)}.$$

We assume that $(\bar{C}^{(n)}, \bar{C}^{(n-1)}, D_+^{(n-1)}, D_-^{(n-1)})$ is a (strict) dynamic compactification and we prove that $(\bar{C}^{(n+1)}, \bar{C}^{(n)}, D_+^{(n)}, D_-^{(n)})$ is a dynamic compactification. We define $D_+^{(n)} = (\bar{p}_1^{(n)})^\star D_+^{(n-1)}$ and $D_-^{(n)} = (\bar{p}_2^{(n)})^\star D_-^{(n-1)}$. We see that $(\bar{p}_2^{(n)})^\star D_+^{(n-1)} \subseteq (\bar{p}_1^{(n)})^\star D_+^{(n-1)}$ implies that

$$(\bar{p}_1^{(n+1)})^\star (\bar{p}_2^{(n)})^\star D_+^{(n-1)} \subseteq (\bar{p}_1^{(n+1)})^\star (\bar{p}_1^{(n)})^\star D_+^{(n-1)}$$

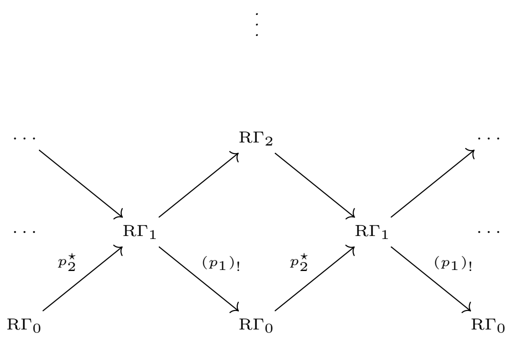
Since $(\bar{p}_1^{(n+1)})^\star (\bar{p}_2^{(n)})^\star D_+^{(n-1)} = (\bar{p}_2^{(n+1)})^\star (\bar{p}_1^{(n)})^\star D_+^{(n-1)}$, we deduce that $(\bar{p}_2^{(n+1)})^\star D_+^{(n)} \subseteq (\bar{p}_1^{(n+1)})^\star D_+^{(n)}$. We proceed similarly to prove that $(\bar{p}_1^{(n+1)})^\star D_-^{(n)} \subseteq (\bar{p}_2^{(n+1)})^\star D_-^{(n)}$. One checks easily that if $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D_+, D_-)$ is strict dynamic or compact dynamic, the same procedure yields a strict dynamic or compact dynamic compactification of $(C^{(n)}, C^{(n-1)}, p_1^{(n)}, p_2^{(n)})$. □

Let $T : p_2^\star \mathscr{O}_X \to p_1^! \mathscr{O}_X$ be a cohomological correspondence. We assume that p_1 and p_2 are flat morphisms. For any n, there is a natural way to construct a cohomological correspondence $T_n : (p_2^{(n)})^\star \mathscr{O}_{C^{(n-1)}} \to (p_1^{(n)})^! \mathscr{O}_{C^{(n-1)}}$. Indeed, consider a diagram:



We assume that we have constructed $T_n : (p_2^{(n)})^\star \mathscr{O}_{C^{(n-1)}} \to (p_1^{(n)})^! \mathscr{O}_{C^{(n-1)}}$ and we proceed to construct T_{n+1}. We can take $T_{n+1} = (p_2^{(n+1)})^\star T_n : (p_2^{(n+1)})^\star (p_2^{(n)})^\star \mathscr{O}_{C^{(n-1)}} \to (p_2^{(n+1)})^\star (p_1^{(n)})^! \mathscr{O}_{C^{(n-1)}}$. We have $(p_2^{(n+1)})^\star (p_2^{(n)})^\star \mathscr{O}_{C^{(n-1)}} = (p_2^{(n+1)})^\star \mathscr{O}_{C^{(n)}}$. Moreover, by flat base change ([CSb], Prop. 11.4) $(p_2^{(n+1)})^\star (p_1^{(n)})^! \mathscr{O}_{C^{(n-1)}} = (p_1^{(n+1)})^! \mathscr{O}_{C^{(n)}}$. Therefore, by theorem 2.5.1, it makes sense to consider $\mathrm{R}\Gamma(C^{(n)}, \mathscr{O}_{C^{(n)}})^{C^{(n+1)}-ord}$. To simplify notation, let us put $\mathrm{R}\Gamma(C^{(n)}, \mathscr{O}_{C^{(n)}})^{C^{(n+1)}-ord} = \mathrm{R}\Gamma_n$.

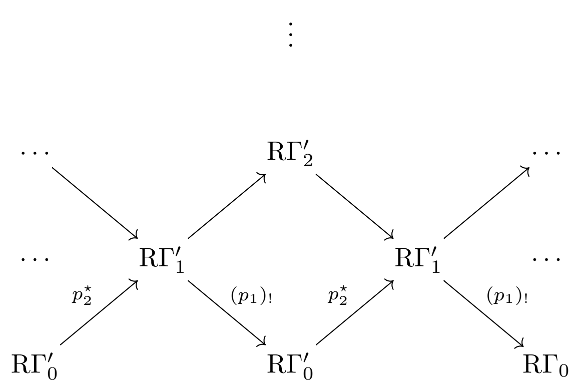
Proposition 2.8.2. *We have a commutative diagram:*



where all maps are quasi-isomorphisms, so that for any $n \geq 0$,

$$\mathrm{R}\Gamma(C^{(n)}, \mathscr{O}_{C^{(n)}})^{C(n+1)-ord} \simeq \mathrm{R}\Gamma(X, \mathscr{O}_X)^{C-ord}.$$

Proof. Let us denote by $\mathrm{R}\Gamma'_n = \mathrm{R}\Gamma_{D_+^{(n)}, D_-^{(n)}}(\bar{C}^{(n)}, \mathscr{O}_{\bar{C}^{(n)}})$. We will prove that the following diagram is commutative:



The map T_n is by definition obtained by going right up via $(\bar{p}_2^{(n+1)})^\star : \mathrm{R}\Gamma'_n \to \mathrm{R}\Gamma'_{n+1}$, and after tracing right down via $(\bar{p}_1^{(n+1)})_! : \mathrm{R}\Gamma'_{n+1} \to \mathrm{R}\Gamma'_n$. Thus, if we can prove the commutativity, we conclude. The commutativity of the diagram boils down to the property that the following squares commutes:

(2.8.A)
$$\begin{array}{ccc} \mathrm{R}\Gamma'_n & \xrightarrow{(\bar{p}_2^{(n+1)})^\star} & \mathrm{R}\Gamma'_{n+1} \\ \Big\downarrow{\scriptstyle (\bar{p}_1^{(n)})_!} & & \Big\downarrow{\scriptstyle (\bar{p}_1^{(n+1)})_!} \\ \mathrm{R}\Gamma'_{n-1} & \xrightarrow{(\bar{p}_2^{(n)})^\star} & \mathrm{R}\Gamma'_n \end{array}$$

This commutativity will be a consequence of the proper base change identity

$$(p_2^{(n+1)})^\star (p_1^{(n+1)})_! = (p_1^{(n)})_! (p_2^{(n)})^\star.$$

Here are some details. We consider the open immersions $C^{(n)} \overset{j_-^{(n)}}{\to} \bar{C}^{(n)} \setminus D_+^{(n)} \overset{j_+^{(n)}}{\to} \bar{C}^{(n)}$. We have maps:

$$\begin{array}{ccc} (\bar{p}_1^{(n+1)})_! (\bar{p}_2^{(n+1)})^\star (j_+^{(n)})_\star (j_-^{(n)})_! \mathscr{O}_{C^{(n)}} & \longrightarrow & (j_+^{(n)})_\star (j_-^{(n)})_! (p_1^{(n+1)})_! (p_2^{(n+1)})^\star \mathscr{O}_{C^{(n)}} \\ & & \Big\| \\ (\bar{p}_2^{(n)})^\star (\bar{p}_1^{(n)})_! (j_+^{(n)})_\star (j_-^{(n)})_! \mathscr{O}_{C^{(n)}} & \longrightarrow & (j_+^{(n)})_\star (j_-^{(n)})_! (\bar{p}_2^{(n})^\star (\bar{p}_1^{(n)})_! \mathscr{O}_{C^{(n)}} \end{array}$$

By construction, the following square commutes (where the second horizontal maps are adjunction):

$$\begin{array}{ccccc} (p_1^{(n+1)})_! (p_2^{(n+1)})^\star \mathscr{O}_{C^{(n)}} & \xrightarrow{T_{n+1}} & (p_1^{(n+1)})_! (p_1^{(n+1)})^! \mathscr{O}_{C^{(n)}} & \longrightarrow & \mathscr{O}_{C^{(n)}} \\ \Big\| & & & & \Big\| \\ (\bar{p}_2^{(n})^\star (\bar{p}_1^{(n)})_! \mathscr{O}_{C^{(n)}} & \xrightarrow{T_n} & (\bar{p}_2^{(n})^\star (\bar{p}_1^{(n)})_! (\bar{p}_1^{(n)})^! \mathscr{O}_{C^{(n-1)}} & \longrightarrow & \mathscr{O}_{C^{(n)}} \end{array}$$

We therefore deduce that the following diagram commutes:

$$\begin{array}{ccccc} & \nearrow & \mathrm{R}\Gamma_{D_+^{(n)}, D_-^{(n)}}(\bar{C}^{(n)}, (p_1^{(n+1)})_! (p_2^{(n+1)})^\star \mathscr{O}_{C^{(n)}}) & \searrow & \\ \mathrm{R}\Gamma'_n & & \Big\| & & \mathrm{R}\Gamma'_n \\ & \searrow & \mathrm{R}\Gamma_{D_+^{(n)}, D_-^{(n)}}(\bar{C}^{(n)}, (p_2^{(n)})^\star (p_1^{(n)})_! \mathscr{O}_{C^{(n)}}) & \nearrow & \end{array}$$

and this is equivalent to the commutative of diagram 2.8.A. □

2.9. **The spectral sequence associated to a filtration.** In this section, we explain how we can associate to a filtration on a space a spectral sequence, called the Cousin spectral sequence. This is a variation of [Har66], IV, p. 227.

2.9.1. *Generalities on localization sequences.* We apply here [CSa], lecture V, to our particular setting. Let $X = \mathrm{Spec}\ R$ be an affine scheme. Let I be an ideal. Let $U = X \setminus V(I)$. Let $j : U \hookrightarrow X$. We let $i : Z \to X$ be the closed complement of U. The closed subset Z can be viewed as a closed subscheme of X with underlying algebra of function R/J for the choice of some ideal J satisfying $\sqrt{J} = \sqrt{I}$. The closed subset Z can also be viewed as a closed subset of the locale of $D(R_\blacksquare)$, which means that the algebra of functions on Z is the idempotent $R_\blacksquare$-algebra $\hat{R}^I = \lim_n R/I^n$ (see lemma 2.9.2 below). When we think of Z as a closed subset of the locale, equipped with the idempotent algebra $\hat{R}^I$, we denote it $\hat{Z}$.

Lemma 2.9.2. *The solid R-algebra $\hat{R}^I = \lim_n R/I^n$ is an idempotent algebra in $D(R_\blacksquare)$.*

Proof. In this proof, all tensor products are derived and are in $D(R_\blacksquare)$. We first assume that I is generated by one element f. We have an exact sequence: $0 \to R[[X]] \stackrel{X-f}{\to} R[[X]] \to \hat{R}^I \to 0$. Since $R[[X]] \otimes_R R[[X']] = R[[X,X']]$, we deduce that $\hat{R}^I \otimes_R R[[X]] = \hat{R}^I[[X]]$. We finally deduce that $\hat{R}^I \otimes_R \hat{R}^I = \hat{R}^I$. We now suppose that $I = (f_1, \cdots, f_r)$. It suffices to show that $\hat{R}^{(f_1,\cdots,f_r)} = \otimes_{i=1}^r \hat{R}^{f_i}$ as the later is idempotent. We argue by induction on r. Assume that $\hat{R}^{(f_1,\cdots,f_{r-1})} = \otimes_{i=1}^{r-1}\hat{R}^{f_i}$. This implies that $\hat{R}^{(f_1,\cdots,f_{r-1})}$ is a compact object in $D(R_\blacksquare)$, represented by a bounded complex, whose terms are of the form $R^{\mathbb{N}}$. We deduce that $\hat{R}^{(f_1,\cdots,f_{r-1})} \otimes_R R[[X]] = \hat{R}^{(f_1,\cdots,f_{r-1})}[[X]]$. This implies that $\hat{R}^{(f_1,\cdots,f_{r-1})} \otimes_R \hat{R}^{f_r} = \left[\hat{R}^{(f_1,\cdots,f_{r-1})}[[X]] \stackrel{X-f_r}{\to} \hat{R}^{(f_1,\cdots,f_{r-1})}[[X]]\right] = \hat{R}^{(f_1,\cdots,f_r)}$. □

We let $\hat{i} : \hat{Z} \to X$ be the "closed immersion" of the locale. This means that we have a functor $\hat{i}_\star \hat{i}^\star : D(R_\blacksquare) \to D(R_\blacksquare)$, $M \mapsto M \otimes_R \hat{R}^I$, whose essential image we denote by $D((\hat{R}^I, R)_\blacksquare)$. For a module $M \in D(R_\blacksquare)$, we get two localization sequences:

$$j_! j^\star M \to M \to \hat{i}_\star \hat{i}^\star M \stackrel{+1}{\to}$$
$$\hat{i}_\star \hat{i}^! M \to M \to j_\star j^\star M \stackrel{+1}{\to}$$

The functors $j_\star, j^!, j^\star$ are part of our six functor formalism for the open immersion of schemes $U \hookrightarrow X$ and have already been used a lot in this paper.

This globalizes. If $i : Z \hookrightarrow X$ is a closed subscheme of X, defined by a sheaf of ideals $\mathscr{I}$, and if $j : U \to X$ is the open complement, then we have a closed subset of the locale of X,$\hat{i} : \hat{Z} \hookrightarrow X$. For any object $\mathscr{F} \in D(\mathscr{O}_{X,\blacksquare})$, we have two localization sequences:

$$j_! j^\star \mathscr{F} \to \mathscr{F} \to \hat{i}_\star \hat{i}^\star \mathscr{F} \stackrel{+1}{\to}$$
$$\hat{i}_\star \hat{i}^! \mathscr{F} \to \mathscr{F} \to j_\star j^\star \mathscr{F} \stackrel{+1}{\to}$$

Lemma 2.9.3. *If $\mathscr{F}$ is pseudo-coherent, $\hat{i}_\star \hat{i}^\star \mathscr{F} = \lim_n (\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X/\mathscr{I}^n)$ and $\mathrm{R}\Gamma(X, \hat{i}_\star \hat{i}^\star \mathscr{F}) = \lim_n \mathrm{R}\Gamma(X, \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X/\mathscr{I}^n)$.*

Proof. By definition, $\hat{i}_\star \hat{i}^\star \mathscr{F} = \mathscr{F} \otimes \lim_n(\mathscr{O}_X/\mathscr{I}^n)$. We check that the natural map $\mathscr{F} \otimes \lim_n(\mathscr{O}_X/\mathscr{I}^n) \to \lim_n(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X/\mathscr{I}^n)$ is an isomorphism. We can reduce to the affine case. So $X = \mathrm{Spec}\, R$ and $\mathscr{I}(X) = I$ and $\mathscr{F}$ corresponds to a pseudo-coherent object $M \in D(R_\blacksquare)$. On the other hand, $\hat{R}^I$ is a compact object of $D(R_\blacksquare)$. For $M = R^J$ for a set J, we deduce that $R^J \otimes_R \hat{R}^I = (\hat{R}^I)^J$. Since M can be represented by a bounded above complex, all whose terms are finite sums of objects of the form R^J, the claim follows. □

Remark 2.9.4. This lemma compares to proposition 2.3.2. Indeed, the triangle $j_! j^\star \mathscr{F} \to \mathscr{F} \to \hat{i}_\star \hat{i}^\star \mathscr{F} \stackrel{+1}{\to}$ writes $\lim_n \mathscr{I}^n \otimes \mathscr{F} \to \mathscr{F} \to \lim_n \mathscr{F} \otimes \mathscr{O}_X/\mathscr{I}^n \stackrel{+1}{\to}$.

2.9.5. *The spectral sequence.* We now let X be a scheme and we let $Z_{-1} = \emptyset \subseteq Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq \cdots \subseteq X = Z_d$ be a filtration by closed subschemes. We let $\hat{i}_l : \hat{Z}_l \hookrightarrow X$. Let $U_l = X \setminus Z_l$ and $j_l : U_l \hookrightarrow X$. We let $\mathrm{R}\Gamma(\hat{Z}_l, \mathscr{F})$ be $\mathrm{R}\Gamma(X, (\hat{i}_l)_\star \hat{i}_l^\star \mathscr{F})$. We have exact triangles: $\mathrm{R}\Gamma_c(U_l, \mathscr{F}) \to \mathrm{R}\Gamma(X, \mathscr{F}) \to \mathrm{R}\Gamma(\hat{Z}_i, \mathscr{F}) \stackrel{+1}{\to}$. We let $\mathrm{R}\Gamma_c(\hat{Z}_l \setminus \hat{Z}_{l-1}, \mathscr{F}) = \mathrm{R}\Gamma(X, (j_{l-1})_!(j_{l-1})^\star(\hat{i}_l)_\star(\hat{i}_l)^\star \mathscr{F})$.

Lemma 2.9.6. *We have that $\mathrm{R}\Gamma_c(\hat{Z}_l \setminus \hat{Z}_{l-1}, \mathscr{F}) = \mathrm{R}\Gamma(X, (\hat{i}_l)_\star(\hat{i}_l)^\star(j_{l-1})_!(j_{l-1})^\star \mathscr{F})$. We have the following exact triangles:*

$$\mathrm{R}\Gamma_c(\hat{Z}_l \setminus \hat{Z}_{l-1}, \mathscr{F}) \to \mathrm{R}\Gamma(\hat{Z}_l, \mathscr{F}) \to \mathrm{R}\Gamma(\hat{Z}_{l-1}, \mathscr{F}) \stackrel{+1}{\to}$$
$$\mathrm{R}\Gamma_c(U_l, \mathscr{F}) \to \mathrm{R}\Gamma_c(U_{l-1}, \mathscr{F}) \to \mathrm{R}\Gamma_c(\hat{Z}_l \setminus \hat{Z}_{l-1}, \mathscr{F}) \stackrel{+1}{\to}$$

Proof. We need to see that $(j_{l-1})_!(j_{l-1})^\star(\hat{i}_l)_\star(\hat{i}_l)^\star = (\hat{i}_l)_\star(\hat{i}_l)^\star(j_{l-1})_!(j_{l-1})^\star$. We consider the following Cartesian diagram:

$$\begin{array}{ccc} U_{l-1} \cap Z_l & \xrightarrow{i'_l} & U_{l-1} \\ \downarrow{\scriptstyle j'_{l-1}} & & \downarrow{\scriptstyle j_{l-1}} \\ Z_l & \xrightarrow{i_l} & X \end{array}$$

Since $(\hat{i}_l)_\star = (\hat{i}_l)_!$, we have by proper base change, $(j_{l-1})^\star(\hat{i}_l)_\star = (j'_{l-1})^\star(\hat{i}'_l)_\star$. The identity follows. The first triangle comes from the triangle $(j_{l-1})_!(j_{l-1})^\star(\hat{i}_l)_\star(\hat{i}_l)^\star\mathscr{F} \to (\hat{i}_l)_\star(\hat{i}_l)^\star\mathscr{F} \to (\hat{i}_l)_\star(\hat{i}_l)^\star\mathscr{F} \to (\hat{i}_{l-1})_\star(\hat{i}_{l-1})^\star\mathscr{F} \stackrel{+1}{\to}$. The second triangle comes from the triangle : $(j_l)_!(j_l)^\star\mathscr{F} \to (j_{l-1})_!(j_{l-1})^\star\mathscr{F} \to (\hat{i}_l)_\star(\hat{i}_l)^\star(j_{l-1})_!(j_{l-1})^\star\mathscr{F} \stackrel{+1}{\to}$. □

Theorem 2.9.7. *Assume that $\mathscr{F} \in D^-(\mathscr{O}_{X,\blacksquare})$. We have a spectral sequence: $E_1^{p,q} = \mathrm{H}_c^{p+q}(\hat{Z}_p \setminus \hat{Z}_{p-1}, \mathscr{F}) \Rightarrow \mathrm{H}^{p+q}(X, \mathscr{F})$.*

Proof. Each of the cohomologies $\mathrm{R}\Gamma_c(U_i, \mathscr{F})$ can be represented by a complex $A_i^\bullet$ in $D^-(A_\blacksquare)$ of projective modules. This uses that X is qcqs. We have maps $A^\bullet_{d-1} \to A^\bullet_{d-2} \to \cdots \to A^\bullet_{-1}$ representing the maps $\mathrm{R}\Gamma_c(U_{d-1}, \mathscr{F}) \to \mathrm{R}\Gamma_c(U_{d-2}, \mathscr{F}) \to \cdots \to \mathrm{R}\Gamma(X, \mathscr{F})$. By applying [Sta22], TAG 014M, we deduce that we can replace $A_i^\bullet$ by homotopic, bounded above complexes with the property that $A_i^\bullet \to A_{i-1}^\bullet$ is a termwise split injection. This means that the complex $A^\bullet_{-1}$ is filtered. The spectral sequence we look for is just the spectral sequence of a filtered complex ([Sta22], TAG 014M). □

2.9.8. *Action of a correspondence on the spectral sequence.* Let (C, X, p_1, p_2) be a correspondence. Let $T : p_2^\star\mathscr{F} \to p_1^!\mathscr{F}$ be a cohomological correspondence. Let $Z_{-1} = \emptyset \subseteq Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq \cdots \subseteq X = Z_d$ be a filtration by closed subschemes on X.

Proposition 2.9.9. *Assume that $p_1^\star(Z_l) \subseteq p_2^\star(Z_l)$ for all $0 \leq l \leq d$ (set theoretically). Then we have an action of T on $\mathrm{R}\Gamma(\hat{Z}_l, \mathscr{F})$, $\mathrm{R}\Gamma_c(U_l, \mathscr{F})$and $\mathrm{R}\Gamma_c(\hat{Z}_l \setminus \hat{Z}_{l-1}, \mathscr{F})$ for all i, compatible with the natural morphisms. The spectral sequence of theorem 2.9.7 is T-equivariant.*

Proof. We need to explain how we can attach to T a map of triangles:

$$\begin{array}{ccccccc}
p_2^\star(j_l)_!j_l^\star\mathscr{F} & \longrightarrow & p_2^\star\mathscr{F} & \longrightarrow & p_2^\star(\hat{i}_l)_\star(\hat{i}_l)^\star\mathscr{F} & \longrightarrow & \\
\downarrow & & \downarrow T & & \downarrow & & \\
p_1^!(j_l)_!j_l^\star\mathscr{F} & \longrightarrow & p_1^!\mathscr{F} & \longrightarrow & (p_1)^!(\hat{i}_l)_\star(\hat{i}_l)^\star\mathscr{F} & \longrightarrow &
\end{array}$$

We have that $p_2^\star(U_l) \subseteq p_1^\star(U_l)$. Let us write $C_{U_l,1} = p_1^\star(U_l)$ and $C_{U_l,2} = p_2^\star(U_l)$. Let $j_{C_{U_l,k}} : C_{U_l,k} \hookrightarrow X$ be the open immersions for $k = 1, 2$. We have a chain of maps:

$$(j_{C_{U_l,2}})_!j^\star_{C_{U_l,2}}p_2^\star\mathscr{F} \to (j_{C_{U_l,2}})_!j^\star_{C_{U_l,2}}p_1^!\mathscr{F} \to (j_{C_{U_l,1}})_!j^\star_{C_{U_l,1}}p_1^!\mathscr{F}.$$

We observe that thanks to the base change theorem we have: $(j_{C_{U_l,2}})_!j^\star_{C_{U_l,2}}p_2^\star\mathscr{F} = p_2^\star(j_l)_!j_l^\star\mathscr{F}$. On the other hand, we claim that we have a map $(j_{C_{U_l,1}})_!j^\star_{C_{U_l,1}}p_1^!\mathscr{F} \to (p_1)^!(j_l)_\star(j_l)^\star\mathscr{F}$. By adjunction this map is equivalent to a map $(p_1)_!(j_{C_{U_l,1}})_!j^\star_{C_{U_l,1}}p_1^!\mathscr{F} \to (j_l)_\star(j_l)^\star\mathscr{F}$. Since $(p_1)_!(j_{C_{U_l,1}})_!j^\star_{C_{U_l,1}}p_1^!\mathscr{F} = (j_l)_!(p'_1)_!(p'_1)^!(j_l)^!\mathscr{F}$ for $p'_1 : C_{U_l,1} \to U_l$, this map is just given by the co-unit of the adjunction $(p'_1)_!(p'_1)^! \to Id$. □

2.10. **Compact dynamic correspondences and finite rank operators.** In this section we investigate the case of compact dynamic correspondences.

Before we move on, let us briefly recall the definition of the cohomology with support in a closed subset. If $Y \to \mathrm{Spec}\, A$ is a scheme, $i : Z \hookrightarrow Y$ is a closed subscheme of Y, and if $\mathscr{F} \in D(\mathscr{O}_{Y,\blacksquare})$, we let $\mathrm{R}\Gamma_Z(Y, \mathscr{F}) = \mathrm{R}\Gamma(Z, i^!\mathscr{F})$. We let $\hat{i} : \hat{Z} \to Y$ be the closed subset of the locale. We let $\mathrm{R}\Gamma_{\hat{Z}}(Y, \mathscr{F}) = \mathrm{R}\Gamma(\hat{Z}, \hat{i}^!\mathscr{F})$

Lemma 2.10.1. (1) *We have a natural map $\mathrm{R}\Gamma_Z(Y, \mathscr{F}) \to \mathrm{R}\Gamma_{\hat{Z}}(Y, \mathscr{F})$.*

(2) *If $Z \subseteq Y' \stackrel{j}{\hookrightarrow} Y$ with Y' open in Y, then $\mathrm{R}\Gamma_Z(Y, \mathscr{F}) = \mathrm{R}\Gamma_Z(Y', j^\star\mathscr{F})$ and $\mathrm{R}\Gamma_{\hat{Z}}(Y, \mathscr{F}) = \mathrm{R}\Gamma_{\hat{Z}}(Y', j^\star\mathscr{F})$.*

(3) *Let $j_U : U \hookrightarrow Y$ and $j_{U'} : U' \hookrightarrow Y$ be open subsets. We assume that $U \subseteq Z \subseteq U' \subseteq Y$. Then we have natural maps: $\mathrm{R}\Gamma(Y, (j_U)_!j_U^\star\mathscr{F}) \to \mathrm{R}\Gamma_Z(Y, \mathscr{F}) \to \mathrm{R}\Gamma_{\hat{Z}}(Y, \mathscr{F}) \to \mathrm{R}\Gamma(Y, (j_{U'})_!j^\star_{U'}\mathscr{F})$.*

(4) *If Z is proper and if $\mathscr{F}$ has coherent cohomology groups, $\mathrm{R}\Gamma_Z(Y, \mathscr{F})$ has finite cohomology groups.*

Proof. For the first point, we have a factorization $i : Z \xrightarrow{u} \hat{Z} \xrightarrow{\hat{i}} Y$. We deduce that there is a map $u_\star i^! \mathscr{F} \to \hat{i}^! \mathscr{F}$ (by adjunction this is given by the identity $i^! \mathscr{F} \to u^! \hat{i}^! \mathscr{F}$). The second point follows immediately from the fact that $j^\star = j^!$. For the third point, we just use the standard adjunctions. For the last point, we observe that if $\mathscr{F}$ has coherent cohomology groups, the same holds for $i^! \mathscr{F}$ and one uses the finiteness of the cohomology of proper schemes. $\square$

Lemma 2.10.2. *Let $\bar{X}$ be a scheme and let D_+, D_-, D'_+, D'_- be closed subschemes. Assume that:*

1. $D_+ \subseteq \overset{\circ}{D}{}'_+ \subseteq D'_+$,
2. $D'_- \subseteq \overset{\circ}{D}_- \subseteq D_-$.

Let $\mathscr{F} \in D(\mathscr{O}_{\bar{X},\blacksquare})$. We have natural maps:

$$\mathrm{R}\Gamma_{D_+,D_-}(\bar{X}, \mathscr{F}) \to \mathrm{R}\Gamma_{\bar{X} \setminus (\overset{\circ}{D}{}'_+ \cup \overset{\circ}{D}_-)}(\bar{X} \setminus \overset{\circ}{D}{}'_+, \mathscr{F}) \to \mathrm{R}\Gamma_{D'_+,D'_-}(\bar{X}, \mathscr{F})$$

$$\mathrm{R}\Gamma_{D_-,D_+}(\bar{X}, \mathscr{F}) \to \mathrm{R}\Gamma_{\bar{X} \setminus (\overset{\circ}{D}{}'_+ \cup \overset{\circ}{D}_-)}(\bar{X} \setminus \overset{\circ}{D}{}'_+, \mathscr{F}) \to \mathrm{R}\Gamma_{D'_-,D'_+}(\bar{X}, \mathscr{F})$$

Proof. For the first line, we consider the composition of the following natural maps. The map $\mathrm{R}\Gamma_{D_+,D_-}(\bar{X}, \mathscr{F}) \to \mathrm{R}\Gamma_{\emptyset,D_-}(\bar{X} \setminus \overset{\circ}{D}{}'_+, \mathscr{F})$ deduced from the map $\bar{X} \setminus \overset{\circ}{D}{}'_+ \to \bar{X}$, by applying lemma 2.2.7. The map $\mathrm{R}\Gamma_{\emptyset,D_-}(\bar{X} \setminus \overset{\circ}{D}{}'_+, \mathscr{F}) \to \mathrm{R}\Gamma_{\bar{X} \setminus (\overset{\circ}{D}{}'_+ \cup \overset{\circ}{D}_-)}(\bar{X} \setminus \overset{\circ}{D}{}'_+, \mathscr{F})$ obtained by applying lemma 2.10.1. The map $\mathrm{R}\Gamma_{\bar{X} \setminus (\overset{\circ}{D}{}'_+ \cup \overset{\circ}{D}_-)}(\bar{X} \setminus \overset{\circ}{D}{}'_+, \mathscr{F}) \to \mathrm{R}\Gamma_{\emptyset,D'_-}(\bar{X} \setminus \overset{\circ}{D}{}'_+, \mathscr{F})$ obtained by applying lemma 2.10.1. The map $\mathrm{R}\Gamma_{\emptyset,D'_-}(\bar{X} \setminus \overset{\circ}{D}{}'_+, \mathscr{F}) = \mathrm{R}\Gamma_{\emptyset,D'_-}(\bar{X}, \mathscr{F}) \to \mathrm{R}\Gamma_{D'_+,D'_-}(\bar{X}, \mathscr{F})$ obtained by applying lemma 2.2.6 to the map $\bar{X} \setminus D'_+ \to \bar{X} \setminus \overset{\circ}{D}{}'_+$.

For the second line, we consider the composite of the following natural maps. The map $\mathrm{R}\Gamma_{D_-,D_+}(\bar{X}, \mathscr{F}) \to \mathrm{R}\Gamma_{D_-,\emptyset}(\bar{X} \setminus \overset{\circ}{D}{}'_+, \mathscr{F})$ deduced by applying lemma 2.2.7 to the map $\bar{X} \setminus \overset{\circ}{D}{}'_+ \to \bar{X}$. The map $\mathrm{R}\Gamma_{D_-,\emptyset}(\bar{X} \setminus \overset{\circ}{D}{}'_+, \mathscr{F}) \to \mathrm{R}\Gamma_{\bar{X} \setminus (\overset{\circ}{D}{}'_+ \cup \overset{\circ}{D}_-)}(\bar{X} \setminus \overset{\circ}{D}{}'_+, \mathscr{F})$ obtained by applying lemma 2.10.1. The map $\mathrm{R}\Gamma_{\bar{X} \setminus (\overset{\circ}{D}{}'_+ \cup \overset{\circ}{D}_-)}(\bar{X} \setminus \overset{\circ}{D}{}'_+, \mathscr{F}) \to \mathrm{R}\Gamma_{D'_-,\emptyset}(\bar{X} \setminus \overset{\circ}{D}{}'_+, \mathscr{F})$ obtained by applying lemma 2.10.1. The natural map $\mathrm{R}\Gamma_{D'_-,\emptyset}(\bar{X} \setminus \overset{\circ}{D}{}'_+, \mathscr{F}) = \mathrm{R}\Gamma_{D'_-,}(\bar{X}, \mathscr{F}) \to \mathrm{R}\Gamma_{D'_-,D'_+}(\bar{X}, \mathscr{F})$ obtained by lemma 2.2.6. $\square$

Let (C, X, p_1, p_2) be a correspondence. Let $T : p_2^\star \mathscr{F} \to p_1^! \mathscr{F}$ be a map in $D(\mathscr{O}_{C,\blacksquare})$.

Proposition 2.10.3. *Let $\mathcal{C} = (\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D_+, D_-)$ be a compact dynamic compactification. Assume that $\mathscr{F}$ has coherent cohomology groups, and that the map T extends to a map over a neighborhood of $\bar{C} \setminus ((\bar{p}_1^\star \overset{\circ}{D}{}^+) \cup (\bar{p}_2^\star \overset{\circ}{D}_-))$ in $\bar{C}$. Then the endomorphism $T_{\mathcal{C}}$ of proposition 2.2.10 has finite rank (see definition 2.4.5).*

Proof. We only consider the case of $\mathrm{R}\Gamma_{D_-,D_+}(\bar{X}, \mathscr{F})$. It suffices to show that the map $\mathrm{R}\Gamma_{\bar{p}_2^\star D_-,\bar{p}_2^\star D_+}(\bar{C}, p_2^\star \mathscr{F}) \to \mathrm{R}\Gamma_{\bar{p}_1^\star D_-,\bar{p}_1^\star D_+}(\bar{C}, p_1^! \mathscr{F})$ has finite rank.

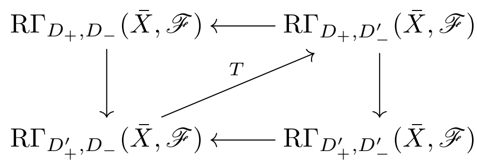
By lemma 2.10.2, we deduce that we have the following factorization of this map:

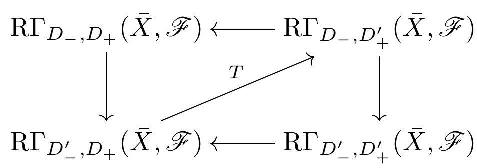
$$\begin{array}{ccc}
\mathrm{R}\Gamma_{\bar{p}_2^\star D_-,\bar{p}_2^\star D_+}(\bar{C}, p_2^\star \mathscr{F}) & \longrightarrow & \mathrm{R}\Gamma_{\bar{p}_1^\star D_-,\bar{p}_1^\star D_+}(\bar{C}, p_1^! \mathscr{F}) \\
\downarrow & & \uparrow \\
\mathrm{R}\Gamma_{\bar{C} \setminus \bar{p}_2^\star \overset{\circ}{D}_- \cup \bar{p}_1^\star \overset{\circ}{D}{}^+}(\bar{C} \setminus \bar{p}_1^\star \overset{\circ}{D}{}^+, p_2^\star \mathscr{F}) & \longrightarrow & \mathrm{R}\Gamma_{\bar{C} \setminus \bar{p}_2^\star \overset{\circ}{D}_- \cup \bar{p}_1^\star \overset{\circ}{D}{}^+}(\bar{C} \setminus \bar{p}_1^\star \overset{\circ}{D}{}^+, p_1^! \mathscr{F})
\end{array}$$

By lemma 2.10.1, the complex $\mathrm{R}\Gamma_{\bar{C} \setminus \bar{p}_2^\star \overset{\circ}{D}_- \cup \bar{p}_1^\star \overset{\circ}{D}{}^+}(\bar{C} \setminus \bar{p}_1^\star \overset{\circ}{D}{}^+, p_2^\star \mathscr{F})$ has finite cohomology. $\square$

Proposition 2.10.4. *Let $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D_+, D_-)$ and $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D'_+, D'_-)$ be two dynamic correspondences. We assume that $D_+ \subseteq \overset{\circ}{D'_+} \subseteq D'_+$ and $D_- \subseteq \overset{\circ}{D'_-} \subseteq D'_-$. Assume also that $\bar{p}_1^\star D_+ \supseteq \bar{p}_2^\star D'_+$ and that $\bar{p}_2^\star D_- \supseteq \bar{p}_1^\star D'_-$. Then the following holds:*

1. *The correspondences $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D_+, D_-)$, $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D_+, D'_-)$, $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D'_+, D_-)$, $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D'_+, D'_-)$ are compact dynamic,*
2. *Let $C = \bar{C} \setminus \bar{p}_1^\star D_+ \cup \bar{p}_2^\star D_-$. Let $T : p_2^\star \mathscr{F} \to p_1^! \mathscr{F}$ be a map. Then there are canonical maps $T : \mathrm{R}\Gamma_{D'_+, D_-}(\bar{X}, \mathscr{F}) \to \mathrm{R}\Gamma_{D_+, D'_-}(\bar{X}, \mathscr{F})$ and $T : \mathrm{R}\Gamma_{D_-, D'_+}(\bar{X}, \mathscr{F}) \to \mathrm{R}\Gamma_{D'_-, D_+}(\bar{X}, \mathscr{F})$.*
3. *We have the following diagrams:*





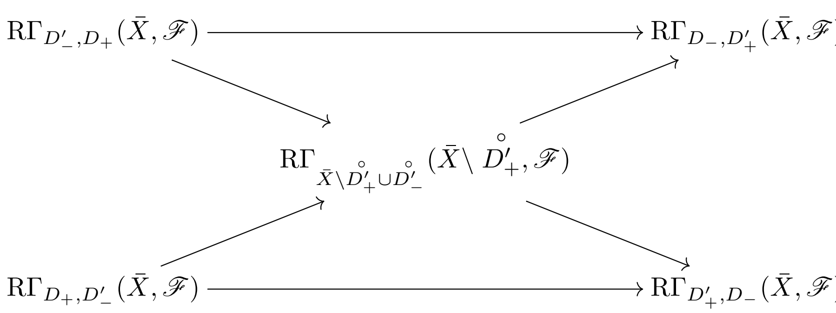
where the horizontal maps and vertical maps are the maps given by lemma 2.2.6 and going around in this diagram yields the map $T_{\mathcal{C}}$ of proposition 2.2.10 acting on the various cohomology with support.

4. *After taking the ordinary part in the above diagrams, all maps become quasi-isomorphisms.*
5. *We have canonical quasi-isomorphisms between $\mathrm{R}\Gamma_{D_+, D_-}(\bar{X}, \mathscr{F})^{C-ord}$ and $\mathrm{R}\Gamma_{D_-, D_+}(\bar{X}, \mathscr{F})^{C-ord}$, $\mathrm{R}\Gamma_{D'_+, D_-}(\bar{X}, \mathscr{F})^{C-ord}$ and $\mathrm{R}\Gamma_{D_-, D'_+}(\bar{X}, \mathscr{F})^{C-ord}$, … .*

Proof. The first point is clear, for example $\bar{p}_2^\star D_+ \subseteq \bar{p}_2^\star \overset{\circ}{D'_+} \subseteq \bar{p}_1^\star D_+$. For the second point, we consider the following composition:

$$T : \mathrm{R}\Gamma_{D'_+, D_-}(\bar{X}, \mathscr{F}) \to \mathrm{R}\Gamma_{p_2^\star D'_+, p_2^\star D_-}(\bar{C}, p_2^\star \mathscr{F}) \to \mathrm{R}\Gamma_{p_1^\star D_+, p_1^\star D'_-}(\bar{C}, p_1^! \mathscr{F}) \to \mathrm{R}\Gamma_{D_+, D'_-}(\bar{X}, \mathscr{F}).$$

One proceeds similarly with $\mathrm{R}\Gamma_{D'_-, D'_+}(\bar{X}, \mathscr{F})$. The third and fourth point are clear. Let us prove the last point. By lemma 2.10.2, we have a commutative diagram :



It follows that we can define an action of T on these 5 complexes using (2). On the ordinary part, all spaces are quasi-isomorphic. □

2.11. **Control theorems and change of support.** In this section we want to investigate a situation where we have both a strict dynamic and a compact dynamic correspondence and we want to compare the ordinary cohomologies. The setting is the following. We consider three dynamic correspondences $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D_+, D_-)$, $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D'_+, D'_-)$ and $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D''_+, D''_-)$. We assume that

$$D''_+ \subseteq \mathring{D}'_+ \subseteq D'_+ \subseteq \mathring{D}_+ \subseteq D_+$$
$$D''_- \subseteq \mathring{D}'_- \subseteq D'_- \subseteq \mathring{D}_- \subseteq D_-$$

We let $X = \bar{X} \setminus D_+ \cup D_-$ and $X' = \bar{X} \setminus D'_+ \cup D'_-$ and $X'' = \bar{X} \setminus D''_+ \cup D''_-$. We let $C = \bar{C} \setminus \bar{p}_1^\star D_+ \cup \bar{p}_2^\star D_-$, $C' = \bar{C} \setminus \bar{p}_1^\star D'_+ \cup \bar{p}_2^\star D'_-$ and $C'' = \bar{C} \setminus \bar{p}_1^\star D''_+ \cup \bar{p}_2^\star D''_-$. We call p'_i and p''_i the restrictions of $\bar{p}_i$ to C' and C'' respectively. We assume that over C'', we have that $s(p''_1)^\star D_+ \geq (p''_2)^\star D_+$ and $s(p''_2)^\star D_- \geq (p''_1)^\star D_-$ for some $0 < s < 1$. We also assume that $\bar{p}_1^\star D'_+ \subseteq \bar{p}_2^\star D''_+$ and that $\bar{p}_2^\star D'_- \subseteq \bar{p}_1^\star D''_-$.

Remark 2.11.1. In the case that $D''_+ = D''_- = \emptyset$, then $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D_+, D_-)$ is simply a strict dynamic correspondence. Here, we consider a situation where the compactification $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D_+, D_-)$ is dynamic and "becomes strict dynamic" if we further restrict to C''.

Remark 2.11.2. The compactifications $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D'_+, D'_-)$ and $(\bar{C}, \bar{X}, \bar{p}_1, \bar{p}_2, D''_+, D''_-)$ are compact dynamic by proposition 2.10.4.

Remark 2.11.3. We see that the closure of X in X' and the closure of C in C' are proper.

We assume that D_+, D_-, D''_+, D''_- are locally principal subschemes. We furthermore assume that we have a coherent sheaf $\mathscr{F}''$ over X'', and a cohomological correspondence $T : (p''_2)^\star \mathscr{F}'' \to (p''_1)^! \mathscr{F}''$. We let $\mathscr{F}$ be the restriction of $\mathscr{F}''$ to X.

Theorem 2.11.4. (1) *The operator T acts on the canonical diagrams:*

$$\begin{array}{ccc}
\mathrm{R}\Gamma_{D''_+, D_-}(\bar{X}, \mathscr{F}'') & \longrightarrow & \mathrm{R}\Gamma_{D_+, D_-}(\bar{X}, \mathscr{F}) \\
\downarrow & & \downarrow \\
\mathrm{R}\Gamma_{D''_+, D''_-}(\bar{X}, \mathscr{F}'') & \longrightarrow & \mathrm{R}\Gamma_{D_+, D''_-}(\bar{X}, \mathscr{F}'')
\end{array}$$

$$\begin{array}{ccc}
\mathrm{R}\Gamma_{D_-, D''_+}(\bar{X}, \mathscr{F}'') & \longrightarrow & \mathrm{R}\Gamma_{D_-, D_+}(\bar{X}, \mathscr{F}) \\
\downarrow & & \downarrow \\
\mathrm{R}\Gamma_{D''_-, D''_+}(\bar{X}, \mathscr{F}'') & \longrightarrow & \mathrm{R}\Gamma_{D''_-, D_+}(\bar{X}, \mathscr{F}'')
\end{array}$$

(2) *The correspondence (C, X, p_1, p_2) admits a strict dynamic compactification and* $\mathrm{R}\Gamma_{D_+, D_-}(\bar{X}, \mathscr{F})^{ord} = \mathrm{R}\Gamma_{D_-, D_+}(\bar{X}, \mathscr{F})^{ord} = \mathrm{R}\Gamma(X, \mathscr{F})^{C-ord}$.

(3) *Assume that the map T factors into a map:*

$$\begin{array}{ccc}
(p''_2)^\star \mathscr{F}''(D_+) & & \\
\uparrow & \searrow & \\
(p''_2)^\star \mathscr{F}'' & \longrightarrow & (p''_1)^! \mathscr{F}'' \\
 & & \uparrow \\
 & & (p''_1)^! \mathscr{F}''(-D_-)
\end{array}$$

and that we have maps $(p''_2)^\star \mathscr{O}_{X''}(D_+) \to (p''_1)^\star \mathscr{O}_{X''}(D_+)$ and $(p''_2)^\star \mathscr{O}_{X''}(-D_-) \to (p''_1)^\star \mathscr{O}_{X''}(-D_-)$. Then in the ordinary part of the diagram in (1)*, all maps become quasi-isomorphisms, and we obtain a canonical quasi-isomorphism:*

$$\mathrm{R}\Gamma_{D''_+, D''_-}(\bar{X}, \mathscr{F}'')^{ord} = \mathrm{R}\Gamma_{D''_-, D''_+}(\bar{X}, \mathscr{F}'')^{ord} \to \mathrm{R}\Gamma(\bar{X}, \mathscr{F})^{C-ord}.$$

(4) *In general, after replacing D_+ and D_- by a multiple nD_+ and nD_- for $n \in \mathbb{Z}_{>0}$, there exists $m \geq 0$ such that we have a quasi-isomorphism:*

$$\mathrm{R}\Gamma_{D''_+, D''_-}(\bar{X}, \mathscr{F}''(mD_+ - mD_-))^{ord} = \mathrm{R}\Gamma_{D''_-, D''_+}(\bar{X}, \mathscr{F}''(mD_+ - mD_-))^{ord} \to \mathrm{R}\Gamma_{D_+, D_-}(\bar{X}, \mathscr{F})^{ord}.$$

Proof. For the first point, the maps are given lemma 2.2.6. The operator T acts equivariently by construction. For the second point, we let $\overline{X}$ be the closure of X in $\bar{X}$. We see that $\overline{X} \subseteq X''$. We let $\overline{C}$ be the closure of C in $\bar{C}$. We observe that $\overline{C} \subseteq C''$. It follows that $(\overline{C}, \overline{X}, \overline{p}_1, \overline{p}_2, D_+ \cap \overline{X}, D_- \cap \overline{X})$ is a strict dynamic compactification. Moreover, the map $\mathrm{R}\Gamma_{D_+,D_-}(\bar{X}, \mathscr{F}) \to \mathrm{R}\Gamma_{D_+\cap\overline{X},D_-\cap\overline{X}}(\overline{X}, \mathscr{F})$ is a quasi-isomorphism by lemma 2.2.8. This map induces a quasi-isomorphism $\mathrm{R}\Gamma_{D_+,D_-}(\bar{X}, \mathscr{F})^{ord} \to \mathrm{R}\Gamma(X, \mathscr{F})^{C-ord}$. Considering $\mathrm{R}\Gamma_{D_-,D_+}(\bar{X}, \mathscr{F})$, one also gets a quasi-isomorphism $\mathrm{R}\Gamma_{D_-,D_+}(\bar{X}, \mathscr{F})^{ord} \to \mathrm{R}\Gamma(X, \mathscr{F})^{C-ord}$.

For the next point, we will first prove that $\mathrm{R}\Gamma_{D''_+,D_-}(\bar{X}, \mathscr{F}'')^{ord} = \mathrm{R}\Gamma_{D_-,D''_+}(\bar{X}, \mathscr{F}'')^{ord}$. The proof is similar to the proof of proposition 2.10.4. We claim that we have a commutative diagram :

$$\begin{array}{ccccc}
\mathrm{R}\Gamma_{D''_+,D_-}(\bar{X}, \mathscr{F}'') & & \longrightarrow & & \mathrm{R}\Gamma_{D'_+,D_-}(\bar{X}, \mathscr{F}'') \\
& \searrow & & \nearrow & \\
& & \mathrm{R}\Gamma_c(\bar{X} \setminus \{\overset{\circ}{D'_+} \cup D_-\}, \mathscr{F}'') & & \\
& \nearrow & & \searrow & \\
\mathrm{R}\Gamma_{D_-,D''_+}(\bar{X}, \mathscr{F}'') & & \longrightarrow & & \mathrm{R}\Gamma_{D_-,D'_+}(\bar{X}, \mathscr{F}'')
\end{array}$$

together with maps (abusively denoted) $T : \mathrm{R}\Gamma_{D'_+,D_-}(\bar{X}, \mathscr{F}'') \to \mathrm{R}\Gamma_{D''_+,D_-}(\bar{X}, \mathscr{F}'')$ as well as $T : \mathrm{R}\Gamma_{D_-,D'_+}(\bar{X}, \mathscr{F}'') \to \mathrm{R}\Gamma_{D_-,D''_+}(\bar{X}, \mathscr{F}'')$ inducing the endomorphism T when you go around the diagram. Passing to the ordinary part we deduce that $\mathrm{R}\Gamma_{D''_+,D_-}(\bar{X}, \mathscr{F}'')^{ord} = \mathrm{R}\Gamma_{D_-,D''_+}(\bar{X}, \mathscr{F}'')^{ord}$.

Now we observe by corollary 2.3.11 that the maps $\mathrm{R}\Gamma_{D''_+,D_-}(\bar{X}, \mathscr{F}'') \to \mathrm{R}\Gamma_{D''_+,D''_-}(\bar{X}, \mathscr{F}''(-mD_-))$ induce an isomorphism $\mathrm{R}\Gamma_{D''_+,D_-}(\bar{X}, \mathscr{F}'') \to \lim_m \mathrm{R}\Gamma_{D''_+,D''_-}(\bar{X}, \mathscr{F}''(-mD_-))$. But by assumption, the map T on the limit factors as maps $\mathrm{R}\Gamma_{D''_+,D''_-}(\bar{X}, \mathscr{F}''(-mD_-)) \to \mathrm{R}\Gamma_{D''_+,D''_-}(\bar{X}, \mathscr{F}''(-(m+1)D_-))$ for any $m \geq 0$. We deduce that $\mathrm{R}\Gamma_{D''_+,D_-}(\bar{X}, \mathscr{F}'')^{ord} = \mathrm{R}\Gamma_{D''_+,D''_-}(\bar{X}, \mathscr{F})^{ord}$

We similarly observe by corollary 2.3.11 that the maps $\mathrm{R}\Gamma_{D_-,D''_+}(\bar{X}, \mathscr{F}''(mD_+)) \to \mathrm{R}\Gamma_{D_-,D_+}(\bar{X}, \mathscr{F})$ induce an isomorphism $\mathrm{colim}_{m\geq 0}\mathrm{R}\Gamma_{D_-,D''_+}(\bar{X}, \mathscr{F}''(mD_+)) \to \mathrm{R}\Gamma_{D_-,D_+}(\bar{X}, \mathscr{F})$. But by assumption, the map T on the colimit factors as maps $\mathrm{R}\Gamma_{D_-,D''_+}(\bar{X}, \mathscr{F}''(mD_+)) \to \mathrm{R}\Gamma_{D_-,D''_+}(\bar{X}, \mathscr{F}''((m-1)D_+))$ for any $m > 0$. It follows that the colimit is constant on the ordinary part.

Regarding the last point, by lemma 2.5.3 and lemma 2.1.17, after replacing D_+ and D_- by a multiple nD_+ and nD_- for $n \in \mathbb{Z}_{>0}$, there exists $m \geq 0$ such that the twisted map $T : (p''_2)^\star \mathscr{F}''(mD_+ - mD_-) \to (p''_1)^! \mathscr{F}''(mD_+ - mD_-)$ satisfies the factorization of (2) and such that we have maps $(p''_2)^\star \mathscr{O}_{X''}(D_+) \to (p''_1)^\star \mathscr{O}_{X''}(D_+)$ and $(p''_2)^\star \mathscr{O}_{X''}(-D_-) \to (p''_1)^\star \mathscr{O}_{X''}(-D_-)$. The last point is now a direct consequence of (2).

□

3. The Siegel Shimura varieties

3.1. **The group G.** We let $G = \mathrm{GSp}_{2g}$ be the symplectic group realized as the group of symplectic similitudes of the $2g$-dimensional free module $\mathbb{Z}^{2g} = V$ with canonical basis $e_1, \cdots, e_{2g}$, and equipped with the symplectic form with matrix

$$\begin{pmatrix} 0 & S \\ -S & 0 \end{pmatrix}$$

with S the antidiagonal matrix with 1 on the antidiagonal. We have $\langle e_i, e_j \rangle = 0$ if $i + j \neq 2g + 1$, $\langle e_i, e_{2g+1-i} \rangle = 1$ for $1 \leq i \leq g$.

We let $h_0 : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{\mathbb{R}}$ be the morphism $z = (x + iy) \mapsto \begin{pmatrix} x & Sy \\ -Sy & x \end{pmatrix}$ and we let X be the $G(\mathbb{R})$ conjugacy class of h_0. This is a Hermitian symmetric domain. The pair (G, X) is the Siegel Shimura datum. We have $(\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m)_{\mathbb{C}} = (\mathbb{G}_m)_{\mathbb{C}} \times (\mathbb{G}_m)_{\mathbb{C}}$, via $z \mapsto (z, \bar{z})$. For any $h \in X$, there is an associated cocharacter $\mu_h : (\mathbb{G}_m)_{\mathbb{C}} \to G_{\mathbb{C}}$ given by the rule $\mu(z) = h(z, 1)$. To fix ideas, we choose the following representative in the conjugacy class of μ_h: $\mu(z) = \mathrm{diag}(\mathrm{Id}_g, z\mathrm{Id}_g)$. We let $P_\mu^{Std} = P^{std}$ be the stabilizer of $\langle e_1, \cdots, e_g\rangle$. We let $P_\mu = P$ be the opposite parabolic, $M_\mu = M$ be the Levi and U_P be the unipotent radical. We let $T = \{\mathrm{diag}(t_1, \cdots, t_{2g}),\ t_i t_{2g+1-i} = c\}$ be the diagonal torus. The maximal torus of the derived group is $T^{der} = \{\mathrm{diag}(t_1, \cdots, t_g, t_g^{-1}, \cdots, t_1^{-1})$. The center Z consists of scalar diagonal elements. We identify the group of characters of T, $X^\star(T)$, with tuples $\kappa = (k_1, \cdots, k_g; k) \in \mathbb{Z}^g \times \mathbb{Z}$ satisfying $\sum_{i=1}^g k_i = k \mod 2$, by

$$\kappa(\mathrm{diag}(zt_1, \cdots, zt_g, zt_g^{-1}, \cdots, zt_1^{-1})) = z^k \prod_{i=1}^{g} t_i^{k_i}.$$

We let $B \subseteq P$ be the Borel which is upper triangular on the diagonal $g \times g$ blocks. We let Φ be the set of roots. The positive roots for B are denoted by Φ^+. We have a decomposition $\Phi^+ = \Phi_M^+ \coprod \Phi^{+,M}$ where Φ_M^+ accounts for the positive roots in $\mathfrak{m} = \mathrm{Lie}(M)$ and $\Phi^{+,M}$ for the positive roots in $\mathfrak{u}_P = \mathrm{Lie}(U_P)$. We let $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$, $\rho_{nc} = \frac{1}{2}\sum_{\alpha\in\Phi^{+,M}}\alpha$.

The dominant cone $X^\star(T)^+$ is given by the condition $0 \geq k_1 \geq \cdots \geq k_g$. The dominant cone for the Levi M, denoted by $X^\star(T)^{+,M}$, is given by the condition $k_1 \geq \cdots \geq k_g$.

We let $X_\star(T)$ be the group of cocharacters of T. We define a map $v : T(\mathbb{Q}_p) \to X_\star(T)$ as follows: $T(\mathbb{Q}_p) = \mathbb{Q}_p^\times \otimes X_\star(T)$, and we compose with the valuation $\mathbb{Q}_p^\times \to \mathbb{Z}$ (normalized by $v(p) = 1$). The map v has a section $X_\star(T) \to T(\mathbb{Q}_p)$, sending λ to $\lambda(p)$, so that $T(\mathbb{Q}_p) = X_\star(T) \times T(\mathbb{Z}_p)$. We write $T^+(\mathbb{Q}_p)$ (resp. $T^{++}(\mathbb{Q}_p)$) for the set of those $t \in T(\mathbb{Q}_p)$ for which $v(t)$ is dominant (resp. dominant regular).

We let W be the Weyl group, and W_M be the Weyl group of the Levi. We let ${}^MW$ be the set of minimal length representatives of the quotient $W_M\backslash W$. The set ${}^MW$ is defined by the condition $w.X^\star(T)^+ \subseteq X^\star(T)^{+,M}$, or equivalently by the condition $w^{-1}\Phi^{+,M} \subseteq \Phi^+$. We have a concrete realization of these groups: W identifies with the subgroup of permutations w of $\{1, \cdots, 2g\}$ such that $w(i) + w(2g+1-i) = 2g+1$, via the formula:

$$w(\mathrm{diag}(t_1, \cdots, t_{2g})) = \mathrm{diag}(t_{w^{-1}(1)}, \cdots, t_{w^{-1}(2g)})$$

Also, $W_M \subseteq W$ is the subgroup of permutation w such that $w(\{1, \cdots, g\}) = \{1, \cdots, g\}$. Finally, ${}^MW$ is the subset of w such that $w^{-1}(g+1) < w^{-1}(g+2) \cdots < w^{-1}(2g)$.

There is a partial order on W and thus a partial order on ${}^MW$, as well as a length function $\ell : {}^MW \to [0, \frac{g(g+1)}{2}]$. The set ${}^MW$, the order and length function have a geometric interpretation that we now recall. Let $FL = P\backslash G$ be the partial flag variety (viewed as a scheme for the moment). There is a G-action on FL by translation, and the stratification by B-orbits is $FL = \coprod_{w\in {}^MW} C_w$ where $C_w = P\backslash PwB$ is a Bruhat cell. We have $\ell(w) = \dim C_w$, and $w' \leq w$ if and only if $C_{w'} \subseteq \overline{C_w}$. To $w \in {}^MW$, we can attach the following set $w^{-1}\{g+1, \cdots, 2g\} \cap \{g+1, \cdots, 2g\}$ (this set actually determines w) and a sequence of numbers $(w_1, \cdots, w_g)$ given by $w_i - w_{i-1} = 0$ if $g+i \notin w^{-1}\{g+1, \cdots, 2g\}$, $w_i - w_{i-1} = 1$ if $g+i \in w^{-1}\{g+1, \cdots, 2g\}$ (with the convention $w_0 = 0$). The sequence $(w_1, \cdots, w_g)$ has the following geometric interpretation: for any $x \in P\backslash G$, we let $x^{-1}Px$ be the corresponding parabolic, equal to the stabilizer of $x^{-1}\langle e_{g+1}, \cdots, e_{2g}\rangle$. If $x \in C_w$, then $w_i = \dim(x^{-1}\langle e_{g+1}, \cdots, e_{2g}\rangle \cap \langle e_{g+1}, \cdots, e_{g+i}\rangle)$.

3.2. **Shimura varieties.** For any neat compact open subgroup $K \subseteq G(\mathbb{A}_f)$ we have the Shimura variety $S_{K,\mathbb{Q}} \to \mathrm{Spec}\ \mathbb{Q}$. At the level of complex points, we have $S_{K,\mathbb{Q}}(\mathbb{C}) = G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_f)/K)$.

For any connected affine $\mathbb{Q}$-scheme Spec R, we have that $S_{K,\mathbb{Q}}(\mathrm{Spec}\ R)$ consists of the following list of data (A, λ, η), up to isomorphism:

(1) $A \to \mathrm{Spec}\ R$ is an abelian scheme of dimension g,
(2) $\lambda : A \to A^D$ is a quasi-polarization,
(3) $\eta : V \otimes_{\mathbb{Z}} \mathbb{A}_f \to \mathrm{H}_1(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a K-orbit of symplectic isomorphism (up to a similitude factor) of pro-étale sheaves.

An isomorphism $\xi : (A, \lambda, \eta) \to (A', \lambda', \eta')$ is a quasi-isogeny $\xi : A \to A'$ for which $\eta' = \mathrm{H}_1(\xi) \circ \eta$ and $\xi^D\lambda\xi = r\lambda$ for some $r \in \mathbb{Q}_{>0}$.

We let $S^{tor}_{K,\mathbb{Q},\Sigma}$ be a toroidal compactification, depending on a suitable choice Σ of K-admissible polyhedral cone decompositions ([FC90], [Lan13]).

3.3. **Hodge-Tate period map.** We now fix a prime p and a neat compact open subgroup $K^p \subseteq G(\mathbb{A}_f^p)$. We also fix a compact open subgroup $K_p \subseteq G(\mathbb{Q}_p)$. We fix a cone decomposition Σ which is K^pK_p-admissible. We let $\mathcal{S}^{tor}_{K^pK_p,\Sigma}$ be the adic space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ attached to the scheme $S^{tor}_{K^pK_p,\Sigma}$. Let $\mathcal{S}^{tor}_{K^p,\Sigma} = \lim_{K'_p \subseteq K_p} \mathcal{S}^{tor}_{K^pK'_p,\Sigma}$ be the perfectoid Siegel moduli space [Sch15], [PS16] (the same cone decomposition is used at each stage of the tower). We let $\mathcal{S}_{K^p}$ be the open complement of the boundary in $\mathcal{S}^{tor}_{K^p,\Sigma}$. We have a $G(\mathbb{Q}_p)$-equivariant map $\pi_{HT} : \mathcal{S}_{K^p} \to \mathcal{FL} = P\backslash G$ which can be extended to a K_p-equivariant map $\pi_{HT} : \mathcal{S}^{tor}_{K^p,\Sigma} \to \mathcal{FL}$ (since only K_p acts on $\mathcal{S}^{tor}_{K^p,\Sigma}$). Let us briefly recall how this map is constructed on a $(\mathbb{C}, \mathcal{O}_\mathbb{C})$-point corresponding to an abelian scheme A together with an isomorphism $\Psi : \mathbb{Q}_p^{2g} \to T_p(A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. We have the Hodge-Tate exact sequence

$$0 \to \mathrm{Lie}_A \to T_p(A) \otimes_{\mathbb{Z}_p} \mathbb{C} \to \omega_{A^t} \to 0.$$

There exists an element $g(\Psi) \in G(\mathbb{C})$ such that $\Psi(\langle g(\Psi)(e_{g+1}), \cdots, g(\Psi)(e_{2g})\rangle) = \mathrm{Lie}_A$. We let $\pi_{HT}((A,\Psi)) = g(\Psi)^{-1}$.

Remark 3.3.1. The map π_{HT} is indeed $G(\mathbb{Q}_p)$-equivariant: for $f \in G(\mathbb{Q}_p)$, we have $f.(A,\Psi) = (A, \Psi \circ f)$ and $g(\Psi \circ f) = f^{-1}g(\Psi)$, so $\pi_{HT}((A, \Psi \circ f)) = \pi_{HT}((A,\Psi))f$.

3.4. **Igusa towers.** We consider the ordinary Igusa tower. We first recall the definition given in [CS17], section 4 (notably def. 4.3.1) of the perfectoid Igusa tower. Then we explain that this perfectoid Igusa tower is actually an inverse limit of finite level Igusa tower which are more classical objects. Finally, we give the relation with the Siegel Shimura variety.

3.4.1. *Definition of Igusa towers.* We consider the p-divisible group over $\mathbb{Z}_p$: $\mathbb{X}_{ord} = (\mu_{p^\infty})^g \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^g$, equipped with its canonical polarization $\lambda_{can} : \mathbb{X}_{ord} \to \mathbb{X}_{ord}^D = (\mathbb{Q}_p/\mathbb{Z}_p)^g \oplus (\mu_{p^\infty})^g$ given by

$$\begin{pmatrix} 0 & S\mathrm{Id}_{(\mathbb{Q}_p/\mathbb{Z}_p)^g} \\ -S\mathrm{Id}_{(\mu_{p^\infty})^g} & 0 \end{pmatrix}.$$

where S is the antidiagonal matrix with 1 on the antidiagonal. The Frobenius of the Dieudonné module of $\mathbb{X}_{ord}|_{\mathbb{F}_p}$ corresponds to the element $b_{ord} = \mathrm{diag}(p^{-1}\mathrm{Id}_g, \mathrm{Id}_g) \in G(\breve{\mathbb{Q}}_p)$. We now consider the Igusa variety corresponding to this element b_{ord}. We let $Nilp/\mathbb{Z}_p$ be the category of $\mathbb{Z}_p$-algebras on which p is nilpotent. We let $\mathfrak{IG}_{K^p}$ be the p-adic formal scheme over $\mathbb{Z}_p$ representing the functor sending $R \in Nilp/\mathbb{Z}_p$ to the set of isomorphism classes of (A, λ, η^p, j) consisting of:

(1) An abelian variety of dimension g, $A \to \mathrm{Spec}\ R$, equipped with a prime-to-p polarization λ, and a K^p-level structure η^p,
(2) An isomorphism $j : \mathbb{X}_{ord} \to A[p^\infty]$ such that there exists $c \in \mathbb{Z}_p^\times$ and a commutative diagram:

$$\begin{array}{ccc} \mathbb{X}_{ord} & \xrightarrow{\ j\ } & A[p^\infty] \\ \big\downarrow{\scriptstyle c\lambda_{can}} & & \big\downarrow{\scriptstyle \lambda} \\ \mathbb{X}_{ord}^D & \xleftarrow[\ j^D\ ]{} & A[p^\infty]^D \end{array}$$

An isomorphism $(A, \lambda, \eta^p, j) \to (A', \lambda', \eta^{p\prime}, j')$ is a prime-to-p quasi-isogeny $\xi : A \to A'$ matching λ and λ' up to an element in $\mathbb{Z}^\times_{(p),>0}$, and matching η^p with $\eta^{p\prime}$ and j with j'.

There is another equivalent formulation of the moduli problem, up to quasi-isogeny. The formal scheme $\mathfrak{IG}_{K^p}$ parametrizes isomorphism classes of quadruples (A, λ, η^p, j) consisting of:

(1)' An abelian variety of dimension g, $A \to \mathrm{Spec}\ R$, equipped with a quasi-polarization λ, and a K^p-level structure η^p, (2)' A quasi-isogeny $j : \mathbb{X}_{ord} \to A[p^\infty]$ such that there exists $c \in \mathbb{Q}_p^\times$ and a commutative diagram:

$$\begin{array}{ccc} \mathbb{X}_{ord} & \xrightarrow{j} & A[p^\infty] \\ \downarrow{\scriptstyle c\lambda_{can}} & & \downarrow{\scriptstyle \lambda} \\ \mathbb{X}_{ord}^D & \xleftarrow[j^D]{} & A[p^\infty]^D \end{array}$$

An isomorphism $(A, \eta^p, \lambda, j) \to (A', \eta^{p\prime}, \lambda', j')$ is a quasi-isogeny $\xi : A \to A'$ matching λ and λ' up to an element in $\mathbb{Q}_{>0}$, and matching η^p with $\eta^{p\prime}$ and j with j'.

3.4.2. *The group J_{ord}.* We define a functor in groups $J_{b_{ord}} := J_{ord}$ on $Nilp/\mathbb{Z}_p$ by the rule: for any $R \in Nilp/\mathbb{Z}_p$, $J_{ord}(R) = \mathrm{Qisog}_{symp}(\mathbb{X}_{ord} \times_{\mathrm{Spec}\ \mathbb{Z}_p} \mathrm{Spec}\ R)$ is the group of self quasi-isogenies of $\mathbb{X}_{ord}$ respecting the polarization up to a similitude factor in $\mathbb{Q}_p^\times$.

Let $\tilde{\mathbb{X}}_{ord} = \lim_{\times p} \mathbb{X}_{ord}$ be the universal cover of $\mathbb{X}_{ord}$ ([SW13], sect. 3.1). Then we recall that $\mathrm{Qisog}_{symp}(\mathbb{X}_{ord} \times_{\mathrm{Spec}\ \mathbb{Z}_p} \mathrm{Spec}\ R) = \mathrm{Aut}_{symp}(\tilde{\mathbb{X}}_{ord} \times_{\mathrm{Spec}\ \mathbb{Z}_p} \mathrm{Spec}\ R)$.

We give a description of J_{ord}. We recall that $M = M_\mu$ is the Levi of $P_\mu = P$.

Lemma 3.4.3. *The group J_{ord} is a representable group scheme over* Spf $\mathbb{Z}_p$. *We have $J_{ord} = M_\mu(\mathbb{Q}_p) \ltimes U_{J_{ord}}$ where $M(\mathbb{Q}_p)$ is a locally profinite group viewed as a locally constant group scheme over* Spf $\mathbb{Z}_p$*, and $U_{J_b} = \tilde{\mu}_{p^\infty} \otimes_{\mathbb{Q}_p} \mathrm{Sym}^2 \mathbb{Q}_p^g$, where $\tilde{\mu}_{p^\infty}$ is the universal cover of μ_{p^∞}. The action of $M(\mathbb{Q}_p)$ on U_{J_b} is given by the usual action of $M(\mathbb{Q}_p)$ on $\mathrm{Sym}^2 \mathbb{Q}_p^g \simeq U_P(\mathbb{Q}_p)$.*

Proof. We first compute the group of all automorphisms of $\tilde{\mathbb{X}}_{ord}$. Since $\mathbb{X}_{ord} = \mathbb{X}_{ord,m} \oplus \mathbb{X}_{ord,et}$ is a direct sum of its multiplicative and étale part, $\mathrm{Aut}(\tilde{\mathbb{X}}_{ord})$ has the shape:

$$\begin{pmatrix} \mathrm{Aut}(\tilde{\mathbb{X}}_{ord,m}) & \mathrm{Hom}(\tilde{\mathbb{X}}_{ord,et}, \tilde{\mathbb{X}}_{ord,m}) \\ 0 & \mathrm{Aut}(\tilde{\mathbb{X}}_{ord,et}) \end{pmatrix}$$

We have that $\mathrm{Aut}(\tilde{\mathbb{X}}_{ord,m}) \simeq \mathrm{Aut}(\tilde{\mathbb{X}}_{ord,et}) \simeq \mathrm{GL}_g(\mathbb{Q}_p)$ and $\mathrm{Hom}(\tilde{\mathbb{X}}_{ord,et}, \tilde{\mathbb{X}}_{ord,m}) = \tilde{\mu}_{p^\infty} \otimes \mathrm{M}_{g\times g}(\mathbb{Q}_p)$. Taking the polarization into account yields the promised description. □

From the second formulation of the moduli problem, it is clear thet J_{ord} acts on the right on $\mathfrak{IG}_{K^p}$. Namely, we have $(A, \lambda, \eta^p, j)g = (A, \lambda, \eta^p, j \circ g)$ for any element $g \in J_{ord}$.

We let $J_{ord} \times_{\mathrm{Spf}\ \mathbb{Z}_p} \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p) =: J_{ord}^{an}$ be the adic generic fiber of J_{ord}. We have an exact sequence

$$0 \to T_p(\mu_{p^\infty}) \to \tilde{\mu}_{p^\infty} \to \mu_{p^\infty} \to 0.$$

We denote as usual $T_p(\mu_{p^\infty}) \times \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p) = \mathbb{Z}_p(1)$ (this is a profinite étale group scheme over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$). We let $\widehat{\mathbb{G}_m} = \mu_{p^\infty} \times \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ be the open unit ball with its multiplicative group structure. We deduce that $\tilde{\mu}_{p^\infty} \times \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ fits in an exact sequence:

$$0 \to \mathbb{Z}_p(1) \to \tilde{\mu}_{p^\infty} \times \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p) \to \widehat{\mathbb{G}_m} \to 0.$$

It follows that J_{ord}^{an} is the semi-direct product of the locally profinite group $M(\mathbb{Q}_p)$ by a unipotent group $U_{J_{ord}} \times \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ which fits in an exact sequence:

$$0 \to \mathrm{Sym}^2 \mathbb{Z}_p^g(1) \to U_{J_{ord}} \times \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p) \to \widehat{\mathbb{G}_m} \otimes_{\mathbb{Z}_p} \mathrm{Sym}^2 \mathbb{Z}_p^g \to 0.$$

Since multiplication by p is an isomorphism in $\tilde{\mu}_{p^\infty} \times \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$, we deduce an embedding $\mathbb{Q}_p(1) \to \tilde{\mu}_{p^\infty} \times \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. We can define a locally profinite sub-group scheme $P'(\mathbb{Q}_p) \hookrightarrow J_{ord}^{an}$, generated by the Levi $M(\mathbb{Q}_p)$ and the unipotent radical $U_P(\mathbb{Q}_p)(1) \simeq \mathbb{Q}_p(1) \otimes_{\mathbb{Z}_p} \mathrm{Sym}^2 \mathbb{Z}_p^g$. Over $\mathrm{Spa}(\mathbb{Q}_p^{cycl}, \mathbb{Z}_p^{cycl})$, the choice of an isomorphism $\mathbb{Z}_p(1) = \mathbb{Z}_p$ identifies $P(\mathbb{Q}_p)$ and $P'(\mathbb{Q}_p)$.

Remark 3.4.4. We note that J_{ord}^{an} is $\frac{g(g+1)}{2}$-dimensional. In that sense it is much bigger than $P'(\mathbb{Q}_p)$ which is zero dimensional.

3.4.5. *Finite level Igusa varieties.* We now want to consider finite level Igusa varieties and we will see that they can be organized in a tower which enjoys very good properties, similar to the tower of Shimura varieties over $\mathbb{Q}$.

The finite level Igusa varieties will be parametrized by compact open subgroups $K_{p,P}$ of $P'(\mathbb{Q}_p)$, or which is the same thing, compact open subgroups of $P(\mathbb{Q}_p)$ which are stable under the Galois action on $P(\mathbb{Q}_p)$ where Galois acts on the unipotent radical via multiplication by the cyclotomic character.

Remark 3.4.6. We give a motivation for considering these compact open subgroups. The canonical choice of compact open subgroup we want to consider is $J_{ord}^{int} := \mathrm{Aut}_{symp}(\mathbb{X}_{ord})$ and its generic fiber $J_{ord}^{int} \times \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p) =: P'(\mathbb{Z}_p)$, the semi direct product of $M(\mathbb{Z}_p)$ by $U_P(\mathbb{Z}_p)(1)$. We also want to allow any finite index subgroup of $P'(\mathbb{Z}_p)$ as well as any conjugate by $M(\mathbb{Q}_p)$. Since $P'(\mathbb{Q}_p)$ is generated by $P'(\mathbb{Z}_p)$ and $M(\mathbb{Q}_p)$, we are exactly led to consider $P'(\mathbb{Q}_p)$ and its compact open subgroups.

We write $U_{K_{p,P}} = K_{p,P} \cap U_P(\mathbb{Q}_p)(1)$ and $M_{K_{p,P}} = K_{p,P} \cap M(\mathrm{Q}_p) = K_{p,P}^{\mathrm{Gal}(\mathbb{Q}_p^{cycl}/\mathbb{Q}_p)}$. When $p > 2$ we in fact have $K_{p,P} = M_{K_{p,P}} \ltimes U_{K_{p,P}}$. Indeed given $g = m \cdot u \in K_{p,P}$ and $\sigma \in \mathrm{Gal}(\mathbb{Q}_p^{cycl}/\mathbb{Q}_p)$ with $\chi_{\mathrm{cycl}}(\sigma) - 1 \in \mathbb{Z}_p^\times$, we see that $\sigma(g)^{-1}g = (1 - \chi_{\mathrm{cycl}}(\sigma))u \in K_{p,P}$ and hence $u \in K_{p,P}$.

Remark 3.4.7. We remark therefore that when $p > 2$ we have a bijection between compact open subgroup of $P(\mathbb{Q}_p)$ which are a semi-direct products $M_{K_{p,P}} \ltimes U_{K_{p,P}}$ and compact open subgroup of $P'(\mathbb{Q}_p)$.

For any compact open subgroup $K_{p,P}$ of $P'(\mathbb{Q}_p)$, we let $\overline{K_{p,P}}$ be the schematic closure in J_{ord}.[2]

Lemma 3.4.8. *The group scheme $\overline{K_{p,P}}$ is a profinite flat group scheme over* Spec $\mathbb{Z}_p$. *Its generic fiber is $K_{p,P}$.*

Proof. We first assume that $K_{p,P} = P'(\mathbb{Z}_p)$. Then $\overline{K_{p,P}} = J_{ord}^{int}$. We then consider the general case. It is easy to see that a general $K_{p,P}$ can always be conjugated by an element $x \in M(\mathbb{Q}_p)$ to a finite index subgroup of $P'(\mathbb{Z}_p)$. Indeed, we first consider the Levi quotient $K_{p,M}$, and use that any maximal compact subgroup of $GL_g(\mathbb{Q}_p)$ is conjugated to $GL_g(\mathbb{Z}_p)$. We can then further conjugate by the element $\mathrm{diag}(1, \cdots, 1, p, \cdots, p)$ in the center of $M(\mathbb{Q}_p)$ to have $K_{p,U_P} \subseteq U_P(\mathbb{Z}_p)(1)$. We thus reduce to the case that $K_{p,P} \subseteq P'(\mathbb{Z}_p)$ is of finite index. We see that $\overline{K_{p,P}} = \lim_n \overline{K_{p,P}}_n$ where $\overline{K_{p,P}}_n$ is the schematic closure of $K_{p,P}$ in $M(\mathbb{Z}/p^n\mathbb{Z}) \ltimes \mu_{p^n} \otimes_{\mathbb{Z}_p} \mathrm{Sym}^2\mathbb{Z}_p^g$. By [Ray74], section 2, each $\overline{K_{p,P}}_n$ is a finite flat group scheme over Spec $\mathbb{Z}_p$. □

Taking the schematic closure of the exact sequence $1 \to U_{K_{p,P}} \to K_{p,P} \to M_{K_{p,P}} \to 1$ yields an exact sequence $1 \to \overline{U_{K_{p,P}}} \to \overline{K_{p,P}} \to \overline{M_{K_{p,P}}} \to 1$ where $\overline{M_{K_{p,P}}}$ is a profinite étale group scheme. We also let $\hat{T}_{K_{p,P}} := \tilde{\mu}_{p^\infty} \otimes \mathrm{Sym}^2\mathbb{Z}_p^g/\overline{U_{K_{p,P}}}$. This is a formal torus, isogenous to $\widehat{\mathbb{G}_m} \otimes \mathrm{Sym}^2\mathbb{Z}_p^g$.

Proposition 3.4.9. (1) *For any compact open subgroup $K_{p,P}$, the fpqc quotient of $\mathfrak{IG}_{K^p}$ by $\overline{K_{p,P}}$ exists as a p-adic formal scheme and is denoted by $\mathfrak{IG}_{K^pK_{p,P}}$.*
(2) *The map $\mathfrak{IG}_{K^p} \to \mathfrak{IG}_{K^pK_{p,P}}$ is a $\overline{K_{p,P}}$-torsor in the fpqc-topology.*
(3) *The formal scheme $\mathfrak{IG}_{K^pK_{p,P}}$ is smooth of finite type over $\mathbb{Z}_p$.*
(4) *For any compact open subgroup $K_{p,P}$, the fpqc quotient of $\mathfrak{IG}_{K^p}$ by $\overline{U_{K_{p,P}}}$ exists as a p-adic formal scheme and is denoted by $\mathfrak{IG}_{K^pU_{K_{p,P}}}$.*
(5) *The map $\mathfrak{IG}_{K^pU_{K_{p,P}}} \to \mathfrak{IG}_{K^pK_{p,P}}$ is a profinite étale map with group $M_{K_{p,P}}$.*
(6) *The action of J_{ord} on $\mathfrak{IG}_{K^p}$ induces an action of the formal torus $\hat{T}_{K_{p,P}}$ on $\mathfrak{IG}_{K^pU_{K_{p,P}}}$ and the formal completions at closed points are homogeneous spaces under the formal torus $\hat{T}_{K_{p,P}}$.*
(7) *Let $K'_{p,P} \subseteq K_{p,P}$ be another compact. The map $\mathfrak{IG}_{K^pU_{K'_{p,P}}} \to \mathfrak{IG}_{K^pU_{K_{p,P}}}$ is finite flat. This map is equivariant with respect to the action of $\hat{T}_{K'_{p,P}}$ on the source, the action of $\hat{T}_{K_{p,P}}$ on the target, and the natural isogeny $\hat{T}_{K'_{p,P}} \to \hat{T}_{K_{p,P}}$. The map $\mathfrak{IG}_{K^pK'_{p,P}} \to \mathfrak{IG}_{K^pK_{p,P}}$ is also finite flat.*

[2]This means that $\overline{K_{p,P}}$ is defined by the sheaf of ideals in $\mathscr{O}_{J_{ord}}$ of sections vanishing on $K_{p,P}$

Remark 3.4.10. The point (6), implies that the completed local rings at a closed point x of $\widehat{\mathscr{O}}_{\mathfrak{IG}_{K^pU_{K_{p,P}}},x}$ is isomorphic to $\mathscr{O}_{\hat{T}_{K_{p,P}}} \otimes_{\mathbb{Z}_p} W(k(x))$. By (5), for any closed point $x \in \mathfrak{IG}_{K^pK_{p,P}}$ and any $x' \in \mathfrak{IG}_{K^pU_{K_{p,P}}}$ above x, we have an isomorphism:

$$\widehat{\mathscr{O}}_{\mathfrak{IG}_{K^pK_{p,P}},x}\hat{\otimes}_{W(k(x))}W(k(x')) \simeq \widehat{\mathscr{O}}_{\mathfrak{IG}_{K^pU_{K_{p,P}}},x'}.$$

Remark 3.4.11. In the situation of (7), assume that $U_{K_{p,P}} = \prod_{\alpha\in\Phi^{+,M}} K_{p,\alpha}$ admits a decomposition as a product for all the roots in $\Phi^{+,M}$ and that $U_{K'_{p,P}} = \prod_{\alpha\in\Phi^{+,M}} p^{n_\alpha}K_{p,\alpha}$. Let x' be a closed point in $\mathfrak{IG}_{K^pK'_{p,P}}$ mapping to a closed point $x \in \mathfrak{IG}_{K^pK_{p,P}}$ Then the induced map between completions at x' and x is isomorphic to:

$$\begin{array}{ccc}\prod_{\alpha\in\Phi^{+,M}} \widehat{\mathbb{G}}_{m,W(k(x'))} & \rightarrow & \prod_{\alpha\in\Phi^{+,M}} \widehat{\mathbb{G}}_{m,W(k(x))} \\ (x_\alpha) & \rightarrow & (x_\alpha^{p^{n_\alpha}})\end{array}$$

Proof. We first consider the case where $K_{p,P} = P'(\mathbb{Z}_p)$. Let us take as an alternative definition of $\mathfrak{IG}_{K^pP'(\mathbb{Z}_p)}$ the ordinary locus in the formal scheme corresponding to the level $K^pG(\mathbb{Z}_p)$-Shimura variety. This is a smooth formal scheme by the deformation theory of abelian varieties. We have a natural map $\mathfrak{IG}_{K^p} \to \mathfrak{IG}_{K^pK_{p,P}}$. Let us prove that this map has a section fpqc-locally by giving another construction of $\mathfrak{IG}_{K^p}$. Over $\mathfrak{IG}_{K^pK_{p,P}}$, the p-divisible group is an extension $0 \to A[p^\infty]^m \to A[p^\infty] \to A[p^\infty]^e \to 0$. We first define a pro-étale covering $\mathfrak{IG}_{K^p,U_P(\mathbb{Z}_p)(1)} \to \mathfrak{IG}_{K^pP'(\mathbb{Z}_p)}$ as the space of isomorphisms $x : \mu_{p^\infty}^g \to A[p^\infty]^m$, $y : (\mathbb{Q}_p/\mathbb{Z}_p)^g \to A[p^\infty]^e$ and $c \in \mathbb{Z}_p^\times$, such that $(x^D, y^D) \circ \lambda \circ (x, y) = c\lambda_{can}$.[3] We have a lift of the Frobenius map $F : \mathfrak{IG}_{K^pP'(\mathbb{Z}_p)} \to \mathfrak{IG}_{K^pP'(\mathbb{Z}_p)}$ and $F : \mathfrak{IG}_{K^pU_P(\mathbb{Z}_p)(1)} \to \mathfrak{IG}_{K^pU_P(\mathbb{Z}_p)(1)}$. We now define $\mathfrak{IG}_{K^p} = \lim_{\times F} \mathfrak{IG}_{K^pU_P(\mathbb{Z}_p)(1)}$. This parametrizes splittings of the sequence $0 \to A[p^\infty]^m \to A[p^\infty] \to A[p^\infty]^e \to 0$ over $\mathfrak{IG}_{K^pU_P(\mathbb{Z}_p)(1)}$, thus we indeed recover our original space $\mathfrak{IG}_{K^p}$. Since $\mathfrak{IG}_{K^pP'(\mathbb{Z}_p)}$ is smooth, the map F is finite flat and thus

$$\mathfrak{IG}_{K^p} \to \mathfrak{IG}_{K^pP'(\mathbb{Z}_p)}$$

is a profinite flat map. The case where $K_{p,P} = P'(\mathbb{Z}_p)_n$ is the principal level congruence subgroup of elements of $P'(\mathbb{Z}_p)$ reducing to 1-mod p^n can be treated similarly. A general $K_{p,P}$ can always be conjugated by an element $x \in M(\mathbb{Q}_p)$ to a finite index subgroup of $P'(\mathbb{Z}_p)$. We thus reduce to the case $P'(\mathbb{Z}_p)_n \subseteq K_{p,P} \subseteq P'(\mathbb{Z}_p)$. Now we need to consider the *fpqc* quotient of $\mathfrak{IG}_{K^pP(\mathbb{Z}_p)_n}$ by the finite flat group scheme $\overline{K_{p,P}}/\overline{P(\mathbb{Z}_p)_n}$. We can invoke [Ray67]. This finishes the proof of (1) and (2). The proof of (4) follows along similar lines: we already treated the case $K_{p,P} = P'(\mathbb{Z}_p)$. The case $K_{p,P} = P'(\mathbb{Z}_p)_n$ follows similarly. We then treat the general case using [Ray67]. We deduce point (5). By Serre-Tate theory ([Kat81]), the action of $\hat{T}_{P'(\mathbb{Z}_p)}$ on the formal completions of the local rings of $\mathfrak{IG}_{K^pU_P(\mathbb{Z}_p)(1)}$ at any closed point x is simply transitive, so we have isomorphisms

$$\hat{T}_{P'(\mathbb{Z}_p)} \times_{\mathrm{Spf}\ \mathbb{Z}_p} \mathrm{Spf}\ W(k(x)) \simeq \mathfrak{IG}_{\widehat{K^pU_P(\mathbb{Z}_p)(1)}}^{\ x}$$

which intertwines the Frobenius map on both sides for any closed point x. We deduce that there are isomorphisms:

$$\hat{T}_{P'(\mathbb{Z}_p)_n} \times_{\mathrm{Spf}\ \mathbb{Z}_p} \mathrm{Spf}\ W(k(x_n)) \simeq \mathfrak{IG}_{\widehat{K^pU_P(\mathbb{Z}_p)(1)_n}}^{\ x_n}$$

for any n and any closed point x_n of $\mathfrak{IG}_{K^pU_P(\mathbb{Z}_p)(1)_n}$. We finally deduce point (6) by passing to the quotient by $\overline{U_{K_{p,P}}}/\overline{U_P(\mathbb{Z}_p)(1)_n}$ for n large enough. This implies (3). Indeed, by (4) the formal schemes $\mathfrak{IG}_{K^pK_{p,P}}$ are formally smooth, and they are also of finite type (as we noticed earlier during the proof of (1) and (2)). The remaining points follow easily. □

[3] In this formula, λ stands for the map $A[p^\infty]^m \oplus A[p^\infty]^{et} \to A^t[p^\infty]^m \oplus A^t[p^\infty]^{et}$ deduced from the map $\lambda : A \to A^t$.

3.4.12. *Variant over* $\mathbb{Z}_p^{cycl}$. When comparing with Shimura varieties, it is useful to work over $\mathbb{Z}_p^{cycl}$ as well. We let $\mathfrak{IG}_{K^p,\mathbb{Z}_p^{cycl}} = \mathfrak{IG}_{K^p} \times_{\mathrm{Spf}\ \mathbb{Z}_p} \mathrm{Spf}\ \mathbb{Z}_p^{cycl}$. For any compact open subgroup $K_{p,P}$ of $P(\mathbb{Q}_p)$, we let $\mathfrak{IG}_{K^pK_{p,P},\mathbb{Z}_p^{cycl}}$ be the quotient of $\mathfrak{IG}_{K^p,\mathbb{Z}_p^{cycl}}$ by $\overline{K_{p,P}}$. If $K_{p,P}$ arises from a subgroup of $P'(\mathbb{Q}_p)$, then $\mathfrak{IG}_{K^pK_{p,P},\mathbb{Z}_p^{cycl}} = \mathfrak{IG}_{K^pK_{p,P}} \times_{\mathrm{Spf}\ \mathbb{Z}_p} \mathrm{Spf}\ \mathbb{Z}_p^{cycl}$.

3.4.13. *Hecke correspondences.* For any $g \in M(\mathbb{Q}_p)$ and any compact open subgroup $K_{p,P}$, $gK_{p,P}g^{-1} \cap K_{p,P}$ is again a compact open subgroup, and we define a Hecke correspondence:

$$\begin{array}{ccc} & \mathfrak{IG}_{K^p.gK_{p,P}g^{-1}\cap K_{p,P}} & \\ \swarrow^{p_2} & & \searrow^{p_1} \\ \mathfrak{IG}_{K^p,K_{p,P}} & & \mathfrak{IG}_{K^pK_{p,P}} \end{array}$$

where p_1 is induced by the inclusion $gK_{p,P}g^{-1} \cap K_{p,P} \subseteq K_{p,P}$, and p_2 is given by the action map $[g] : \mathfrak{IG}_{K^p.gK_{p,P}g^{-1}\cap K_{p,P}} \to \mathfrak{IG}_{K^p.K_{p,P}\cap g^{-1}K_{p,P}g}$, followed by the projection induced by the inclusion $g^{-1}K_{p,P}g \cap K_{p,P} \subseteq K_{p,P}$. The projection maps are finite flat by proposition 3.4.9.

Remark 3.4.14. We can also work over $\mathbb{Z}_p^{cycl}$, in which case we can consider any element $g \in P(\mathbb{Q}_p)$.

3.4.15. *Partial toroidal compactifications.* We construct partial toroidal compactification of our Igusa variety. This is a special case of [CS19], sect. 3.2. Let Σ be a $K^pG(\mathbb{Z}_p)$-admissible cone decomposition. We construct $\mathfrak{IG}^{tor}_{K^p,\Sigma}$, a toroidal (partial) compactification of $\mathfrak{IG}_{K^p}$ as follows. We first let $\mathfrak{IG}^{tor}_{K^pP'(\mathbb{Z}_p),\Sigma}$ be the ordinary locus in the toroidal compactification of the Shimura variety of level $K^pG(\mathbb{Z}_p)$. We next construct the pro-étale torsor: $\mathfrak{IG}^{tor}_{K^pU_P(\mathbb{Z}_p)(1),\Sigma} \to \mathfrak{IG}^{tor}_{K^pP(\mathbb{Z}_p),\Sigma}$ as the space of isomorphisms $\mu^g_{p^\infty} \to A[p^\infty]^m$, $c \in \mathbb{Z}_p^\times$. This is justified because the connected part of the p-divisible group extends over the toroidal compactification.

We have a lifting of the Frobenius map $F : \mathfrak{IG}^{tor}_{K^pP'(\mathbb{Z}_p),\Sigma} \to \mathfrak{IG}^{tor}_{K^pP'(\mathbb{Z}_p),\Sigma}$ and $F : \mathfrak{IG}^{tor}_{K^pU_P(\mathbb{Z}_p)(1),\Sigma} \to \mathfrak{IG}^{tor}_{K^pU_P(\mathbb{Z}_p)(1),\Sigma}$. We let $\mathfrak{IG}^{tor}_{K^p,\Sigma} = \lim_F \mathfrak{IG}^{tor}_{K^pU_P(\mathbb{Z}_p)(1),\Sigma}$. By construction, we have an open immersion $\mathfrak{IG}_{K^p} \hookrightarrow \mathfrak{IG}^{tor}_{K^p,\Sigma}$.

Proposition 3.4.16. *The space $\mathfrak{IG}^{tor}_{K^p,\Sigma}$ carries an action of $P'(\mathbb{Z}_p)$, extending the action on $\mathfrak{IG}_{K^p}$. For any $K_{p,P} \subseteq P'(\mathbb{Z}_p)$, the categorical quotient of $\mathfrak{IG}^{tor}_{K^p,\Sigma}$ by $K_{p,P}$ exists and is denoted by $\mathfrak{IG}^{tor}_{K^pK_{p,P},\Sigma}$.*

Proof. The map $\mathfrak{IG}^{tor}_{K^p,\Sigma} \to \mathfrak{IG}^{tor}_{K^pP(\mathbb{Z}_p),\Sigma}$ is profinite flat. Moreover, by normality the action of $P'(\mathbb{Z}_p)$ extends to $\mathfrak{IG}^{tor}_{K^pU_P(\mathbb{Z}_p)(1),\Sigma}$ and then to $\lim_F \mathfrak{IG}^{tor}_{K^pU_P(\mathbb{Z}_p)(1),\Sigma}$. Since the map $\mathfrak{IG}^{tor}_{K^p,\Sigma} \to \mathfrak{IG}^{tor}_{K^pP'(\mathbb{Z}_p),\Sigma}$ is affine, we can cover $\mathfrak{IG}^{tor}_{K^p,\Sigma}$ by open affine $\cup_i \mathrm{Spf}\ A_i$, stable under the action of $P'(\mathbb{Z}_p)$. For each i, $A_i^{K_{p,P}}$ is finite over $A_i^{P'(\mathbb{Z}_p)}$. We deduce that the Spf $A_i^{K_{p,P}}$ glue to give the desired space. □

We have by construction an open immersion $\mathfrak{IG}_{K^pK_{p,P}} \hookrightarrow \mathfrak{IG}^{tor}_{K^pK_{p,P},\Sigma}$. We say that $\mathfrak{IG}^{tor}_{K^pK_{p,P},\Sigma}$ is a partial toroidal compactification of $\mathfrak{IG}_{K^pK_{p,P}}$.

By blowing up at the boundary as in [Lan17], we can obtain more toroidal compactifications for more general cone decompositions. In the sequel, we ignore the dependence on the cone decomposition and simply denote by $\mathfrak{IG}^{tor}_{K^pK_{p,P}}$ a partial toroidal compactification for a suitable cone decomposition. The coherent cohomology doesn't depend on the choice of the cone decomposition.

3.4.17. *Relation to Shimura varieties.* We work over $\mathbb{Z}_p^{cycl}$ and fix an isomorphism $\mathbb{Z}_p(1) = \mathbb{Z}_p$. Let $\mathcal{IG}_{K^p,\mathbb{Z}_p^{cycl}} = \mathfrak{IG}_{K^p,\mathbb{Z}_p^{cycl}} \times \mathrm{Spa}(\mathbb{Q}_p^{cycl}, \mathbb{Z}_p^{cycl})$. We have a $P(\mathbb{Q}_p)$-equivariant map $\mathcal{IG}_{K^p,\mathbb{Z}_p^{cycl}} \to \mathcal{S}_{K^p} \times \mathrm{Spa}(\mathbb{Q}_p^{cycl}, \mathbb{Z}_p^{cycl})$ which sends A equipped with the isomorphism $(\mathbb{Q}_p/\mathbb{Z}_p)^g \oplus \mu^g_{p^\infty} \to A[p^\infty]$ to the level structure $\mathbb{Z}_p^g \oplus \mathbb{Z}_p^g(1) = \mathbb{Z}_p^{2g} \to T_p(A)$ (using our fixed isomorphism $\mathbb{Z}_p(1) = \mathbb{Z}_p$). We deduce that we have an injective, $G(\mathbb{Q}_p)$-equivariant map $\mathcal{IG}_{K^p,\mathbb{Z}_p^{cycl}} \times^{P(\mathbb{Q}_p)} G(\mathbb{Q}_p) \to \mathcal{S}_{K^p} \times \mathrm{Spa}(\mathbb{Q}_p^{cycl}, \mathbb{Z}_p^{cycl})$. Let $x \in G(\mathbb{Q}_p)$, and let K_p be a compact open subgroup of $G(\mathbb{Q}_p)$. Let $K_{p,P} =$

$P(\mathbb{Q}_p) \cap xK_px^{-1}$. The map $\mathcal{IG}_{K^p} \times^{P(\mathbb{Q}_p)} G(\mathbb{Q}_p) \to \mathcal{S}_{K^p} \times \mathrm{Spa}(\mathbb{Q}_p^{cycl}, \mathbb{Z}_p^{cycl})$ induces an open immersion:

$$\mathcal{IG}_{K^pK_{p,P},\mathbb{Z}_p^{cycl}} \xrightarrow{x} \mathcal{S}_{K^pK_p} \times \mathrm{Spa}(\mathbb{Q}_p^{cycl}, \mathbb{Z}_p^{cycl}).$$

Remark 3.4.18. In general this map doesn't descend to a map: $\mathcal{IG}_{K^pK_{p,P}} \xrightarrow{x} \mathcal{S}_{K^pK_p}$ because the level structure on both sides are formulated in a different way. One uses isomorphisms with $\mu_{p^\infty}^g \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^g$ on the left, and isomorphisms with $(\mathbb{Q}_p/\mathbb{Z}_p)^{2g}$ on the right. For certain level structure, it does however descend. See the section 4.1 below.

3.4.19. *Integral model of Shimura varieties.* We review the standard method to obtain integral models of the Shimura variety of deep level by normalization (following for example [Lan16] and [MP19]). We first consider Shimura varieties with level given by maximal compact open subgroups. In that case, we have a natural moduli theoretic definition of the integral structure. For a general level, we consider maps towards Shimura varieties whose level is a maximal compact open subgroup, and we take a normalization.

We let $A_{g,d} \to \mathrm{Spec}\ \mathbb{Z}$ be the moduli stack of abelian varieties of dimension g, with a degree d^2 polarization (see [dJ93]). For any $\delta = (\delta_1, \cdots, \delta_g) \in \mathbb{Z}_{\geq 0}^g$ with $(\delta_1 \mid \delta_2 \mid \cdots \mid \delta_g)$ and $\prod_{i=1}^g \delta_i = d$, we let $M_\delta \in \mathrm{GL}_{2g}(\mathbb{Q})$ be the matrix with diagonal coefficients $(\delta_1, \cdots, \delta_g, 1, \cdots, 1)$. We let $K_\delta = M_\delta \mathrm{GL}_{2g}(\hat{\mathbb{Z}}) M_\delta^{-1} \cap G(\mathbb{A}_f)$. We let $A_{g,\delta} \to A_{g,d}$ be the locally closed substack where the kernel of the polarization is isomorphic, locally in the étale topology, to $\prod_{i=1}^g \mathbb{Z}/\delta_i\mathbb{Z} \times \mu_{\delta_i}$. By definition, we have $A_{g,\delta,\mathbb{Q}} = S_{K_\delta,\mathbb{Q}}$. We also see easily that we have the decomposition

$$A_{g,d,\mathbb{Q}} = \coprod_\delta S_{K_\delta,\mathbb{Q}}.$$

We let $\mathfrak{A}_{g,d}$ be the p-adic formal completion of $A_{g,d}$ and we let $\mathfrak{A}_{g,d}^{ord}$ be the ordinary locus. We let $\mathfrak{A}_{g,\delta}$ be the p-adic formal completion of $A_{g,\delta}$ and we let $\mathfrak{A}_{g,\delta}^{ord}$ be the ordinary locus. We have $\mathfrak{A}_{g,d}^{ord} = \coprod_\delta \mathfrak{A}_{g,\delta}^{ord}$.

Lemma 3.4.20. *We have* $\mathfrak{A}_{g,\delta}^{ord} = \mathfrak{IG}_{K_{\delta,P}}$ *where* $K_{\delta,P} = K_\delta^p(K_{\delta,p} \cap P(\mathbb{Q}_p))$.

Proof. Clear from the definition. □

Let us fix $K = K_pK^p$. We now give the construction of certain integral models for the Shimura variety of level K, depending on some auxiliary data. Let $d_1, \cdots, d_r$ be positive integers and let $\delta_1, \cdots, \delta_r$ be sequences $(\delta_{i,1}, \cdots, \delta_{i,g})$ as above with $\prod_{j=1}^g \delta_{i,j} = d_i$. We also fix maps $K \hookrightarrow K_{\delta_i}$ induced by conjugation by elements of $G(\mathbb{A}_f)$ followed by inclusion. They induce finite maps $S_{K^pK_p,\mathbb{Q}} \to \prod_{i=1}^r A_{g,d_i,\mathbb{Q}}$. We let $S_{K^pK_p} \to \mathrm{Spec}\ \mathbb{Z}_p$ be the normalization of $\prod_{i=1}^r A_{g,d_i} \times \mathrm{Spec}\ \mathbb{Z}_p$ in $S_{K^pK_p,\mathbb{Q}_p}$.

Remark 3.4.21. The space $S_{K^pK_p} \to \mathrm{Spec}\ \mathbb{Z}_p$ will in general depend on the choice of the auxiliary spaces A_{g,d_i}.

One can perform a similar construction over $\mathbb{Z}_p^{cycl}$. We let $S_{K^pK_p,\mathbb{Z}_p^{cycl}} \to \mathrm{Spec}\ \mathbb{Z}_p^{cycl}$ be the normalization of $\prod_{i=1}^r A_{g,d_i} \times \mathrm{Spec}\ \mathbb{Z}_p^{cycl}$ in $S_{K^pK_p,\mathbb{Q}_p^{cycl}}$.

Remark 3.4.22. In general, the natural map $S_{K^pK_p,\mathbb{Z}_p^{cycl}} \to S_{K^pK_p} \times_{\mathrm{Spec}\ \mathbb{Z}_p} \mathrm{Spec}\ \mathbb{Z}_p^{cycl}$ is not an isomorphism.

3.4.23. *Toroidal compactifications.* We obtain integral models of toroidal compactifications in the same way. We consider a finite map $S_{K^pK_p,\mathbb{Q}}^{tor} \to \prod_{i=1}^r A_{g,d_i,\mathbb{Q}}^{tor}$. We let $S_{K^pK_p}^{tor} \to \mathrm{Spec}\ \mathbb{Z}_p$ be the normalization of $\prod_{i=1}^r A_{g,d_i}^{tor} \times \mathrm{Spec}\ \mathbb{Z}_p$ in $S_{K^pK_p,\mathbb{Q}_p}^{tor}$. A similar variant holds over $\mathbb{Z}_p^{cycl}$. The following lemma shows that despite the fact that the integral model depends on choices, the ordinary locus is independent on the choices.

Lemma 3.4.24. *The map of section 3.4.17*

$$\mathcal{IG}_{K^pK_{p,P},\mathbb{Z}_p^{cycl}} \xrightarrow{x} \mathcal{S}_{K^pK_p,\mathbb{Z}_p^{cycl}}$$

comes from an open immersion of formal schemes:

$$\mathfrak{IG}_{K^pK_{p,P},\mathbb{Z}_p^{cycl}} \hookrightarrow \mathfrak{S}_{K^pK_p,\mathbb{Z}_p^{cycl}}$$

Proof. The maps $\mathfrak{IG}_{K^pK_{p,P},\mathbb{Z}_p^{cycl}} \to \prod \mathfrak{A}^{ord}_{g,\delta_i,\mathbb{Z}_p^{cycl}}$ are finite and the source is normal by proposition 3.4.9. □

The above map extends to maps between toroidal compactifications $\mathfrak{IG}^{tor}_{K^pK_{p,P},\mathbb{Z}_p^{cycl}} \hookrightarrow \mathfrak{S}^{tor}_{K^pK_p,\mathbb{Z}_p^{cycl}}$ for suitable choices of cone decomposition.

Remark 3.4.25. In general, the map does not descend to a map $\mathfrak{IG}^{tor}_{K^pK_{p,P}} \hookrightarrow \mathfrak{S}^{tor}_{K^pK_p}$.

4. Higher Hida theory in fixed weight and the Cousin complex

4.1. **The Siegel variety of Iwahori level.** Let $n \in \mathbb{Z}_{\geq 0}$. Let $K_{p,n} \subseteq G(\mathbb{Q}_p)$ be the Iwahori subgroup of matrices in $G(\mathbb{Z}_p)$, reducing modulo p^n to $B(\mathbb{Z}/p^n\mathbb{Z})$.

4.1.1. *Definition and integral models.* Let $S^{tor}_{K^pK_{p,n},\mathbb{Q}_p}$ be a toroidal compactification of the Siegel moduli space of level $K_{p,n}K^p$ over Spec $\mathbb{Q}_p$. Away from the boundary, we have a self-dual chain of isogenies:

$$A = A_g \to A_{g+1} \to \cdots \to A_{2g} \to A_1 \to \cdots \to A_g = A.$$

For $0 \leq i \leq g$, the group $\mathrm{Ker}(A_g \to A_{g+i})$ is a totally isotropic subgroup of $A[p^n]$, locally isomorphic to $(\mathbb{Z}/p^n\mathbb{Z})^i$. The total map $A \to A$ is multiplication by p^n.

We now construct an integral model $S^{tor}_{K^pK_{p,n}} \to$ Spec $\mathbb{Z}_p$. We observe that A_{g+i} is equipped with a polarization of degree $p^{2ng-2in} = d_i$ for $1 \leq i \leq g$. We also let $d_0 = 1$.

We therefore get a map $S_{K^pK_{p,n},\mathbb{Q}} \to \prod_{i=0}^g A_{g,d_i}$. We define $S_{K^pK_{p,n}} \to$ Spec $\mathbb{Z}_p$ by normalizing $\prod_{i=0}^g A_{g,d_i} \times$ Spec $\mathbb{Z}_p$ in $S_{K^pK_{p,n},\mathbb{Q}_p}$. We construct similarly $S^{tor}_{K^pK_{p,n}}$. See section 3.4.19.

By [FC90], I, prop. 2.7, we have a chain of isogenies of semi-abelian schemes

$$A = A_g \to A_{g+1} \to \cdots \to A_{2g} \to A_1 \to \cdots \to A_g = A$$

where the total map $A \to A$ is multiplication by p^n.

Remark 4.1.2. When $n = 1$, $S^{tor}_{K^pK_{p,1}}$ is the usual integral model of Iwahori level.

We let $0 \subseteq H_{g+1} \subseteq H_{g+2} \subseteq \cdots \subseteq H_{2g} = H_{2g}^{\perp} \subseteq H_1 \subseteq H_2 \subseteq \cdots \subseteq A[p^n] = H_g$ be the full flag of subgroups of $A[p^n]$, given by the kernel of the maps $A \to A_{g+i}$, and we let $\mathrm{Gr}_i = H_i/H_{i-1}$. They are finite flat group scheme of order p^n over the interior of the Shimura variety and $Gr_{2g-i+1} = \mathrm{Gr}_i^D$.

4.1.3. *Special fiber.* We let $S^{tor}_{K^pK_{p,n},\mathbb{F}_p}$ be the special fiber. For each $w \in {}^MW$, we let $S^{tor}_{K^pK_{p,n},\mathbb{F}_p,w}$ be the locus where $\mathrm{Gr}_{w^{-1}(i)}$ is a multiplicative group scheme for $i = g+1, \cdots, 2g$.

Remark 4.1.4. Clearly, $S^{tor}_{K^pK_{p,n},\mathbb{F}_p,w}$ is included in the ordinary locus. However, unless $n = 1$, the ordinary locus is bigger than $\cup_{w\in {}^MW} S^{tor}_{K^pK_{p,n},\mathbb{F}_p,w}$.

4.1.5. *Formal schemes and adic spaces.* Let $\mathfrak{S}^{tor}_{K^pK_{p,n}}$ be the formal completion of $S^{tor}_{K^pK_{p,n}}$. We also define $\mathfrak{S}^{tor}_{K^pK_{p,n},w}$ to be the open formal scheme corresponding to $S^{tor}_{K^pK_{p,n},\mathbb{F}_p,w}$.

We let $\mathcal{S}^{tor}_{K^pK_{p,n}}$ be the associated adic space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. We have a set theoretical map $\pi_{HT,K_{p,n}} : |\mathcal{S}^{tor}_{K^pK_{p,n}}| \to |\mathcal{FL}|/K_{p,n}$, obtained by passing the Hodge-Tate period map to the quotient (see section 4.5 of [BP21]). Let $\mathcal{S}^{tor}_{K^pK_{p,n},w}$ be the adic space associated to $\mathfrak{S}^{tor}_{K^pK_{p,n},w}$.

Lemma 4.1.6. *We have* $\pi^{-1}_{HT,K_{p,n}}(w \cdot K_{p,n}) = \overline{\mathcal{S}^{tor}_{K^pK_{p,n},w}}$ *(the closure of* $\mathcal{S}^{tor}_{K^pK_{p,n},w}$*).*

Proof. We know that $\pi^{-1}_{HT,K_{p,n}}(\mathcal{FL}(\mathbb{Q}_p)) = \overline{\mathcal{S}^{tor}_{K^pK_{p,n},ord}}$, is the closure of the ordinary locus. On $\pi^{-1}_{HT}(w)$ we have that $\mathrm{Lie}(A) = \langle e_{w^{-1}(i)},\ g+1 \leq i \leq 2g\rangle$, so that $e_{w^{-1}(i)}$, $g+1 \leq i \leq 2g$ span the multiplicative part of the Tate module. Passing to $K_{p,n}$ orbits, we have that H_{g+i} for $1 \leq i \leq g$ corresponds to $\langle e_{g+1}, \ldots, e_{g+i}\rangle$ mod p^n, from which we deduce that Gr_{g+i} is multiplicative if $g + i \in w^{-1}\{g+1, \ldots, 2g\}$ and étale otherwise. □

Lemma 4.1.7. *We have* $\mathfrak{S}^{tor}_{K^pK_{p,n},w} = \mathfrak{IG}^{tor}_{K_{p,P,w}K^p}$ *for* $K_{p,P,w} = wK_{p,n}w^{-1} \cap P(\mathbb{Q}_p)$.

Proof. This follows from lemma 3.4.24. Note that here the embedding is defined over $\mathbb{Z}_p$, not only $\mathbb{Z}_p^{cycl}$. □

4.1.8. *Cartier divisors.* For all $1 \leq i \leq 2g$, we let ℓ_{Gr_i} be the co-Lie complex of Gr_i ([Far10], sect. 2). We let $\delta_i = \det(\ell_{\mathrm{Gr}_i})$. We have a canonical section $\mathscr{O}_{S^{tor}_{K^pK_{p,n}}} \to \delta_i$. It follows that $V(\delta_i^{-1})$ defines an effective Cartier divisor D_i on $S^{tor}_{K^pK_{p,n}}$, supported on the special fiber. The complement of D_i is the locus where Gr_i is étale.

We have a canonical isomorphism of Cartier divisors $\delta_i\delta_{2g+1-i} = p^{-n}\mathscr{O}_{S^{tor}}$ so we get the important identity: $D_i + D_{2g+1-i} = (p^n)$ for all $1 \leq i \leq g$.

Lemma 4.1.9. *We have* $S^{tor}_{K^pK_{p,n},\mathbb{F}_p,w} = S^{tor}_{K^pK_{p,n},\mathbb{F}_p} \backslash \sum_{i=1}^{g} D_{w^{-1}(i)}$. *We have* $\cup_{w\in {}^MW} S^{tor}_{K^pK_{p,n},\mathbb{F}_p,w} = S^{tor}_{K^pK_{p,n},\mathbb{F}_p} \setminus \sum_{i=1}^{g} D_i \cap D_{2g+1-i}$

Proof. $S^{tor}_{K^pK_{p,n},\mathbb{F}_p,w}$ is given by the condition that $\mathrm{Gr}_{w^{-1}(g+i)}$ is multiplicative for $1 \leq i \leq g$, equivalently by the condition that $\mathrm{Gr}_{w^{-1}(g+1-i)}$ is étale for $1 \leq i \leq g$. □

4.1.10. *Interlude: blow-ups and Cartier divisors.* Let $\bar{X}$ be a scheme and let D_1, D_2 be effective Cartier divisors.

Proposition-construction 4.1.11. *There exists a blow-up* $p : \tilde{\bar{X}} \to \bar{X}$ *and effective Cartier divisors* D'_1, D'_2 *on* $\tilde{\bar{X}}$ *with disjoint support and with the property that* $p^\star D_1 + D'_2 = p^\star D_2 + D'_1$.

Proof. We let $\tilde{\bar{X}}$ be the blow-up of $\bar{X}$ along the ideal generated by $\mathscr{O}_{\bar{X}}(-D_1)$ and $\mathscr{O}_{\bar{X}}(-D_2)$. We now check that $\tilde{\bar{X}}$ has the claimed properties. The question is local on $\bar{X}$, so we may assume that $\bar{X} = \mathrm{Spec}\, A$ is affine and that $D_1 = V(x_1)$, $D_2 = V(x_2)$ for $x_1, x_2 \in A$. Then we let $I = (x_1, x_2)$ and $\tilde{\bar{X}} = \mathrm{Proj}(\bigoplus_n I^n)$. We have maps $x'_i : \bigoplus_n I^{n-1} \stackrel{x_i}{\to} \bigoplus_n I^n$, which define sections of $\mathscr{O}_{\tilde{\bar{X}}}(1)$ and Cartier Divisors D'_i. We also have maps $x_i : \bigoplus_n I^n \stackrel{x_i}{\to} \bigoplus_n I^n$ which correspond to $p^\star D_i$. We have the relation $x'_1 x_2 = x'_2 x_1$. Finally $V(x'_1, x'_2) = \mathrm{Proj}(\oplus_n I^n/(x_1,x_2)I^{n-1}) = \emptyset$. □

4.1.12. *A certain blow-up.* We will now perform a suitable blow-up of our space. We let $Z_i = \sum_{k=1}^{i} D_{g+k}$.

Proposition 4.1.13. *There exists a blow-up* $\tilde{S}^{tor}_{K^pK_{p,n}} \to S^{tor}_{K^pK_{p,n}}$ *satisfying the following properties:*

(1) *for all* $0 \leq i \leq g$ *and all* $0 \leq j \leq i$ *we have effective Cartier divisors* $Z_{i,>j}$ *and* $Z_{i,<j}$ *on* $\tilde{S}^{tor}$ *with disjoint support, and* $Z_{i,>j} - Z_{i,<j} = Z_i - V(p^{nj})$.

(2) *For any increasing sequence* $j_1 \leq \cdots \leq j_k$, *we can consider the open subset*

$$\tilde{S}^{tor}_{K^pK_{p,n},=j_1,\cdots,=j_k} = \cap_{i=1}^{k} Z^c_{i,>j_i} \cap Z^c_{i,<j_i}$$

of $\tilde{S}^{tor}_{K^pK_{p,n}}$. *This open subset is empty unless* $j_i - j_{i-1} \in \{0,1\}$ *(where* $j_0 = 0$ *by convention). The map* $\tilde{S}^{tor}_{K^pK_{p,n},=j_1,\cdots,=j_k} \to S^{tor}_{K^pK_{p,n}}$ *is an open immersion, and the image of* $\tilde{S}^{tor}_{K^pK_{p,n},=j_1,\cdots,=j_k}$ *identifies with the open sub-scheme where* Gr_{g+i} *is multiplicative if* $j_i - j_{i-1} = 1$, *and étale if* $j_i - j_{i-1} = 0$.

(3) *Over* $\tilde{S}^{tor}_{K^pK_{p,n},=j_1,\cdots,=j_k}$, *we have* $Z_{k+1,>j_k} = D_{g+k+1}$ *and* $Z_{k+1,<j_k+1} = D_{g-k} = V(p^n) - D_{g+k+1}$.

Proof. We construct $\tilde{S}^{tor}_{K^pK_{p,n}}$ inductively. For each $1 \leq k \leq g$, we construct a tower of blow-ups $\tilde{S}^{tor}_{K^pK_{p,n},k} \to \tilde{S}^{tor}_{K^pK_{p,n},k-1} \to \cdots \to S^{tor}_{K^pK_{p,n}}$ where $\tilde{S}^{tor}_{K^pK_{p,n},k}$ satisfies the properties of the proposition for all indices $i \leq k$. We first consider $k = 1$. We observe that $Z_{1,>0} = D_{g+1}$ and $Z_{1,<1} = D_g$. Therefore, we can take $\tilde{S}^{tor}_{K^pK_{p,n},1} = S^{tor}_{K^pK_{p,n},1}$. We assume that we have constructed $\tilde{S}^{tor}_{K^pK_{p,n},k}$. We apply proposition-construction 4.1.11 to the divisors Z_{k+1} and $V(p^{nj})$ for $0 \leq j \leq k+1$ to obtain a blow-up $\tilde{S}^{tor}_{K^pK_{p,n},k+1} \to \tilde{S}^{tor}_{K^pK_{p,n},k}$ with divisors with disjoint support $Z_{k+1,>j}$ and $Z_{k+1,<j}$ with the property that $Z_{k+1,>j} - Z_{k+1,<j} = Z_{k+1} - V(p^{nj})$. We see that if we restrict to $\tilde{S}^{tor}_{k,=j_1,\cdots,=j_k}$ then $Z_k = V(p^{nj_k})$ so that $Z_{k+1} - V(p^{nj_k}) = D_{g+k+1} = Z_{k+1,>j_k}$, $Z_{k+1} - V(p^{n(j_k+1)}) = -D_{g-k-1} = -Z_{k+1,<j_k+1}$ and $Z_{k+1,>j} = \tilde{S}^{tor}_{k,=j_1,\cdots,=j_k}$ for $j < j_k$, $Z_{k+1,<j} = \tilde{S}^{tor}_{k,=j_1,\cdots,=j_k}$ for $j > j_k + 1$. It follows that the map $\tilde{S}^{tor}_{K^pK_{p,n},k+1} \to \tilde{S}^{tor}_{K^pK_{p,n},k}$ induces an isomorphism

$$\tilde{S}^{tor}_{K^pK_{p,n},k+1} \times_{\tilde{S}^{tor}_{K^pK_{p,n},k}} \tilde{S}^{tor}_{k,=j_1,\cdots,=j_k} \to \tilde{S}^{tor}_{K^pK_{p,n},k,=j_1,\cdots,=j_k}.$$

□

Recall that to each $w \in {}^MW$ we have attached a sequence $(w_1, \cdots, w_g)$ of integers at the bottom of section 3.1.

Corollary 4.1.14. *Let $w \in {}^MW$. The map $\tilde{S}^{tor}_{K^pK_{p,n},=w_1,\cdots,=w_g} \to S^{tor}$ is an open immersion and $\tilde{S}^{tor}_{K^pK_{p,n},=w_1,\cdots,=w_g} \times \mathrm{Spec}\,\mathbb{F}_p$ identifies with $S^{tor}_{K^pK_{p,n},\mathbb{F}_p,w}$.*

Form now on, we simplify our notation and let S^{tor} denote $\tilde{S}^{tor}_{K^pK_{p,n}}$.

Remark 4.1.15. We are about to define some integral cohomology over S^{tor}, and prove a control theorem relating the ordinary cohomology with the ordinary cohomology of the ordinary locus in theorem 4.5.2. We want to insist that we could perform any further blow-up of S^{tor}, whose center is is disjoint from $\cup_w S^{tor}_{\mathbb{F}_p,w}$. We would still be able to prove a version of theorem 4.5.2 for this other model.

4.1.16. *A filtration.* Let $w \in {}^MW$. We let $Z_{>w-1} = \cap_{i=1}^g Z_{i,>w_i-1}$

Lemma 4.1.17. (1) *We have $Z_{i,>j} \leq Z_{i,>j'}$ if $j' \leq j$ and $Z_{i,<j} \leq Z_{i,<j'}$ if $j' \geq j$.*
(2) *We have $Z_{>w-1} \subseteq Z_{>w'-1}$ if $w \leq w'$.*

Proof. We have $Z_i - V(p^{nj}) \leq Z_i - V(p^{nj'})$. Passing to effective divisors gives the first item. For the second point, we observe that $w' \leq w$ implies that $w'_i \geq w_i$ for all i. □

We construct a filtration by closed subsets: $F_{-1} = \emptyset \subseteq F_0 \subseteq \cdots \subseteq F_{\frac{g(g+1)}{2}} = S^{tor}_{\mathbb{F}_p}$, by letting $F_i = \cup_{w\in {}^MW,\ell(w)\leq i} Z_{>w-1}$.

Lemma 4.1.18. *We have that $F_i \setminus F_{i-1} = \coprod_{w,\ell(w)=i} \cap_{i=1}^g (Z_{i,>w_i-1} \cap Z^c_{i,>w_i})$.*

Proof. This follows easily from the definitions. □

We note that $S^{tor}_{K^pK_{p,n},\mathbb{F}_p,w} \subseteq \cap_{i=1}^g (Z_{i,>w_i-1} \cap Z^c_{i,>w_i})$.

4.2. **The Hecke correspondences.** Let $\mathcal{C}_c(K_{p,n}\backslash G(\mathbb{Q}_p)/K_{p,n},\mathbb{Z})$ be the Hecke algebra of level $K_{p,n}$. By [Cas], lemma 4.1.5, we have an algebra map $\mathbb{Z}[T^+(\mathbb{Q}_p)/T(\mathbb{Z}_p)] \to \mathcal{C}_c(K_{p,n}\backslash G(\mathbb{Q}_p)/K_{p,n},\mathbb{Z})$ mapping $t \in T^+(\mathbb{Q}_p)$ to the characteristic function of $[K_{p,n}tK_{p,n}]$. We now attach to each $[K_{p,n}tK_{p,n}]$ a Hecke correspondence and an integral model.

4.2.1. *Definition and integral model.* Let $t \in T^+(\mathbb{Q}_p)$. Let $C^{tor}_{t,\mathbb{Q}_p} = S^{tor}_{tK_{p,n}t^{-1}\cap K_{p,n}K^p,\mathbb{Q}_p,\Sigma'}$ (for a suitable choice of Σ'). Let $p_1 \times p_2 : C^{tor}_{t,\mathbb{Q}_p} \to S^{tor}_{\mathbb{Q}_p} \times S^{tor}_{\mathbb{Q}_p}$ be the Hecke correspondence (which is finite for good choices of cone decompositions). We let C^{tor}_t be the normalization of $S^{tor} \times S^{tor}$ in $C^{tor}_{t,\mathbb{Q}_p}$.

Lemma 4.2.2. *The universal quasi-isogeny $p_1^\star A \to p_2^\star A$ over $C^{tor}_{t,\mathbb{Q}_p}$ extends uniquely to a quasi-isogeny $p_1^\star A \to p_2^\star A$ over C^{tor}_t.*

Proof. This is [FC90], I, prop. 2.7. □

4.2.3. *Restricting the Hecke correspondence.* We let $\mathfrak{C}^{tor}_t$ be the p-adic formal completion of C^{tor}_t. We let $\mathfrak{C}^{tor}_{t,w} = p_1^{-1}\mathfrak{S}^{tor}_w \cap p_2^{-1}\mathfrak{S}^{tor}_w$.

Proposition 4.2.4. *We have an isomorphism of correspondences:*

$$\mathfrak{C}^{tor}_{t,w} \simeq \mathfrak{IG}^{tor}_{w(t)K_{p,P,w}w(t)^{-1}\cap K_{p,P,w}K^p}$$

over $\mathfrak{S}^{tor}_w \simeq \mathfrak{IG}_{K_{p,P,w}K^p}$ (see lemma 4.1.7).

Proof. We let $\mathcal{C}^{tor}_{t,w}$ be the generic fiber of $\mathfrak{C}^{tor}_{t,w}$. This space is a union of components in the ordinary locus of $\mathcal{C}^{tor}_t$. We also see that $\mathcal{IG}^{tor}_{w(t)K_{p,P,w}w(t)^{-1}\cap K_{p,P,w}K^p} \xrightarrow{w} \mathcal{C}^{tor}_t$ is a union of component of the ordinary locus by lemma 3.4.24. It suffices to prove that they agree (since then the formal schemes are identified by lemma 3.4.24). We have $P(\mathbb{Q}_p)wK_{p,n} = \coprod_{x\in I} P(\mathbb{Q}_p)wx(tK_{p,n}t^{-1}\cap K_{p,n})$ for some finite set I of elements of $K_{p,n}$. We see that $p_1^\star \mathcal{S}^{tor}_w = \cup_x \mathcal{IG}^{tor}_{P(\mathbb{Q}_p)\cap wx(K_{p,n}\cap tK_{p,n}t^{-1})(wx)^{-1}K^p}$ where

$\mathcal{IG}^{tor}_{P(\mathbb{Q}_p)\cap wx(K_{p,n}\cap tK_{p,n}t^{-1})(wx)^{-1}K^p} \overset{wx}{\to} \mathcal{C}^{tor}_t$. For a component $\mathcal{IG}^{tor}_{P(\mathbb{Q}_p)\cap wx(K_{p,n}\cap tK_{p,n}t^{-1})(wx)^{-1}K^p}$, the condition that $p_2(\mathcal{IG}^{tor}_{P(\mathbb{Q}_p)\cap wx(K_{p,n}\cap tK_{p,n}t^{-1})(wx)^{-1}K^p}) \subseteq \mathcal{S}^{tor}_w$ translates into the condition that $wxt \in P(\mathbb{Q}_p)wK_{p,n}$. We have a unique decomposition

$$P(\mathbb{Q}_p)wK_{p,n} = P(\mathbb{Q}_p)w \prod_{\alpha\in w^{-1}\Phi^{-,M}\cap\phi^+} U_{\alpha,n} \prod_{\alpha\in w^{-1}\Phi^{-,M}\cap\phi^-} U_{\alpha,0}$$

where U_α is the root group corresponding to α and $U_{\alpha,n}$ is the subgroup of elements reducing to 1 modulo p^n. Let us take $x \in \prod_{\alpha\in w^{-1}\Phi^{-,M}\cap\phi^+} U_{\alpha,n}\prod_{\alpha\in w^{-1}\Phi^{-,M}\cap\phi^-} U_{\alpha,0}$. We have $wxt = wtw^{-1}wt^{-1}xt$. We deduce that $wxt \in P(\mathbb{Q}_p)wK_{p,n}$ if and only if

$$t^{-1}xt \in \prod_{\alpha\in w^{-1}\Phi^{-,M}\cap\phi^+} U_{\alpha,n} \prod_{\alpha\in w^{-1}\Phi^{-,M}\cap\phi^-} U_{\alpha,0}.$$

This is equivalent to $x \in tK_{p,n}t^{-1}$. Thus, we deduce that $p_2(\mathcal{IG}^{tor}_{P(\mathbb{Q}_p)\cap wx(K_{p,n}\cap tK_{p,n}t^{-1})(wx)^{-1}K^p}) \subseteq \mathcal{S}^{tor}_w$ implies that $x = 1 \mod tK_{p,n}t^{-1}\cap K_{p,n}$. Therefore,

$$p_1^{-1}\mathfrak{S}^{tor}_w \cap p_2^{-1}\mathfrak{S}^{tor}_w = \mathcal{IG}^{tor}_{P(\mathbb{Q}_p)\cap w(K_{p,n}\cap tK_{p,n}t^{-1})(w)^{-1}K^p} \overset{w}{\to} \mathcal{C}^{tor}_t.$$

We finally remark that $P(\mathbb{Q}_p)\cap w(K_{p,n}\cap tK_{p,n}t^{-1})(w)^{-1} = K_{p,P,w}\cap w(t)K_{p,P,w}w(t)^{-1}$. □

4.2.5. *Carrying the level structure over the generic fiber of C^{tor}_t.* We now describe how the Iwahori level structure is taken from $p_1^\star A$ to $p_2^\star A$ over the generic fiber. If $t = \mathrm{diag}(p^{n_1},\cdots,p^{n_{2g}})$ with $n_i \leq 0$, then the quasi-isogeny is an isogeny with kernel L. Then we consider the filtration $L[p] \subseteq L[p^2] \subset \cdots \subseteq L[p^k] = L$. We can factor this isogeny as a composition of isogenies with p-torsion kernel, say $p_1^\star A = G_0 \to G_1 \cdots \to G_k = p_2^\star A$ where $\ker(G_i \to G_{i+1}) = L[p^{i+1}]/L[p^i]$.

Remark 4.2.6. We have $t \in T^{++}(\mathbb{Q}_p)$ if and only if the ranks of the groups $L[p^{i+1}]/L[p^i]$ take all the values p, p^2, $\cdots$, p^{2g-1}.

The Iwahori level structure is a flag of subgroups $p_1^\star H_{g+1} \subseteq p_1^\star H_{g+2}\cdots$ on $G_0[p^n]$, and we define a flag inductively on $G_1[p^n]$, $G_2[p^n]$, ... , $p_2^\star A[p^n]$ by the procedure we now describe. Let G, G' be Barsotti-Tate group schemes of height $2g$ over a scheme of characteristic 0. Let $\phi : G \to G'$ be an isogeny, whose kernel $L \subseteq G[p]$ has rank p^s. Let $H_{g+i} \subseteq G[p^n]$ be a finite flat group scheme of rank p^{in}. We assume that locally in the étale topology, $H_{g+i} \simeq (\mathbb{Z}/p^n\mathbb{Z})^i$. We assume that H_{g+i} and L are in generic position. This means that $H_{g+i}\cap L$ is a group scheme of rank $p^{\sup\{0,i+s-2g\}}$. We define $H'_{g+i} \subseteq G'$ as follows:

(1) If $i+s \leq 2g$, we let $H'_{g+i} = \phi(H_{g+i})$.
(2) If $i+s > 2g$, we let $H'_{g+i} = \phi(p^{-1}H_{g+i}\cap p^{-n}L)$.

We now exhibit an isomorphism: $Gr_{g+i} \to Gr'_{g+i}$. If $i+s \leq 2g$, this map is induced by the isomorphism $H_{g+i} \to H'_{g+i}$. If $i+s > 2g$, we give a construction.

The multiplication by p gives a map $p^{-1}H_{g+i}\cap p^{-n}L \to H_{g+i}$. We let H''_{g+i} be its image. The map $H''_{g+i} \to H_{g+i}$ induces an isomorphism $H''_{g+i}/(H''_{g+i}\cap H_{g+i-1}) \to Gr_{g+i}$.

The map $\phi : p^{-1}H_{g+i}\cap p^{-n}L \to H'_{g+i} \to Gr'_{g+i}$ factors over multiplication by p and induces a map: $H''_{g+i} \to Gr'_{g+i}$, factorizing into an isomorphism $H''_{g+i}/(H''_{g+i}\cap H_{g+i-1}) \to Gr'_{g+i}$. In summary, we have a diagram:

$$\begin{array}{ccc} H''_{g+i}/(H''_{g+i}\cap H_{g+i-1}) & \longrightarrow & Gr_{g+i} \\ \downarrow & & \\ Gr'_{g+i} & & \end{array}$$

where both maps are isomorphisms.

4.2.7. *Dynamics: preliminaries.* Let V be a complete valuation ring for a rank 1 valuation v extending the p-adic valuation (we normalize v by $v(p)=1$). We let K be its fraction field. Let H be a finite flat group scheme over Spec V. We recall the definition of $\deg H$, following [Far10], section 2. We let $\delta_H = \det(\ell_H)$, with ℓ_H the co-Lie complex of H. We have a canonical section $V \to \delta_H$. Since $\delta_H \simeq V$, the canonical section is given by an element $x \in V$, well defined up to an element in $V^\times$. We let $\deg\ H = v(x)$.

Let G, G' be Barsotti-Tate group schemes of height h over Spec V. Let $\phi : G \to G'$ be an isogeny, whose kernel $L \subseteq G[p]$ has rank p^s. Let $H \subseteq G[p^n]$ be a finite flat group scheme of rank p^{in}. We assume that locally in the étale topology, $H_K \simeq (\mathbb{Z}/p^n\mathbb{Z})^i$. We assume that H and L are in generic position. This means that $H_K \cap L_K$ is a group scheme of rank $p^{\sup\{0,i+s-h\}}$. We define $H' \subseteq G'$ as follows:

(1) If $i+s \leq h$, we let H' be the schematic closure of $\phi(H_K)$.
(2) If $i+s \geq h$, we let H' be the schematic closure of $\phi(p^{-1}H_K \cap p^{-n}L_K)$.

Proposition 4.2.8. *We have* $\deg H' \geq \deg H$*. If* $i+s=h$ *and* $\deg H = \deg H'$*, then* L *is a* BT_1 *and* H *is a* BT_n*.*

Proof. If $i+s \leq h$, then $H \to H'$ is a generic isomorphism, so $\deg H' \geq \deg H$ by [Far10], section 3, corollaire 3. Assume that $i+s=h$, and $\deg H' = \deg H$, so that $H \simeq H'$. Let $H+L$ be the subgroup of G equal to the schematic closure of $H_K + L_K$. We have an exact sequence $0 \to L_K \to (H+L)_K \to H'_K \to 0$. We deduce that $\deg(H+L) \leq \deg L + \deg H'$ by [Far10], section 3, corollaire 3. We also have a generic isomorphism $H \times L \to H+L$. We deduce that $\deg H + L \geq \deg L + \deg H$. It follows that $H \times L \to H+L$ is an isomorphism. We deduce that $H[p] \cap L = \{0\}$, and therefore $H[p] \oplus L = G[p]$. It follows that $H[p]$ is a BT_1. By induction on l, we deduce that $H[p^l]/H[p^{l-1}] \times L = (G/H[p^{l-1}])[p]$. So it follows that $H[p^l]/H[p^{l-1}]$ is a BT_1 of degree $\deg H[p] = \dim G - \deg L$. We deduce that H is a BT_n. Assume that $i+s \geq h$. Let $p^{-1}H \cap p^{-n}L$ be the schematic closure of $p^{-1}H_K \cap p^{-n}L_K$. We have an exact sequence

$$0 \to p^{-1}H_K \cap p^{-n}L_K \to p^{-1}H_K \times p^{-n}L_K \to G[p^{n+1}]_K \to 0.$$

We deduce that $\deg p^{-1}H + \deg p^{-n}L_K \leq \deg p^{-1}H \cap p^{-n}L + \deg G[p^{n+1}]$. We have exact sequences: $0 \to H[p] \to p^{-1}H \to H \to 0$, as well as $0 \to G[p^n] \to p^{-n}L \to L \to 0$, so that $\deg p^{-1}H = \deg H + \deg G[p]$, $\deg p^{-n}L = \deg L + \deg G[p^n]$. We finally deduce that $\deg H' = \deg p^{-1}H \cap p^{-n}L - \deg L \geq \deg H$. □

4.2.9. *Dynamic of Hecke correspondences.* We recall that $Z_i = \sum_{k=1}^{i} D_{g+k}$.

Corollary 4.2.10. *Over* C_t^{tor}*, we have:* $p_2^\star Z_i \geq p_1^\star Z_i$ *for all* $1 \leq i \leq g$*. Moreover, if* $t \in T^{++}(\mathbb{Q}_p)$*, the support of* $p_2^\star Z_i - p_1^\star Z_i$ *contains* $p_1^\star \cap_{j=0}^{i} (Z_{i<j} \cup Z_{i>j})$ *and* $p_2^\star \cap_{j=0}^{i} (Z_{i<j} \cup Z_{i>j})$*.*

Proof. By normality, it suffices to check this in codimension 1. Let $V = \mathscr{O}_{C_t^{tor},x}$ where x is a generic point of the special fiber (indeed, both divisors are supported on the special fiber). The isogeny $p_1^\star A \to p_2^\star A$ can be written as a composition of isogenies whose kernel is a finite flat group scheme killed by p (see section 4.2.1). We may therefore apply proposition 4.2.8 and deduce that $\deg(p_2^\star H_{g+i,x}) \geq \deg(p_1^\star H_{g+i,x})$. Moreover, in case $t \in T^{++}(\mathbb{Q}_p)$ and we have equality, the groups $p_1^\star H_{g+i,x}$ and $p_1^\star H_{g+i,x}$ are BT_n's and their degree is a multiple of n. □

Lemma 4.2.11. *We have* $p_1^\star Z_{i,>j} \leq p_2^\star Z_{i,>j}$ *and* $p_2^\star Z_{i,<j} \leq p_1^\star Z_{i,<j}$*.*

Proof. We have $p_1^\star \sum_{1\leq k\leq i} D_{g+k} \subseteq p_2^\star \sum_{1\leq k\leq i} D_{g+k}$ and $p_1^\star V(p^{nj}) = p_2^\star V(p^{nj})$. We deduce that $p_1^\star \sum_{1\leq k\leq i} D_{g+k} - V(p^{nj}) \subseteq p_2^\star \sum_{1\leq k\leq i} D_{g+k} - V(p^{nj})$. Since $\sum_{1\leq k\leq i} D_{g+k} - V(p^{nj}) = Z_{i,>j} - Z_{i,<j}$ and they have disjoint support (see proposition 4.1.13). Taking effective divisors on both sides yields the promised inequalities. □

Lemma 4.2.12. *For any* $w \in W$*, we let* $I_{w,1} = \{1, \cdots, g\} \cap w\{1, \cdots, g\}$ *and* $I_{w,2} = \{1, \cdots, g\} \cap w\{g+1, \cdots, 2g\}$*. We have:*

$$p_1^\star \sum_{i\in I_{w,1}} D_{w^{-1}(i)} + p_2^\star \sum_{i\in I_{w,2}} D_{w^{-1}(i)} \geq p_2^\star \sum_{i\in I_{w,1}} D_{w^{-1}(i)} + p_1^\star \sum_{i\in I_{w,2}} D_{w^{-1}(i)}$$

Proof. Granting that $D_i + D_{2g+1-i} = (p^n)$, this is just another way of writing the identity $p_2^\star \sum_{i=1}^k D_{g+i} \geq p_1^\star \sum_{i=1}^k D_{g+i}$. □

Lemma 4.2.13. *We have that* $\mathfrak{C}_{t,w}^{tor} = \mathfrak{C}_t^{tor} \setminus \sum_{i \in I_{w,1}} p_1^\star D_{w^{-1}(i)} + \sum_{i \in I_{w,2}} p_2^\star D_{w^{-1}(i)}$.

Proof. By definition, $\mathfrak{C}_{t,w}^{tor} = \mathfrak{C}_t^{tor} \setminus \sum_{i=1}^g p_1^\star D_{w^{-1}(i)} + \sum_{i=1}^g p_2^\star D_{w^{-1}(i)}$. We can use the last lemma. □

The following proposition illustrates that the correspondence is extremely well behaved over $\cup_{w \in {}^M W} \mathfrak{C}_{t,w}^{tor}$.

Proposition 4.2.14.
(1) *The open formal scheme equal to the complement of* $\sum_{i=1}^g p_1^\star D_i \cap p_2^\star D_{2g+1-i}$ *is* $\cup_{w \in {}^M W} \mathfrak{C}_{t,w}^{tor}$.
(2) *Over* $\cup_{w \in {}^M W} \mathfrak{C}_{t,w}^{tor}$ *we have* $p_2^\star Z_i = p_1^\star Z_i$ *for all* $1 \leq i \leq g$.
(3) *Over* $\cup_{w \in {}^M W} \mathfrak{C}_{t,w}^{tor}$ *there are natural isomorphisms* $p_1^\star Gr_i \to p_2^\star Gr_i$ *for all* $1 \leq i \leq 2g$.

Proof. By lemma 4.2.13, we see that $\cup_{w \in {}^M W} \mathfrak{C}_{t,w}^{tor}$ is included in the complement of $\sum_{i=1}^g p_1^\star D_i \cap p_2^\star D_{2g+1-i}$. Let us denote by U the complement of $\sum_{i=1}^g p_1^\star D_i \cap p_2^\star D_{2g+1-i}$. Let $x \in U$. We first see that either $x \notin p_1^\star D_g$ or $x \notin p_2^\star D_{g+1}$. In the first case, $p_1^\star H_{g+1}$ is multiplicative, in the second case $p_2^\star H_{g+1}$ is étale. We consider a discrete valuation ring V of mixed characteristic and a map $\tilde{x} : \mathrm{Spec}\, V \to U$ mapping the special point of Spec V to x. We have $n \geq \deg p_2^\star H_{g+1,x} \geq \deg p_1^\star H_{g+1,x} \geq 0$. If $p_2^\star H_{g+1,x}$ is étale then $\deg p_2^\star H_{g+1,x} = \deg p_1^\star H_{g+1,x} = 0$ and all groups are étale. If $p_1^\star H_{g+1,x}$ is multiplicative then $\deg p_2^\star H_{g+1,x} = \deg p_1^\star H_{g+1,x} = n$ and all groups are multiplicative. Proceeding by induction in the same way, we prove that $p_2^\star Gr_{g+i,x}$ and $p_1^\star Gr_{g+i,x}$ are either both étale or both multiplicative. It remains to see that we have a natural isomorphism between both groups. In order to see this we go back to the construction of section 4.2.5 and one sees that all the construction give integral isomorphisms since the groups involved are extensions of étale and multiplicative groups. □

The following proposition shows in particular that if $t \in T^{++}(\mathbb{Q}_p)$, then the correspondence is well behaved exactly over $\cup_{w \in {}^M W} \mathfrak{C}_{t,w}^{tor}$.

Proposition 4.2.15. *Let* $t \in T^{++}(\mathbb{Q}_p)$ *and let* $1 \leq k \leq g$.

(1) *The complement of* $p_2^\star \sum_{i=1}^k Z_i - p_1^\star \sum_{i=1}^k Z_i$ *is the locus where* $p_1^\star Gr_{g+i}$ *and* $p_2^\star Gr_{g+i}$ *are multiplicative or étale BT's of the same type for all* $1 \leq i \leq k$.
(2) *On the complement of* $p_2^\star \sum_{i=1}^k Z_i - p_1^\star \sum_{i=1}^k Z_i$, *we have a natural isomorphism* $p_1^\star Gr_{g+i} \to p_2^\star Gr_{g+i}$.
(3) *The support of* $p_2^\star \sum_{i=1}^k Z_i - p_1^\star \sum_{i=1}^k Z_i$ *is equal to the support of* $\sum_{i=1}^k p_1^\star D_{g-i+1} \cap p_2^\star D_{g+i}$.

Proof. On the locus where $p_1^\star Gr_{g+i}$ and $p_2^\star Gr_{g+i}$ are multiplicative or étale BT's of the same type for all $1 \leq i \leq k$ we have $p_2^\star D_{g+i} = p_1^\star D_{g+i}$. Conversely, let us consider the open subscheme where $p_2^\star \sum_{i=1}^k (k+1-i) D_{g+i} = p_1^\star \sum_{i=1}^k (k+1-i) D_{g+i}$, or what amounts to the same: $p_2^\star \sum_{i=1}^{k'} D_{g+i} = p_1^\star \sum_{i=1}^{k'} D_{g+i}$ for all $k' \leq k$. We consider a valuation ring V of mixed characteristic and a map $x : \mathrm{Spec}\, V \to C_t^{tor}$ mapping to this open subscheme. We therefore assume that for all $g+1 \leq k' \leq k$, $\deg(p_2^\star H_{k',x}) = \deg(p_1^\star H_{k',x})$. We deduce from proposition 4.2.8 that $p_1^\star H_{k',x}$ is a BT_n. Since this is true for all k', we deduce that $p_1^\star Gr_{k',x} \simeq p_2^\star Gr_{k',x}$ is a multiplicative or an étale BT_n for all k'. This proves the first point. The second point follows from the construction of section 4.2.5. Regarding the last point, we check that if $p_1^\star Gr_{g+i}$ and $p_2^\star Gr_{g+i}$ are multiplicative or étale BT's of the same type for all $1 \leq i < k'$ then $p_1^\star Gr_{g+k'}$ and $p_2^\star Gr_{g+k'}$ are not multiplicative or étale BT's of the same type if and only if $p_1^\star Gr_{g+k'}$ is not multiplicative and $p_2^\star Gr_{g+k'}$ is not étale. □

4.3. **The cohomological correspondence.**

4.3.1. *Automorphic sheaves.* Over S^{tor}, we can attach to the semi-abelian scheme A a M-torsor M_{dR}. We consider the relative logarithmic de Rham homology $\mathrm{H}_{1,dR}(A)$, and its filtration $0 \to \omega_{A^t} \to \mathrm{H}_{1,dR}(A) \to \mathrm{Lie}(A) \to 0$. The torsor $M_{dR} \to S^{tor}$ parametrizes triples (ψ_1, ψ_2, c), where $\psi_1 : \mathscr{O}^g_{S^{tor}} \to \mathrm{Lie}_A$, $\psi_2 : \mathscr{O}^g_{S^{tor}} \to \omega(A^t)$, $c \in \mathscr{O}^\times_{S^{tor}}$ is such that $\psi_2 = c(\psi_1^t)^{-1}$. Let $\mathrm{Rep}(M)$ be the category of representations of M over $\mathbb{Z}_p$-modules. We therefore get a functor $\mathrm{Rep}(M) \to Coh(S^{tor})$. Let us try to make this functor explicit.

Let $\kappa \in X^\star(T)^{M,+}$. We can attach to κ the following representation V_κ of M over $\mathbb{Z}_p$, which is a model of the highest weight κ representation. We let V_κ be the space of functions $f : M \to \mathbb{A}^1$ such that $f(mb) = w_{0,M}\kappa(b^{-1})f(m)$, $\forall (m, b) \in M \times B \cap M$, equipped with the action $m'.f(m) = f(m^{-1}m')$.

To the representation V_κ, we attach the sheaf $\omega^\kappa = \pi_\star \mathscr{O}_{M_{dR}}[-w_{0,M}\kappa]$, where $\pi : M_{dR} \to S^{tor}$ is the projection and $[-w_{0,M}\kappa]$ stands for the isotypic part for the action of $B \cap M$.

Remark 4.3.2. The torsor M_{dR} depends on our choice of A. By definition of the moduli problem, we could have considered another isogenous semi-abelian scheme (any A_i for example). We would have obtained another natural torsor, identified with M_{dR} over $\mathbb{Q}_p$. The sheaf ω^κ depends on the choice of A, and also on the choice of the model V_κ.

Remark 4.3.3. We normalize the construction in such a way that the standard representation of M, which has highest weight $((0, \cdots, 0, -1); 1)$, corresponds to $\mathrm{Lie}(A)$. Therefore, ω_A corresponds to $\kappa = ((1, 0, \cdots, 0); -1)$.

We have an isomorphism $p_2^\star\omega^\kappa[1/p] \to p_1^\star\omega^\kappa[1/p]$ over $C_{t,\mathbb{Q}_p}$ induced by the isomorphism of torsors $p_1^\star M_{dR} = p_2^\star M_{dR}$ over $C_{t,\mathbb{Q}_p}$.

Lemma 4.3.4. *Over $\mathfrak{C}^{tor}_{t,w}$, the map $p_2^\star\omega^\kappa[1/p] \to p_1^\star\omega^\kappa[1/p]$ has the property that $p_2^\star\omega^\kappa \subseteq \langle w_{0,M}\kappa, w(t)\rangle p_1^\star\omega^\kappa$.*

Proof. We have a natural isomorphism $p_1^\star T_p(A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to p_2^\star T_p(A) \otimes \mathbb{Q}_p$, induced by the isogeny $p_1^\star A \to p_2^\star A$. We let x_1 be a local section of $p_1^\star(T_p(A)^m \oplus T_p(A)^{et})$ (compatible with the $K_{p,P,w}$-level structure), then $x_2 = x_1 \circ w(t)$ is a local section of $p_2^\star(T_p(A)^m \oplus T_p(A)^{et})$ by proposition 4.2.4. Both x_1 and x_2 induce sections of M_{dR}, via the Hodge-Tate map, $T_p(A)^m \to \mathrm{Lie}(A)$ and $T_p(A)^{et} \to \omega_{A^t}$. There is an isomorphism $p_2^\star\omega^\kappa[1/p] \to p_1^\star\omega^\kappa[1/p]$ mapping a function $f(x_2 \circ m)$ to the function $f(x_1 w(t) \circ m) = w(t)^{-1}f(x_1 \circ m)$. The lattice $p_2^\star\omega^\kappa$ is given by sections $g(x_2 \circ m)$ taking integral values. The lattice $p_1^\star\omega^\kappa$ is given by sections $g(x_1 \circ m)$ taking integral values. We deduce that $w(t)^{-1}f(x_1 \circ m)$ takes integral values. As the eigenvalue with greater p-adic valuation of $w(t)^{-1}$ acting on the representation V_κ is $\langle -w_{0,M}\kappa, w(t)\rangle$, we deduce that $f(x_1 \circ m) \in \langle w_{0,M}\kappa, w(t)\rangle p_1^\star\omega^\kappa$. □

4.3.5. *Nebentypus.* Let $K_{p,1,n} \subseteq K_{p,n}$ be the subgroup of elements congruent to $U(\mathbb{Z}_p)$ modulo p^n. We have an isomorphism $K_{p,n}/K_{p,1,n} = T(\mathbb{Z}/p^n\mathbb{Z})$. We now work over $\mathbb{Z}_p(\zeta_{p^n})$. We have a map $S^{tor}_{K^pK_{p,1,n},\mathbb{Q}_p(\zeta_{p^n})} \to S^{tor}_{K^pK_{p,n},\mathbb{Q}_p(\zeta_{p^n})}$. We define an integral model $S^{tor}_{K^pK_{p,1,n},\mathbb{Z}_p(\zeta_{p^n})} \to S^{tor}_{\mathbb{Z}_p(\zeta_{p^n})}$ by normalization.

Lemma 4.3.6. *The restriction of the map $\pi : S^{tor}_{K^pK_{p,1,n},\mathbb{Z}_p(\zeta_{p^n})} \to S^{tor}_{\mathbb{Z}_p(\zeta_{p^n})}$ to $S^{tor}_{\mathbb{Z}_p(\zeta_{p^n})} \setminus \sum_{i=1}^g D_i \cap D_{2g+1-i}$ is a finite étale map with Galois group $T(\mathbb{Z}/p^n\mathbb{Z})$. Moreover,*

$$S^{tor}_{K^pK_{p,1,n},\mathbb{Z}_p(\zeta_{p^n})} \times_{S^{tor}_{\mathbb{Z}_p(\zeta_{p^n})}} \mathfrak{IG}^{tor}_{K_{p,P,w}K^p,\mathbb{Z}_p(\zeta_{p^n})} = \mathfrak{IG}^{tor}_{K_{p,P,1,w}K^p,\mathbb{Z}_p(\zeta_{p^n})}$$

where $K_{p,P,1,w} = wK_{p,1,n}w^{-1} \cap P(\mathbb{Q}_p)$.

Proof. We first argue that the group schemes Gr_i extend to finite flat group schemes over $S^{tor}_{\mathbb{Z}_p(\zeta_{p^n})} \setminus \sum_{i=1}^d D_i \cap D_{2g+1-i}$. Indeed, over $S^{tor}_{\mathbb{Z}_p(\zeta_{p^n})} \setminus \sum_{i=1}^d D_{w^{-1}(i)}$, the groups $\{Gr_{w^{-1}(g+l)}\}_{1\leq l\leq g}$ are multiplicative, thus finite flat. Since $Gr_{w^{-1}(l)} = Gr^D_{w^{-1}(2g+1-l)}$ for $1 \leq l \leq g$, it is also finite flat. We recall that integral closure commutes with étale localization ([Sta22], TAG 03GE). We take an étale map $U \to S^{tor}_{\mathbb{Z}_p(\zeta_{p^n})} \setminus \sum_{i=1}^d D_{w^{-1}(i)}$ such that $Gr_{w^{-1}(l)}|_U$ is isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$ and $Gr_{w^{-1}(g+l)}$ is isomorphic to μ_{p^n}. It is clear that

$$S^{tor}_{K^pK_{p,1,n},\mathbb{Q}_p(\zeta_{p^n})} \times_{S^{tor}_{\mathbb{Z}_p(\zeta_{p^n})}} U_{\mathbb{Q}_p} = T(\mathbb{Z}/p^n\mathbb{Z}) \times U_{\mathbb{Q}_p}.$$

We deduce that

$$S^{tor}_{K^pK_{p,1,n},\mathbb{Z}_p(\zeta_{p^n})} \times_{S^{tor}_{\mathbb{Z}_p(\zeta_{p^n})}} U = T(\mathbb{Z}/p^n\mathbb{Z}) \times_{\mathrm{Spec}\ \mathbb{Z}_p(\zeta_{p^n})} U.$$

□

Let $\chi : T(\mathbb{Z}/p^n\mathbb{Z}) \to \mathbb{Z}_p(\zeta_{p^n})^\times$ be a character. We let $\mathscr{O}(\chi)$ be the subsheaf of

$$\pi_\star \mathscr{O}_{S^{tor}_{K^pK_{p,1,n},\mathbb{Z}_p(\zeta_{p^n})}}$$

of sections s which satisfy $ts = \chi(t)s$ for any $t \in T(\mathbb{Z}/p^n\mathbb{Z})$. This is a coherent sheaf and an invertible sheaf over $S^{tor}_{\mathbb{Z}_p(\zeta_{p^n})} \setminus \sum_{i=1}^d D_i \cap D_{2g+1-i}$. We let $\omega^\kappa(\chi) = \omega^\kappa \otimes \mathscr{O}(\chi)$.

4.3.7. *Model of the Hecke correspondence.* Let $w \in {}^MW$. Let $t \in T^+(\mathbb{Q}_p)$. Consider the correspondence:

$$\begin{array}{ccc} & \mathfrak{C}^{tor}_{t,w} & \\ \swarrow & & \searrow \\ \mathfrak{S}^{tor}_w & & \mathfrak{S}^{tor}_w \end{array}$$

Lemma 4.3.8. *The map* $\mathrm{Tr}_{p_1} : (p_1)_\star \mathscr{O}_{\mathfrak{C}^{tor}_{t,w}} \to \mathscr{O}_{\mathfrak{S}^{tor}_w}$ *factors through* $\langle w^{-1}w_{0,M}\rho + \rho, t\rangle \mathscr{O}_{\mathfrak{S}^{tor}_w}$.

Proof. We apply proposition 3.4.9 as in remark 3.4.11. The unipotent part of $w(t)K_{p,P,w}w(t)^{-1} \cap K_{p,P,w}$ is

$$\prod_{\alpha \in \Phi^{+,M}} p^{\sup\{0,\alpha(w(v(t)))\}} K_p\alpha.$$

It is a simple exercise to see that $w^{-1}w_{0,M}\rho + \rho = \sum_{\alpha \in (w^{-1}\Phi^{+,M}) \cap \Phi^+} \alpha$ so that the trace map factors through $\langle w^{-1}w_{0,M}\rho + \rho, t\rangle \mathscr{O}_{\mathfrak{S}^{tor}_w}$. □

Let $\kappa \in X^\star(T)^{M,+}$.

Proposition 4.3.9. *We have a natural cohomological correspondence over* $\mathfrak{C}^{tor}_{t,w}$*:* $t_w : p_2^\star\omega^\kappa \to p_1^!\omega^\kappa$ *which is* $\langle -w^{-1}w_{0,M}(\kappa+\rho) - \rho, t\rangle t_w^{naive}$ *where* t_w^{naive} *is the rational map obtained as the tensor product of* $p_2^\star\omega^\kappa[1/p] \to p_1^\star\omega^\kappa[1/p]$ *and the fundamental class:* $p_2^\star\mathscr{O}_{\mathfrak{S}^{tor}_w} \to p_1^!\mathscr{O}_{\mathfrak{S}^{tor}_w}$ *given by the trace of* p_1*.*

Proof. This follows from lemma 4.3.4 and lemma 4.3.8. □

Remark 4.3.10. The normalization of the map depends on $t \in T^+(\mathbb{Q}_p)$ and not just on its class modulo $T(\mathbb{Z}_p)$.

We let $C(\kappa) = \{w \in {}^MW, w^{-1}w_{0,M}(\kappa+\rho) \in X^\star(T)^-_{\mathbb{Q}}\}$. As $X^\star(T)^-_{\mathbb{Q}}$ is a fundamental domain for the action of W on $X^\star(T)_{\mathbb{Q}}$, $C(\kappa)$ is nonempty and $w^{-1}w_{0,M}(\kappa+\rho)$ is independent of $w \in C(\kappa)$.

Proposition 4.3.11. *We have a cohomological correspondence* $t : p_2^\star\omega^\kappa \to p_1^!\omega^\kappa$ *over* $C_t^{tor} \setminus \sum_{i=1}^g p_1^\star D_i \cap p_2^\star D_{2g+1-i}$ *which is* $p^{-\langle w^{-1}w_{0,M}(\kappa+\rho)-\rho,t\rangle} t^{naive}$ *for any* $w \in C(\kappa)$*, where* t^{naive} *is the rational map obtained as the tensor product of* $p_2^\star\omega^\kappa[1/p] \to p_1^\star\omega^\kappa[1/p]$ *and the fundamental class given by the map (induced from the trace):*

$$p_2^\star\mathscr{O}_{S^{tor}\setminus\sum_{i=1}^g D_i\cap D_{2g+i-1}} \to p_1^!\mathscr{O}_{S^{tor}\setminus\sum_{i=1}^g D_i\cap D_{2g+i-1}}.$$

Proof. We first observe that $p_1 : C_t^{tor} \setminus \sum_{i=1}^g p_1^\star D_i \cap p_2^\star D_{2g+1-i} \to S^{tor} \setminus \sum_{i=1}^g D_i \cap D_{2g+i-1}$ is an lci map, as both the source and target are smooth over Spec $\mathbb{Z}_p$. We deduce that $p_1^!\mathscr{O}_{S^{tor}\setminus\sum_{i=1}^g D_i\cap D_{2g+i-1}}$ is an invertible sheaf and we have a fundamental class $p_2^\star\mathscr{O}_{S^{tor}\setminus\sum_{i=1}^g D_i\cap D_{2g+i-1}} \to p_1^!\mathscr{O}_{S^{tor}\setminus\sum_{i=1}^g D_i\cap D_{2g+i-1}}$ (see [FP21], prop. 2.6). We therefore get a rational map t^{naive} of locally free sheaves t^{naive} : $p_2^\star\omega^\kappa \dashrightarrow p_1^!\omega^\kappa$. We claim that t is a true map. It suffices to prove this at each generic point of the special fiber, and this follows from proposition 4.3.9. □

Remark 4.3.12. The restriction of t to $\mathfrak{C}^{tor}_{t,w}$ is t_w for all $w \in C(\kappa)$. More generally, for all $w' \in {}^MW$, it restricts to $p^{n_{w'}}t_{w'}$, where $n_{w'} = \langle (w')^{-1}(w_{0,M}(\kappa+\rho)+\rho, t\rangle - \langle w^{-1}(w_{0,M}(\kappa+\rho)+\rho, t\rangle$. If $t \in T^{++}(\mathbb{Q}_p)$, $n_{w'} > 0$ for $w' \notin C(\kappa)$.

Proposition 4.3.13. *Assume that $t \in T^{++}(\mathbb{Q}_p)$. There exists $k \in \mathbb{Z}_{\geq 0}$ (depending on κ) such that t can be extended to a cohomological correspondence $\tilde{t} : p_2^\star\omega^\kappa(-k\sum_{i=1}^g Z_i) \to p_1^!\omega^\kappa(-k\sum_{i=1}^g Z_i)$ over C^{tor}.*

Proof. Let us denote by $\mathscr{I}$ the ideal corresponding to the effective Cartier divisor $p_2^\star\sum_{i=1}^g Z_i - p_1^\star\sum_{i=1}^g Z_i$. By proposition 4.3.11, we have a map $p_2^\star\omega^\kappa \to \mathrm{colim}_k \mathscr{I}^{-k}p_1^!\omega^\kappa$. Since $p_2^\star\omega^\kappa$ is pseudo-coherent, there exists k such that this map factors through a map $p_2^\star\omega^\kappa \to \mathscr{I}^{-k}p_1^!\omega^\kappa$. □

From now on, we fix $t \in T^{++}(\mathbb{Q}_p)$ and we let $\tilde{\omega}^\kappa = \omega^\kappa(-k'\sum_{i=1}^g Z_i)$ for $k' \geq k$, where k is given by proposition 4.3.13 and k' is taken to be large enough (how large we need to take k' will be explained in the proof of proposition 4.5.8).

Lemma 4.3.14. *We have a factorization:*

$$\tilde{t} : p_2^\star\tilde{\omega}^\kappa \to \mathscr{O}_{C_t^{tor}}((k'-k)(\sum_{i=1}^g p_2^\star Z_i - \sum_{i=1}^g p_1^\star Z_i)) \otimes p_2^\star\tilde{\omega}^\kappa \to p_1^!\tilde{\omega}^\kappa.$$

Proof. We consider the sequence of maps

$$\mathscr{O}_{C_t^{tor}}((k'-k)(\sum_{i=1}^g p_2^\star Z_i - \sum_{i=1}^g p_1^\star Z_i)) \otimes p_2^\star\tilde{\omega}^\kappa \to p_2^\star\omega^\kappa(-k\sum_{i=1}^g Z_i) \otimes p_1^\star\mathscr{O}_{S^{tor}}((k-k')\sum_{i=1}^g Z_i) \to$$
$$p_1^!\omega^\kappa(-k\sum_{i=1}^g Z_i) \otimes p_1^\star\mathscr{O}_{S^{tor}}((k-k')\sum_{i=1}^g Z_i) \to p_1^!\tilde{\omega}^\kappa.$$

□

4.3.15. *Nebentypus.* It is also possible to consider a Nebentypus. Let $\chi : T(\mathbb{Z}/p^n\mathbb{Z}) \to \mathbb{Z}_p(\zeta_{p^n})^\times$.

Lemma 4.3.16. *Over $C^{tor}_{t,\mathbb{Z}_p(\zeta_{p^n})} \setminus \sum_{i=1}^d p_1^\star D_i \cap p_2^\star D_{2g+1-i}$ we have canonical isomorphisms $p_1^\star Gr_i = p_2^\star Gr_i$, inducing an isomorphism $p_2^\star\mathscr{O}(\chi) \to p_1^\star\mathscr{O}(\chi)$.*

Proof. This follows from proposition 4.2.14. □

Proposition 4.3.17. *Over $\mathfrak{C}^{tor}_{t,w,\mathbb{Z}_p(\zeta_{p^n})}$, we have a natural cohomological correspondence $t_w : p_2^\star\omega^\kappa(\chi) \to p_1^!\omega^\kappa(\chi)$ which is $\langle -w^{-1}(w_{0,M}(\kappa+\rho)-\rho, t\rangle t_w^{naive}$ where t_w^{naive} is the rational map obtained as the tensor product of $p_2^\star\omega^\kappa[1/p] \to p_1^\star\omega^\kappa[1/p]$, of $p_2^\star\mathscr{O}(\chi) \to p_1^\star\mathscr{O}(\chi)$, and the fundamental class: $p_2^\star\mathscr{O}_{\mathfrak{S}^{tor}_{w,\mathbb{Z}_p(\zeta_{p^n})}} \to p_1^!\mathscr{O}_{\mathfrak{S}^{tor}_{w,\mathbb{Z}_p(\zeta_{p^n})}}$ given by the trace of p_1.*

Proof. This is analogue to proposition 4.3.9. □

We define a cohomological correspondence: $t : p_2^\star\omega^\kappa(\chi) \to p_1^!\omega^\kappa(\chi)$ over $C^{tor}_{t,\mathbb{Z}_p(\zeta_{p^n})} \setminus \sum_{i=1}^g p_1^\star D_i \cap p_2^\star D_{2g+1-i}$ which is $\langle -w^{-1}w_{0,M}(\kappa+\rho)-\rho, t\rangle t^{naive}$ for any $w \in C(\kappa)$, where t^{naive} is the rational map obtained as the tensor product of $p_2^\star\omega^\kappa(\chi)[1/p] \to p_1^\star\omega^\kappa(\chi)[1/p]$, the isomorphism $p_2^\star\mathscr{O}(\chi) \to p_1^\star\mathscr{O}(\chi)$ and the fundamental class given by the map (induced from the trace):

$$p_2^\star\mathscr{O}_{S^{tor}_{\mathbb{Z}_p(\zeta_{p^n})}\setminus\sum_{i=1}^g D_i\cap D_{2g+i-1}} \to p_1^!\mathscr{O}_{S^{tor}_{\mathbb{Z}_p(\zeta_{p^n})}\setminus\sum_{i=1}^g D_i\cap D_{2g+i-1}}.$$

Proposition 4.3.18. *Assume that $t \in T^{++}(\mathbb{Q}_p)$. There exists $k \in \mathbb{Z}_{\geq 0}$ (depending on κ and χ) such that t can be extended to a cohomological correspondence over $C^{tor}_{t,\mathbb{Z}_p(\zeta_{p^n})}$:*

$$\tilde{t} : p_2^\star\omega^\kappa(\chi)(-k\sum_{i=1}^g Z_i) \to p_1^!\omega^\kappa(\chi)(-k\sum_{i=1}^g Z_i).$$

Proof. This is analogue to proposition 4.3.13. □

For simplicity, we let $\tilde{\omega}^\kappa(\chi) = \omega^\kappa(\chi)(-k'\sum_{i=1}^g Z_i)$ for some $k' \geq k$ where k is given by the proposition.

Remark 4.3.19. We remark that the existence of normalized cohomological correspondences in propositions 4.3.13 and 4.3.18 imply that the normalized Hecke operators for $t \in T^{++}(\mathbb{Q}_p)$ preserve a lattice in rational coherent cohomology, and so the eigenvalues are p-adically integral. This proves conjecture 5.10.7 of [BP21] in the Siegel case.

4.4. **Higher Hida cohomology groups in weight** κ**.** Let $w \in {}^M W$. We construct higher Hida cohomology groups labeled by w.

4.4.1. *A strict dynamical correspondence.* Let $t \in T^{++}(\mathbb{Q}_p)$. We study the correspondence $(\mathfrak{C}^{tor}_{t,w}, \mathfrak{S}^{tor}_w)$. We let $S^{tor}_{\mathbb{F}_p,(w-1,w+1)} = \cap_i Z_{i,<w_i+1} \cap Z_{i,>w_i+1}$. We $D_{w,-} = \cup Z_{i,>w_i}$ and $D_{w,+} = \cup_i Z_{i,<w_i}$.

Lemma 4.4.2. *Let* $C^{tor}_{t,\mathbb{F}_p,(w-1,w+1)} = p_1^{-1} S^{tor}_{\mathbb{F}_p,(w-1,w+1)} \cap p_2^{-1} S^{tor}_{\mathbb{F}_p,(w-1,w+1)}$*. Then*

$$(C^{tor}_{t,\mathbb{F}_p,(w-1,w+1)}, S^{tor}_{\mathbb{F}_p,(w-1,w+1)}, p_1, p_2, D_{w,+}, D_{w,-})$$

is a strict dynamic compactification of $(C^{tor}_{t,\mathbb{F}_p,w}, S^{tor}_{\mathbb{F}_p,w})$.

Proof. We first work on C_t. By lemma 4.2.11, we see that $p_2^\star Z_{i,>w_i} \geq p_1^\star Z_{i,>w_i}$ and $p_1^\star Z_{i,<w_i} \geq p_1^\star Z_{i,<w_i}$. By corollary 4.2.10, we find that the support of $p_2^\star(Z_{i,>w_i} - Z_{i,<w_i}) - p_1^\star(Z_{i,>w_i} - Z_{i,<w_i})$ contains $p_1^\star \cap_{j=0}^i (Z_{i<j} \cup Z_{i>j})$ and $p_2^\star \cap_{j=0}^i (Z_{i<j} \cup Z_{i>j})$. We now consider this property in $C^{tor}_{t,\mathbb{F}_p,(w-1,w+1)}$. We deduce that the support of $p_2^\star(Z_{i,>w_i} - Z_{i,<w_i}) - p_1^\star(Z_{i,>w_i} - Z_{i,<w_i})$ contains $p_1^\star Z_{i,>w_i}$ and $p_2^\star Z_{i,<w_i}$. In particular, the support of $p_1^\star Z_{i,<w_i} - p_2^\star Z_{i,<w_i} + p_2^\star Z_{i,>w_i}$ contains $p_1^\star Z_{i,>w_i}$.

Taking local generators f_+ and f_- of $Z_{i,<w_i}$ and $Z_{i,>w_i}$, we deduce that there exists an identity: $p_1^\star f_+ p_2^\star f_- = h p_2^\star f_+$ where h vanishes (at least) on $p_1^\star Z_{i,<w_i}$. Since $Z_{i,>w_i}$ and $Z_{i,<w_i}$ have disjoint support, we deduce that there exists an identity $af_+ + bf_- = 1$. It follows that:

$$\begin{aligned} p_1^\star f_+ &= p_1^\star f_+(p_2^\star a f_+ + p_2^\star b f_-) \\ &= (p_1^\star f_+ p_2^\star a + h p_2^\star b) p_2^\star f_+ \end{aligned}$$

The function $p_1^\star f_+ p_2^\star a + h p_2^\star b$ vanishes on $p_1^\star Z_{i,<w_i}$. We deduce that there exists $0 < s < 1$ such that $s p_1^\star Z_{i,<w_i} \geq p_2^\star Z_{i,<w_i}$. One argues similarly with $Z_{i,>w_i}$ and then sum over all i. □

We record the following corollary for future use:

Corollary 4.4.3. *Let* $i \in \{1,2\}$*. There exists* $m \in \mathbb{Z}_{\geq 0}$ *(depending on* w *and* i*) such that the canonical map* $\mathscr{O}_{C_t^{tor}} \to \mathscr{O}_{C_t^{tor}}(m(p_2^\star \sum_{i=1}^g Z_i - p_1^\star \sum_{i=1}^g Z_i))$ *can be factored over* $p_i^{-1} S^{tor}_{\mathbb{F}_p,(w-1,w+1)}$ *through maps:*

$$\mathscr{O}_{p_i^{-1} S^{tor}_{\mathbb{F}_p,(w-1,w+1)}} \to \mathscr{O}_{p_i^{-1} S^{tor}_{\mathbb{F}_p,(w-1,w+1)}}(p_1^\star D_{w,+} + p_2^\star D_{w,+}) \to \mathscr{O}_{p_i^{-1} S^{tor}_{\mathbb{F}_p,(w-1,w+1)}}(m(p_2^\star \sum_{i=1}^g Z_i - p_1^\star \sum_{i=1}^g Z_i)).$$

Proof. We construct the dual maps

$$\mathscr{O}_{p_i^{-1} S^{tor}_{\mathbb{F}_p,(w-1,w+1)}}(-m(p_2^\star \sum_{i=1}^g Z_i - p_1^\star \sum_{i=1}^g Z_i)) \to \mathscr{O}_{p_i^{-1} S^{tor}_{\mathbb{F}_p,(w-1,w+1)}}(-p_1^\star D_{w,+} - p_2^\star D_{w,+}) \to \mathscr{O}_{p_i^{-1} S^{tor}_{\mathbb{F}_p,(w-1,w+1)}}.$$

We first do some local computations. Let $\{U_i\}_i$ be a finite affine open cover of $p_i^{-1} S^{tor}_{\mathbb{F}_p,(w-1,w+1)}$ which trivializes $\mathscr{O}_{p_i^{-1} S^{tor}_{\mathbb{F}_p,(w-1,w+1)}}(-(p_2^\star \sum_{i=1}^g Z_i - p_1^\star \sum_{i=1}^g Z_i))$ and $\mathscr{O}_{p_i^{-1} S^{tor}_{\mathbb{F}_p,(w-1,w+1)}}(-p_1^\star D_{w,+} - p_2^\star D_{w,+})$. A generator X_i of $\mathscr{O}_{p_i^{-1} S^{tor}_{\mathbb{F}_p,(w-1,w+1)}}(-p_1^\star D_{w,+} - p_2^\star D_{w,+})(U_i)$ maps to $g_i \in \mathscr{O}_{p_i^{-1} S^{tor}_{\mathbb{F}_p,(w-1,w+1)}}(U_i)$ and a generator Y_i of $\mathscr{O}_{p_i^{-1} S^{tor}_{\mathbb{F}_p,(w-1,w+1)}}(-(p_2^\star \sum_{i=1}^g Z_i - p_1^\star \sum_{i=1}^g Z_i))$ maps to $f_i \in \mathscr{O}_{p_i^{-1} S^{tor}_{\mathbb{F}_p,(w-1,w+1)}}(U_i)$. Over $p_i^{-1} S^{tor}_{\mathbb{F}_p,(w-1,w+1)}$, $p_2^\star \sum_{i=1}^g Z_i - p_1^\star \sum_{i=1}^g Z_i$ and $p_1^\star D_{w,+} + p_2^\star D_{w,+}$ have the same support. Indeed, the complement of $p_1^\star D_{w,+} + p_2^\star D_{w,+}$ is $C^{tor}_{t,\mathbb{F}_p,w}$. On the other hand, the complement of $p_2^\star \sum_{i=1}^g Z_i - p_1^\star \sum_{i=1}^g Z_i$ in $C^{tor}_{t,\mathbb{F}_p}$ is $\cup_{w'} C^{tor}_{t,\mathbb{F}_p,w'}$ and $(\cup_{w'} C^{tor}_{t,\mathbb{F}_p,w'}) \cap p_i^{-1} S^{tor}_{\mathbb{F}_p,(w-1,w+1)} = C^{tor}_{t,\mathbb{F}_p,w}$. We deduce that there exists n such that $f_i^n = h_i g_i$ for some $h_i \in \mathscr{O}_{p_i^{-1} S^{tor}_{\mathbb{F}_p,(w-1,w+1)}}(U_i)$ and for all i. Let $U_{i,j} = U_i \cap U_j$. There exists units $a_{i,j}$ and $b_{i,j} \in \mathscr{O}_{p_i^{-1} S^{tor}_{\mathbb{F}_p,(w-1,w+1)}}(U_{i,j})^\times$ such that $X_i = b_{i,j} X_j$ and $Y_i^n = a_{i,j} Y_j^n$. We deduce that $f_i = a_{i,j} f_j$ and $g_i = b_{i,j} g_j$. It follows that

$g_j(h_j - b_{i,j}a_{i,j}^{-1}h_i) = 0$ in $\mathscr{O}_{p_i^{-1}S^{tor}_{\mathbb{F}_p,(w-1,w+1)}}(U_{i,j})$. We deduce that $f_i^{2n} = (h_i^2 g_i)g_i$. We observe that $g_jh_j^2 - b_{i,j}a_{i,j}^{-2}h_i^2 g_i = g_jh_j^2 - b_{i,j}^2 a_{i,j}^{-2}h_i^2 g_j = g_j(h_j - b_{i,j}a_{i,j}^{-1}h_i)(h_j + b_{i,j}a_{i,j}^{-1}h_i) = 0$. We now set $m = 2n$. It follows that the map defined locally by sending Y_i^m to $g_ih_i^2X_i$ glues to give a map $\mathscr{O}_{p_i^{-1}S^{tor}_{\mathbb{F}_p,(w-1,w+1)}}(-2m(p_2^\star \sum_{i=1}^g Z_i - p_1^\star \sum_{i=1}^g Z_i)) \to \mathscr{O}_{p_i^{-1}S^{tor}_{\mathbb{F}_p,(w-1,w+1)}}(-p_1^\star D_{w,+} - p_2^\star D_{w,+})$. □

Proposition 4.4.4. *For all $s \geq 0$, the correspondence $(S^{tor}_{w,\mathbb{Z}/p^s\mathbb{Z}}, C^{tor}_{t,w,\mathbb{Z}/p^s\mathbb{Z}})$ admits a strict dynamical compactification.*

Proof. We let $\bar{S}^{tor}_{w,\mathbb{Z}/p^s\mathbb{Z}}$ be the the closure of $S^{tor}_{w,\mathbb{Z}/p^s\mathbb{Z}}$ in $S^{tor}_{\mathbb{Z}/p^s\mathbb{Z}}$ and $\bar{C}^{tor}_{t,w,\mathbb{Z}/p^s\mathbb{Z}}$ be $p_1^\star \bar{S}^{tor}_{w,\mathbb{Z}/p^s\mathbb{Z}} \cap p_2^\star \bar{S}^{tor}_{w,\mathbb{Z}/p^s\mathbb{Z}}$. When $s = 1$, lemma 4.4.2 proves that this is a strict dynamical compactification. The general case is deduced by lemma 2.1.34. □

Using the cohomological correspondence of proposition 4.3.9, $t_w : p_2^\star \omega^\kappa \to p_1^! \omega^\kappa$, proposition 4.4.4, theorem 2.5.1, and proposition 2.6.13 we can define (recall that the support condition is hidden in the notation):

$$\begin{aligned} \mathrm{R}\Gamma_w(K^pK_{w,p,P}, \kappa) &= \lim_s \mathrm{R}\Gamma(S^{tor}_{\mathbb{Z}/p^s\mathbb{Z},w}, \omega^\kappa)^{C^{tor}_{t,\mathbb{Z}/p^s\mathbb{Z}}-ord} \\ \mathrm{R}\Gamma_w(K^pK_{w,p,P}, \kappa, cusp) &= \lim_s \mathrm{R}\Gamma(S^{tor}_{\mathbb{Z}/p^s\mathbb{Z},w}, \omega^\kappa(-D))^{C^{tor}_{t,\mathbb{Z}/p^s\mathbb{Z}}-ord} \end{aligned}$$

We can also allow a Nebentypus χ and define (over $\mathbb{Z}_p(\zeta_{p^n})$):

$$\begin{aligned} \mathrm{R}\Gamma_w(K^pK_{w,p,P}, \kappa, \chi) &= \lim_s \mathrm{R}\Gamma(S^{tor}_{\mathbb{Z}_p(\zeta_{p^n})/p^s,w}, \omega^\kappa(\chi))^{C^{tor}_{t,\mathbb{Z}/p^s\mathbb{Z}}-ord} \\ \mathrm{R}\Gamma_w(K^pK_{w,p,P}, \kappa, \chi, cusp) &= \lim_s \mathrm{R}\Gamma(S^{tor}_{\mathbb{Z}_p(\zeta_{p^n})/p^s,w}, \omega^\kappa(\chi)(-D))^{C^{tor}_{t,\mathbb{Z}/p^s\mathbb{Z}}-ord} \end{aligned}$$

Proposition 4.4.5. *The cohomologies $\mathrm{R}\Gamma_w(K^pK_{w,p,P}, \kappa)$, $\mathrm{R}\Gamma_w(K^pK_{w,p,P}, \kappa, cusp)$, $\mathrm{R}\Gamma_w(K^pK_{w,p,P}, \kappa, \chi)$ and $\mathrm{R}\Gamma_w(K^pK_{w,p,P}, \kappa, \chi, cusp)$ are independent of a specific choice of element $t \in T^{++}(\mathbb{Q}_p)$. They are perfect complexes of $\mathbb{Z}_p$-modules and carry an action of $T(\mathbb{Q}_p)$. Moreover, the action of $T(\mathbb{Q}_p)$ is locally algebraic, with algebraic part given by $\nu = -w^{-1}w_{0,M}(\kappa + \rho) - \rho$.*

Remark 4.4.6. On $\mathrm{R}\Gamma_w(K_pK_{w,p,P}, \kappa)$, $\mathrm{R}\Gamma_w(K^pK_{w,p,P}, \kappa, cusp)$, the action of $T(\mathbb{Z}_p)$ is given by ν. On $\mathrm{R}\Gamma_w(K_pK_{w,p,P}, \kappa, \chi)$, $\mathrm{R}\Gamma_w(K^pK_{w,p,P}, \kappa, \chi, cusp)$, the action of $T(\mathbb{Z}_p)$ is given by $\nu\chi$.

Proof. We only consider $\mathrm{R}\Gamma_w(K^pK_{w,p,P}, \kappa)$. The other cases are similar. In proposition 4.4.4, the construction of a compactification $\bar{S}^{tor}_{w,\mathbb{Z}/p^s\mathbb{Z}}$ of $S^{tor}_{w,\mathbb{Z}/p^s\mathbb{Z}}$, with boundary $D = D_+ \cup D_-$ is independent of the choice of $t \in T^+(\mathbb{Q}_p)$. We deduce that $T^+(\mathbb{Q}_p)$ acts on $\mathrm{R}\Gamma_{D_+,D_-}(\bar{S}^{tor}_{w,\mathbb{Z}/p^s\mathbb{Z}}, \omega^\kappa)$. Let t and t' be two elements in $T^{++}(\mathbb{Q}_p)$. We see that $\mathrm{R}\Gamma(S^{tor}_{\mathbb{Z}/p^s,w}, \omega^\kappa)^{C^{tor}_{t,\mathbb{Z}/p^s\mathbb{Z}}-ord}$ and $\mathrm{R}\Gamma(S^{tor}_{\mathbb{Z}_p/p^s,w}, \omega^\kappa)^{C^{tor}_{t',\mathbb{Z}/p^s\mathbb{Z}}-ord}$ are direct factors of the cohomology with support $\mathrm{R}\Gamma_{D_+,D_-}(\bar{S}^{tor}_{w,\mathbb{Z}/p^s\mathbb{Z}}, \omega^\kappa)$. Since there exists n and $u, u' \in T^+(\mathbb{Q}_p)$ such that $t^n = t'u'$ and $(t')^n = tu$ it is easy to deduce that these direct factors are quasi-isomorphic. If follows from theorem 2.6.8 that $\mathrm{R}\Gamma_w(K^pK_{w,p,P}, \kappa)$ is a perfect complex. The property that $T(\mathbb{Z}_p)$ acts via ν follows from the construction (and the normalization of the Hecke correspondences in proposition 4.3.9). □

4.5. **The Cousin complex.** In this section we construct a spectral sequence from higher Hida cohomology to classical cohomology.

4.5.1. *The Cousin spectral sequence.* Let $t \in T^{++}(\mathbb{Q}_p)$. Let $\tilde{\omega}^\kappa$ and $\tilde{\omega}^\kappa$ be as in proposition 4.3.13 and 4.3.18.

Theorem 4.5.2. *There are spectral sequences:*

(1) $\oplus_{w \in C(\kappa),\ \ell(w)=p} \mathrm{H}^{p+q}_w(K^pK_{w,p,P}, \kappa) \Rightarrow \mathrm{H}^{p+q}(S^{tor}, \tilde{\omega}^\kappa)^{t-ord}$
(2) $\oplus_{w \in C(\kappa),\ \ell(w)=p} \mathrm{H}^{p+q}_w(K^pK_{w,p,P}, \kappa, cusp) \Rightarrow \mathrm{H}^{p+q}(S^{tor}, \tilde{\omega}^\kappa(-D))^{t-ord}$

More generally, let $\chi : T(\mathbb{Z}/p^n\mathbb{Z}) \to \mathbb{Z}_p(\zeta_{p^n})^\times$. We have spectral sequences:

(1) $\oplus_{w \in C(\kappa),\ \ell(w)=p} \mathrm{H}^{p+q}_w(K^pK_{w,p,P}, \kappa, \chi) \Rightarrow \mathrm{H}^{p+q}(S^{tor}_{\mathbb{Z}_p(\zeta_{p^n})}, \tilde{\omega}^\kappa(\chi))^{t-ord}$

(2) $\oplus_{w\in C(\kappa),\ \ell(w)=p}\mathrm{H}^{p+q}_w(K^pK_{w,p,P},\kappa,\chi,cusp) \Rightarrow \mathrm{H}^{p+q}(S^{tor}_{\mathbb{Z}_p(\zeta_{p^n})},\tilde{\omega}^\kappa(-D))^{t-ord}$

Remark 4.5.3. The differentials and the abutment in the spectral sequence of theorem 4.5.2 depend in principle on the choice of the integral models S^{tor} and $\tilde{w}^\kappa$. The higher Hida complexes are however canonical.

Corollary 4.5.4. *Let $\kappa \in X^\star(T)^+$ be such that $C(\kappa) = \{w\}$. There is a canonical isomorphism:*

$$\mathrm{R}\Gamma_w(K^pK_{w,p,P},\kappa) \simeq \mathrm{R}\Gamma(S^{tor},\tilde{\omega}^\kappa)^{t-ord}$$

and similarly for cuspidal cohomology. Let $\chi : T(\mathbb{Z}/p^n\mathbb{Z}) \to \mathbb{Z}_p(\zeta_{p^n})^\times$ be a character. There is a canonical isomorphism:

$$\mathrm{R}\Gamma_w(K^pK_{w,p,P},\kappa,\chi) \simeq \mathrm{R}\Gamma(S^{tor}_{\mathbb{Z}_p(\zeta_{p^n})},\tilde{\omega}^\kappa(\chi))^{t-ord}$$

and similarly for cuspidal cohomology.

Corollary 4.5.5. *Let $\kappa \in X^\star(T)^+$ be such that $C(\kappa) = \{w\}$. Let $\nu = -w^{-1}w_{0,M}(\kappa+\rho)-\rho$. There is a canonical isomorphism:*

$$\mathrm{R}\Gamma_w(K^pK_{w,p,P},\kappa)\otimes\mathbb{Q}_p(-\nu) \simeq \mathrm{R}\Gamma(S^{tor}_{\mathbb{Q}_p},\omega^\kappa)^{ord}$$

of $T(\mathbb{Q}_p)$-modules, and similarly for cuspidal cohomology. Let $\chi : T(\mathbb{Z}/p^n\mathbb{Z}) \to \mathbb{Z}_p(\zeta_{p^n})^\times$ be a character. There is an isomorphism of $T(\mathbb{Q}_p)$-modules:

$$\mathrm{R}\Gamma_w(K^pK_{w,p,P},\kappa,\chi)\otimes\mathbb{Q}_p(\zeta_{p^n})(-\nu) \simeq \mathrm{R}\Gamma(S^{tor}_{\mathbb{Q}_p(\zeta_{p^n})},\omega^\kappa(\chi))^{ord}.$$

4.5.6. *proof of the main result.* We recall the filtration by closed subsets defined in section 4.1.16: $F_{-1} = \emptyset \subseteq F_0 \subseteq \cdots \subseteq F_{\frac{g(g+1)}{2}} = S^{tor}_{\mathbb{F}_p}$, where $F_i = \cup_{w,\ell(w)\leq i}Z_{>w-1}$. Let us also recall the definitions of $D_{-,w} = \sum_{i=1}^g Z_{i,>w_i}$ and $D_{+,w} = \sum_{i=1}^g Z_{i,<w_i}$.

Lemma 4.5.7. *We have that $F_i\setminus F_{i-1} = \coprod_{w,\ell(w)=i}(Z_{>w-1}\setminus D_{-,w})$.*

Proof. This follows from lemma 4.1.18. □

By theorem 2.9.7, we have a spectral sequence:

$$\mathrm{H}^{p+q}_c(\hat{F}_p\setminus\hat{F}_{p-1},\tilde{\omega}^\kappa) \Rightarrow \mathrm{H}^{p+q}(S^{tor},\tilde{\omega}^\kappa).$$

We see by lemma 2.9.3 and lemma 4.5.7 that

$$\mathrm{R}\Gamma_c(\hat{F}_p\setminus\hat{F}_{p-1},\tilde{\omega}^\kappa) = \oplus_{w,\ell(w)=p}\mathrm{R}\Gamma(S^{tor},\lim_m\lim_n\tilde{\omega}^\kappa(-mD_{-,w})/\mathscr{I}^n_{Z,>w-1}).$$

We have an action of $t\in T^{++}(\mathbb{Q}_p)$ on the spectral sequence by proposition 2.9.9 and corollary 4.2.10.

The key proposition which implies theorem 4.5.2 is the following (given that the ordinary part of the right hand side is $\mathrm{R}\Gamma_w(K_pK_{w,p,P},\kappa)$ by corollary 2.3.11):

Proposition 4.5.8. *The map*

$$\mathrm{R}\Gamma(S^{tor},\lim_m\lim_n\tilde{\omega}^\kappa(-mD_{-,w})/\mathscr{I}^n_{Z,>w-1}) \to \mathrm{R}\Gamma(S^{tor},\lim_n\operatorname{colim}_r\lim_m\tilde{\omega}^\kappa(-mD_{-,w}+rD_{+,w})/p^n)$$

induces a quasi-isomorphism on the ordinary part.

Proof. We consider the cohomological correspondence:

$$p_2^\star\lim_m\lim_n\left(\tilde{\omega}^\kappa(-mD_{-,w})/\mathscr{I}^n_{Z,>w-1}\right)\to p_1^!\lim_m\lim_n\left(\tilde{\omega}^\kappa(-mD_{-,w})/\mathscr{I}^n_{Z,>w-1}\right)$$

We tensor with $p_2^\star\mathscr{I}^{n-1}_{Z,>w-1}/\mathscr{I}^n_{Z,>w-1}\to p_1^\star\mathscr{I}^{n-1}_{Z,>w-1}/\mathscr{I}^n_{Z,>w-1}$, we see that the cohomological correspondence induces a map:

$$p_2^\star\lim_m\left(\tilde{\omega}^\kappa(-mD_{-,w})\otimes\mathscr{I}^{n-1}_{Z,>w-1}/\mathscr{I}^n_{Z,>w-1}\right)\to p_1^!\lim_m\left(\tilde{\omega}^\kappa(-mD_{-,w})\otimes\mathscr{I}^{n-1}_{Z,>w-1}/\mathscr{I}^n_{Z,>w-1}\right)$$

We claim that it factors into a map:

$$\begin{array}{ccc} p_2^\star \lim_m \left(\tilde{\omega}^\kappa(-mD_{-,w}+D_{+,w}) \otimes \mathscr{I}^{n-1}_{Z,>w-1}/\mathscr{I}^n_{Z,>w-1}\right) & \searrow & \\ \uparrow & & \\ p_2^\star \lim_m \left(\tilde{\omega}^\kappa(-mD_{-,w}) \otimes \mathscr{I}^{n-1}_{Z,>w-1}/\mathscr{I}^n_{Z,>w-1}\right) & \longrightarrow & p_1^! \lim_m \left(\tilde{\omega}^\kappa(-mD_{-,w}) \otimes \mathscr{I}^{n-1}_{Z,>w-1}/\mathscr{I}^n_{Z,>w-1}\right) \end{array}$$

This follows from the property that the original cohomological correspondence factors into

$$p_2^\star\tilde{\omega}^\kappa \to \mathscr{O}_{C_t^{tor}}((k'-k)\sum_{i=1}^g p_2^\star Z_i - \sum_{i=1}^g p_1^\star Z_i) \otimes p_2^\star\tilde{\omega}^\kappa \to p_1^!\tilde{\omega}^\kappa$$

by lemma 4.3.14, and an application of corollary 4.4.3 (at this place we choose k' large enough).

An application of lemma 2.4.4 implies that the map:

$$\mathrm{R}\Gamma(S^{tor}, \lim_m \lim_n \tilde{\omega}^\kappa(-mD_{-,w}) \otimes \mathscr{I}^{n-1}_{Z,>w-1}/\mathscr{I}^n_{Z,>w-1}) \to$$

$$\mathrm{R}\Gamma(S^{tor}, \lim_n \mathrm{colim}_r \lim_m \tilde{\omega}^\kappa(-mD_{-,w}+rD_{+,w}) \otimes \mathscr{I}^{n-1}_{Z,>w-1}/\mathscr{I}^n_{Z,>w-1})$$

is a quasi-isomorphism on the ordinary part. This implies that

$$\mathrm{R}\Gamma(S^{tor}, \lim_m \lim_n \tilde{\omega}^\kappa(-mD_{-,w})/\mathscr{I}^n_{Z,>w-1}) \to \mathrm{R}\Gamma(S^{tor}, \lim_n \mathrm{colim}_r \lim_m \tilde{\omega}^\kappa(-mD_{-,w}+rD_{+,w})/\mathscr{I}^n_{Z,>w-1})$$

is a quasi-isomorphism on the ordinary part. Finally we can pass to the limit over n and observe that the inclusion $(p) \subseteq \mathscr{I}_{Z,>w-1}$ is an equality over $\mathfrak{S}^{tor}_w$.

$\square$

5. Higher Hida theory in p-adic families

5.1. **Strict dynamic compactifications of Hecke correspondences.** Let $w \in {}^MW$. We let $B_w = wBw^{-1}$ and let $T_w = wTw^{-1} = T$ and $N_w = wNw^{-1}$ be its maximal torus and unipotent radical. Let $K_{p,P} \subseteq P'(\mathbb{Q}_p)$ be a compact open subgroup. We say that a compact $K_{p,P}$ is decomposable with respect to w if:

1. We have $K_{p,P} = M_{K_{p,P}} \ltimes U_{K_{p,P}}$ (this condition is automatic if $p > 2$, see remark 3.4.7).
2. $K_{p,P}$ has an Iwahori decomposition with respect to B_w:
$$K_{p,P} = N_{w,K_{p,P}} \times T_{w,K_{p,P}} \times \bar{N}_{w,K_{p,P}}.$$
3. $M_{K_{p,P}}$ is a subgroup of the Iwahori subgroup of $M(\mathbb{Q}_p)$ (with respect to $B \cap M = B_w \cap M$) and admits an Iwahori decomposition $N_{M_{K_{p,P}}} \times T_{w,K_{p,P}} \times \bar{N}_{M_{K_{p,P}}}$.

Remark 5.1.1. These compact open subgroups are cofinal among all compact open subgroups. We could probably use more general compacts, but it simplifies some arguments to restrict to these compact open subgroups.

Remark 5.1.2. Let $t_0 = \mathrm{diag}(1_g, p1_g)$. Let $K_{p,P}$ be a decomposable compact. For n large enough, $t_0^{-n} K_{p,P} t_0^n \subseteq P'(\mathbb{Z}_p)$.

Remark 5.1.3. Note that $K_{p,P}$ also has an Iwahori decomposition with respect to B:

$$K_{p,P} = N_{K_{p,P}} \times T_{w,K_{p,P}} \times \bar{N}_{K_{p,P}}$$

with $\bar{N}_{K_{p,P}} = \bar{N}_{M_{K_{p,P}}}$, $N_{K_{p,P}} = N_{M_{K_{p,P}}} \times U_{K_{p,P}}$.

Elements $t \in T_w(\mathbb{Q}_p)^{++} = wT(\mathbb{Q}_p)^{++}w^{-1}$ define Hecke correspondences $\mathfrak{IG}^{tor}_{K^p tK_{p,P}t^{-1} \cap K_{p,P}}$ over $\mathfrak{IG}^{tor}_{K^pK_{p,P}}$. Given a formal scheme $\mathfrak{X}$ over Spf $\mathbb{Z}_p$, we let $X_{\mathbb{Z}/p^n\mathbb{Z}}$ be the corresponding scheme over Spec $\mathbb{Z}/p^n\mathbb{Z}$.

Theorem 5.1.4. *For any decomposable compact open subgroup $K_{p,P} \subseteq P'(\mathbb{Q}_p)$, for any $n \in \mathbb{Z}_{\geq 0}$, and any $t \in T_w(\mathbb{Q}_p)^{++}$, the correspondence $((IG^{tor}_{K^p(tK_{p,P}t^{-1} \cap K_{p,P})})_{\mathbb{Z}/p^n\mathbb{Z}}, (IG^{tor}_{K^pK_{p,P}})_{\mathbb{Z}/p^n\mathbb{Z}}, p_1, p_2)$ admits a strict dynamic compactification.*

Proof. We first let $K''_p \subseteq G(\mathbb{Q}_p)$ be the Iwahori level subgroup and let $K'_{p,P} = wK''_p w^{-1} \cap P(\mathbb{Q}_p)$. In this case, the theorem follows from proposition 4.4.4.Now the theorem holds for any compact open subgroup satisfying $K_{p,P} \subseteq K'_{p,P}$ by lemma 2.1.33. Also, if the theorem is satisfied by a compact open subgroup $K_{p,P}$, it is satisfied by any conjugate by $t_0 = \mathrm{diag}(1_g, p1_g)$ and for r large enough, $t_0^{-r} K_{p,P} t_0^r \subseteq K'_{p,P}$. □

5.2. **Definition of higher Hida complexes.** We define higher Hida complexes by taking a suitable limit of cohomology of Igusa varieties with coefficients in the structural sheaf.

5.2.1. *Working at finite level.* We first generalize the definition given in section 4.4 to the case of an arbitrary decomposable compact open subgroup, in weight 0.

Lemma 5.2.2. *Let $t \in T_w(\mathbb{Q}_p)^+$ and consider the correspondence $(\mathfrak{IG}^{tor}_{K^p(tK_{p,P}t^{-1} \cap K_{p,P})}, \mathfrak{IG}^{tor}_{K^pK_{p,P}}, p_1, p_2)$. We have a cohomological correspondence $p_2^\star \mathscr{O}_{\mathfrak{IG}_{K^pK_{p,P}}} \to p_1^! \mathscr{O}_{\mathfrak{IG}_{K^pK_{p,P}}}$, given by $\langle -w_{0,M}\rho - w\rho, t\rangle \mathrm{Tr}_{p_1}$.*

Proof. Analogue to lemma 4.3.8. □

We let

$$\mathrm{R}\Gamma_w(K^pK_{p,P}) = \mathrm{R}\Gamma(\mathfrak{IG}^{tor}_{K^p,K_{p,P}}, \mathscr{O}_{\mathfrak{IG}_{K^p,K_{p,P}}})^{ord} = \lim_s \mathrm{R}\Gamma((IG^{tor}_{K^p,K_{p,P}})_{\mathbb{Z}/p^s\mathbb{Z}}, \mathscr{O}_{(IG_{K^p,K_{p,P}})_{\mathbb{Z}/p^s\mathbb{Z}}})^{ord}.$$

The ordinary part is taken for the action of any $t \in T_w^{++}(\mathbb{Q}_p)$ (the ordinary part does not depend on such choice). It is legitimate to take the ordinary part because of theorem 5.1.4 (of course, taking the ordinary part hides a support condition). Also, note that we have transition maps in the limit by proposition 2.6.13. We have a similar definition for cuspidal cohomology:

$$\mathrm{R}\Gamma_w(K^pK_{p,P}, cusp) = \mathrm{R}\Gamma(\mathfrak{IG}^{tor}_{K^p,K_{p,P}}, \mathscr{O}_{\mathfrak{IG}^{tor}_{K^p,K_{p,P}}}(-D))^{ord} = \lim_s \mathrm{R}\Gamma((IG^{tor}_{K^p,K_{p,P}})_{\mathbb{Z}/p^s\mathbb{Z}}, \mathscr{O}_{(IG_{K^p,K_{p,P}}(-D))_{\mathbb{Z}/p^s\mathbb{Z}}})^{ord}.$$

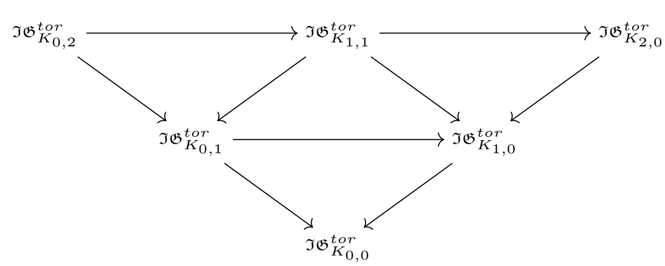
Proposition 5.2.3. *$\mathrm{R}\Gamma_w(K^pK_{p,P})$ and $\mathrm{R}\Gamma_w(K^pK_{p,P}, cusp)$ belong to $D(\mathbb{Z}_p[T(\mathbb{Q}_p)])$ and are perfect complexes of $\mathbb{Z}_p[T_w(\mathbb{Z}_p)/T_{w,K_{p,P}}]$-modules.*

Proof. This is a consequence of theorem 2.6.10. □

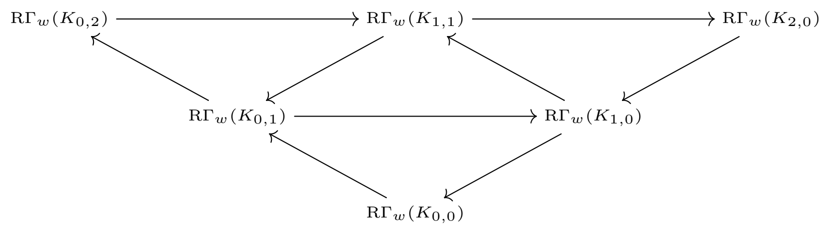
5.2.4. *Independence of the compact.* The ordinary cohomology of a decomposable compact open subgroup only depends on the torus part of the compact. This generalizes a standard result in the theory of Jacquet modules (see [Cas], sect. 4.1, the canonical lifting).

Proposition 5.2.5. *If $K_{p,P}$ and $K'_{p,P}$ are two decomposable compact open subgroups and if $T_{w,K_{p,P}} = T_{w,K'_{p,P}}$, then there is a natural quasi-isomorphism $\mathrm{R}\Gamma_w(K^pK_{p,P}) = \mathrm{R}\Gamma_w(K^pK'_{p,P})$ and similarly for cuspidal cohomology.*

Proof. Let $K_{p,P} = N_{w,K_{p,P}} \times T_{w,K_{p,P}} \times \bar{N}_{w,K_{p,P}}$ and $K'_{p,P} = N'_{w,K_{p,P}} \times T_{w,K_{p,P}} \times \bar{N}'_{w,K_{p,P}}$. We first claim that there exists $N''_{w,K_{p,P}} \subseteq N_{w,K_{p,P}} \cap N'_{w,K_{p,P}}$ such that $K''_{p,P} = N''_{w,K_{p,P}} \times T_{w,K_{p,P}} \times \bar{N}_{w,K_{p,P}}$ and $K'''_{p,P} = N''_{w,K_{p,P}} \times T_{w,K_{p,P}} \times \bar{N}'_{w,K_{p,P}}$ are decomposable compact open subgroups. Indeed, by conjugating by t_0, we may assume that $K_{p,P} \subseteq P'(\mathbb{Z}_p)$ and $K'_{p,P} \subseteq P'(\mathbb{Z}_p)$. One can take $K''_{p,P}$ (resp. $K'''_{p,P}$) to be the intersection of $K_{p,P}$ (resp. $K'_{p,P}$) with $\{M \in P'(\mathbb{Z}_p),\ M \mod p^n\mathbb{Z}_p \in B(\mathbb{Z}/p^n\mathbb{Z})\}$ for n large enough. We will show that the theorem holds for $K_{p,P}$ and $K''_{p,P}$, $K'_{p,P}$ and $K'''_{p,P}$ and finally $K''_{p,P}$ and $K'''_{p,P}$. Let us denote by $K_{n,m} = K^p.t^{-m}K_{p,P}t^m \cap t^nK_{p,P}t^{-n}$. We can consider the following infinite diagram:



This induces a diagram on cohomology where all maps are quasi-isomorphisms by proposition 2.8.2.



If we let $K''_{n,m} = K^p.t^{-m}K''_{p,P}t^m \cap t^n K''_{p,P}t^{-n}$ we see that there exists n, m, l, r such that $K''_{n,0} \subseteq K_{m,0} \subseteq K''_{l,0} \subseteq K_{r,0}$. If we consider the trace maps $\mathrm{R}\Gamma_w(K''_{n,0}) \to \mathrm{R}\Gamma_w(K_{m,0}) \to \mathrm{R}\Gamma_w(K''_{l,0}) \to \mathrm{R}\Gamma_w(K_{r,0})$, then we deduce that all maps are quasi-isomorphisms. Therefore $\mathrm{R}\Gamma_w(K^pK_{p,P}) = \mathrm{R}\Gamma_w(K^pK''_{p,P})$. We prove similarly that $\mathrm{R}\Gamma_w(K^pK'_{p,P}) = \mathrm{R}\Gamma_w(K^pK'''_{p,P})$. The proof that $\mathrm{R}\Gamma_w(K^pK''_{p,P}) = \mathrm{R}\Gamma_w(K^pK'''_{p,P})$ is similar, using pull back maps instead of traces. □

5.2.6. *Passing to the limit.* Let $K_{p,P}$ be a decomposable compact open subgroup. We assume $K_{p,P} \subseteq P'(\mathbb{Z}_p)$ for simplicity. We now let $(K_{p,P})_n = \{M \in K_{p,P}, M \mod p^n \in N(\mathbb{Z}/p^n)\}$. We let $\mathrm{R}\Gamma_w(K^p) = \lim_m \mathrm{colim}_n \mathrm{R}\Gamma_w(K^p(K_{p,P})_n) \otimes \mathbb{Z}/p^m\mathbb{Z}$. We define similarly $\mathrm{R}\Gamma_w(K^p, cusp)$.

Proposition 5.2.7. *The complexes $\mathrm{R}\Gamma_w(K^p)$ and $\mathrm{R}\Gamma_w(K^p, cusp)$ are admissible complexes (see definition 1.4.1) and their construction is independent of the choice of $K_{p,P}$.*

Proof. The independence on the compact follows as in proposition 5.2.5 (one checks easily that the quasi-isomorphism is functorial in n). We let $T'(\mathbb{Z}_p/p^n\mathbb{Z}_p)$ be the p-subgroup of $T(\mathbb{Z}_p/p^n\mathbb{Z}_p)$. For n large enough, the complex $\mathrm{R}\Gamma_w(K^p(K_{p,P})_n)$ is a perfect complex of $\mathbb{Z}_p[T'(\mathbb{Z}_p/p^n\mathbb{Z}_p)]$-modules by proposition 5.2.3. We can let $M_n^\bullet$ be a minimal representative of this complex (see [KT17], sect. 2.2). This means that $M_n^i = \mathbb{Z}_p[T'(\mathbb{Z}_p/p^n\mathbb{Z}_p)]^{k_{n,i}}$ and the differentials in this complex are zero modulo the maximal ideal of $\mathbb{Z}_p[T'(\mathbb{Z}_p/p^n\mathbb{Z}_p)]$. We have quasi-isomorphisms $M_n^\bullet \to (M_{n+1}^\bullet)^{T_{w,(K_{p,P})_n}}$. By minimality, we deduce that $k_{n,i} = k_{n+1,i}$ and the map $M_n^i \to M_{n+1}^i$ identifies with the canonical injection $\mathbb{Z}_p[T'(\mathbb{Z}_p/p^n\mathbb{Z}_p)]^{k_{n,i}} \hookrightarrow \mathbb{Z}_p[T'(\mathbb{Z}_p/p^{n+1}\mathbb{Z}_p)]^{k_{n+1,i}}$ (we think of $\mathbb{Z}_p[T'(\mathbb{Z}_p/p^n\mathbb{Z}_p)]$ as functions on $T'(\mathbb{Z}_p/p^n\mathbb{Z}_p)$). The proposition boils down to the fact that $\lim_m \mathrm{colim}_n \mathbb{Z}/p^m\mathbb{Z}[T'(\mathbb{Z}_p/p^n\mathbb{Z}_p)] = \mathcal{C}^0(T'(\mathbb{Z}_p), \mathbb{Z}_p)$. □

The complexes $\mathrm{R}\Gamma_w(K^p)$ and $\mathrm{R}\Gamma_w(K^p, cusp)$ belong to $D(\mathbb{Z}_p[T(\mathbb{Q}_p)])$. We now twist the action of $T(\mathbb{Q}_p)$ by w. This twist is motivated by section 4.1.5: the relevant Igusa variety is embedded via w inside the global Shimura variety. Also, note that we have dropped any reference to the compact $K_{p,P}$ in the notation.

5.3. **Control theorem.** We now study the specialization of these complexes in locally algebraic weights.

5.3.1. *Higher Hida complexes in fixed weight.* We revisit the construction of section 4.4 for more general compact open subgroups. Let $K_{p,P}$ be a decomposable compact open subgroup. After conjugation, we can assume that $K_{p,P} \subseteq P'(\mathbb{Z}_p)$ (this assumption is not really crucial but we keep it for simplicity). There is a map $\mathfrak{IG}^{tor}_{K^pK_{p,P}} \to A_g^{tor}$ providing a natural semi-abelian scheme A and therefore there is a natural M-torsor M_{dR} over $\mathfrak{IG}^{tor}_{K^pK_{p,P}}$. For any $\kappa \in X^\star(T)^{M,+}$ we have a corresponding sheaf ω^κ over $\mathfrak{IG}^{tor}_{K^pK_{p,P}}$. We let $\mathrm{R}\Gamma_w(K^pK_{p,P}, \kappa) = \mathrm{R}\Gamma(\mathfrak{IG}^{tor}_{K^pK_{p,P}}, \omega^\kappa)^{ord} = \lim_n \mathrm{R}\Gamma(IG^{tor}_{K^pK_{p,P},\mathbb{Z}/p^n\mathbb{Z}}, \omega^\kappa)^{ord}$ (where the ordinary part is taken for the action of some $t \in T_w^{++}(\mathbb{Q}_p)$). We define similarly $\mathrm{R}\Gamma_w(K^pK_{p,P}, \kappa, cusp)$. If κ is the trivial character, we simply recover the cohomology groups considered in section 5.2.1. These are $T(\mathbb{Q}_p)$-modules. The action of $T(\mathbb{Q}_p)$ is locally algebraic with weight $\nu = -w^{-1}w_{0,M}(\kappa + \rho) - \rho$. We now explain how we can recover these cohomologies from $\mathrm{R}\Gamma_w(K^p)$ and $\mathrm{R}\Gamma_w(K^p, cusp)$.

5.3.2. *Reduction of the torsor.* We have an exact sequence $1 \to U_{K_{p,P}} \to K_{p,P} \to M_{K_{p,P}} \to 1$ and there is a $M_{K_{p,P}}$-torsor $\mathfrak{IG}^{tor}_{K^pU_{K_{p,P}}} \to \mathfrak{IG}^{tor}_{K^pK_{p,P}}$.

Proposition 5.3.3. *There is a map of torsors, equivariant for the map $M_{K_{p,P}} \to M_\mu$:*

$$\begin{array}{ccc} \mathfrak{IG}^{tor}_{K^pU_{K_{p,P}}} & \longrightarrow & M_{dR} \\ & \searrow^{\pi} & \downarrow \\ & & \mathfrak{IG}^{tor}_{K^pK_{p,P}} \end{array}$$

Proof. If A is the semi-abelian scheme over $\mathfrak{IG}_{K_{p,P}}$, we have a Hodge-Tate map $T_p(A[p^\infty]^{et}) \to \omega_{A^t}$ (of pro-étale sheaves over $\mathfrak{IG}_{K_{p,P}}$), inducing an isomorphism: $T_p(A[p^\infty]^{et}) \otimes_{\mathbb{Z}_p} \mathscr{O}_{\mathfrak{IG}_{U_{K_{p,P}}}} \to \omega_{A^t} \otimes \mathscr{O}_{\mathfrak{IG}_{K_{p,P}}}$. This provides an equivariant map $\mathfrak{IG}_{U_{K_{p,P}}} \to M_{dR}$. $\square$

5.3.4. *Control.* Recall that $K_{p,P}$ has an Iwahori decomposition with respect to B : $N_{K_{p,P}} \times T_{w,K_{p,P}} \times \overline{N}_{K_{p,P}}$. We let $T_{K_{p,P}} = w^{-1}T_{w,K_{p,P}}w$. Recall that we have twisted the action of $T(\mathbb{Q}_p)$ by w on $\mathrm{R}\Gamma_w(K^p)$.

Theorem 5.3.5. *We have a canonical isomorphism:*

$$\mathrm{RHom}_{T_{K_{p,P}}}(\nu, \mathrm{R}\Gamma_w(K^p)) = \mathrm{R}\Gamma_w(K^pK_{p,P}, \kappa)$$

and similarly for cuspidal cohomology.

Proof. We let $(K_{p,P})''_n = \{M \in K_{p,P}, M \in B(\mathbb{Z}/p^n) \mod p^n\}$. We have

$$\begin{aligned} \mathrm{R}\Gamma_w(K^pK_{p,P}, \kappa) &= \mathrm{R}\Gamma(\mathfrak{IG}^{tor}_{K^pK_{p,P}}, \omega^\kappa)^{ord} \\ &= \mathrm{R}\Gamma(\mathfrak{IG}^{tor}_{K^p(K_{p,P})''_n}, \omega^\kappa)^{ord} \text{ by a version of proposition 5.2.5,} \\ &= \mathrm{R}\Gamma(\mathfrak{IG}^{tor}_{K^pB_{K_{p,P}}}, \omega^\kappa)^{ord} \text{ by passing to the limit over } n. \end{aligned}$$

We have by definition:

$$\begin{aligned} \mathrm{R}\Gamma_w(K^p) &= \mathrm{R}\Gamma(\mathfrak{IG}^{tor}_{K^pN_{K_{p,P}}}, \mathscr{O}_{\mathfrak{IG}^{tor}_{K^pN_{K_{p,P}}}})^{ord} \\ &= \mathrm{R}\Gamma(\mathfrak{IG}^{tor}_{K^pB_{K_{p,P}}}, \pi_\star\mathscr{O}_{\mathfrak{IG}^{tor}_{K^pN_{K_{p,P}}}})^{ord} \text{ because } \mathfrak{IG}^{tor}_{K^pN_{K_{p,P}}} \to \mathfrak{IG}^{tor}_{K^pB_{K_{p,P}}} \text{ is affine.} \end{aligned}$$

By proposition 5.3.3, we have a map $\mathscr{O}_{M_{dR}} \to \mathscr{O}_{\mathfrak{IG}_{U_{K_{p,P}}}}$. It induces a surjective map which is thought as the projection to the lowest weight vector: $\omega^\kappa \to (\pi_\star\mathscr{O}_{\mathfrak{IG}_{N_{K_{p,P}}}} \otimes \omega_{0,M}\kappa)^{T_{w,K_{p,P}}}$.

Let K be the kernel of this map. One checks that $\mathrm{R}\Gamma(\mathfrak{IG}^{tor}_{K^pB_{K_{p,P}}}, K)^{ord} = 0$ as in [Pil20], sect. 10.7 for example. It follows that

$$\mathrm{R}\Gamma(\mathfrak{IG}^{tor}_{K^pB_{K_{p,P}}}, \omega^\kappa)^{ord} = \mathrm{R}\Gamma(\mathfrak{IG}^{tor}_{K^pB_{K_{p,P}}}, (\pi_\star\mathscr{O}_{\mathfrak{IG}_{K^pU(\mathbb{Z}_p)}} \otimes \omega_{0,M}\kappa)^{T_{w,K_{p,P}}})^{ord}.$$

We then check that

$$\begin{aligned} \mathrm{RHom}_{T_{K_{p,P}}}(\nu, \mathrm{R}\Gamma_w(K^p)) &= \mathrm{R}\Gamma(\mathfrak{IG}^{tor}_{K^pB_{K_{p,P}}}, (\pi_\star\mathscr{O}_{\mathfrak{IG}_{K^pN_{K_{p,P}}}} \otimes w^{-1}\omega_{0,M}\kappa)^{T_{K_{p,P}}})^{ord} \\ &= \mathrm{R}\Gamma(\mathfrak{IG}^{tor}_{K^pB_{K_{p,P}}}, (\pi_\star\mathscr{O}_{\mathfrak{IG}_{K^pN_{K_{p,P}}}} \otimes \omega_{0,M}\kappa)^{T_{w,K_{p,P}}})^{ord} \end{aligned}$$

In the first equality we exchange sheaf cohomology and group cohomology. In this process, $\nu = -w^{-1}w_{0,M}(\kappa+\rho) - \rho$ becomes $-w^{-1}w_{0,M}(\kappa)$ since the factor $-w^{-1}w_{0,M}(\rho) - \rho$ is absorbed by the normalization of the trace map. $\square$

Let us fix some definitions. Let $\nu = \nu_{alg}\chi : T(\mathbb{Z}_p) \to \mathbb{Z}_p(\zeta_{p^n})^\times$, with $\nu_{alg} \in X^\star(T)$ and χ a finite order character. Let $\kappa = -w_{0,M}w(\nu+\rho) - \rho$. We let $\mathrm{R}\Gamma_w(K^p, \kappa, \chi) = \mathrm{RHom}_{T(\mathbb{Z}_p)}(\nu, \mathrm{R}\Gamma_w(K^p) \otimes \mathbb{Z}_p(\zeta_{p^n}))$ and we define similarly the cuspidal part. If χ is trivial, we also simply use $\mathrm{R}\Gamma_w(K^p, \kappa)$ for $\mathrm{R}\Gamma_w(K^p, \kappa, 0)$.

We can compare with section 4.4. We have $\mathrm{R}\Gamma_w(K^pK_{w,p,P}, \kappa) = \mathrm{R}\Gamma_w(K^p, \kappa)$ and $\mathrm{R}\Gamma_w(K^pK_{w,p,P}, \kappa, \chi) = \mathrm{R}\Gamma_w(K^p, \kappa, \chi)$. It is justified to drop $K_{w,p,P}$ from the notation since the ordinary cohomology is independent of the choice of the compact $K_{p,P}$ as explained in theorem 5.2.7.

5.4. **Ordinary Hecke algebras.** Let $\Lambda = \mathbb{Z}_p[[T(\mathbb{Z}_p)]]$. We let $M^\bullet_w = \mathrm{RHom}_{T(\mathbb{Z}_p)}(1, \mathrm{R}\Gamma_w(K^p) \otimes \Lambda)$ and $M^\bullet_{w,cusp} = \mathrm{RHom}_{T(\mathbb{Z}_p)}(1, \mathrm{R}\Gamma_w(K^p, cusp) \otimes \Lambda)$

Lemma 5.4.1. *The complexes $M^\bullet_w$ and $M^\bullet_{w,cusp}$ are perfect complexes of Λ-modules.*

Proof. Let $T'(\mathbb{Z}_p)$ be the pro-p subgroup of $T(\mathbb{Z}_p)$. We observe that $\mathrm{RHom}_{T'(\mathbb{Z}_p)}(1, \mathcal{C}^0(T'(\mathbb{Z}_p), \mathbb{Z}_p)\otimes\Lambda) = \Lambda[0]$. We deduce from proposition 5.2.7 that $\mathrm{RHom}_{T'(\mathbb{Z}_p)}(1, \mathrm{R}\Gamma_w(K^p) \otimes \Lambda)$ is a perfect complex of Λ-modules. Since $T(\mathbb{Z}_p)/T'(\mathbb{Z}_p) = T(\mathbb{F}_p)$ is a group of order prime to p, we deduce that $\mathrm{RHom}_{T(\mathbb{Z}_p)}(1, \mathrm{R}\Gamma_w(K^p) \otimes \Lambda)$ is a direct factor of $\mathrm{RHom}_{T'(\mathbb{Z}_p)}(1, \mathrm{R}\Gamma_w(K^p) \otimes \Lambda)$ and also a perfect complex. □

We clarify the relation between both complexes.

Lemma 5.4.2. *We have*

$$\mathrm{R}\Gamma_w(K^p) = \mathrm{RHom}_{\mathbb{Z}_p}(\mathrm{RHom}_\Lambda(M^\bullet_w, \Lambda), \mathbb{Z}_p)$$

and similarly for cuspidal cohomology.

Proof. There is an anti-equivalence of categories from admissible complexes to perfect complexes of Λ-modules, given by $i : M \mapsto \mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$. This uses that Λ is the algebra of measures on $T(\mathbb{Z}_p)$. Let i^{-1} be a quasi-inverse given by $\mathrm{Hom}_{\mathbb{Z}_p}(-, \mathbb{Z}_p)$. We have a second functor given by $j : M \mapsto (M \otimes \Lambda)^{T(\mathbb{Z}_p)}$. We claim that $j \circ i^{-1}$ is simply the duality functor $\mathrm{Hom}_\Lambda(-, \Lambda)$ on perfect complexes of Λ-modules. It suffices to check this on $\Lambda[0]$. We have a natural map $\mathrm{Hom}_{\mathbb{Z}_p}(\Lambda, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \Lambda \to \mathrm{Hom}_{\mathbb{Z}_p}(\Lambda, \Lambda)$, which induces an isomorphism $(\mathrm{Hom}_{\mathbb{Z}_p}(\Lambda, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \Lambda)^{T(\mathbb{Z}_p)} \to \mathrm{Hom}_\Lambda(\Lambda, \Lambda)$. We deduce that one can construct $\mathrm{R}\Gamma_w(K^p)$ form M_w by applying $\mathrm{Hom}_\Lambda(-, \Lambda)$ and the $\mathrm{Hom}_{\mathbb{Z}_p}(-, \mathbb{Z}_p)$. □

Remark 5.4.3. Let $K_{p,P}$ be a decomposable compact with $T_{w,K_{p,P}} = T(\mathbb{Z}_p)$. We have the formula:

$$M^\bullet_w = \mathrm{R}\Gamma(\mathfrak{I}\mathfrak{G}^{tor}_{K^p B_{K_{p,P}}}, (\pi_\star \mathscr{O}_{\mathfrak{I}\mathfrak{G}^{tor}_{K^p N_{K_{p,P}}}} \otimes \Lambda)^{T_w(\mathbb{Z}_p)})^{ord}.$$

Let $\nu^{univ} : T(\mathbb{Z}_p) \to \Lambda^\times$ be the universal character. Let κ^{univ} be defined by the formula $\kappa^{univ} = -w_{0,M} w(\nu^{univ} + \rho) - \rho$. In this last formula, $T_w(\mathbb{Z}_p)$ acts on $\pi_\star \mathscr{O}_{\mathfrak{I}\mathfrak{G}^{tor}_{K^p N_{K_{p,P}}}}$ since $T_w(\mathbb{Z}_p)$ is the diagonal torus of $B_{K_{p,P}}$. It acts on $\Lambda = \mathbb{Z}_p[[T(\mathbb{Z}_p)]]$ via the character $w_{0,M}\kappa^{univ}$ (this is the composition of $T_w(\mathbb{Z}_p) \overset{w^{-1}}{\to} T(\mathbb{Z}_p) \overset{w^{-1}w_{0,M}\kappa^{univ}}{\to} \Lambda^\times$).

For any $\nu = \nu_{alg}\chi : T(\mathbb{Z}_p) \to \mathbb{Z}_p(\zeta_{p^n})^\times$, with $\nu_{alg} \in X^\star(T)$ and χ a finite order character, and for $\kappa = -w_{0,M} w(\nu + \rho) - \rho$, we have:

$$M^\bullet_w \otimes_{\Lambda, -\nu} \mathbb{Z}_p(\zeta_{p^n}) = \mathrm{R}\Gamma_w(K^p, \kappa, \chi).$$

We let $\mathbb{T}_w$ be the image of the Hecke algebra in $\mathrm{End}(M^\bullet_w)$. This is a finite Λ-algebra. We let $E_w = \mathrm{Spec}\ \mathbb{T}_w \to \mathrm{Spec}\ \Lambda$. This is the ordinary Hecke eigenvariety. We define similarly $\mathbb{T}_{w,cusp}$.

5.5. **Duality.**

Theorem 5.5.1. *We have a duality, for which the adjoint of $t \in T(\mathbb{Q}_p)$ is $w_0 t^{-1} \in T(\mathbb{Q}_p)$.*

$$\mathrm{R}\Gamma_w(K^p, \kappa, \chi) \otimes \mathrm{R}\Gamma_{w_{0,M}ww_0}(K^p, -w_{0,M}\kappa - 2\rho_{nc}, w_0\chi^{-1}) \to \mathbb{Z}_p(\zeta_{p^n})[-d]$$

Proof. We take the compact $K_{p,P} = \{M = 1 \mod p^n\}$. By theorem 2.7.4, the duality is a perfect pairing

$$\mathrm{R}\Gamma(\mathfrak{I}\mathfrak{G}^{tor}_{K^p K_{p,P}}, \omega^\kappa)^{ord} \otimes \mathrm{R}\Gamma(\mathfrak{I}\mathfrak{G}^{tor}_{K^p K_{p,P}}, \omega^{\kappa - 2\rho_{nc}})^{-,ord} \to \mathbb{Z}_p(\zeta_{p^n})[-d]$$

where on $\mathrm{R}\Gamma(\mathfrak{I}\mathfrak{G}^{tor}_{K^p K_{p,P}}, \omega^{\kappa-2\rho_{nc}})^{-,ord}$ the ordinary part is taken with respect to the map $T^-(\mathbb{Q}_p) \to T(\mathbb{Q}_p) \overset{w}{\to} M(\mathbb{Q}_p)$. The adjoint of t is t^{-1}. Now we can consider the following diagram:

$$\begin{array}{ccc} T^-(\mathbb{Q}_p) & \overset{w}{\longrightarrow} & M(\mathbb{Q}_p) \\ {\scriptstyle w_0}\uparrow & & \downarrow{\scriptstyle w_{0,M}} \\ T^+(\mathbb{Q}_p) & \overset{w_{0,M}ww_0}{\longrightarrow} & M(\mathbb{Q}_p) \end{array}$$

We deduce that conjugation by $w_{0,M}$ induces an isomorphism

$$\mathrm{R}\Gamma(\mathfrak{IG}^{tor}_{K^pK_{p,P}}, \omega^{\kappa-2\rho_{nc}})^{-,ord} \simeq \mathrm{R}\Gamma(\mathfrak{IG}^{tor}_{K^pK_{p,P}}, \omega^{\kappa-2\rho_{nc}})^{ord}.$$

□

Theorem 5.5.2. *We have a perfect pairing* $M^\bullet_w \otimes M^\bullet_{w_{0,M}ww_0,cusp} \to \Lambda[-d]$*, for which the adjoint of* t *is* w_0t^{-1}*.*

Proof. This follows as in the previous theorem. □

5.6. **The case of** GSp_4**.** In this subsection we state a conjecture and prove an optimal vanishing results for our cohomology in the case of the group GSp_4.

Conjecture 5.6.1. *The cohomology* $\mathrm{R}\Gamma_w(K^p)$ *is an admissible complex of amplitude* $[\ell(w), \frac{g(g+1)}{2}]$ *and* $\mathrm{R}\Gamma_w(K^p, cusp)$ *is an admissible complex of amplitude* $[0, \ell(w)]$*.*

Remark 5.6.2. The conjecture is equivalent to the conjecture that $M^\bullet_w$ is a perfect complex of Λ-modules of amplitude $[\ell(w), \frac{g(g+1)}{2}]$ and $M^\bullet_{w,cusp}$ is a perfect complex of amplitude $[0, \ell(w)]$ (see lemma 5.4.2). The conjecture is compatible with duality between the cuspidal and non-cuspidal part.

Proposition 5.6.3. *The conjecture holds when* $g = 1$ *and* $g = 2$*.*

Proof. The case $g = 1$ is easy and was treated in [BP20]. Actually in that case the stronger assertion holds that the cuspidal and non-cuspidal theories for $w = Id$ are in degree 0 and dually, the cuspidal and non-cuspidal theories for $w = w_0^M$ are in degree 1. We now consider the case that $g = 2$. It suffices to prove that $M^\bullet_w \otimes_\Lambda \mathbb{F}_p$ and $M^\bullet_{w,cusp} \otimes_\Lambda \mathbb{F}_p$ have the correct amplitude as $\mathbb{F}_p$-vector spaces for all the maps $\Lambda \to \mathbb{F}_p$ corresponding to all the residual characters κ. We now use the notations from section 4.4. Let $K_p \subseteq G(\mathbb{Q}_p)$ be the Iwahori subgroup. We want to prove that

$$M^\bullet_{w,cusp} \otimes_\Lambda \mathbb{F}_p = \mathrm{R}\Gamma_w(K^pK_{w,p,P}, \kappa, cusp) \otimes \mathbb{Z}/p\mathbb{Z} = \mathrm{R}\Gamma_w(K^p, \kappa, cusp) \otimes \mathbb{Z}/p\mathbb{Z}$$

has amplitude $[0, \ell(w)]$. We let $(Ig_{K_pK_{w,p,P}})_{\mathbb{Z}/p\mathbb{Z}}$ be the corresponding component of the ordinary locus. When $w = Id$, $\mathrm{R}\Gamma_w(K^pK_{w,p,P}, \kappa, cusp) \otimes \mathbb{Z}/p\mathbb{Z}$ is a direct summand of

$$\mathrm{R}\Gamma((Ig_{K_pK_{w,p,P}})_{\mathbb{Z}/p\mathbb{Z}}, \omega^\kappa(-cusp))$$

(in that case the support condition has $D_- = \emptyset$, so we consider usual cohomology) which is concentrated in degree 0 since the image of $(Ig_{K_pK_{w,p,P}})_{\mathbb{Z}/p\mathbb{Z}}$ in the minimal compactification is affine and the relative cohomology to the minimal compactification vanishes ([Lan17], thm. 8.6). When w is the length one element, we consider $S^{tor}_{\mathbb{F}_p,(w-1,w+1)}$ which is a compactification of $(Ig_{K_pK_{w,p,P}})_{\mathbb{Z}/p\mathbb{Z}}$. Concretely, $S^{tor}_{(w-1,w+1)} = Z_{2,<2} \cap Z_{1,>0}$. We have locally principal subschemes $D_{w,+} = Z_{1,<1}$ and $D_{w,-} = Z_{2,>1}$. By lemma 4.4.2, we know that $(C^{tor}_{t,\mathbb{F}_p,(w-1,w+1)}, S^{tor}_{\mathbb{F}_p,(w-1,w+1)}, p_1, p_2, D_{w,+}, D_{w,-})$ is a strict dynamic compactification of $((Ig_{K_pw(t)K_{w,p,P}w(t)^{-1}\cap K_{w,p,P}})_{\mathbb{Z}/p\mathbb{Z}}, (Ig_{K_pK_{w,p,P}})_{\mathbb{Z}/p\mathbb{Z}}, p_1, p_2)$. We deduce that the cohomology $\mathrm{R}\Gamma_w(K^pK_{w,p,P}, \kappa, cusp)\otimes\mathbb{Z}/p\mathbb{Z}$ is a direct summand of $\mathrm{R}\Gamma(S^{tor}_{(w-1,w+1)} \setminus D_{w,+}, \omega^\kappa(-cusp))$. We observe that $S^{tor}_{(w-1,w+1)} \setminus D_{w,+}$ is contained in the p-rank at least one locus, which has cohomological dimension 1 in the minimal compactification (being covered by two affines). When w is the length two element, it is easier to prove the dual statement for the non-cuspidal cohomology, namely that the cohomology of the length one element is in the range $[1, 3]$. This is clear as there is no non-zero section with non-trivial support.

□

Remark 5.6.4. The same kind argument would apply to prove the expected vanishing of $\mathrm{R}\Gamma_w(K^p)$ or $\mathrm{R}\Gamma_w(K^p, cusp)$ for certain particular w and all values of g, but we were not able to find a general argument.

6. Higher Coleman theory and Higher Hida theory

6.1. **Overconvergent cohomology with support.**

6.1.1. *Cohomology with support.* We first recall some definitions of cohomology with support as considered in [BP21].

Let $\bar{\mathcal{X}}$ be a quasi-compact rigid space. Let $\bar{p}_1, \bar{p}_2 : \bar{\mathcal{C}} \to \bar{\mathcal{X}}$ be a correspondence over $\bar{\mathcal{X}}$ with $\bar{p}_1$ and $\bar{p}_2$ finite flat maps.

Remark 6.1.2. If $\bar{\mathcal{X}}$ is a toroidal compactification of a Shimura variety, we can rarely impose simultaneously that $\bar{p}_1$ and $\bar{p}_2$ are finite flat (for the same choice of cone decomposition). However, all the support conditions we will consider behave nicely at the boundary and typically make sense for any choice of cone decomposition. We therefore keep this assumption for simplicity.

We let $T = \bar{p}_2\bar{p}_1^{-1}(-)$ and $T^t = \bar{p}_1\bar{p}_2^{-1}(-)$.

Definition 6.1.3. *A open/closed support condition, for the correspondence $\bar{\mathcal{C}}$ is a pair $(\mathcal{U}, \mathcal{Z})$ where $\mathcal{U}$ is a finite union of quasi-Stein open subspaces and $\mathcal{Z}$ is a closed with complement a finite union of quasi-Stein open subspaces such that $T(\overline{\mathcal{U}}) \subseteq \mathcal{U}$ and $T^t(\mathcal{Z}) \subseteq \mathring{\mathcal{Z}}$. The open/closed support condition is called a quasi-compact open/closed support condition if $\mathcal{U}$ is a quasi-compact open subset and $\mathcal{Z}$ is the complement of a quasi-compact open subset.*

Let $\mathcal{U}_m = T^m(\mathcal{U})$ and $\mathcal{Z}_n = (T^t)^n(\mathcal{Z})$. Let $\mathscr{F}$ be a coherent sheaf (or even a Banach sheaf [BP21], def. 2.5.2) defined in a neighborhood of $\mathcal{U} \cap \mathcal{Z}$ in $\bar{\mathcal{X}}$, equipped with a cohomological correspondence $T : p_2^\star\mathscr{F} \to p_1^!\mathscr{F}$ which is assumed to be compact if $\mathscr{F}$ is a Banach sheaf ([BP21], def. 2.5.3). This is automatic if $\mathscr{F}$ is coherent.

We can consider the cohomology with support $\mathrm{R}\Gamma_{m,n} := \mathrm{R}\Gamma_{\mathcal{U}_m \cap \mathcal{Z}_n}(\mathcal{U}_m, \mathscr{F})$. We have natural maps $\mathrm{R}\Gamma_{m,n} \to \mathrm{R}\Gamma_{m+1,n}$ and $\mathrm{R}\Gamma_{m,n+1} \to \mathrm{R}\Gamma_{m,n}$.

In the quasi-compact case, these cohomologies are represented by complexes of Banach modules. In general they are represented by projective systems of complexes of Banach modules ([BP21], lem. 2.5.20).

Proposition 6.1.4 ([BP21], thm 5.3.7)**.** *The operator T acts on the cohomology $\mathrm{R}\Gamma_{m,n}$ and is potent compact (T^2 is compact). Moreover, the finite slope part $\mathrm{R}\Gamma_{m,n}^{fs}$ are quasi-isomorphic for varying m and n.*

One can actually always reduce to working with quasi-compact support conditions by the following lemma:

Lemma 6.1.5. *Let $(\mathcal{U}, \mathcal{Z})$ be an open/closed support condition. Then there exists a quasi-compact open/closed support condition $(\mathcal{U}', \mathcal{Z}')$ such that $T(\mathcal{U}) \subseteq \mathcal{U}' \subseteq \mathcal{U}$ and $T^t(\mathcal{Z}) \subseteq \mathcal{Z}' \subseteq \mathcal{Z}$.*

Proof. Observe that $T(\overline{\mathcal{U}}) \subseteq \mathcal{U}$. We can write $\mathcal{U}$ as a union $\cup_i \mathcal{U}_i$ of quasi-compact opens. Since $T(\overline{\mathcal{U}})$ is closed, hence compact in the constructible topology, we deduce that there exists a quasi-compact open $\mathcal{U}'$ such that $T(\mathcal{U}) \subseteq \mathcal{U}' \subseteq \mathcal{U}$. We similarly find a $\mathcal{Z}'$. □

Corollary 6.1.6. *In the setting of lemma 6.1.5, the finite slope part of $\mathrm{R}\Gamma_{\mathcal{U}\cap\mathcal{Z}}(\mathcal{U}, \mathscr{F})$ and $\mathrm{R}\Gamma_{\mathcal{U}'\cap\mathcal{Z}'}(\mathcal{U}', \mathscr{F})$ are quasi-isomorphic.*

Proof. We claim that the finite slope part of cohomology for each of the following support conditions are quasi-isomorphic: $(\mathcal{U}, \mathcal{Z})$, $(\mathcal{U}', \mathcal{Z})$, $(\mathcal{U}, \mathcal{Z}')$, $(\mathcal{U}', \mathcal{Z}')$. Let us compare $(\mathcal{U}, \mathcal{Z})$ and $(\mathcal{U}', \mathcal{Z})$ and leave the remaining cases to the reader. We have restriction maps: $\mathrm{R}\Gamma_{\mathcal{U}\cap\mathcal{Z}}(\mathcal{U}, \mathscr{F}) \to \mathrm{R}\Gamma_{\mathcal{U}'\cap\mathcal{Z}}(\mathcal{U}', \mathscr{F}) \to \mathrm{R}\Gamma_{T(\mathcal{U})\cap\mathcal{Z}}(T(\mathcal{U}), \mathscr{F})$. The correspondence provides a map $T : \mathrm{R}\Gamma_{T(\mathcal{U})\cap\mathcal{Z}}(T(\mathcal{U}), \mathscr{F}) \to \mathrm{R}\Gamma_{\mathcal{U}\cap\mathcal{Z}}(\mathcal{U}, \mathscr{F})$. Composing the map T with the restriction maps yields an endomorphism of the various cohomologies. It is obvious that on the finite slope part, the restriction map induce quasi-isomorphisms. □

Proposition 6.1.7. *Let $(\mathcal{U}, \mathcal{Z})$ and $(\mathcal{U}', \mathcal{Z}')$ be two open/closed support conditions. Assume that $\cap_{m,n} T^m(\mathcal{U}) \cap (T^t)^n(\mathcal{Z}) = \cap_{m,n} T^m(\mathcal{U}') \cap (T^t)^n(\mathcal{Z}')$. Then $\mathrm{R}\Gamma_{\mathcal{U}\cap\mathcal{Z}}(\mathcal{U}, \mathscr{F})^{fs}$ and $\mathrm{R}\Gamma_{\mathcal{U}'\cap\mathcal{Z}'}(\mathcal{U}', \mathscr{F})^{fs}$ are canonically quasi-isomorphic.*

Proof. We can assume that the support conditions are quasi-compact by lemma 6.1.5. If $(\mathcal{U}, \mathcal{Z})$ and $(\mathcal{U}', \mathcal{Z}')$ are two support conditions, then $(\mathcal{U} \cup \mathcal{U}', \mathcal{Z} \cup \mathcal{Z}')$ is also a support condition. We therefore reduce to the situation that $\mathcal{U} \subseteq \mathcal{U}'$ and $\mathcal{Z} \subseteq \mathcal{Z}'$. Let us assume that $\mathcal{U} \subseteq \mathcal{U}'$ and $\mathcal{Z} = \mathcal{Z}'$. One treats similarly the case that $\mathcal{U} = \mathcal{U}'$ and $\mathcal{Z} \subseteq \mathcal{Z}'$.

We see that there exists m, n such that $\mathcal{U}'_m \cap \mathcal{Z}_n \subseteq \mathcal{U}$. Indeed, we know that $\cap_{m,n} \mathcal{U}'_m \cap \mathcal{Z}_n \bigcap \mathcal{U}^c = \emptyset$. But all these sets are compact in the constructible topology. Therefore there must exist m, n such that $\mathcal{U}'_m \cap \mathcal{Z}_n \subseteq \mathcal{U}$.

We may further replace $\mathcal{Z}$ by $\mathcal{Z}_n$. We thus reduce to the case that $\mathcal{U}'_m \cap \mathcal{Z} \subseteq \mathcal{U}$. We now consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{R}\Gamma_{\mathcal{U}' \cap \mathcal{Z}}(\mathcal{U}', \mathscr{F}) & \longrightarrow & \mathrm{R}\Gamma_{\mathcal{U} \cap \mathcal{Z}}(\mathcal{U}, \mathscr{F}) \\ \downarrow & \swarrow & \downarrow \\ \mathrm{R}\Gamma_{\mathcal{U}'_m \cap \mathcal{Z}}(\mathcal{U}'_m, \mathscr{F}) & \longrightarrow & \mathrm{R}\Gamma_{\mathcal{U}_m \cap \mathcal{Z}}(\mathcal{U}_m, \mathscr{F}) \end{array}$$

We know that on the finite slope part, the two vertical arrows are quasi-isomorphisms. This implies that all maps in this diagram are quasi-isomorphisms. $\square$

6.1.8. *Application to Siegel varieties.* We now apply this to our Shimura variety $\mathcal{S}^{tor}$ of level $K_{p,n}$, viewed as an adic space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. We consider two kinds of support conditions. We let $w \in {}^M W$. Let $\epsilon \in \mathbb{Q}$. We let $\mathcal{U}_{\geq w - \epsilon} = \{x \in \mathcal{S}^{tor},\ \forall i, \deg H_{g+i}(x) \geq n(w_i - \epsilon)\}$ and $\mathcal{U}_{\leq w+\epsilon} = \{x \in \mathcal{S}^{tor},\ \forall i, \deg H_{g+i}(x) \leq n(w_i + \epsilon)\}$. These are quasi-compact open subsets. We let $\mathcal{Z}_{<w+\epsilon} = \overline{\cup_{\epsilon' < \epsilon} \mathcal{U}_{\leq w + \epsilon'}}$ and $\mathcal{Z}_{>w-\epsilon} = \overline{\cup_{\epsilon' > \epsilon} \mathcal{U}_{\geq w - \epsilon'}}$. These are closed subsets with quasi-compact complement. Let $t \in T^{++}(\mathbb{Q}_p)$. We write T for the correspondence $\mathcal{C}_t^{tor}$.

Proposition 6.1.9. (1) *For any $\epsilon \in]0, 1[\cap \mathbb{Q}$, there exists $\epsilon' < \epsilon$ such that $T(\mathcal{U}_{\geq w-\epsilon}) \subseteq \mathcal{U}_{\geq w - \epsilon'}$.*
(2) *For any $\epsilon \in]0, 1[\cap \mathbb{Q}$, there exists $\epsilon' < \epsilon$ such that $T^t(\mathcal{U}_{\leq w+\epsilon}) \subseteq \mathcal{U}_{\leq w+\epsilon'}$.*
(3) *For any $0 < \epsilon' < \epsilon < 1$, there exists n such that $T^n(\mathcal{U}_{\geq w-\epsilon}) \subseteq \mathcal{U}_{\geq w-\epsilon'}$.*
(4) *For any $0 < \epsilon' < \epsilon < 1$, there exists n such that $(T^t)^n(\mathcal{U}_{\leq w+\epsilon}) \subseteq \mathcal{U}_{\leq w+\epsilon'}$.*

Proof. It follows from proposition 4.2.15 that for a point $x \in \mathcal{C}_t^{tor}$, $\deg H_{g+i}(p_1(x)) \leq \deg H_{g+i}(p_2(x))$ and if equality holds they are multiples of n. Points (1) and (2) follow from this and quasi-compactness. Points (3) and (4) are proved as in [Pil11], prop. 2.5. See also [Pil11], thm. 3.1. $\square$

Proposition 6.1.10. *For any $\epsilon \in]0, 1[\cap \mathbb{Q}$, the pair $(\mathcal{U}_{\geq w-\epsilon}, \mathcal{Z}_{<w+\epsilon})$ is a quasi-compact open/closed support condition. Moreover, $\cap_{m,n} T^m(\mathcal{U}_{\geq w-\epsilon}) \cap (T^t)^n(\mathcal{Z}_{<w+\epsilon}) = \overline{\mathcal{S}^{tor}_w}$.*

Proof. This follows from proposition 6.1.9. $\square$

We now give another kind of support conditions which are defined using the Hodge-Tate period map. These are the support conditions used in [BP21]. Let $w \in {}^M W$. Let $FL = P \backslash G \to \mathrm{Spec}\ \mathbb{Z}_p$ be the flag variety viewed as a scheme. We let $C_w = P \backslash PwB$ be the Bruhat cell in $P \backslash G$. We let $X_w = \overline{C_w} = \cup_{w' \leq w} C_{w'}$ be the Schubert cell. We let X^w be the opposite Schubert cell, equal to the closure of $P \backslash PwB^{op}$ where B^{op} is the opposite Borel. We let $Y_w = \cup_{w' \geq w} C_{w'}$. We have a natural inclusion $X^w \hookrightarrow Y_w$. We recall that there is a continuous map $\pi_{HT, K_{p,n}} : |\mathrm{S}^{tor}| \to |\mathcal{FL}|/K_{p,n}$. Therefore, we can pull back any $K_{p,n}$-stable subset of $\mathcal{FL}$ to S^{tor}.

Proposition 6.1.11. *The pair $(\pi^{-1}_{HT,K_{p,n}}]X_{w,\mathbb{F}_p}[, \pi^{-1}_{HT,K_{p,n}} \overline{]Y_{w,\mathbb{F}_p}[})$ is an open/closed support condition. Moreover*

$$\cap T^m(\pi^{-1}_{HT,K_{p,n}}]X_{w,\mathbb{F}_p}[) \cap (T^t)^n (\pi^{-1}_{HT,K_{p,n}} \overline{]Y_{w,\mathbb{F}_p}[}) = \overline{\mathcal{S}^{tor}_w}.$$

Proof. We first consider the closed analytic spaces $\mathcal{X}_w \hookrightarrow \mathcal{FL}$ and $\mathcal{X}^w \hookrightarrow \mathcal{FL}$. We see that $\mathcal{X}_w \hookrightarrow]X_{w,\mathbb{F}_p}[$ and $\mathcal{X}^w \hookrightarrow]Y_{w,\mathbb{F}_p}[$ ([BP21], lemma 3.1.2). We let $\mathcal{B}_n$ (resp. $\mathcal{B}_n^{op}$) be the subgroups of the Borel (resp. the opposite Borel) of elements reducing modulo p^n to the identity. We let $\mathcal{X}_{w,n} = \mathcal{X}_w \mathcal{B}_n^{op}$ and $\mathcal{X}^w_n = \mathcal{X}^w \mathcal{B}_n$. These are open subsets of the flag variety equal to the tubular neighborhood of width p^n of $\mathcal{X}_w$ and $\mathcal{X}^w$. We see easily that $\mathcal{X}_{w,n} \subseteq]X_{w,\mathbb{F}_p}[$ and $\mathcal{X}^w_n \subseteq]X^w_{\mathbb{F}_p}[$, and that $\overline{\mathcal{X}_{w,n}} \subseteq \mathcal{X}_{w,n-1}$, $\overline{\mathcal{X}^w_n} \subseteq \mathcal{X}^w_{n-1}$. We claim that $]X_{w,\mathbb{F}_p}[.t^n \subseteq \mathcal{X}_{w,n}$ and $]Y_{w,\mathbb{F}_p}[.t^{-n} \subseteq \mathcal{X}^{w,n}$ (see [BP21], lemma 3.4.1 and its proof). We deduce that $\overline{]X_{w,\mathbb{F}_p}[} K_{p,n} t K_{p,n} \subseteq]X_{w,\mathbb{F}_p}[$ and $\overline{]Y_{w,\mathbb{F}_p}[} K_{p,n} t^{-1} K_{p,n} \subseteq \overset{\circ}{\overline{]Y_{w,\mathbb{F}_p}[}} =]Y_{w,\mathbb{F}_p}[$. This implies that we have an open/closed support condition. We now compute $T^m(\pi^{-1}_{HT,K_{p,n}}]X_{w,\mathbb{F}_p}[) \cap (T^t)^n (\pi^{-1}_{HT,K_{p,n}} \overline{]Y_{w,\mathbb{F}_p}[})$. By [BP21], lem. 3.5.10.5 this set is contained in $]C_{w,\mathbb{F}_p}[_{m,\overline{0}}K_{p,n}\cap]C_{w,\mathbb{F}_p}[_{0,\overline{n}}K_{p,n}$ (see [BP21], sect. 3.3.6 for the definition of these sets). Consider the Iwahori decomposition $K_{p,n} = B_{K_{p,n}} \times U^{op}_{K_{p,n}}$. We deduce easily that

$$]C_{w,\mathbb{F}_p}[_{m,\overline{0}}K_{p,n}\cap]C_{w,\mathbb{F}_p}[_{0,\overline{n}}K_{p,n} = \big((]C_{w,\mathbb{F}_p}[_{m,\overline{0}}B_{K_{p,n}}) \cap (]C_{w,\mathbb{F}_p}[_{0,\overline{n}}U^{op}_{K_{p,n}})\big)K_{p,n}.$$

Now, by [BP21], coro. 3.3.14, $(]C_{w,\mathbb{F}_p}[_{m,\overline{0}}B_{K_{p,n}}) =]C_{w,\mathbb{F}_p}[_{m,\overline{0}}$ and $(]C_{w,\mathbb{F}_p}[_{0,\overline{n}}U^{op}_{K_{p,n}}) =]C_{w,\mathbb{F}_p}[_{0,\overline{n}}$ so that finally $]C_{w,\mathbb{F}_p}[_{m,\overline{0}}K_{p,n}\cap]C_{w,\mathbb{F}_p}[_{0,\overline{n}}K_{p,n} =]C_{w,\mathbb{F}_p}[_{m,\overline{n}}K_{p,n}$. □

6.2. **Integral overconvergent cohomology and comparison with higher Hida theory.** Recall that we have fixed $w \in {}^MW$ and $t \in T^{++}(\mathbb{Q}_p)$.

6.2.1. *Some integral models and blow-ups of the Shimura variety.* Let $\epsilon \in]0,1[\cap\mathbb{Q}$. Let $\epsilon' \in]0,\epsilon[\cap\mathbb{Q}$ such that $T(\mathcal{U}_{\geq w-\epsilon}) \subseteq \mathcal{U}_{\geq w-\epsilon'}$ and $T^t(\mathcal{U}_{\leq w+\epsilon}) \subseteq \mathcal{U}_{\leq w+\epsilon'}$. We first revisit section 4.1.12.

Proposition 6.2.2. *There exists a blow-up $\pi : \tilde{S}^{tor} \to S^{tor}$ satisfying the following properties:*

1. *for all $0 \leq i \leq g$ we have effective divisors $Z_{i,>w_j}$ and $Z_{i,<w_j}$ on $\tilde{S}^{tor}$ with disjoint support, and $Z_{i,>w_j} - Z_{i,<w_j} = Z_i - V(p^{nw_j})$.*
2. *for all $0 \leq i \leq g$ and all $\eta \in \{\epsilon, -\epsilon, \epsilon', -\epsilon'\}$, we have effective divisors $Z_{i,>w_j+\eta}$ and $Z_{i,<w_j+\eta}$ on $\tilde{S}^{tor}$ with disjoint support, and $Z_{i,>w_j+\eta} - Z_{i,<w_j+\eta} = Z_i - V(p^{n(w_j+\eta)})$.*
3. *The map $\pi : \tilde{S}^{tor} \to S^{tor}$ induces an isomorphism $\pi^{-1}S^{tor}_{w,\mathbb{F}_p} \to S^{tor}_{w,\mathbb{F}_p}$.*

Proof. This is identical to the proof of proposition 4.1.13. □

We change our notation and let S^{tor} be $\tilde{S}^{tor}$. We let $D_+ = \sum_i Z_{i,<w_j}$, $D_- = \sum_i Z_{i,>w_j}$, $D_{+,\epsilon} = \sum_i Z_{i,<w_j-\epsilon}$, $D_{-,\epsilon} = \sum_i Z_{i,>w_j+\epsilon}$ $D_{+,\epsilon'} = \sum_i Z_{i,<w_j-\epsilon'}$, $D_{-,\epsilon'} = \sum_i Z_{i,>w_j+\epsilon'}$.

Proposition 6.2.3. *The following holds:*

1. *We have $D_+ \geq D_{+,\epsilon'} \geq D_{+,\epsilon}$ and $D_- \geq D_{-,\epsilon'} \geq D_{-,\epsilon}$.*
2. *We have $p_1^\star D_+ \geq p_2^\star D_+$, $p_1^\star D_{+,\epsilon} \geq p_2^\star D_{+,\epsilon'}$.*
3. *We have $p_2^\star D_- \geq p_1^\star D_-$, $p_2^\star D_{-,\epsilon} \geq p_1^\star D_{-,\epsilon'}$.*
4. *Over $C_t^{tor} \setminus p_1^\star D_{+,\epsilon} \cup p_2^\star D_{-,\epsilon}$, there exists $s \in]0,1[\cap\mathbb{Q}$ such that we have $sp_1^\star D_+ \geq p_2^\star D_+$ and $sp_2^\star D_- \geq p_1^\star D_-$.*

Proof. The first point is obvious. The second and third point follow from proposition 6.1.9 and lemma 4.2.11. The last point follows as in lemma 4.4.2. □

6.2.4. *Integral model for the Hecke correspondence.*

Proposition 6.2.5. *There is a fundamental class $p_2^\star \mathscr{O}_{C_t^{tor}\setminus p_1^\star D_+ \cup p_2^\star D_-} \to \langle w^{-1}w_{0,M}\rho + \rho, t\rangle p_1^! \mathscr{O}_{C_t^{tor}\setminus p_1^\star D_+ \cup p_2^\star D_-}$ given by the trace map.*

Proof. We observe that $S^{tor}\setminus D_+ \cup D_-$ and $C_t^{tor}\setminus p_1^\star D_+ \cup p_2^\star D_-$ are smooth. Therefore $p_1^! \mathscr{O}_{C_t^{tor}\setminus p_1^\star D_+ \cup p_2^\star D_-}$ is an invertible sheaf. The trace map gives a rational map $p_2^\star \mathscr{O}_{C_t^{tor}\setminus p_1^\star D_+ \cup p_2^\star D_-} \to p_1^! \mathscr{O}_{C_t^{tor}\setminus p_1^\star D_+ \cup p_2^\star D_-}$. One checks that it induces a true map $p_2^\star \mathscr{O}_{C_t^{tor}\setminus p_1^\star D_+ \cup p_2^\star D_-} \to \langle w^{-1}w_{0,M}\rho + \rho, t\rangle p_1^! \mathscr{O}_{C_t^{tor}\setminus p_1^\star D_+ \cup p_2^\star D_-}$ by lemma 4.3.8. □

Proposition 6.2.6. *Let $\kappa \in X^\star(T)^{M_\mu,+}$ and $\chi : T(\mathbb{Z}_p) \to \mathbb{Z}_p(\zeta_{p^n})^\times$ be a finite order character. We have a cohomological correspondence $t : p_2^\star \omega^\kappa(\chi) \to p_1^! \omega^\kappa(\chi)$ over $C_t^{tor} \setminus p_1^\star D_+ \cup p_2^\star D_-$ which is $p^{-\langle w^{-1}w_{0,M}(\kappa+\rho)-\rho,t\rangle} t^{naive}$ where t^{naive} is the rational map obtained as the tensor product of $p_2^\star \omega^\kappa(\chi)[1/p] \to p_1^\star \omega^\kappa(\chi)[1/p]$ and the fundamental class given by the map of proposition 6.2.5.*

Proof. This follows as in proposition 4.3.11. □

Proposition 6.2.7. *There is an integral model $\tilde{\omega}^\kappa(\chi) = \omega^\kappa(\chi)(m(D_+ - D_-))$ for some $m \geq 0$, such that the map t extends to a cohomological correspondence $\tilde{t} : p_2^\star \tilde{\omega}^\kappa(\chi) \to p_1^! \tilde{\omega}^\kappa(\chi)$ over $C_t^{tor} \setminus p_1^\star D_{+,\epsilon} \cup p_2^\star D_{-,\epsilon}$. Moreover, we have the following factorization: $p_2^\star \tilde{\omega}^\kappa(\chi) \to p_2^\star \tilde{\omega}^\kappa(\chi)(D_+) \to p_1^! \tilde{\omega}^\kappa(\chi)(-D_-) \to p_1^! \tilde{\omega}^\kappa(\chi)$.*

Proof. This is analogue to proposition 4.3.13 and lemma 4.3.14. □

6.2.8. *Integral overconvergent cohomology.* Let $Z_{<w+\epsilon} = \cap Z_{i,<w_i+\epsilon}$. We view $\hat{i} : \hat{Z}_{<w+\epsilon} \hookrightarrow S^{tor}$ as a closed subset of the locale of $D(\mathscr{O}_{S^{tor},\blacksquare})$. We let $U_{\geq w-\epsilon} = S^{tor} \setminus D_{+,\epsilon} \hookrightarrow S^{tor}$. We let $\hat{i}' : \hat{Z}_{<w+\epsilon} \cap U_{\geq w-\epsilon} \hookrightarrow U_{\geq w-\epsilon}$. For any s, we consider the cohomology $\mathrm{R}\Gamma_{\hat{Z}_{<w+\epsilon}\cap U_{\geq w-\epsilon,\mathbb{Z}/p^s\mathbb{Z}}}(U_{\geq w-\epsilon,\mathbb{Z}/p^s\mathbb{Z}}, \tilde{\omega}^\kappa(\chi)) = \mathrm{R}\Gamma(U_{\geq w-\epsilon,\mathbb{Z}/p^s\mathbb{Z}}, \hat{i}'_\star(\hat{i}')^!\tilde{\omega}^\kappa(\chi))$. This cohomology fits in an exact triangle:

$$\mathrm{R}\Gamma_{\hat{Z}_{<w+\epsilon}\cap U_{\geq w-\epsilon,\mathbb{Z}/p^s\mathbb{Z}}}(U_{\geq w-\epsilon,\mathbb{Z}/p^s\mathbb{Z}}, \tilde{\omega}^\kappa(\chi)) \to \mathrm{R}\Gamma(U_{\geq w-\epsilon,\mathbb{Z}/p^s\mathbb{Z}}, \tilde{\omega}^\kappa(\chi)) \to \mathrm{R}\Gamma(U_{\geq w-\epsilon,\mathbb{Z}/p^s\mathbb{Z}} \backslash Z_{<w+\epsilon}, \tilde{\omega}^\kappa(\chi)) \stackrel{+1}{\to}$$

Let $\mathrm{R}\Gamma_{\mathcal{Z}_{<w+\epsilon}\cap\mathcal{U}_{\geq w-\epsilon}}(\mathcal{U}_{\geq w+\epsilon}, \omega^\kappa(\chi))^{int} = \lim_s \mathrm{R}\Gamma_{\hat{Z}_{<w+\epsilon}\cap U_{\geq w-\epsilon,\mathbb{Z}/p^s\mathbb{Z}}}(U_{\geq w+\epsilon,\mathbb{Z}/p^s\mathbb{Z}}, \tilde{\omega}^\kappa(\chi))$.

Theorem 6.2.9. (1) *The cohomology* $\mathrm{R}\Gamma_{\mathcal{Z}_{<w+\epsilon}\cap\mathcal{U}_{\geq w-\epsilon}}(\mathcal{U}_{\geq w-\epsilon}, \omega^\kappa(\chi))^{int}$ *carries an action of* t.
(2) *The ordinary part identifies with higher Hida theory* $\mathrm{R}\Gamma_w(K^p, \kappa, \chi)$.
(3) *We have* $\mathrm{R}\Gamma_{\mathcal{Z}_{<w+\epsilon}\cap\mathcal{U}_{\geq w-\epsilon}}(\mathcal{U}_{\geq w-\epsilon}, \omega^\kappa(\chi))^{int} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathrm{R}\Gamma_{\mathcal{Z}_{<w+\epsilon}\cap\mathcal{U}_{\geq w-\epsilon}}(\mathcal{U}_{\geq w-\epsilon}, \omega^\kappa(\chi))$ *and the image of* $\mathrm{R}\Gamma_{\mathcal{Z}_{<w+\epsilon}\cap\mathcal{U}_{\geq w-\epsilon}}(\mathcal{U}_{\geq w-\epsilon}, \omega^\kappa(\chi))^{int}$ *in* $\mathrm{R}\Gamma_{\mathcal{Z}_{<w+\epsilon}\cap\mathcal{U}_{\geq w-\epsilon}}(\mathcal{U}_{\geq w-\epsilon}, \omega^\kappa(\chi))^{fs}$ *is a* $\mathbb{Z}_p$*-lattice.*

Proof. By lemma 2.10.1, we have changing the support maps:

$$\mathrm{R}\Gamma_{D_{+,\epsilon},D_{-,\epsilon'}}(S^{tor}_{\mathbb{Z}/p^s\mathbb{Z}}, \tilde{\omega}^\kappa(\chi)) \to \mathrm{R}\Gamma_{\hat{Z}_{<w+\epsilon}\cap U_{\geq w-\epsilon,\mathbb{Z}/p^s\mathbb{Z}}}(U_{\geq w-\epsilon,\mathbb{Z}/p^s\mathbb{Z}}, \tilde{\omega}^\kappa(\chi)) \to \mathrm{R}\Gamma_{D_{+,\epsilon'},D_{-,\epsilon}}(S^{tor}_{\mathbb{Z}/p^s\mathbb{Z}}, \tilde{\omega}^\kappa(\chi))$$

By proposition 2.10.4, we have a map $\tilde{t} : \mathrm{R}\Gamma_{D_{+,\epsilon'},D_{-,\epsilon}}(S^{tor}_{\mathbb{Z}/p^s\mathbb{Z}}, \tilde{\omega}^\kappa(\chi)) \to \mathrm{R}\Gamma_{D_{+,\epsilon},D_{-,\epsilon'}}(S^{tor}_{\mathbb{Z}/p^s\mathbb{Z}}, \tilde{\omega}^\kappa(\chi))$ such that when we compose with the above changing the support maps we get endomorphisms of all the cohomologies. Moreover, they are all quasi-isomorphic when taking the ordinary part. By theorem 2.11.4, the ordinary part of these cohomologies further identifies with $\mathrm{R}\Gamma_w(K^p, \kappa, \chi)\otimes_{\mathbb{Z}_p}\mathbb{Z}/p^s\mathbb{Z}$. The quasi-isomorphism $\mathrm{R}\Gamma_{\mathcal{Z}_{<w+\epsilon}\cap\mathcal{U}_{\geq w-\epsilon}}(\mathcal{U}_{\geq w-\epsilon}, \omega^\kappa(\chi))^{int}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p = \mathrm{R}\Gamma_{\mathcal{Z}_{<w+\epsilon}\cap\mathcal{U}_{\geq w-\epsilon}}(\mathcal{U}_{\geq w-\epsilon}, \omega^\kappa(\chi))$ is clear. Using the triangle

$$\mathrm{R}\Gamma_{\hat{Z}_{<w+\epsilon}\cap U_{\geq w-\epsilon,\mathbb{Z}/p^s\mathbb{Z}}}(U_{\geq w-\epsilon,\mathbb{Z}/p^s\mathbb{Z}}, \tilde{\omega}^\kappa(\chi)) \to \mathrm{R}\Gamma(U_{\geq w-\epsilon,\mathbb{Z}/p^s\mathbb{Z}}, \tilde{\omega}^\kappa(\chi)) \to \mathrm{R}\Gamma(U_{\geq w-\epsilon,\mathbb{Z}/p^s\mathbb{Z}} \backslash Z_{<w+\epsilon}, \tilde{\omega}^\kappa(\chi)) \stackrel{+1}{\to}$$

and the analogous one for $\mathrm{R}\Gamma_{\mathcal{Z}_{<w+\epsilon}\cap\mathcal{U}_{\geq w-\epsilon}}(\mathcal{U}_{\geq w-\epsilon}, \omega^\kappa(\chi))$ we are left to observe that:

$$\begin{aligned} \lim_s \mathrm{R}\Gamma(U_{\geq w-\epsilon,\mathbb{Z}/p^s\mathbb{Z}}, \tilde{\omega}^\kappa(\chi)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p &= \mathrm{R}\Gamma(\mathcal{U}_{\geq w-\epsilon}, \omega^\kappa(\chi)) \\ \lim_s \mathrm{R}\Gamma(U_{\geq w-\epsilon,\mathbb{Z}/p^s\mathbb{Z}} \setminus Z_{<w+\epsilon}, \tilde{\omega}^\kappa(\chi)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p &= \mathrm{R}\Gamma(\mathcal{U}_{\geq w-\epsilon} \setminus \mathcal{Z}_{<w+\epsilon}, \omega^\kappa(\chi)). \end{aligned}$$

That this gives a lattice in the finite slope part of cohomology is a consequence of [BP21] lemma 5.9.10 (which explains how to construct lattices). □

In [BP21] we conjectured a lower bound on slopes of overconvergent modular forms of a given weight.

Corollary 6.2.10. *Conjecture 5.9.2 of* [BP21] *holds in the Siegel case.*

Proof. The conjecture is the statement that elements $t \in T^+(\mathbb{Q}_p)$ acting on the finite slope cohomology have bounded below slope by a precise bound. This slope bound is equivalent to the property that the normalized action of t stabilizes a lattice. It is also enough to check this for all $t \in T^{++}(\mathbb{Q}_p)$ as they generate $T(\mathbb{Q}_p)$ as a group. This follows from theorem 6.2.9. □

We remark that as a consequence, several of the results from [BP21] can be improved in the Siegel case. In particular we can replace the "strongly small slope" conditions with the weaker "small slope" conditions in the following theorems of [BP21]: the classicality theorem 5.12.3 and the vanishing theorems 5.12.5 and 5.12.11 for rational coherent and Betti cohomology. As a consequence we obtain theorem 1.5.1 of the introduction.

6.2.11. *Big sheaves.* We finally prove a version of the theorem 6.2.9 above in p-adic families. Let us briefly indicate the strategy. Higher Hida theory is computed as the ordinary cohomology of a certain "topological induction" sheaf over some finite level Igusa variety. We show that one can replace the "topological induction" sheaf by a "locally analytic induction" sheaf. The advantage is that this "locally analytic induction" sheaf can be defined over some formal model of a strict neighborhood of the Igusa variety and provides an integral model for the locally analytic overconvergent cohomology considered in section 6 of [BP21].

Recall that $\Lambda = \mathbb{Z}_p[[T(\mathbb{Z}_p)]]$. Let $\mathcal{W} = \mathrm{Spa}(\Lambda, \Lambda) \times_{\mathrm{Spa}(\mathbb{Z}_p,\mathbb{Z}_p)} \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ be the analytic weight space over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. Let $\mathrm{Spa}(A[1/p], A)$ be a quasi-compact open subset of $\mathcal{W}$. Let $\nu_A : T(\mathbb{Z}_p) \to A^\times$ be the restriction of the universal character.

Let $\mathcal{T}_n \to \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ be the group defined by $\mathcal{T}_n(C, C^+) = \{x \in T(C^+), x \mod p^n \in T(\mathbb{Z}/pn\mathbb{Z}) \hookrightarrow T(C^+/p^nC^+)\}$. This is an affinoid group scheme. We let $\mathfrak{T}_n = \mathrm{Spf}\ \mathscr{O}^+_{\mathcal{T}_n}$. This is a formal group scheme. For n large enough, the character ν_A extends to a character $\mathfrak{T}_n \to \mathfrak{G}_m \times \mathrm{Spf}\ A$. We let $\kappa_A = -w_{0,M}w(\nu_A + \rho) - \rho$. We also let $K_{p,P} = wK_{p,n}w^{-1} \cap P(\mathbb{Q}_p)$. We let $\mathcal{IW}_n \to \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ be the group defined by $\mathcal{IW}_n(C, C^+) = \{x \in M(C^+), x \mod p^n \in M_{K_{p,P}}\}$. This is an affinoid group scheme. We let $\mathfrak{IW}_n = \mathrm{Spf}\ \mathscr{O}^+_{\mathcal{IW}_n}$. We let $\mathfrak{BIW}_n$ be its "Borel" subgroup, with torus $\mathfrak{T}_n$.

The $M_{K_{p,P}}$-torsor $\pi : \mathfrak{IG}_{U_{K_{p,P}}} \to \mathfrak{IG}_{K_{p,P}} = \mathfrak{S}^{tor}_w$ can be pushed to a $\mathfrak{IW}_n$-torsor $\pi' : \mathfrak{IG}_{U_{K_p}} \times^{M_{K_{p,n}}} \mathfrak{IW}_n \to \mathfrak{IG}_{K_{p,n}}$. We let

$$\mathfrak{F}^{\nu_A} = (\pi_\star \mathscr{O}_{\mathfrak{IG}_{U_{K_{p,P}}}} \hat{\otimes} A)^{B_{K_{p,n}} = -w_{0,M}\kappa_A}.$$

We let

$$\mathfrak{F}^{\nu_A,n} = (\pi'_\star \mathscr{O}_{\mathfrak{IG}_{U_{K_p}} \times^{M_{K_{p,n}}} \mathfrak{IW}_n} \hat{\otimes} A)^{\mathfrak{BIW}_n = -w_{0,M}\kappa_A}.$$

We have a canonical restriction map $\mathfrak{F}^{\nu_A,n} \to \mathfrak{F}^{\nu_A}$.

We let $\mathscr{F}^{\kappa_A,n}_r$ and $\mathscr{F}^{\nu_A}_r$ be the sheaves obtained by reduction modulo p^r.

Proposition 6.2.12. *For any $r \geq 0$, we have cohomological correspondences:*

$$\begin{array}{ccc} t : p_2^\star \mathscr{F}^{\nu_A,n}_r & \longrightarrow & p_1^! \mathscr{F}^{\nu_A,n}_r \\ \downarrow & & \downarrow \\ t : p_2^\star \mathscr{F}^{\nu_A}_r & \longrightarrow & p_1^! \mathscr{F}^{\nu_A}_r \end{array}$$

Proof. We only need to define maps

$$\begin{array}{ccc} p_2^\star \mathscr{F}^{\nu_A,n}_r & \longrightarrow & p_1^\star \mathscr{F}^{\nu_A,n}_r \\ \downarrow & & \downarrow \\ p_2^\star \mathscr{F}^{\nu_A}_r & \longrightarrow & p_1^\star \mathscr{F}^{\nu_A}_r \end{array}$$

The cohomological correspondences are simply obtained by tensoring with the normalized fundamental class of proposition 6.2.5. See for example section 6.3.10 of [BP21]. ☐

Proposition 6.2.13. *The map $\mathrm{R}\Gamma_{D_+,D_-}(S^{tor}_{w,\mathbb{Z}/p^r\mathbb{Z}}, \mathscr{F}^{\nu_A,n}_r) \to \mathrm{R}\Gamma_{D_+,D_-}(S^{tor}_{w,\mathbb{Z}/p^r\mathbb{Z}}, \mathscr{F}^{\nu_A}_r)$ induces a quasi-isomorphism on the ordinary part. Moreover, $\mathrm{R}\Gamma_{D_+,D_-}(S^{tor}_{w,\mathbb{Z}/p^r\mathbb{Z}}, \mathscr{F}^{\nu_A}_r)^{ord} = M^\bullet_w \otimes A/p^n$*

Proof. For the first point, one needs to study the cohomology of the kernel of the map $\mathscr{F}^{\nu_A,n}_r \to \mathscr{F}^{\nu_A}_r$. It is easy to see that the Hecke operator is nilpotent on this kernel. Compare with [Pil20], sect. 10.7. The property that $\mathrm{R}\Gamma_{D_+,D_-}(S^{tor}_{w,\mathbb{Z}/p^r\mathbb{Z}}, \mathscr{F}^{\nu_A}_r)^{ord} = M^\bullet_w \otimes A/p^r$ follows from the definition. See remark 5.4.3. ☐

Proposition 6.2.14. *For ε sufficiently small, the sheaf $\mathfrak{F}^{\nu_A,n}$ extends to a sheaf $\tilde{\mathfrak{F}}^{\nu_A,n}$ over $\tilde{\mathfrak{U}}_{[w-\epsilon,w+\epsilon]}$ where $\tilde{\mathfrak{U}}_{[w-\epsilon,w+\epsilon]} \to \mathfrak{U}_{[w-\epsilon,w+\epsilon]}$ is a further blow-up away from $\mathfrak{S}^{tor}_w$, and moreover for all $r > 0$ we have cohomological correspondences $p_2^\star \tilde{\mathscr{F}}^{\nu_A,n}_r \to p_1^! \tilde{\mathscr{F}}^{\nu_A,n}_r$ which furthermore factor as*

$$p_2^\star \tilde{\mathscr{F}}^{\nu_A,n}_r \to p_2^\star \tilde{\mathscr{F}}^{\nu_A,n}_r(D_+) \to p_1^! \tilde{\mathscr{F}}^{\nu_A,n}_r(-D_-) \to p_1^! \tilde{\mathscr{F}}^{\nu_A,n}_r.$$

Proof. The blow-up $\tilde{\mathfrak{U}}_{[w-\epsilon,w+\epsilon]}$ and the sheaf $\tilde{\mathfrak{F}}^{\nu_A,n}$ over $\tilde{\mathfrak{U}}_{[w-\epsilon,w+\epsilon]}$ are constructed as in the proof of lem. 6.6.2 of [BP21]. We remind the reader that the purpose of making this blow-up is to have an integral model for the Hodge-Tate period map which is used to produce an extension a good extension of the $\mathfrak{IW}_n$-torsor on $\mathfrak{S}^{tor}_w$. Maps $p_2^\star \mathfrak{F}^{\nu_A,n}[1/p] \to p_1^\star \mathfrak{F}^{\nu_A,n}[1/p]$ are constructed in lem. 6.3.10 of [BP21], and it follows from the local description given there that they extend to a map $p_2^\star \mathfrak{F}^{\nu_A,n} \to p_1^\star \mathfrak{F}^{\nu_A,n}$.

We now tensor with the fundamental class to get a map $p_2^\star \mathscr{F}^{\nu_A,n}_r(-mD_-) \to p_1^! \mathscr{F}^{\nu_A,n}(mD_+)$ for some $m \geq 0$. We emphasize that m is independent of r as $p_2^\star \mathfrak{F}^{\nu_A,n} \to p_1^\star \mathfrak{F}^{\nu_A,n}$ is regular and the poles only come from the fundamental class. We may now let $\tilde{\mathfrak{F}}^{\nu_A,n} = \tilde{\mathfrak{F}}^{\nu_A,n}(m'(D_+ - D_-))$ for m' large enough and use the dynamical property to deduce that we have a cohomological correspondence $p_2^\star \tilde{\mathscr{F}}_r^{\nu_A,n} \to p_1^! \tilde{\mathscr{F}}_r^{\nu_A,n}$ which moreover has the expected factorization. □

Let $\mathrm{R}\Gamma_{\mathcal{Z}_{<w+\epsilon}\cap\mathcal{U}_{\geq w-\epsilon}}(\mathcal{U}_{\geq w-\epsilon}, \tilde{\mathfrak{F}}^{\nu_A,n})^{int} = \lim_r \mathrm{R}\Gamma_{\hat{Z}_{<w+\epsilon}\cap U_{\geq w-\epsilon,\mathbb{Z}/p^r\mathbb{Z}}}(U_{\geq w-\epsilon,\mathbb{Z}/p^r\mathbb{Z}}, \tilde{\mathscr{F}}_r^{\nu_A,n})$. We also want to consider locally analytic overconvergent cohomology in a fixed p-adic weight, so let $\nu : T(\mathbb{Z}_p) \to \mathcal{O}_{\mathbb{C}}^\times$ be a continuous character inducing a map $\nu : A \to \mathcal{O}_{\mathbb{C}}$. We let $\tilde{\mathfrak{F}}^{\nu,n} = \tilde{\mathfrak{F}}^{\nu_A,n} \otimes_{A,\nu} \mathcal{O}_{\mathbb{C}}$.

Theorem 6.2.15. (1) *The operator t acts on $\mathrm{R}\Gamma_{\mathcal{Z}_{<w+\epsilon}\cap\mathcal{U}_{\geq w-\epsilon}}(\mathcal{U}_{\geq w-\epsilon}, \tilde{\mathfrak{F}}^{\nu_A,n})^{int}$ and the ordinary part identifies with higher Hida theory $M_w^\bullet \otimes_\Lambda A$.*

(2) *We have $\mathrm{R}\Gamma_{\mathcal{Z}_{<w+\epsilon}\cap\mathcal{U}_{\geq w-\epsilon}}(\mathcal{U}_{\geq w-\epsilon}, \tilde{\mathfrak{F}}^{\nu,n})^{int} \otimes_{\mathcal{O}_{\mathbb{C}}} \mathbb{C} = \mathrm{R}\Gamma_{\mathcal{Z}_{<w+\epsilon}\cap\mathcal{U}_{\geq w-\epsilon}}(\mathcal{U}_{\geq w-\epsilon}, \tilde{\mathfrak{F}}^{\nu,n}[1/p])$ and the image of $\mathrm{R}\Gamma_{\mathcal{Z}_{<w+\epsilon}\cap\mathcal{U}_{\geq w-\epsilon}}(\mathcal{U}_{\geq w-\epsilon}, \tilde{\mathfrak{F}}^{\nu,n})^{int}$ in $\mathrm{R}\Gamma_{\mathcal{Z}_{<w+\epsilon}\cap\mathcal{U}_{\geq w-\epsilon}}(\mathcal{U}_{\geq w-\epsilon}, \tilde{\mathfrak{F}}^{\nu,n})^{fs}$ is a lattice.*

Proof. This is similar to the proof of theorem 6.2.9. □

In conjecture 6.8.1 of [BP21], we conjectured that the slopes of $t \in T^+(\mathbb{Q}_p)$ are nonnegative on finite slope locally analytic overconvergent cohomologies, or equivalently that these operators preserve a lattice.

Corollary 6.2.16. *Conjecture 6.8.1 of* [BP21] *holds in the Siegel case.*

Proof. It suffices to check this for $t \in T^{++}(\mathbb{Q}_p)$ which then follows from 6.2.15 (2). □

We remark that again this implies that results from [BP21] can be improved in the Siegel case. In particular we can replace the "strongly small slope" conditions with the weaker "small slope" conditions in corollary 6.8.4 and theorem 6.9.3 (2) of [BP21].

Finally we can use the vanishing results of [BP21] to deduce cohomological vanishing in higher Hida theory after inverting p:

Corollary 6.2.17. *The complexes of $\Lambda \otimes \mathbb{Q}_p$-modules $M_w^\bullet \otimes \mathbb{Q}_p$ have amplitude $[\ell(w), \frac{g(g+1)}{2}]$ and the complexes $M_{w,cusp}^\bullet \otimes \mathbb{Q}_p$ have amplitude $[0, \ell(w)]$.*

Proof. This follows from 6.2.15 combined with [BP21], thm. 6.7.3. □

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