THE RAMANUJAN AND SATO–TATE CONJECTURES FOR BIANCHI MODULAR FORMS

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Abstract. We prove the Ramanujan and Sato–Tate conjectures for Bianchi modular forms of weight at least 2. More generally, we prove these conjectures for all regular algebraic cuspidal automorphic representations of $\mathrm{GL}_2(\mathbf{A}_F)$ of parallel weight, where F is any CM field. We deduce these theorems from a new potential automorphy theorem for the symmetric powers of 2-dimensional compatible systems of Galois representations of parallel weight.

Contents

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1. Introduction

Let $f = \sum_{n=1}^{\infty} a_n q^n$ be a cuspidal modular form of weight $k \geq 2$ and level $\Gamma_1(N) \subset \mathrm{SL}_2(\mathbf{Z})$ which is an eigenform for all the Hecke operators T_p for $(p, N) = 1$ and normalized so that $a_1 = 1$. The Ramanujan conjecture for f — proved by Deligne [Del71] as a consequence of the Weil conjectures — is the claim that

$$|a_p| \leq 2 \cdot p^{(k-1)/2}.$$

Suppose that the coefficients of f are real. The Sato–Tate conjecture (proved in a sequence of papers [CHT08, Tay08, HSBT10, BLGHT11]) is the theorem that the normalized values $a_p/2p^{(k-1)/2} \in [-1, 1]$ are equidistributed with respect to the Sato–Tate measure $2/\pi \cdot \sqrt{1 - x^2}dx$ unless f is a so-called CM form, in which case the corresponding measure is the average of the atomic measure with support zero and the measure $1/\pi \cdot 1/\sqrt{1 - x^2}dx$ (the proof in this CM case is much easier and follows from [Hec1920]). (If the coefficients a_p are not real, some minor modifications are required to formulate the conjecture properly.) These conjectures were originally made for the particular (non-CM) form $f = \Delta = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n$ of level $\mathrm{SL}_2(\mathbf{Z})$ and weight $k = 12$ studied by Ramanujan; this particular case turns out to be no easier than the general case.

Both of these conjectures have an equivalent reformulation in the language of automorphic representations. Associated to a cuspidal modular eigenform f (as above) is an automorphic representation π for $\mathrm{GL}(2)/\mathbf{Q}$. The data of π includes irreducible admissible infinite dimensional complex representations π_p of $\mathrm{GL}_2(\mathbf{Q}_p)$ for all p. For $(p, N) = 1$, the representations π_p satisfy the additional property of being so-called *spherical*, and are in particular classified by a pair of complex numbers $\{\alpha_p, \beta_p\}$ known as Satake parameters, which are related to the original coefficients a_p via the equation

$$x^2 - a_p x + p^{k-1}\chi(p) = (x - \alpha_p)(x - \beta_p),$$

where $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$ is the Nebentypus character of f. The Ramanujan conjecture is equivalent to the equality $|\alpha_p| = |\beta_p| = p^{(k-1)/2}$, which can be reformulated as saying that the representation π_p is tempered. The Sato–Tate conjecture is equivalent (for non-CM forms) to the claim that the conjugacy classes of the matrices

$$\frac{1}{p^{(k-1)/2}} \cdot \begin{pmatrix} \alpha_p & 0 \\ 0 & \beta_p \end{pmatrix}$$

are equidistributed in $\mathrm{SU}(2)$/conjugacy with respect to the probability Haar measure.

One advantage of these reformulations is that they can be generalized; the original Ramanujan conjecture becomes the statement that if π is a regular algebraic cuspidal automorphic representation for $\mathrm{GL}(2)/\mathbf{Q}$, then π_p is tempered for all p. The general Ramanujan conjecture is the statement that if π is a cuspidal automorphic representation for $\mathrm{GL}(n)/F$ for any number field F, then π_v is tempered for all primes v of F. (One can generalize further to groups beyond $\mathrm{GL}(n)$ but then the formulation becomes more subtle.) This conjecture is still open in the case of $\mathrm{GL}(2)/\mathbf{Q}$; after one drops the adjectives "regular algebraic" (or even just "regular"), one then allows Maass forms, which seem beyond the reach of all current techniques. On the other hand, one can consider regular algebraic automorphic representations π for $\mathrm{GL}(2)/F$ for number fields F. If F is a totally real field, then these correspond to Hilbert modular forms of weight $(k_i)_{i=1}^d$ (with $d = [F : \mathbf{Q}]$) with all weights k_i at least 2, and parity independent of i; the theory here is close to the original setting of classical modular forms. One point of similarity is that Hilbert modular forms can also be written as q-series (now in more variables). Moreover, just as for classical modular forms, there is a direct link between Hilbert modular forms and the étale cohomology of certain algebraic (Shimura) varieties, which allows one to deduce the Ramanujan conjecture in these cases as a consequence of the Weil conjectures ([BL84, Bla06]). The Sato–Tate conjecture can also be proved in these cases by arguments generalizing those used for modular forms [BLGG11].

In this paper, we consider the Ramanujan and Sato–Tate conjectures for regular algebraic cuspidal automorphic representations for $\mathrm{GL}(2)/F$ where F is now an imaginary quadratic field (or more generally an imaginary CM field). In this case, the classical interpretation of these objects (sometimes called Bianchi modular forms when F is imaginary quadratic) looks quite different from the familiar q-expansions associated to classical or Hilbert modular forms; for example, if F is an imaginary quadratic field, they can be thought of as vector valued differential one-forms on arithmetic hyperbolic three manifolds. The Eichler–Shimura map allows one to relate classical modular forms of weight $k \geq 2$ to the cohomology of local systems for congruence subgroups of $\mathrm{SL}_2(\mathbf{Z})$; the analogous theorem also allows one to relate Bianchi modular forms to the cohomology of local systems for subgroups of $\mathrm{SL}_2(\mathcal{O}_F)$. However, what is missing in this setting is that there is now no longer any direct link to the cohomology of algebraic varieties. Despite this, in this paper, we prove the Ramanujan conjecture for regular algebraic cuspidal automorphic representations in full for all quadratic fields and with the parallel weight condition for arbitrary imaginary CM fields.

For a precise clarification of what parallel weight k means, see Definition 1.6.1. The meaning of 'regular algebraic' is also recalled immediately before this definition. When F is imaginary quadratic, all regular algebraic cuspidal automorphic representations for $\mathrm{GL}(2)/F$ have parallel weight.

Theorem A (Ramanujan Conjecture, Theorem 7.1.1)**.** *Let $F/\mathbf{Q}$ be an imaginary CM field. Let π be a cuspidal algebraic automorphic representation for $\mathrm{GL}(2)/F$ of parallel weight $k \geq 2$. Then π_v is tempered for all finite places v; in particular, for places v prime to the level of π, the Satake parameters $\{\alpha_v, \beta_v\}$ of π_v satisfy $|\alpha_v| = |\beta_v| = N(v)^{(k-1)/2}$.*

Theorem B (Sato–Tate Conjecture, Theorem 7.2.3)**.** *Let $F/\mathbf{Q}$ be an imaginary CM field. Let π be a cuspidal algebraic automorphic representation for $\mathrm{GL}(2)/F$ of parallel weight $k \geq 2$. Assume that π does not have CM, equivalently, π is not the automorphic induction of an algebraic Hecke character from a quadratic CM extension F'/F. For each finite place v prime to the level of π, let $a_v = (\alpha_v + \beta_v)/(2N(v)^{(k-1)/2})$ denote the normalized parameter, and suppose that the a_v are real. Then the a_v are uniformly distributed with respect to the Sato–Tate measure $2/\pi \cdot \sqrt{1-x^2}dx$.*

As in the case $F = \mathbf{Q}$, a minor modification of the statement is needed when the a_v are not real; we relegate the details of this to Section 7.2. We also discuss some alternate formulations of Theorem A when $F/\mathbf{Q}$ is an imaginary quadratic field in Section 1.3.

To prove Theorems A and B, we prove the potential automorphy of the symmetric powers of the compatible systems of Galois representations associated to a cuspidal, regular algebraic automorphic representation π of $\mathrm{GL}_2(\mathbf{A}_F)$. Here again is a simplified version of our main result in this direction. (To orient the reader, the integer $k \geq 2$ parametrizing the weight in this discussion above is related the integer $m \geq 1$ below via the relation $m = k - 1$. This mirrors the fact that the Hodge–Tate weights of p-adic Galois representations associated to modular forms of weight k are equal to $\{0, k-1\}$.)

Theorem C (Potential automorphy of symmetric powers, Theorem 7.2.1)**.** *Let F be a CM field, let M be a number field, and let $m \geq 1$ be an integer. Suppose we have a system of Galois representations*

$$\rho_\lambda : G_F \to \mathrm{GL}_2(\overline{M}_\lambda)$$

indexed by primes λ of M with the following compatibilities:

(1) ρ_λ is unramified outside a finite set of primes $\{v \in S\} \cup \{v | N(\lambda)\}$ where S is independent of λ. For any v not in this set, the characteristic polynomial $P_v(X) = X^2 + a_v X + b_v$ of $\rho_\lambda(\mathrm{Frob}_v)$ lies in $M[X]$ and is independent of v.

(2) For all but finitely many λ, the representations $\rho_\lambda|_{G_v}$ for primes $v|N(\lambda)$ and $v \notin S$ are crystalline with Hodge–Tate weights $H = \{0, m\}$ for every embedding of F into $\overline{\mathbf{Q}}_p$.

Assume that at least one ρ_λ is irreducible. Then:

(1) **Purity:** *for any embedding of $M \hookrightarrow \mathbf{C}$, the roots α_v and β_v of $X^2 + a_v X + b_v$ have absolute value $q^{m/2}$ where $q = N(v)$.*

(2) **Potential automorphy:** *There is a number field F'/F such that the restrictions $\rho_\lambda|_{G_{F'}}$ are all automorphic and associated to a fixed cuspidal algebraic π for $\mathrm{GL}(2)/F'$.*

(3) **Potential automorphy of symmetric powers:** *Fix $n - 1 \geq 2$. Either:*

  (a) The ρ_λ are all induced from a compatible system associated to an algebraic Hecke character χ of some quadratic extension F'/F. Then $\mathrm{Sym}^{n-1} \rho_\lambda$ is reducible and decomposes into representations of dimension two and one which are all automorphic over F.

  (b) There is a number field F'/F such that the representations $\mathrm{Sym}^{n-1} \rho_\lambda|_{G_{F'}}$ are all irreducible and automorphic, associated to a fixed cuspidal algebraic Π for $\mathrm{GL}(n)/F'$.

The Galois representations associated to cuspidal, regular algebraic automorphic representations of $\mathrm{GL}_2(\mathbf{A}_F)$ are not yet known to satisfy the conditions of Theorem C, but rather a weaker condition (they form a 'very weakly compatible system'). We establish potential automorphy of symmetric powers also under this weaker condition. Once again, we refer to the statement of Theorem 7.2.1 in the main body of the paper for the precise statement that is used to deduce Theorems 7.1.1 and 7.2.3 (and therefore Theorems A and B above).

1.1. **The new ideas in this paper.** When $m = 1$, Theorems A, B and C were proved in [ACC+23] (see [ACC+23, Thm 1.01, Thm 1.0.2, Thm 7.1.14]). The deduction of Theorems A and B from Theorem C exactly parallels the arguments in [ACC+23], so we now focus on explaining the proof of Theorem C.

Unsurprisingly, our arguments build on those of [ACC+23]: in particular, we prove the potential automorphy of the compatible system of symmetric powers $\mathrm{Sym}^{n-1}\mathcal{R} = \{\mathrm{Sym}^{n-1}\rho_\lambda\}$ by checking the residual automorphy over some extension F'/F and then applying an automorphy lifting theorem. We would like to highlight three new ingredients which appear here:

(1) A result on generic reducedness of special fibres of weight 0 (local) crystalline deformation rings (see §1.2 for a further introductory discussion). Using the local-global compatibility result of [CN23], this leads to a new automorphy lifting theorem in the setting of arbitrary ramification (Theorem 3.2.1).
(2) An application of a theorem of Drinfeld and Kedlaya [DK17], showing generic ordinarity of families of Dwork motives (Proposition 4.2.6). This makes it possible to verify the potential residual automorphy of certain residual representations by an automorphic motive which is crystalline ordinary at some set of p-adic places.
(3) A "p-q-r" switch including a version of the "Harris tensor product trick" which incorporates an additional congruence between two tensor products of compatible families. One is a tensor product of $\mathrm{Sym}^{n-1}\mathcal{R}$ with an induction of a character, as usual. The other is a tensor product of $\mathrm{Sym}^{n-1}\mathcal{R}$ with a different auxiliary compatible family, which gives us more flexibility to realise different local properties at places related by complex conjugation. We discuss in the remainder of the introduction the need for this argument, and give a more detailed sketch in §6.2 below.

To explain in more detail the need for these innovations, suppose given a compatible system $\mathcal{R} = \{\rho_\lambda\}$ as in the statement of Theorem C, therefore of Hodge–Tate weights $\{0, m\}$ for some $m \geq 1$ (and with $m \geq 2$ if we hope to go beyond the cases treated in [ACC+23]). The general strategy for proving potential automorphy (the so-called "p-q switch") is as follows:

(1) After making some CM base extension H/F (depending on n), find an auxiliary n-dimensional compatible system $\mathcal{S} = \{s_\lambda\}$ such that:
   (a) For one prime λ, the residual representations $\mathrm{Sym}^{n-1}\overline{\rho}_\lambda|_{G_H}$ and $\overline{s}_\lambda$ coincide, and moreover satisfy a number of standard "Taylor–Wiles" conditions.
   (b) For a second prime λ', the residual representation $\overline{s}_{\lambda'}$ is induced from a character and is thus residually automorphic.
   (c) The Hodge–Tate weights of the compatible system $\mathcal{S}$ coincide with those of $\mathrm{Sym}^{n-1}\mathcal{R}|_{G_H}$.
(2) Apply an automorphy lifting theorem at λ' to deduce that the compatible system $\mathcal{S}$ is automorphic. Then deduce that the residual representation $\overline{s}_\lambda$ is automorphic, and use automorphy lifting theorems again to deduce that $\mathrm{Sym}^{n-1}\mathcal{R}|_{G_H}$ is automorphic.

In our setting, both of these steps cause problems, but those affecting the second step are more serious.

The issue in the first step is the requirement (1)(c) on the Hodge–Tate weights. The most natural source of compatible systems $\mathcal{S}$ are those arising from motives, and a geometrically varying family of motives cannot have Hodge–Tate weights

$0, m, \ldots, m(n-1)$ with $m \geq 2$ by Griffiths transversality. (This difficulty is already present if $F = \mathbf{Q}$ and one wants to prove the Sato–Tate conjecture for a classical modular form of weight greater than 2, such as Δ.) The now-standard resolution to this problem is to employ the "Harris tensor product trick" [Har09], and replace $\mathrm{Sym}^{n-1}\mathcal{R}$ by $\mathrm{Sym}^{n-1}\mathcal{R} \otimes \mathrm{Ind}_{G_L}^{G_F}\mathcal{X}$ for some cyclic CM extension L/F, where $\mathcal{X}$ is a compatible system of algebraic Hecke characters chosen sufficiently carefully so that this new compatible system has consecutive Hodge–Tate weights. Now in the second step, one wants to prove this new compatible system is potentially automorphic (using for $\mathcal{S}$ a compatible system coming from the cohomology of the Dwork family). The potential automorphy of $\mathrm{Sym}^{n-1}\mathcal{R}$ can then be deduced using cyclic base change [AC89].

For the second step, applying an automorphy lifting theorem typically requires that the compatible systems $\mathcal{S}$ and $\mathrm{Sym}^{n-1}\mathcal{R} \otimes \mathrm{Ind}_{G_L}^{G_F}\mathcal{X}$ have "the same" behaviour at places $v|p$. There are two problems with this. Firstly, we will need an automorphy lifting theorem that applies to arbitrarily ramified F, including non-ordinary representations. Secondly, the compatible system of characters $\mathcal{X}$ has a restricted form, and in particular its local behaviour can't be chosen arbitrarily at a pair of conjugate places. We explain more about these difficulties and their resolution below.

For context, we first recall the situation when $F = \mathbf{Q}$. For example, one might try to demand that the p-adic representations in the compatible systems $\mathcal{S}$ and $\mathrm{Sym}^{n-1}\mathcal{R} \otimes \mathrm{Ind}_{G_L}^{G_F}\mathcal{X}$ are both ordinary, and indeed it is straightforward (at least after a ramified base change) to find ordinary representations in the Dwork family, and presumably difficult to understand the non-ordinary representations in any generality. This means that one would like to show that many of the representations in the compatible system $\mathcal{R}$ are ordinary.

For a weight 2 modular form, it is relatively easy to prove that there are infinitely many primes p for which the p-adic Galois representation is ordinary at p. However, the existence of infinitely many ordinary primes for Δ (or for any non-CM form of weight $k \geq 4$) remains an open question, so one also has to consider the possibility that the residual representation $\overline{r}_\lambda|_{G_{F_v}}$ is locally of the form $\omega_2^m \oplus \omega_2^{mp}$ on inertia at p. This problem was resolved for classical modular forms in [BLGHT11], via a further study of the Dwork family; in particular, showing that certain residual representations of the shape $\mathrm{Sym}^{n-1}(\omega_2 \oplus \omega_2^p)$ arise (locally on inertia) as residual representations in that family.

We now consider the case of an imaginary CM field F, and explain why we need an automorphy lifting theorem allowing ramification at non-ordinary places. Given the automorphy lifting theorems for CM fields proved in [ACC+23], the most serious difficulty in adapting the strategy of [BLGHT11] is that there is no way to avoid the possibility that a representation ρ_λ can be simultaneously ordinary at one prime $v|p$ and non-ordinary at the complex conjugate place v^c. (One might hope to avoid this by considering places with $v = v^c$, but then we would have to show that certain residual representations of $G_{\mathbf{Q}_{p^2}}$ occur in the Dwork family which seemed to us to be a difficult task.) This is a problem because the automorphy lifting theorems in [ACC+23] for non-ordinary representations require F to be unramified at our non-ordinary prime v^c; while at the ordinary prime v, we need to be able to make a highly ramified base change of (imaginary) CM fields F'/F to find an appropriate representation in the Dwork family. It is however impossible to arrange that such an extension of CM fields is unramified at v^c and ramified at v. One of the key innovations in this paper is to prove an automorphy lifting theorem that allows us to make a ramified base change at v^c (Theorem 3.2.1). This was done in the two-dimensional case in [CN23]; we discuss the difficulties in extending this result to higher dimensions and how we overcome them in Section 1.2 below. Note that, even when making a ramified base change, it is still important for us to keep track of the inertial type of residual representations in the Dwork family in order to show that the representations of interest are connected in the deformation space.

There turns out to be one final wrinkle, where the second problem mentioned above arises. The p-adic representations in our compatible system $\mathcal{S}$ will satisfy one of two, mutually exclusive, local conditions at each p-adic place: they are crystalline at p and are either ordinary or are (on the same component of a local crystalline deformation ring as) a symmetric power of an induction of a Lubin–Tate character of $G_{\mathbf{Q}_{p^2}}$. It turns out that we can't always arrange for a tensor product $\operatorname{Sym}^{n-1}\mathcal{R}\otimes\operatorname{Ind}_{G_L}^{G_F}\mathcal{X}$ to have local p-adic representations of this shape. The problem is that algebraic Hecke characters have a very restricted form, and the fact that F is an imaginary CM field implies that a suitable choice of $\mathcal{X}$ will exist only if ρ_λ is either both ordinary or non-ordinary at each pair of places $\{v, v^c\}$ permuted by complex conjugation in $\operatorname{Gal}(F/F^+)$.

Our solution is to instead consider tensor products of the form $(\operatorname{Sym}^{n-1}\mathcal{R})\otimes\mathcal{R}_{\mathrm{aux}}$, where $\mathcal{R}_{\mathrm{aux}}$ is a compatible system coming from (part of) the cohomology of the Dwork hypersurface [Qia23]. We will be able to choose $\mathcal{R}_{\mathrm{aux}}$ so that one of the local conditions mentioned in the previous paragraph will be satisfied by $\mathcal{S}_{\mathrm{aux}} = (\operatorname{Sym}^{n-1}\mathcal{R})\otimes\mathcal{R}_{\mathrm{aux}}$ at each p-adic place. It is now no longer possible to directly deduce the potential automorphy of $\operatorname{Sym}^{n-1}\mathcal{R}$ from the potential automorphy of this product. This is not necessary to prove the Ramanujan conjecture — already the automorphy of this tensor product combined with the Jacquet–Shalika bounds (and the fact that $\mathcal{R}_{\mathrm{aux}}$ is pure) is enough to deduce purity — but it is necessary to prove the Sato–Tate conjecture. However, once the potential automorphy of $\mathcal{S}_{\mathrm{aux}}$ is established, we can (having chosen $\mathcal{R}_{\mathrm{aux}}$ carefully to begin with) find a third compatible system $\mathcal{R}_{\mathrm{CM}}$ such that $\mathcal{S}_{\mathrm{aux}} = (\operatorname{Sym}^{n-1}\mathcal{R})\otimes\mathcal{R}_{\mathrm{aux}}$ and $\mathcal{S}_{\mathrm{CM}} = (\operatorname{Sym}^{n-1}\mathcal{R})\otimes\mathcal{R}_{\mathrm{CM}}$ are residually the same at a *third* prime r, and $\mathcal{R}_{\mathrm{CM}}$ is induced from a character. Even though we do not have any control over the r-adic representation associated to $\operatorname{Sym}^{n-1}\mathcal{R}$ locally at $v|r$, the fact that it occurs as the same tensor factor in the r-adic representations of both $\mathcal{S}_{\mathrm{aux}}$ and $\mathcal{S}_{\mathrm{CM}}$ means we can still put ourselves in a situation where both r-adic Galois representations lie on the same component of a local deformation ring at $v|r$. From this p-q-r switch, we can show that $\mathcal{S}_{\mathrm{CM}} = (\operatorname{Sym}^{n-1}\mathcal{R})\otimes\mathcal{R}_{\mathrm{CM}}$ is potentially automorphic, from which we deduce that $\operatorname{Sym}^{n-1}\mathcal{R}$ is potentially automorphic.

One might also ask whether for general CM fields F one can drop the hypothesis that $\mathcal{R}$ has parallel weight. The difficulty in doing so is as follows: in order to pass from $\operatorname{Sym}^{n-1}\mathcal{R}$ to a compatible system with consecutive Hodge–Tate weights, one needs to tensor this compatible system with a second compatible system with certain prescribed local properties. If $\mathcal{R}$ does not have parallel weight, this auxiliary compatible system cannot have consecutive Hodge–Tate weights and for reasons explained above also cannot be induced from a compatible system of characters. It is very hard to construct such compatible systems because of the constraints on families of geometric local systems imposed by Griffiths transversality. The existence of even a *single* regular algebraic cuspidal automorphic representation for $\mathrm{GL}_2(\mathbf{A}_F)$ for some CM field F which is neither of parallel weight 2, nor of CM type, nor arising from base change from the totally real subfield F^+ was only found (by a computation) in [CM09, Lemma 8.11(2)] (see also [RŞ13]).

1.2. **Ihara avoidance and the Emerton–Gee stack.** There are two main difficulties in proving automorphy lifting liftings for p-adic representations with p ramified in F. One is having local–global compatibility theorems at the places dividing p; this was resolved in the recent work of Caraiani–Newton [CN23]. The other difficulty was alluded to above: the usual Taylor–Wiles method for automorphy lifting only allows us to deduce the automorphy of a p-adic representation r from the automorphy of a congruent representation r' if we know that for all finite places v, the representations $r|_{G_{F_v}}$ and $r'|_{G_{F_v}}$ are "connected", in the sense that they lie on the same component of the appropriate local deformation ring.

As we have sketched above, in the particular cases that we consider in this paper, we have arranged this property at the places $v|p$ by considering the ordinary and non-ordinary cases separately. (It was this construction that required us to pass to a situation where p is highly ramified in F.) We are not, however, able to arrange that our representations are connected at all the places $v \nmid p$. Fortunately, Taylor [Tay08] found a way to prove automorphy lifting theorems when the representations fail to be connected at some places $v \nmid p$, using his so-called "Ihara avoidance" argument. This argument makes an ingenious use of two different local deformation problems at places $v \nmid p$, which are congruent modulo p, and relates two corresponding patched modules of automorphic forms. The key point which makes this argument possible is to work with local deformation rings having the following "unique generalization" property: any generic point of their special fibre has a unique generalization to the generic fibre. More geometrically, we need to avoid having two distinct irreducible components in characteristic zero which specialize to a common irreducible component in the special fibre.

In order to apply this argument one also needs the unique generalization property for the deformation rings at the places $v|p$. This was previously only known in the Fontaine–Laffaille and ordinary contexts, in which case the crystalline deformation rings can be understood completely explicitly (and in the former case, there is even a unique irreducible component). (This problem was sidestepped to some extent in [BLGG11, BLGGT14], but the approach there combines the Ihara avoidance argument with the Khare–Wintenberger lifting argument to produce characteristic zero lifts of residual representations of the prescribed weight and level. In our $\ell_0 > 0$ situation (in the language of [CG18]) such lifts do not always exist.)

One way to establish the unique generalization property (when it holds) would be to explicitly compute the irreducible components of the generic fibres of the deformation rings, but this appears to be hopeless for crystalline deformation rings in any generality. However, as was already observed in [Tay08, §3] in the case $v \nmid p$, an alternative approach is to consider an appropriate moduli stack of Galois representations, for which the deformation rings are versal rings at closed points. One shows that its special fibre is generically reduced (or even generically smooth), for example by showing that the deformation rings for generic choices of the residual Galois representation are formally smooth. It then follows that for an arbitrary residual representation, the special fibres of the ($\mathbf{Z}_p$-flat quotients of) the deformation rings are generically reduced, which implies the unique generalization property.

(While [Tay08, §3] does not explicitly work with moduli stacks of Galois representations, [Tay08, Lem. 3.2] is easily reformulated in these terms; and while [Tay08, Prop. 3.1(3)] does not explicitly state that the $\mathbf{Z}_p$-flat quotient of the deformation ring has generically reduced special fibre, this follows from the argument, as in [ANT20, Prop. 3.1].)

The unique generalization property has subsequently been used by Thorne in a context with $v \nmid p$ in the proof of [Tho12, Thm. 8.6] (in order to avoid any additional hypotheses when introducing an auxiliary prime to make the level structure sufficiently small), and in the case that $v|p$ by Caraiani–Newton [CN23], who used the results of [CEGS22a], which establish the generic reducedness of the special fibres of the crystalline deformation rings in the 2-dimensional (tamely potentially) Barsotti–Tate case, by an analysis of the corresponding Emerton–Gee stacks.

Unfortunately an (unconditional) argument with the Breuil–Mézard conjecture shows that generic reducedness is extremely rare when $v|p$ (see Remark 2.5.6). We are however able to prove the following theorem.

Theorem D (Theorem 2.5.5)**.** *Suppose that $p > n$, that $K/\mathbf{Q}_p$ and $\mathbf{F}/\mathbf{F}_p$ are finite extensions, and that $\overline{\rho} : G_K \to \mathrm{GL}_n(\mathbf{F})$ is a continuous representation. Let $R^{\mathrm{crys},0}$ be the universal lifting ring for crystalline lifts of $\overline{\rho}$ of parallel Hodge–Tate weights $0, 1, \ldots, n-1$. Then the special fibre of $\operatorname{Spec} R^{\mathrm{crys},0}$ is generically reduced.*

We refer the reader to the introduction to Section 2 for a detailed overview of the proof of Theorem D, which as above relies on proving the corresponding property of the relevant Emerton–Gee stacks [EG23] (whose versal rings are the crystalline lifting rings). The irreducible components of the special fibres of these stacks were described in [EG23], and we prove our result by combining this description with a computation of extensions of rank 1 Breuil modules. An amusing feature of this argument is that we prove a result about the deformation rings of arbitrary n-dimensional mod p representations by reducing to a calculation for reducible 2-dimensional representations.

1.3. **Bianchi Modular Forms.** Let us specialize to the case when $F/\mathbf{Q}$ is an imaginary quadratic field. Let π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$. Let χ be the central character of π. By definition, the representation π occurs in $L^2_{\mathrm{cusp}}(\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbf{A}_F))$. Let $\mathfrak{g}$ be the Lie algebra of $\mathrm{GL}_2(\mathbf{C})$ as a real group. The assumption that π is regular algebraic is equivalent to the condition that the infinitesimal character of π_∞ is the same as $V^\vee$ for an algebraic representation V of $\mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_2$. The assumption that π is cuspidal places a restriction on V corresponding to the fact (noted earlier) that such π has parallel weight; the corresponding representations are parametrized (up to twist) by an integer $k \geq 2$, where $k = 2$ corresponds to the case when V is trivial. This choice of k determines the action of $Z(\mathfrak{g})$ on π_∞, and by taking functions which are suitable eigenvectors under $Z(\mathfrak{g})$, we may arrive at certain vector valued Hecke eigenfunctions Φ on $\mathrm{GL}_2(\mathbf{A}_F)$ with Fourier expansions ([Wil17, §1.2], [Hid94, §6])

$$f\left[\begin{pmatrix} t & z \\ 0 & 1 \end{pmatrix}\right] = |t|_F \sum_{\alpha \in F^\times} c(\alpha t \delta_F, f) W(\alpha t_\infty) e_F(\alpha z),$$

where $\delta = \delta_F$ is the different, $\alpha t \delta$ can be interpreted as a fractional ideal of $\mathcal{O}_F$, $c(I, f)$ is a Fourier coefficient which vanishes unless $I \subset \mathcal{O}_F$ and which we may assume is normalized so that $c(\mathcal{O}_F, f) = 1$, W is an explicit Whittaker function which is vector valued in some explicit representation of SU(2) depending on k, and e_F is an explicit additive character of $F\backslash\mathbf{A}_F$. This has a direct translation into more classical language, and can be interpreted as a collection of h_F functions on a finite union of hyperbolic spaces $\mathbf{H}^3$. The explicit functions f (either adelically or classically) are known as Bianchi modular forms. For a normalized Bianchi eigenform f of weight k and level prime to $\mathfrak{p}$, Theorem 7.1.1 implies the following bound:

Theorem E. *Let f be a cuspidal Bianchi modular eigenform of level $\mathfrak{n}$ and weight k. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_F$ not dividing $\mathfrak{n}$, and let $c(\mathfrak{p}, f)$ be an eigenvalue of $T_\mathfrak{p}$ on H. Then*

$$|c(\mathfrak{p}, f)| \leq 2N(\mathfrak{p})^{(k-1)/2}. \tag{1.3.0}$$

This connects our theorem with the more classical version of the Ramanujan conjecture for modular forms [Del71] as discussed earlier in the introduction.

The eigenvalues $c(\mathfrak{p}, f)$ associated to f have a second interpretation in terms of the cohomology of arithmetic groups and arithmetic hyperbolic 3-manifolds. The algebraic representations V of $\operatorname{Res}_{F/\mathbf{Q}} \mathrm{GL}_2$ are all, up to twist, given on real points $\operatorname{Res}_{F/\mathbf{Q}} \mathrm{GL}_2(\mathbf{R}) = \mathrm{GL}_2(\mathbf{C})$ by the representations $\operatorname{Sym}^{k-2}\mathbf{C}^2 \otimes \overline{\operatorname{Sym}^{l-2}\mathbf{C}^2}$ for a pair of integers $k, l \geq 2$. Let $\mathfrak{n} \leq \mathcal{O}_F$ be a non-zero ideal. Having fixed k and l, we can form the group cohomology

$$H = H^1(\Gamma_1(\mathfrak{n}), \operatorname{Sym}^{k-2}\mathbf{C}^2 \otimes \overline{\operatorname{Sym}^{l-2}\mathbf{C}^2})$$

of the standard congruence subgroup $\Gamma_1(\mathfrak{n}) \leq \mathrm{GL}_2(\mathcal{O}_F)$. Then H is a finite-dimensional $\mathbf{C}$-vector space. Let $H_{\mathrm{par}} \subset H$ denote the subgroup consisting of classes which vanish under the restriction of H to $H^1(P, \operatorname{Sym}^{k-2}\mathbf{C}^2 \otimes \overline{\operatorname{Sym}^{l-2}\mathbf{C}^2})$ for any parabolic subgroup $P \subset \Gamma_1(\mathfrak{n})$. More geometrically, one can interpret H as the cohomology of a local system on the Bianchi manifold $Y_1(\mathfrak{n}) = \mathbf{H}^3/\Gamma_1(\mathfrak{n})$. If $X_1(\mathfrak{n})$ is the Borel–Serre compactification of $Y_1(\mathfrak{n})$, then parabolic cohomology consists of classes which are trivial on the boundary $X_1(\mathfrak{n}) \smallsetminus Y_1(\mathfrak{n})$; this boundary may be identified (when $\Gamma_1(\mathfrak{n})$ is torsion free) with a finite disjoint union of complex tori. The spaces H and H_{par} are equipped with a commuting family of linear operators, the unramified Hecke operators $T_\mathfrak{p}$, indexed by the principal ideals $\mathfrak{p} \leq \mathcal{O}_F$ not dividing $\mathfrak{n}$. More precisely, if one writes $\mathfrak{p} = (\pi)$, then the group $A\Gamma_1(\mathfrak{n})A^{-1} \cap \Gamma_1(\mathfrak{n}) = \Gamma_1(\mathfrak{n}, \mathfrak{p})$ has finite index in $\Gamma_1(\mathfrak{n})$, where

$$A = \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix};$$

the map $T_\mathfrak{p}$ is induced by composing (in a suitable order) a restriction map, a conjugation by A map, and a trace map respectively. Note that the existence of (a large family of) such operators comes from the fact that $\Gamma = \mathrm{GL}_2(\mathcal{O}_F)$ has infinite index inside its commensurator in $\mathrm{GL}_2(F)$; as shown by Margulis [Mar91, Thm. IX.1.13], this characterizes the arithmeticity of Γ.

In order to obtain an action of $T_\mathfrak{p}$ for more general prime ideals $\mathfrak{p}$, one needs to replace $Y_1(\mathfrak{n})$ by a disconnected union of h_F commensurable arithmetic hyperbolic manifolds $Y_1(\mathfrak{n}; \mathfrak{a}) = \mathbf{H}/\Gamma_1(\mathfrak{n}; \mathfrak{a})$ indexed by ideals $\mathfrak{a}$ in the class group $\mathrm{Cl}(\mathcal{O}_F)$ of F prime to $\mathfrak{n}$, and where $Y_1(\mathfrak{n}; \mathcal{O}_F) = Y_1(\mathfrak{n})$. The group $\Gamma(\mathcal{O}_F; \mathfrak{a})$ is the automorphism group of the $\mathcal{O}_F$-module $\mathcal{O}_F \oplus \mathfrak{a}$, which consists explicitly of matrices of the form

$$\begin{pmatrix} \mathcal{O}_F & \mathfrak{a}^{-1} \\ \mathfrak{a} & \mathcal{O}_F \end{pmatrix} \cap \mathrm{GL}_2(F)$$

with determinant in $\mathcal{O}_F^\times$. The space H_{par} vanishes unless $k = l$ ([Har87, 3.6.1]), which we now assume. If $h_F = 1$, the Eichler–Shimura isomorphism ([Har87, §3.6]) gives a map from Bianchi cuspidal modular eigenforms f of weight k as described above and cohomology classes $\eta_f \in H_{\mathrm{par}}$ which are simultaneous eigenforms for all the Hecke operators. Moreover, the eigenvalues of $T_\mathfrak{p}$ on η_f are given exactly by $c(\mathfrak{p}, f)$. If $h_F > 1$, one must replace H and h_{par} by the direct sum of the corresponding cohomology groups over $Y_1(\mathfrak{n}; \mathfrak{a})$ for $\mathfrak{a} \in \mathrm{Cl}(\mathcal{O}_F)$. Theorem E now implies:

Theorem F. *Let $\mathfrak{p}$ be a principal prime ideal of $\mathcal{O}_F$ not dividing $\mathfrak{n}$, and let $a_\mathfrak{p}$ be an eigenvalue of $T_\mathfrak{p}$ on H_{par}. Then $|a_\mathfrak{p}| \le 2N(\mathfrak{p})^{(k-1)/2}$.*

These explicit formulations of our theorems can be generalized in a number of ways. Remaining in the setting of arithmetic hyperbolic 3-manifolds (or orbifolds), we can replace $\mathrm{GL}_2(\mathcal{O}_F)$ by a congruence subgroup Γ of the norm one units in a maximal order $\mathcal{O}$ of a division algebra D/F where $F \hookrightarrow \mathbf{C}$ is a number field with one complex place and D is definite at all real places of F. When $[F : \mathbf{Q}] = 2$, we obtain the Bianchi manifolds as above (when D/F is split) but also certain compact hyperbolic arithmetic three manifolds; our theorem applies equally well in the latter case (note that $H = H_{\mathrm{par}}$ in this setting). On the other hand, suppose that F has at least one real place; for example, take $F = \mathbf{Q}[\theta]/(\theta^3 - \theta + 1)$, let $k = 2$, let D/F be ramified at the real place and the unique prime of norm 5. Now $H = H_{\mathrm{par}}$ is the first cohomology group of a congruence cover of the Weeks manifold. The generalized Ramanujan conjecture still predicts a bound of the shape $|a_\mathfrak{p}| \le 2N(\mathfrak{p})^{1/2}$ for the eigenvalues of the Hecke operators $T_\mathfrak{p}$. However, our methods do not apply in this situation, and the best current bounds remain those of the form $|a_\mathfrak{p}| \le 2N(\mathfrak{p})^{1/2+7/64}$ proved using analytic methods (see [Sar05]).

We finish with an application of a different sort. Let $\Gamma = \mathrm{SL}_2(\mathcal{O}_F)$. The quotients $\Gamma\backslash\mathbf{H}^3$ were first investigated by Bianchi [Bia1892], and for that reason they are known as Bianchi orbifolds. For a Bianchi modular form f of level one, one may [Mar11, §3] associate to f a normalized measure μ_f on $\Gamma\backslash\mathbf{H}^3$. One then has the following [Mar11, Cor 3]:

Theorem G. *Assume that F has class number one*[1]*. For any sequence of Bianchi modular eigenforms f of weight tending to ∞, the measures μ_f converge weakly to the hyperbolic volume on $Y = \mathrm{SL}_2(\mathcal{O}_F)\backslash\mathbf{H}^3$.*

Proof. As noted in [Mar12], the proof given in [Mar11] assumes the Ramanujan conjecture for Bianchi modular forms — this is now a consequence of Theorem E. □

[1]The paper [Mar11] has the following to say about this assumption: "For simplicity, we assume our fields to have narrow class number one throughout the paper, but this is not essential." One might therefore expect it to be possible to prove the more general adelic statement for all imaginary quadratic F under the additional hypothesis (as explained in [Nel12, §1]) that, when h_F is even, one avoids certain dihedral forms which vanish identically on half of the connected components of the adelic quotient. Similarly, concerning the assumption on the level, the paper [Mar11] says "The proof may easily be modified to allow a nontrivial level in any case."

1.4. **Recent work of Matsumoto.** A few months after the first preprint version of this work was circulated, a remarkable new work by Matsumoto appeared [Mat24] which proves Theorems A and B with no parallel weight condition. Matsumoto's approach introduces several new ideas of a global nature, whilst our approach here is based on refining our understanding of the local ingredients in the potential automorphy argument. For this reason, one might hope that the two approaches could be profitably combined in the future.

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1.6. **Notation.** Let $K/\mathbf{Q}_p$ be a finite extension. If $\sigma : K \hookrightarrow \overline{\mathbf{Q}}_p$ is a continuous embedding of fields then we will write $\mathrm{HT}_\sigma(\rho)$ for the multiset of Hodge–Tate numbers of ρ with respect to σ, which by definition contains i with multiplicity $\dim_{\overline{\mathbf{Q}}_p}(W \otimes_{\sigma,K} \widehat{\overline{K}}(i))^{G_K}$. We write ε for the p-adic cyclotomic character, which is a crystalline representation with $\mathrm{HT}_\sigma(\varepsilon) = \{-1\}$ for each σ.

We say that ρ has weight 0 if for each $\sigma : K \hookrightarrow \overline{\mathbf{Q}}_p$ we have $\mathrm{HT}_\sigma(\rho) = \{0, 1, \ldots, d-1\}$. We often somewhat abusively write that a representation $\rho : G_K \to \mathrm{GL}_d(\overline{\mathbf{Z}}_p)$ is crystalline of weight 0 if the corresponding representation $\rho : G_K \to \mathrm{GL}_d(\overline{\mathbf{Q}}_p)$ is crystalline of weight 0.

Let $\mathcal{O}$ be the ring of integers in some finite extension $E/\mathbf{Q}_p$, and suppose that E is large enough that it contains the images of all embeddings $\sigma : K \hookrightarrow \overline{\mathbf{Q}}_p$. Write ϖ for a uniformizer of $\mathcal{O}$, and $\mathcal{O}/\varpi = \mathbf{F}$ for its residue field. We write $\mathrm{Art}_K : K^\times \to W_K^{\mathrm{ab}}$ for the isomorphism of local class field theory, normalized so that uniformizers correspond to geometric Frobenius elements.

Let $\overline{\rho} : G_K \to \mathrm{GL}_d(\overline{\mathbf{F}}_p)$ be a continuous representation. Then after enlarging E and thus $\mathbf{F}$ if necessary, we may assume that the image of $\overline{\rho}$ is contained in $\mathrm{GL}_d(\mathbf{F})$. We write $R_{\overline{\rho}}^{\square,\mathcal{O}}$ for the universal lifting $\mathcal{O}$-algebra of $\overline{\rho}$; by definition, this (pro-)represents the functor $\mathcal{D}_{\overline{\rho}}^{\square,\mathcal{O}}$ given by lifts of $\overline{\rho}$ to representations $\rho : G_K \to \mathrm{GL}_d(A)$, for A an Artin local $\mathcal{O}$-algebra with residue field $\mathbf{F}$. The precise choice of E is unimportant, in the sense that if $\mathcal{O}'$ is the ring of integers in a finite extension E'/E, then by [BLGGT14, Lem. 1.2.1] we have $R_{\overline{\rho}}^{\square,\mathcal{O}'} = R_{\overline{\rho}}^{\square,\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{O}'$.

We write $R_{\overline{\rho}}^{\mathrm{crys},\underline{0},\mathcal{O}}$ for the unique $\mathcal{O}$-flat quotient of $R_{\overline{\rho}}^{\square,\mathcal{O}}$ with the property that if B is a finite flat E-algebra, then an $\mathcal{O}$-algebra homomorphism $R_{\overline{\rho}}^{\square,\mathcal{O}} \to B$ factors through $R_{\overline{\rho}}^{\mathrm{crys},0,\mathcal{O}}$ if and only if the corresponding representation of G_K is crystalline of weight 0.

We will let rec_K be the local Langlands correspondence of [HT01], so that if π is an irreducible complex admissible representation of $\mathrm{GL}_n(K)$, then $\mathrm{rec}_K(\pi)$ is a Frobenius semi-simple Weil–Deligne representation of the Weil group W_K. We write rec_K^T for the arithmetic normalization of the local Langlands correspondence, as defined in e.g. [CT14, §2.1]; it is defined on irreducible admissible representations of $\mathrm{GL}_n(K)$ defined over any field which is abstractly isomorphic to $\mathbf{C}$ (e.g. $\overline{\mathbf{Q}}_l$).

Let F be a number field. If v is a finite place of F then we write $k(v)$ for the residue field of F_v. We identify dominant weights λ of $\mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_n$ with sets of tuples of integers $(\lambda_{\tau,1} \geq \lambda_{\tau,2} \geq \cdots \geq \lambda_{\tau,n})_{\tau:F\hookrightarrow \mathbf{C}}$ indexed by complex embeddings of F (cf. [ACC+23, §2.2.1]). If π is an irreducible admissible representation of $\mathrm{GL}_n(\mathbf{A}_F)$ and λ is a dominant weight, we say that π is regular algebraic of weight λ if the infinitesimal character of π_∞ is the same as that of $V_\lambda^\vee$, where V_λ is the algebraic representation of $\mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_n$ of highest weight λ. We say that π is regular algebraic if it is regular algebraic of some weight.

Definition 1.6.1 (Parallel Weight)**.** Suppose π is regular algebraic of weight λ. We say that π is of parallel weight if $\lambda_{\tau,1} - \lambda_{\tau,2}$ is independent of τ; equivalently, if π admits a regular algebraic twist of weight $\mu = (m-1, 0)_\tau$ for some $m \geq 1$ in $\mathbf{Z}$. We say that π has parallel weight k for some integer $k \geq 2$ if $\mu = (k-2, 0)_\tau$.

Let F be an imaginary CM field, and let π be a cuspidal, regular algebraic weight λ automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$. The weight $\lambda = (\lambda_{\tau,1}, \lambda_{\tau,2})_\tau \in (\mathbf{Z}^2)^{\mathrm{Hom}(F,\mathbf{C})}$ satisfies:

- There is an integer $w \in \mathbf{Z}$ such that for all τ, we have $\lambda_{\tau,1} + \lambda_{\tau c,2} = w$. In particular, for all τ we have $\lambda_{\tau,1} - \lambda_{\tau,2} = \lambda_{\tau c,1} - \lambda_{\tau c,2}$.

This is a consequence of Clozel's purity lemma [Clo90, Lemma 4.9]. In particular, if F is imaginary quadratic, π is necessarily of parallel weight.

2. The special fibres of weight 0 crystalline lifting rings are generically reduced

The goal of this section is to prove Theorem 2.5.5, which shows that if $p > n$, then for any finite extension $K/\mathbf{Q}_p$ and any $\overline{\rho} : G_K \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$, the special fibre of the corresponding weight 0 crystalline lifting ring is generically reduced. We deduce this from the corresponding statement for the special fibre of the weight 0 crystalline Emerton–Gee stack. This stack was introduced in [EG23]. We recall the results from [EG23] that we need in Section 2.4 below, but for this introduction the key points are as follows: the full Emerton–Gee stack $\mathcal{X}$ is a stack of (φ, Γ)-modules which sees all $\overline{\rho}$ at once, and whose versal ring at any $\overline{\rho}$ is the corresponding unrestricted lifting ring; and the weight 0 crystalline Emerton–Gee stack $\mathcal{X}^{\underline{0}}$ is a closed substack whose versal ring at any $\overline{\rho}$ is the corresponding weight 0 crystalline lifting ring.

Generic reducedness for the (special fibre of the) stack $\mathcal{X}^{\underline{0}}$ is equivalent to the generic reducedness for the special fibres of the crystalline lifting rings, as we show by a direct argument below (in the proofs of Theorem 2.5.2 and Theorem 2.5.5). Working on the stack allows us to argue more geometrically, and in particular one of the main theorems of [EG23] classifies the irreducible components of the underlying reduced substack of $\mathcal{X}$, and shows that the underlying reduced substack of the special fibre $\overline{\mathcal{X}}^{\underline{0}}$ of $\mathcal{X}^{\underline{0}}$ is a union of these irreducible components.

In order to show that the $\overline{\mathcal{X}}^{\underline{0}}$ is generically reduced, it therefore suffices to determine which irreducible components are contained in this special fibre, and to show that $\overline{\mathcal{X}}^{\underline{0}}$ is reduced at a generic point of each such component.

The classification in [EG23] of the irreducible components is via a description of the generic $\overline{\rho}$ which occur on that component. These are all of the form

$$\overline{\rho} \cong \begin{pmatrix} \overline{\chi}_1 & * & \dots & * \\ 0 & \overline{\chi}_2 & \dots & * \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & \overline{\chi}_n \end{pmatrix}$$

where the $\overline{\chi}_i : G_K \to \overline{\mathbf{F}}_p^\times$ are characters and the extension classes $*$ are in generic position (in particular nonsplit). The characters $\overline{\chi}_i|_{I_K}$ are fixed on each irreducible component, and the components are usually determined by the data of the $\overline{\chi}_i|_{I_K}$ (see Theorem 2.4.3 (3) for a precise statement).

In order to prove our results, we show that the condition that a generic such $\overline{\rho}$ has a crystalline lift of weight 0 seriously constrains the possible $\overline{\chi}_i$. We use a theorem of Tong Liu [Liu08], which in particular shows (under the assumption that $p > n$) that $\overline{\rho}$ is obtained from a (crystalline) Breuil module. In the case $n = 2$, we can then argue as follows: we can compute the possible extensions of rank 1 Breuil modules, and we find that a sufficiently generic extension of $\overline{\chi}_2$ by $\overline{\chi}_1$ can only have a crystalline lift of weight 0 if it is ordinary, in the sense that $\overline{\chi}_1$ is unramified and $\overline{\chi}_2$ is an unramified twist of $\overline{\varepsilon}^{-1}$, where $\overline{\varepsilon}$ is the mod p cyclotomic character.

Furthermore, a generic such extension arises from a unique Breuil module, which is ordinary. (Since two-dimensional weight 0 crystalline representations are given by the generic fibres of p-divisible groups, we can alternatively phrase this result as showing that $\overline{\rho}$ comes from a unique finite flat group scheme over $\mathcal{O}_K$, which is ordinary in the sense that it is an extension of a multiplicative by an étale group scheme.) By an argument of Kisin [Kis09, Prop. 2.4.14], the deformations of this Breuil module are also ordinary, so that the weight 0 crystalline lifting rings for these generic $\overline{\rho}$'s are also ordinary. It is then easy to show that the crystalline ordinary lifting ring for a generic $\overline{\rho}$ is formally smooth, and thus has reduced special fibre, which completes the argument.

Perhaps surprisingly, we are able to make a similar argument for general n, without making any additional calculations. For each i, we apply our computation of extension classes of Breuil modules to the extension of $\overline{\chi}_{i+1}$ by $\overline{\chi}_i$ arising as a subquotient of $\overline{\rho}$. If all of these extensions are sufficiently generic, we show that $\overline{\rho}$ can only admit crystalline lifts of weight 0 if $\overline{\chi}_i|_{I_K} = \overline{\varepsilon}^{1-i}$ for all i. Furthermore, we also see that a generic such $\overline{\rho}$ can only arise from an ordinary Breuil module, and again deduce that all weight 0 crystalline lifts of $\overline{\rho}$ are ordinary. From this we deduce the formal smoothness of the corresponding lifting rings for generic $\overline{\rho}$, and conclude as above.

The organization of the proof is as follows. In Section 2.1 we recall Liu's results [Liu08] on strongly divisible modules and lattices in semistable representations, and deduce the results that we need on crystalline Breuil modules. In Section 2.2 we compute extensions of rank one Breuil modules. This is essentially elementary, using only semilinear algebra and some combinatorics. We deduce from this in Section 2.3 that sufficiently generic crystalline representations of weight 0 are ordinary. In Section 2.4 we give a brief introduction to Emerton–Gee stacks, and prove some slight generalizations of some results of [EG23], before deducing our generic reducedness results in Section 2.5.

2.1. **Breuil modules and strongly divisible modules.** We begin by recalling some standard results about Breuil modules and Breuil–Kisin modules. The results we use are largely due to Breuil, Kisin and Liu, but for convenience we mostly cite the papers [EGH13, HLM17] which deduce versions of these results with coefficients and prove some exactness properties of the functors to Galois representations which we will make use of in our main arguments. (Note that [HLM17, App. A] makes a running assumption on the ramification of the field $K/\mathbf{Q}_p$, but this is only made in order to discuss tame descent data and compare to Fontaine–Laffaille theory, and it is easy to check that all of the results we cite from there are valid for general $K/\mathbf{Q}_p$ with trivial descent data, with identical proofs (or often with simpler proofs, as there is no need to consider the descent data).)

Let $K/\mathbf{Q}_p$ be a finite extension for some $p > 2$, with ring of integers $\mathcal{O}_K$ and residue field k. Write e for the absolute ramification degree of K, and f for its inertial degree $[k : \mathbf{F}_p]$. Fix a uniformizer $\pi \in K$ with Eisenstein polynomial $E(u)$, which we choose so that $E(0) = p$. Fix also a compatible choice $(\pi^{1/p^n})_{n\geq 1}$ of p-power roots of π in $\overline{\mathbf{Q}}_p$, and set $K_n := K(\pi^{1/p^n})$ and $K_\infty := \cup_{n\geq 1} K_n$.

Let $E/\mathbf{Q}_p$ be a finite extension containing the normal closure of K, with ring of integers $\mathcal{O}$ and residue field $\mathbf{F}$. We will consider various semilinear algebra objects with coefficients in a finite $\mathcal{O}$-algebra A, and it is trivial to verify that all of our definitions are compatible with extension of scalars of A in an obvious way. In particular, we often take $A = \mathbf{F}$, but we are free to replace $\mathbf{F}$ by an arbitrary finite extension, or (after passing to a limit in an obvious fashion) by $\overline{\mathbf{F}}_p$.

For any finite $\mathcal{O}$-algebra A we let $\mathfrak{S}_A := (W(k) \otimes_{\mathbf{Z}_p} A)[\![u]\!]$, equipped with the usual A-linear, $W(k)$-semilinear Frobenius endomorphism φ, with $\varphi(u) = u^p$. For any integer $h \geq 0$, a *Breuil–Kisin module with A-coefficients* of height at most h is a finite free $\mathfrak{S}_A$-module $\mathfrak{M}$ equipped with a φ-semilinear map $\varphi : \mathfrak{M} \to \mathfrak{M}$ such that the cokernel of the linearized Frobenius $\varphi^*\mathfrak{M} \xrightarrow{1\otimes\varphi} \mathfrak{M}$ is killed by $E(u)^h$, where as usual $\varphi^*\mathfrak{M}$ denotes the Frobenius pullback $\mathfrak{S}_A \otimes_{\varphi,\mathfrak{S}_A} \mathfrak{M}$. (Here we indulge in a standard abuse of notation in writing φ for both the endomorphism of $\mathfrak{S}_A$ and of $\mathfrak{M}$, which should not cause any confusion.)

Suppose that $A = \mathbf{F}$, and let $\overline{S}_{\mathbf{F}} := \mathfrak{S}_{\mathbf{F}}/u^{ep}$. If $h \leq p-2$, then a *quasi-Breuil module with* $\mathbf{F}$*-coefficients* $\mathcal{M}$ of height h is a finite free $\overline{S}_{\mathbf{F}}$ module $\mathcal{M}$ equipped with a $\overline{S}_{\mathbf{F}}$-submodule

$$u^{eh}\mathcal{M} \subseteq \mathcal{M}^h \subseteq \mathcal{M}$$

and a φ-semilinear map $\varphi : \mathcal{M}^h \to \mathcal{M}$ such that

$$\overline{S}_{\mathbf{F}} \cdot \varphi(\mathcal{M}^h) = \mathcal{M}.$$

(The morphism φ is usually denoted φ_h in the literature, but we will shortly fix the choice $h = p-2$ for the rest of the paper, so we have omitted the subscript for the sake of cleaner notation.)

For each $0 \leq h \leq p-2$, there is by [Bre99, Thm. 4.1.1] an equivalence of categories between the category of Breuil–Kisin modules with $\mathbf{F}$-coefficients of height at most h and the category of quasi-Breuil modules with $\mathbf{F}$-coefficients of height at most h. Explicitly, a Breuil–Kisin module $\mathfrak{M}$ of height $h \leq p-2$ determines a quasi-Breuil module as follows. Write $\mathfrak{M}^h := (1\otimes\varphi)^{-1}(u^{eh}\mathfrak{M}) \subseteq \varphi^*\mathfrak{M}$. Set $\mathcal{M} := \varphi^*\mathfrak{M}/u^{pe}$, and $\mathcal{M}^h = \mathfrak{M}^h/u^{pe}\varphi^*\mathfrak{M}$. Then $\varphi : \mathcal{M}^h \to \mathcal{M}$ is defined by the composite

$$\mathcal{M}^h \xrightarrow{1\otimes\varphi} u^{eh}\overline{S}_{\mathbf{F}} \otimes_{\mathfrak{S}_{\mathbf{F}}} \mathfrak{M} \xrightarrow{\varphi_h\otimes 1} \overline{S}_{\mathbf{F}} \otimes_{\varphi,\mathfrak{S}_{\mathbf{F}}} \mathfrak{M} = \mathcal{M},$$

where $\varphi_h : u^{eh}\overline{S}_{\mathbf{F}} \to \overline{S}_{\mathbf{F}}$ is the φ-semilinear morphism $\varphi_h(u^{eh}x) := \varphi(x)$. (Note that this is well-defined because if x is divisible by $u^{e(p-h)}$, then $\varphi(x)$ is divisible by $u^{ep(p-h)}$ and in particular by $u^{ep} = 0$.) We will often say that the Breuil–Kisin module $\mathfrak{M}$ *underlies* the quasi-Breuil module $\mathcal{M}$.

We define $c \in (\overline{S}_{\mathbf{F}})^{\times}$ to be the image of $\varphi(E(u))/p$ in $\overline{S}_{\mathbf{F}}$. We note that $c = \varphi(d)$ for $d \in (\overline{S}_{\mathbf{F}})^{\times}$.

Remark 2.1.1. The equivalence between Breuil–Kisin modules and quasi-Breuil modules recalled above is usually defined with the map $\varphi_h(u^{eh}x) := c^h\varphi(x)$. It is easily checked that multiplying by d^h gives an isomorphism between the quasi-Breuil modules with differently scaled φ.

Write $N : \overline{S}_{\mathbf{F}} \to \overline{S}_{\mathbf{F}}$ for the $(k \otimes_{\mathbf{F}_p} \mathbf{F})$-linear derivation $-u\frac{\partial}{\partial u}$. A *Breuil module with* $\mathbf{F}$*-coefficients* $\mathcal{M}$ of height h is a quasi-Breuil module equipped with the additional data of a map $N : \mathcal{M} \to \mathcal{M}$ which satisfies:

- $N(sx) = sN(x) + N(s)x$ for all $s \in \overline{S}_{\mathbf{F}}, x \in \mathcal{M}$,
- $u^eN(\mathcal{M}^h) \subseteq \mathcal{M}^h$,
- and $\varphi(u^eN(x)) = cN(\varphi(x))$ for all $x \in \mathcal{M}^h$.

We say that a Breuil module $\mathcal{M}$ is *crystalline* if $N(\mathcal{M}) \subseteq u\mathcal{M}$.

Remark 2.1.2. While we will not explicitly need this below, it can be checked that if $\mathcal{M}$ is crystalline, then $u^eN(\mathcal{M}^h) \subseteq uN(\mathcal{M}^h)$. To see this, note that since $\overline{S}_{\mathbf{F}} \cdot \varphi(\mathcal{M}^h) = \mathcal{M}$, there is an induced $\mathbf{F}_p$-linear surjection $\mathcal{M}^h/u\mathcal{M}^h \to \mathcal{M}/u\mathcal{M}$, which is in fact an isomorphism (comparing dimensions as in [Bre98, Lem. 2.2.1.1]). If $\mathcal{M}$ is crystalline then N acts by 0 on $\mathcal{M}/u\mathcal{M}$, and the commutation relation between N and φ then shows that u^eN acts by 0 on $\mathcal{M}^h/u\mathcal{M}^h$, as required.

We now define the Galois representations associated to Breuil modules and to Breuil–Kisin modules, beginning with the latter. An *étale* φ*-module with* $\mathbf{F}$*-coefficients* is by definition a finite free $(k \otimes_{\mathbf{F}_p} \mathbf{F})((u))$-module M with a semilinear endomorphism $\varphi : M \to M$ such that the linearized Frobenius $\varphi^*M \xrightarrow{1\otimes\varphi} M$ is an isomorphism. Note that by definition, if $\mathfrak{M}$ is a Breuil–Kisin module with $\mathbf{F}$-coefficients, then $\mathfrak{M}[1/u]$ is an étale φ-module with $\mathbf{F}$-coefficients. Let $k((u))^{\text{sep}}$ denote a separable closure of $k((u))$. By the results of [Fon90] (see e.g. [Kis09, 1.1.12]), the functor

$$T_\infty : M \mapsto (M \otimes_{k((u))} k((u))^{\text{sep}})^{\varphi=1}$$

is an equivalence of categories between the category of étale φ-modules with $\mathbf{F}$-coefficients and the category of continuous representations of G_{K_∞} on $\mathbf{F}$-vector spaces, and we have $\dim_{\mathbf{F}} T_\infty(M) = \text{rank}_{(k\otimes_{\mathbf{F}_p}\mathbf{F})((u))} M$. We also write T_∞ for the induced functor from Breuil–Kisin modules to G_{K_∞}-representations given by $\mathfrak{M} \mapsto T_\infty(\mathfrak{M}[1/u])$. Similarly, if $\mathfrak{M}$ is the Breuil–Kisin module underlying a quasi-Breuil module $\mathcal{M}$, we write $T_\infty(\mathcal{M})$ for $T_\infty(\mathfrak{M})$.

Similarly, there is a functor T from the category of Breuil modules of height at most h with $\mathbf{F}$-coefficients to the category of continuous representations of G_K on $\mathbf{F}$-vector spaces defined by

$$T(\mathcal{M}) := \text{Hom}_{k[u]/u^{ep},\varphi,N}(\mathcal{M}, \widehat{A})^\vee,$$

where $\widehat{A} := \widehat{A}_{\text{st}} \otimes_S k[u]/u^{ep}$ is defined for example in [HLM17, (A.3.1)]. Again we have $\dim_{\mathbf{F}} T(\mathcal{M}) = \text{rank}_{\overline{S}_{\mathbf{F}}} \mathcal{M}$. Furthermore, by [HLM17, Prop. A.3.2] the forgetful functor from Breuil modules to quasi-Breuil modules induces an isomorphism

$$T(\mathcal{M})|_{G_{K_\infty}} \xrightarrow{\sim} T_\infty(\mathcal{M}).$$

From now on all of our Breuil modules will be crystalline and have height $(p-2)$. We write $\mathbf{F}\mathrm{BrMod}^{\mathrm{cr}}$ for the category of crystalline Breuil modules of height $(p-2)$ with $\mathbf{F}$-coefficients, and $\mathbf{F}\mathrm{qBrMod}$ for the category of quasi-Breuil modules of height $(p-2)$ with $\mathbf{F}$-coefficients, which we identify with the category of Breuil–Kisin modules of height at most $(p-2)$ with $\mathbf{F}$-coefficients. We say that a complex

$$0 \to \mathcal{M}_1 \to \mathcal{M} \to \mathcal{M}_2 \to 0 \tag{2.1.3}$$

in $\mathbf{F}\mathrm{BrMod}^{\mathrm{cr}}$ or $\mathbf{F}\mathrm{qBrMod}$ is exact if it induces exact sequences of $\overline{S}_{\mathbf{F}}$-modules $0 \to \mathcal{M}_1 \to \mathcal{M} \to \mathcal{M}_2 \to 0$ and

$$0 \to \mathcal{M}_1^{p-2} \to \mathcal{M}^{p-2} \to \mathcal{M}_2^{p-2} \to 0.$$

It is easily checked that a complex of quasi-Breuil modules is exact if and only if the corresponding complex of Breuil–Kisin modules is exact (as a complex of $\mathfrak{S}_{\mathbf{F}}$-modules).

If $\mathcal{M}$ is an object of $\mathbf{F}\mathrm{BrMod}^{\mathrm{cr}}$, then an $\overline{S}_{\mathbf{F}}$-submodule $\mathcal{N} \subseteq \mathcal{M}$ is a *Breuil submodule of* $\mathcal{M}$ if it is a direct summand of $\mathcal{M}$ as a $k[u]/u^{ep}$-module, and we furthermore have $N(\mathcal{N}) \subseteq \mathcal{N}$ and $\varphi(\mathcal{N} \cap \mathcal{M}^r) \subseteq \mathcal{N}$. Then $\mathcal{N}$ inherits the structure of a crystalline Breuil module from $\mathcal{M}$, as does the quotient $\mathcal{M}/\mathcal{N}$, and by [HLM17, Lem. 2.3.2], the complex of crystalline Breuil modules

$$0 \to \mathcal{N} \to \mathcal{M} \to \mathcal{M}/\mathcal{N} \to 0$$

is exact; and conversely if (2.1.3) is exact, then $\mathcal{M}_1$ is a Breuil submodule of $\mathcal{M}$.

Theorem 2.1.4.

1. *The categories* $\mathbf{F}\mathrm{BrMod}^{\mathrm{cr}}$ *and* $\mathbf{F}\mathrm{qBrMod}$ *are exact categories in the sense of* [Qui10]*, and the functors* T *and* T_∞ *are exact.*
2. *For any object* $\mathcal{M}$ *of* $\mathbf{F}\mathrm{BrMod}^{\mathrm{cr}}$*, there is an order preserving bijection* Θ *between the Breuil submodules of* $\mathcal{M}$ *and the* G_K*-subrepresentations of* $T(\mathcal{M})$*, taking* $\mathcal{N}$ *to the image of* $T(\mathcal{N}) \hookrightarrow T(\mathcal{M})$*. Furthermore if* $\mathcal{M}_1 \subseteq \mathcal{M}_2$ *are Breuil submodules of* $\mathcal{M}$*, then* $\Theta(\mathcal{M}_2)/\Theta(\mathcal{M}_1) \cong T(\mathcal{M}_2/\mathcal{M}_1)$*.*

Proof. The statement for quasi-Breuil modules is [Car11, Thm. 2.1.2]. The rest of the theorem for not-necessarily-crystalline Breuil modules is [HLM17, Prop. 2.3.4, 2.3.5]. The case of crystalline Breuil modules follows formally, because (as noted above) a Breuil submodule of a crystalline Breuil module is automatically crystalline (as is the corresponding quotient submodule). Alternatively, it is straightforward to check that the proofs of [HLM17, Prop. 2.3.4, 2.3.5] go through unchanged, once one notes that the duality on Breuil modules [EGH13, Defn. 3.2.8] by definition preserves the subcategory of crystalline Breuil modules. □

We now show that any Galois representation obtained as the reduction mod p of a lattice in a crystalline representation with Hodge–Tate weights in the range $[0, p-2]$ comes from a crystalline Breuil module. This is essentially an immediate consequence of the main theorem of Liu's paper [Liu08], which proves an equivalence of categories between G_K-stable lattices inside semistable representations with Hodge–Tate weights in the range $[0, p-2]$ and strongly divisible modules. From this one can easily deduce an equivalence of categories between G_K-stable lattices inside crystalline representations with Hodge–Tate weights in the range $[0, p-2]$ and an appropriate category of "crystalline strongly divisible lattices", but since we do not need this, we avoid recalling the definitions of strongly divisible lattices and leave it to the interested reader.

Recall that by the results of [Kis06], if $\rho : G_K \to \mathrm{GL}_n(\mathcal{O})$ is a lattice in a crystalline representation with non-negative Hodge–Tate weights, there is a Breuil–Kisin module with $\mathcal{O}$-coefficients $\mathfrak{M}_{\mathcal{O}}$ associated to $\rho|_{G_{K_\infty}}$ (see e.g. [GLS14, Thm. 3.2(3), Prop. 3.4(3)] for a precise reference allowing $\mathcal{O}$-coefficients).

Theorem 2.1.5. *Let $\rho : G_K \to \mathrm{GL}_n(\mathcal{O})$ be a lattice in a crystalline representation with Hodge–Tate weights in $[0, h]$ for some integer $0 \le h \le p-2$, and write $\overline{\rho} : G_K \to \mathrm{GL}_n(\mathbf{F})$ for its reduction modulo $\mathfrak{m}_{\mathcal{O}}$. Then there is a crystalline Breuil module $\mathcal{M}$ with $\mathbf{F}$-coefficients such that $\overline{\rho} \cong T(\mathcal{M})$. Furthermore, the underlying Breuil–Kisin module of $\mathcal{M}$ has height at most h, and is the reduction modulo $\mathfrak{m}_{\mathcal{O}}$ of the Breuil–Kisin module $\mathfrak{M}_{\mathcal{O}}$ with $\mathcal{O}$-coefficients associated to $\rho|_{G_{K_\infty}}$.*

Proof. Since crystalline representations are in particular semistable, it is immediate from [EGH13, Prop. 3.1.4, Lem. 3.2.2] that there is a not-necessarily crystalline Breuil module $\mathcal{M}$ with $\mathbf{F}$-coefficients such that $\overline{\rho} \cong T(\mathcal{M})$, whose underlying Breuil–Kisin module has height at most h (note that our h is the integer r in the statement of [EGH13, Prop. 3.1.4]). We claim that the Breuil module provided by these results is necessarily crystalline. To see this, note first that (in the case at hand, with no descent data) [EGH13, Prop. 3.1.4] is a trivial consequence of the main result [Liu08, Thm. 2.3.5] of Liu's paper [Liu08], and gives an equivalence of categories between G_K-stable $\mathcal{O}$-lattices inside semistable E-representations of G_K with Hodge–Tate weights in the range $[0, p-2]$ and strongly divisible modules with $\mathcal{O}$-coefficients. In particular, there is a strongly divisible module with $\mathcal{O}$-coefficients $\widehat{\mathcal{M}}$ corresponding to ρ.

We do not recall the notion of a strongly divisible module here, but we note that they are by definition modules over a coefficient ring $S_{\mathcal{O}}$, equipped with a Frobenius, a filtration and a monodromy operator N, and by [EGH13, Lem. 3.2.2], the Breuil module $\mathcal{M}$ is obtained from the strongly divisible module $\widehat{\mathcal{M}}$ by tensoring over $S_{\mathcal{O}}$ with $\overline{S}_{\mathbf{F}}$. By the commutative diagram at the end of [Liu08, §3.4], the strongly divisible module $\widehat{\mathcal{M}}$ has underlying Breuil–Kisin module $\mathfrak{M}_{\mathcal{O}}$ (via the fully faithful functor of [Liu08, Cor. 3.3.2]). It follows immediately that the underlying Breuil–Kisin module of $\mathcal{M}$ is the reduction modulo $\mathfrak{m}_{\mathcal{O}}$ of $\mathfrak{M}_{\mathcal{O}}$, as claimed.

It remains to show that if ρ is crystalline, the monodromy operator N on $\mathcal{M}$ vanishes mod u. This follows immediately from the compatibility between $\widehat{\mathcal{M}}$ and the weakly admissible module D associated to ρ, for which see [Liu08, §3.2]. □

2.2. **Extensions of rank one Breuil modules.** In this section we make a computation of the possible extensions of rank one Breuil modules, and prove the crucial Lemma 2.2.19, which gives a constraint on the Breuil modules which can witness sufficiently generic extensions of characters. A key input to the proof of this Lemma is Lemma 2.2.7, which constrains the shapes of extensions of rank one Breuil modules. To prove Lemma 2.2.19, we simply write down an explicit extension of characters (after restriction to G_{K_∞}) and show that it cannot arise from a Breuil module satisfying these constraints. These calculations are elementary, but are complicated in the case of a general field $K/\mathbf{Q}_p$, and the reader may find it helpful to firstly work through the case that $K/\mathbf{Q}_p$ is totally ramified, where the calculations simplify dramatically; if furthermore $n = 2$, then the monodromy condition is automatic and the calculations simplify further to a basic exercise with Breuil–Kisin modules.

Let $\overline{\sigma}_0 : k \hookrightarrow \mathbf{F}$ be a fixed embedding. Inductively define $\overline{\sigma}_1, \dots, \overline{\sigma}_{f-1}$ by $\overline{\sigma}_{i+1} = \overline{\sigma}_i \circ \varphi^{-1}$, where φ is the arithmetic Frobenius on k; we will often consider the numbering to be cyclic, so that $\overline{\sigma}_f = \overline{\sigma}_0$. There are idempotents $\epsilon_i \in k \otimes_{\mathbf{F}_p} \mathbf{F}$ such that if M is any $k \otimes_{\mathbf{F}_p} \mathbf{F}$-module, then $M = \bigoplus_i M_i$, where $M_i := \epsilon_i M$ is the subset of M consisting of elements m for which $(x \otimes 1)m = (1 \otimes \overline{\sigma}_i(x))m$ for all $x \in k$. Note that $(\varphi \otimes 1)(\epsilon_i) = \epsilon_{i+1}$ for all i.

As explained above, we are free to work with coefficients in $\overline{\mathbf{F}}_p$ rather than $\mathbf{F}$, and for convenience we do so throughout this section. (To be precise, this means that we apply the definitions above with $\overline{S}_{\mathbf{F}}$ replaced by $(k \otimes_{\mathbf{F}_p} \overline{\mathbf{F}}_p)[u]/u^{ep}$.) It will be clear to the reader that the coefficients do not intervene in any way in the calculations, and we could equally well work with coefficients in any finite extension of $\mathbf{F}$.

Definition 2.2.1. Let $s_0, \dots, s_{f-1}$ be non-negative integers, and let $a \in \overline{\mathbf{F}}_p^{\times}$. Let $\mathfrak{M}(\underline{s}; a)$ be the rank one Breuil–Kisin module with $\overline{\mathbf{F}}_p$ coefficients such that $\mathfrak{M}(\underline{s}; a)_i$ is generated by e_i with

$$\varphi(e_{i-1}) = (a)_i u^{s_i} e_i.$$

Here and below, $(a)_0 = a$ and $(a)_i = 1$ if $i \neq 0$.

By [GLS14, Lem. 6.2], any rank one Breuil–Kisin module is isomorphic to (exactly) one of the form $\mathfrak{M}(\underline{s}; a)$.

Definition 2.2.2. Set $\alpha_i(\mathfrak{M}(\underline{s}; a)) := \frac{1}{p^f-1} \sum_{j=1}^{f} p^{f-j} s_{j+i}$.

By [GLS15, Lem. 5.1.2], there exists a nonzero map $\mathfrak{M}(\underline{s}; a) \to \mathfrak{M}(\underline{t}; b)$ if and only if $\alpha_i(\mathfrak{M}(\underline{s}; a)) - \alpha_i(\mathfrak{M}(\underline{t}; b)) \in \mathbf{Z}_{\geq 0}$ for all i, and $a = b$. We now show that each rank one Breuil–Kisin module of height at most $(p-2)$ underlies a unique rank one (crystalline) Breuil module. (In particular, all rank one Breuil modules are crystalline.)

Lemma 2.2.3. *If each $s_i \in [0, e(p-2)]$ then the rank one Breuil–Kisin module $\mathfrak{M}(\underline{s}; a)$ underlies a unique height $(p-2)$ Breuil module $\mathcal{M} = \mathcal{M}(\underline{s}; a)$ with*

$$\mathcal{M}_j^{p-2} = \langle u^{e(p-2)-s_j}(1 \otimes e_{j-1}) \rangle,$$
$$\varphi(u^{e(p-2)-s_j}(1 \otimes e_{j-1})) = (a)_j (1 \otimes e_j),$$
$$N((1 \otimes e_{j-1})) = 0.$$

Proof. We begin by noting that since $(\varphi^*\mathfrak{M}(\underline{s}; a))_i$ is generated by $(1 \otimes e_{i-1})$, the quasi-Breuil module $\mathcal{M} = \varphi^*\mathfrak{M}(\underline{s}; a)/u^{ep}$ corresponding to $\mathfrak{M}(\underline{s}; a)$ has the given form. It is easy to see that taking $N((1 \otimes e_{j-1})) = 0$ gives $\mathcal{M}$ the structure of a Breuil module. To see that this is the only possibility, write $N((1 \otimes e_{j-1})) = \nu_j (1 \otimes e_{j-1})$. Then we have

$$N(u^{e(p-2)-s_j}(1 \otimes e_{j-1})) = u^{e(p-2)-s_j} \left(s_j - e(p-2) + \nu_j\right)(1 \otimes e_{j-1}) \in \mathcal{M}_j^{p-2},$$

so that

$$\varphi(u^e N(u^{e(p-2)-s_j}(1 \otimes e_{j-1}))) \in u^{ep} \varphi(\mathcal{M}_j) = 0,$$

and the equation $\varphi(u^e N(u^{e(p-2)-s_j}(1 \otimes e_{j-1}))) = cN\varphi(u^{e(p-2)-s_j}(1 \otimes e_{j-1}))$ gives $\nu_{j+1} = 0$ for each j, as required. □

The extensions of rank one Breuil–Kisin modules are computed as follows.

Proposition 2.2.4. *Let $\mathfrak{M}$ be an extension of $\mathfrak{M}(\underline{s};a)$ by $\mathfrak{M}(\underline{t};b)$. Then we can choose bases e_i, f_i of the $\mathfrak{M}_i$ so that φ has the form*

$$\begin{aligned}\varphi(e_{i-1}) &= (b)_i u^{t_i} e_i \\ \varphi(f_{i-1}) &= (a)_i u^{s_i} f_i + y_i e_i\end{aligned} \tag{2.2.5}$$

with $y_i \in \mathbf{F}[\![u]\!]$ a polynomial with $\deg(y_i) < s_i$, except that when there is a nonzero map $\mathfrak{M}(\underline{s};a) \to \mathfrak{M}(\underline{t};b)$ we must also allow y_j to have a term of degree $s_j + \alpha_j(\mathfrak{M}(\underline{s};a)) - \alpha_j(\mathfrak{M}(\underline{t};b))$ for any one choice of j.

Proof. This is [GLS15, Prop. 5.1.3]. □

Remark 2.2.6.

1. In our application to 'generic' $\overline{\rho}$, we could avoid considering the special case where there is a nonzero map $\mathfrak{M}(\underline{s};a) \to \mathfrak{M}(\underline{t};b)$ (for example by ensuring that $a \neq b$), but we have included it for completeness.
2. While this is not claimed in [GLS15, Prop. 5.1.3], we expect that it is possible to show that distinct choices of the y_i in 2.2.4 give distinct extensions of Breuil–Kisin modules.

We now compute a constraint on extension classes of rank 1 Breuil modules.

Lemma 2.2.7. *Let $\mathcal{M}$ be a crystalline Breuil module which is an extension of $\mathcal{M}(\underline{s};a)$ by $\mathcal{M}(\underline{t};b)$, with underlying Breuil–Kisin module $\mathfrak{M}$ as in Proposition 2.2.4. For each i we set*

$$n_i := \frac{1}{p^f - 1} \sum_{j=1}^{f} p^{f-j}(s_{j+i-1} - t_{j+i-1} - e) \in \mathbf{Q}. \tag{2.2.8}$$

Then the y_j in Proposition 2.2.4 cannot have any terms of degree $l < s_j - e + \max(n_{j+1}, 1)$ with $l \not\equiv t_j \pmod p$.

Proof. The quasi-Breuil module corresponding to $\mathfrak{M}$ has $\mathcal{M}_j$ generated by $(1 \otimes e_{j-1}), (1 \otimes f_{j-1})$ and $\mathcal{M}_j^{p-2}$ generated by $E_j := u^{e(p-2)-t_j}(1 \otimes e_{j-1})$ and

$$F_j := u^{e(p-2)-s_j}(1 \otimes f_{j-1}) - u^{e(p-2)-s_j-t_j}(b^{-1})_j y_j (1 \otimes e_{j-1}).$$

The map $\varphi : \mathcal{M}^{p-2} \to \mathcal{M}$ is given by

$$\varphi(E_j) = (b)_j(1 \otimes e_j), \varphi(F_j) = (a)_j(1 \otimes f_j).$$

(Note that $\mathfrak{M}$ must have height at most $(p-2)$, since it underlies a Breuil module, so y_j is indeed divisible by $u^{s_j+t_j-e(p-2)}$.)

We have $N(1 \otimes e_{j-1}) = 0$, and we write $N(1 \otimes f_{j-1}) = \mu_j(1 \otimes e_{j-1})$ with $\mu_j \in u\overline{\mathbf{F}}_p[u]$ (since $\mathcal{M}$ is crystalline), where for each j we must have

$$u^e N(F_j) \in \mathcal{M}_j^{p-2}$$

by the second property of N required in the definition of a Breuil module. Given this, the third property of N gives the commutation relation

$$\begin{aligned}\varphi(u^e N(F_j)) &= cN\varphi(F_j) \\ &= cN((a)_j(1 \otimes f_j)) \\ &= c(a)_j \mu_{j+1}(1 \otimes e_j).\end{aligned}$$

We have

$$\begin{aligned}N(F_j) &= (s_j - e(p-2))u^{e(p-2)-s_j}(1\otimes f_{j-1}) + u^{e(p-2)-s_j}\mu_j(1\otimes e_{j-1})\\ &\quad + N(-u^{e(p-2)-s_j-t_j}(b^{-1})_j y_j)(1\otimes e_{j-1})\\ &= (s_j - e(p-2))(u^{e(p-2)-s_j}(1\otimes f_{j-1}) - u^{e(p-2)-s_j-t_j}(b^{-1})_j y_j(1\otimes e_{j-1}))\\ &\quad + (u^{e(p-2)-s_j}\mu_j + N(-u^{e(p-2)-s_j-t_j}(b^{-1})_j y_j))(1\otimes e_{j-1})\\ &\quad + ((s_j - e(p-2))u^{e(p-2)-s_j-t_j}(b^{-1})_j y_j)(1\otimes e_{j-1}),\end{aligned}$$

so we need the quantity

$$u^e(u^{e(p-2)-s_j}\mu_j + N(-u^{e(p-2)-s_j-t_j}(b^{-1})_j y_j) + (s_j - e(p-2))u^{e(p-2)-s_j-t_j}(b^{-1})_j y_j)$$

to be divisible by $u^{e(p-2)-t_j}$; assuming this holds, the commutation relation with φ reads

$$\begin{aligned}c(a)_j\mu_{j+1} = (b)_j u^{p(e-e(p-2)+t_j)}\varphi\Big(&u^{e(p-2)-s_j}\mu_j\\ &+ N(-u^{e(p-2)-s_j-t_j}(b^{-1})_j y_j) + (s_j - e(p-2))u^{e(p-2)-s_j-t_j}(b^{-1})_j y_j\Big).\end{aligned}$$

In particular, since $c = \varphi(d) \in \operatorname{im}(\varphi)$, we see that $\mu_{j+1} \in \operatorname{im}\varphi$. Writing $\mu_{j+1} = \varphi(\mu'_{j+1})$ and rearranging, we obtain

$$u^{e-s_j+t_j}\left((b)_j\varphi(\mu'_j) - u^{-t_j}(t_j y_j + N(y_j))\right) = d(a)_j\mu'_{j+1}. \tag{2.2.9}$$

(Strictly speaking, since $\varphi(u^e) = u^{ep} = 0$, this is only an equation modulo u^e; but it is easily checked that all terms have degree less than e, so it holds literally.)

Examining the left hand side of (2.2.9), we note that there can be no cancellation between the terms in $\varphi(\mu'_j)$ and $u^{-t_j}(t_j y_j + N(y_j))$, as the exponents of u in $\varphi(\mu'_j)$ are all divisible by p, while none of the exponents of u in $u^{-t_j}(t_j y_j + N(y_j))$ are divisible by p (the terms in $t_j y_j$ with exponent $\equiv t_j \mod p$ cancel with terms in $N(y_j)$). Let $d_j \geq 1$ be the u-adic valuation of μ'_j (setting $d_j = e$ if μ'_j is divisible by u^e). Then, since d is a u-adic unit, (2.2.9) gives us the inequality

$$pd_j - (s_j - t_j - e) \geq d_{j+1}. \tag{2.2.10}$$

(To see this, note that if the left hand side is at least e, there is nothing to prove; and if $d_j = e$, then since $s_j - t_j - e \leq e(p-3)$, the left hand side is at least $3e > e$. Otherwise the term $u^{e-s_j+t_j}(b)_j\varphi(\mu'_j)$ means that the left hand side of (2.2.9) has a term of degree $pd_j - (s_j - t_j - e)$, because of the lack of cancellation.) Multiplying the inequalities (2.2.10) by suitable powers of p and summing, we have

$$\sum_{j=1}^{f} p^{f-j}(pd_{j+i-1} - d_{j+i}) \geq \sum_{j=1}^{f} p^{f-j}(s_{j+i-1} - t_{j+i-1} - e),$$

which simplifies to $d_i \geq n_i$, where n_i is as in (2.2.8). Since $\mathcal{M}$ is crystalline by assumption, we also have $d_i \geq 1$, so that $d_i \geq \max(1, n_i)$ for all i.

Returning to (2.2.9), since the right hand side has valuation d_{j+1}, the lack of cancellation implies that the term $u^{e-s_j}(t_j y_j + N(y_j))$ on the left hand side is divisible by $u^{\max(1,n_{j+1})}$; equivalently, the terms in y_j of degree less than $s_j - e + \max(1, n_{j+1})$ and not congruent to t_j modulo p vanish, as claimed. □

Remark 2.2.11. It follows easily from the definitions that if we have two extensions of $\mathcal{M}(\underline{s};a)$ by $\mathcal{M}(\underline{t};b)$ as in Lemma 2.2.7, then their Baer sum corresponds to the extension obtained by summing the y_j (and has N given by summing the μ_j).

In the following arguments it will be useful to note that by the definition of the n_j in (2.2.8), we have

$$n_j + (s_{j-1} - t_{j-1} - e) = pn_{j-1} \tag{2.2.12}$$

for all j.

Lemma 2.2.13. *Write*

$$r_i := s_i - t_i - e + \lfloor n_{i+1} \rfloor - p\lfloor n_i \rfloor + 1. \tag{2.2.14}$$

Then $r_i \in [1,p]$ for each i.

Proof. Using (2.2.12), we have $r_i - 1 = p(n_i - \lfloor n_i \rfloor) - (n_{i+1} - \lfloor n_{i+1} \rfloor)$ and since for any real number x we have $(x - \lfloor x \rfloor) \in [0,1)$ we have $r_i - 1 \in (-1,p)$. Since r_i is an integer, the result follows. □

Write

$$\mathrm{Ext}^1_{\mathrm{BrMod}^{\mathrm{cr}}}(\mathcal{M}(\underline{s};a), \mathcal{M}(\underline{t};b))$$

for the Ext^1 group computed in the exact category $\overline{\mathbf{F}}_p\mathrm{BrMod}^{\mathrm{cr}}$. Write

$$\overline{\chi}_1 := T(\mathcal{M}(\underline{t};b)),$$
$$\overline{\chi}_2 := T(\mathcal{M}(\underline{s};a)).$$

Then the restriction maps

$$\mathrm{Ext}^1_{\mathrm{BrMod}^{\mathrm{cr}}}(\mathcal{M}(\underline{s};a), \mathcal{M}(\underline{t};b)) \xrightarrow{\mathrm{res}_K} \mathrm{Ext}^1_{G_K}(\overline{\chi}_2, \overline{\chi}_1) \xrightarrow{\mathrm{res}_{K_\infty}} \mathrm{Ext}^1_{G_{K_\infty}}(\overline{\chi}_2, \overline{\chi}_1)$$

are homomorphisms of $\overline{\mathbf{F}}_p$-vector spaces. Regarding elements of $\mathrm{Ext}^1_{G_{K_\infty}}(\overline{\chi}_2, \overline{\chi}_1)$ as étale φ-modules, we have the following description of the image of the restriction map res_{K_∞}. (In our key Lemma 2.2.19, we will show that the composition $\mathrm{res}_{K_\infty} \circ \mathrm{res}_K$ has smaller image.)

Lemma 2.2.15.

(1) *The restriction map res_{K_∞} is injective unless $\overline{\chi}_1\overline{\chi}_2^{-1} = \overline{\varepsilon}$, in which case its kernel is 1-dimensional, and is generated by the très ramifiée line given by the Kummer extension corresponding to the chosen uniformizer π of K.*
(2) *The image of res_{K_∞} has dimension $[K:\mathbf{Q}_p]$, unless $\overline{\chi}_1 = \overline{\chi}_2$, in which case it has dimension $[K:\mathbf{Q}_p]+1$.*
(3) *The étale φ-modules M in the image of res_{K_∞} are precisely those for which we can choose a basis e_i, f_i of M_i so that φ has the form*

$$\begin{aligned} \varphi(e_{i-1}) &= (b)_i u^{t_i} e_i \\ \varphi(f_{i-1}) &= (a)_i u^{s_i} f_i + y_i e_i \end{aligned} \tag{2.2.16}$$

where $y_i \in \overline{\mathbf{F}}_p[u,u^{-1}]$ has nonzero terms only in degrees $[s_i + \lfloor n_{i+1} \rfloor - e + 1, \dots, s_i + \lfloor n_{i+1} \rfloor]$; except that when $\overline{\chi}_1 = \overline{\chi}_2$ we also allow y_i to have a term of degree

$$s_i + \frac{1}{p^f - 1}\sum_{j=1}^{f} p^{f-j}(s_{j+i} - t_{j+i})$$

(necessarily an integer in this case) for any one choice of i.

Proof. The first part is [GLS15, Lem. 5.4.2]. The second part then follows from the usual computation of the dimension of $\mathrm{Ext}^1_{G_K}(\overline{\chi}_2, \overline{\chi}_1)$ via Tate's Euler characteristic formula and local duality.

The final part can presumably be proved in an elementary way, but for convenience we explain how to deduce it from the results of [GLS15] on the Breuil–Kisin modules associated to certain crystalline representations with small Hodge–Tate weights. This was first explained in [CEGM17, Thm. 3.3.2] in the case $K/\mathbf{Q}_p$ unramified (which employed the earlier results [GLS14]), and in general [Ste22, Thm. 4.2], under the assumption that $\overline{\chi}_1\overline{\chi}_2^{-1} \neq \overline{\varepsilon}$. However the comparison to the notation used in [Ste22, Thm. 4.2] is not immediate, and we need to treat the case $\overline{\chi}_1\overline{\chi}_2^{-1} = \overline{\varepsilon}$, so for the convenience of the reader we explain in our notation how to extract the claim from [GLS15].

It is easy to check that the étale φ-modules in (2.2.16) span a space of the dimension computed in part (2). In particular, in the case when $\overline{\chi}_1 = \overline{\chi}_2$, considering the change of basis adding on a suitable multiple of e_i to f_i shows that the additional φ-modules in that case do not depend on the choice of i for which y_i is allowed to have an extra term. It now suffices to show that all of the possibilities in (2.2.16) do indeed arise from G_K-representations. We can and do twist so that that $t_i = 0$ for all i. (This has the effect of replacing s_i by $s_i - t_i$, and leaving n_i unchanged, so the general statement follows immediately by twisting back.) Then our strategy is to show that our étale φ-modules arise from the reductions of certain crystalline representations. In fact, we will see that they arise from the reductions of crystalline extensions of p-adic characters.

We make the change of variables $f_i' = u^{-\lfloor n_{i+1}\rfloor} f_i$, and write $y_i' := u^{-p\lfloor n_i\rfloor} y_i$. Then we have

$$\varphi(e_{i-1}) = (b)_i e_i,$$
$$\varphi(f_{i-1}') = (a)_i u^{r_i+e-1} f_i' + y_i' e_i.$$

By (2.2.14) and our assumption that $t_i = 0$, we have

$$r_i + p\lfloor n_i\rfloor = r_i + t_i + p\lfloor n_i\rfloor = s_i - e + \lfloor n_{i+1}\rfloor + 1,$$

so we need to show that every choice of y_i' having nonzero terms in degrees $[r_i, r_i+e-1]$ occurs (together with the additional term in the statement in the case that $\overline{\chi}_1 = \overline{\chi}_2$). If we make a further change of variables to replace f_i' with $f_i' + z_i e_i$ for all i, with $z_i \in \overline{\mathbf{F}}_p$, then we may exchange the terms in y_i' of degree $r_i + e - 1$ with terms in y_{i-1} of degree 0 (cf. (2.2.23)), so it suffices in turn to show that every choice of y_i' having nonzero terms in degrees 0 and $[r_i, r_i + e - 2]$ occurs in the image of res_{K_∞} (again, together with the additional term in the statement in the case that $\overline{\chi}_1 = \overline{\chi}_2$).

Recall [GLS15, Defn. 2.3.1] that a pseudo-Barsotti–Tate representation of weight $\{r_i\}$ is a 2-dimensional crystalline representation whose labelled Hodge–Tate weights are $\{0, 1\}$, except at a chosen set of f embeddings lifting the embeddings $\sigma_i : k \hookrightarrow \overline{\mathbf{F}}_p$, where they are $\{0, r_i\}$. By [GLS15, Defn. 4.1.3], these are the representations which have $\sigma_{r-1,0} := \otimes_i \mathrm{Sym}^{r_i-1} k^2 \otimes_{k,\sigma_i} \overline{\mathbf{F}}_p$ as a Serre weight.

Now consider [GLS15, Thm. 5.1.5], taking the t_i there to be zero, the x_i to be $e-1$, and the s_i there to be our r_i+e-1 (which are not necessarily equal to our s_i – we apologize for this temporary notation). Note that with this choice, the Breuil–Kisin modules spanned by our basis e_i, f_i' are precisely the extensions of Breuil–Kisin modules in [GLS15, Thm. 5.1.5], for the rank one Breuil–Kisin modules which are the minimal and maximal models of $\overline{\chi}_1, \overline{\chi}_2$ as in the statement of [GLS15, Prop. 5.3.4]. So by [GLS15, Prop. 5.3.4, Thm. 5.1.5], if $\psi \in \mathrm{Ext}^1_{G_K}(\overline{\chi}_2, \overline{\chi}_1)$ comes from the reduction of a pseudo-Barsotti–Tate representations of weight $\{r_i\}$, then $\mathrm{res}_{K_\infty}(\psi)$ is given by an étale φ-module as in (2.2.16). It therefore suffices to show that these classes $\mathrm{res}_{K_\infty}(\psi)$ span the image of res_{K_∞}.

To see this, we consider the reductions of reducible crystalline representations. As in the proof of [GLS15, Thm. 5.4.1], we choose crystalline characters $\chi_{1,\min}, \chi_{2,\max}$ which lift $\overline{\chi}_1, \overline{\chi}_2$ respectively. More precisely, these characters are determined (up to unramified twist, which we do not specify) by their Hodge–Tate weights, which (recalling that $t_i = 0$ for all i) we can and do choose so that $\chi_{2,\max}$ is unramified, and so that any crystalline extension of $\chi_{2,\max}$ by $\chi_{1,\min}$ is pseudo-Barsotti–Tate of weight $\{r_i\}$.

The space of crystalline extensions of $\chi_{2,\max}$ by $\chi_{1,\min}$ is identified with the Galois cohomology group $H^1_f(G_K, \chi_{1,\min}\chi_{2,\max}^{-1})$, and as in the proof of [GLS15, Thm. 5.4.1], one immediately computes that the dimension of the image of the reduction map

$$H^1_f(G_K, \chi_{1,\min}\chi_{2,\max}^{-1}) \to H^1(G_K, \overline{\chi}_1\overline{\chi}_2^{-1}) \tag{2.2.17}$$

is $[K : \mathbf{Q}_p]$, unless $\overline{\chi}_1 = \overline{\chi}_2$ in which case it is $[K : \mathbf{Q}_p] + 1$; so by part (2), this image has the same dimension as the image of res_{K_∞}.

In particular we see that we are done if the restriction of res_{K_∞} to the image of (2.2.17) is injective. If $\overline{\chi}_1\overline{\chi}_2^{-1} \neq \overline{\varepsilon}$ then this is automatic by part (1), so we may suppose that $\overline{\chi}_1\overline{\chi}_2^{-1} = \overline{\varepsilon}$. If some $r_i \neq p$, then by [CEGS22b, Lem. A.4] the image of (2.2.17) is contained in the peu ramifiée subspace, so we again conclude by part (1). Finally if $r_i = p$ for all i, then as in the proof of [GLS15, Thm. 6.1.18], the union of the images of (2.2.17) as $\chi_{1,\min}, \chi_{2,\max}$ range over their twists by unramified characters with trivial reduction is all of $H^1(G_K, \overline{\chi}_1\overline{\chi}_2^{-1})$, so we are done. □

Lemma 2.2.18. *Suppose that* $\sum_{j=1}^f (s_j - t_j - e) < 0$. *Then either there exists an* i *with* $\lfloor n_{i+1} \rfloor = -1$ *and* $r_i \neq p$, *or there exists an* i *with* $\lfloor n_{i+1} \rfloor \leq -2$.

Proof. Summing (2.2.14) over all j, we have

$$\sum_{j=1}^f (s_j - t_j - e) = \sum_{j=1}^f \left((p-1)\lfloor n_{j+1} \rfloor + (r_j - 1)\right).$$

If this is negative, there exists an i with

$$(p-1)\lfloor n_{i+1} \rfloor + (r_i - 1) < 0.$$

Since $r_i \geq 1$, we must have $\lfloor n_{i+1} \rfloor < 0$. If $\lfloor n_{i+1} \rfloor = -1$ then we have $1 - p + r_i - 1 < 0$, so $r_i < p$, as required. □

Lemma 2.2.19. *Suppose that* $\sum_{j=1}^f (s_j - t_j - e) < 0$. *Then the restriction map*

$$\mathrm{Ext}^1_{\mathrm{BrMod}^{\mathrm{cr}}}(\mathcal{M}(\underline{s}; a), \mathcal{M}(\underline{t}; b)) \xrightarrow{\mathrm{res}_K} \mathrm{Ext}^1_{G_K}(\overline{\chi}_2, \overline{\chi}_1)$$

is not surjective.

Proof. It suffices to show that $\mathrm{im}(\mathrm{res}_{K_\infty} \circ \mathrm{res}_K)$ is a proper subspace of $\mathrm{im}(\mathrm{res}_{K_\infty})$. Viewing classes in $\mathrm{Ext}^1_{G_{K_\infty}}(\overline{\chi}_2, \overline{\chi}_1)$ as étale φ-modules, it therefore suffices to exhibit an étale φ-module as in the statement of Lemma 2.2.15 (3) which is not in the image of $\mathrm{res}_{K_\infty} \circ \mathrm{res}_K$.

By Lemma 2.2.18, we may assume that for some i we either have $\lfloor n_{i+1} \rfloor = -1$ and $r_i \neq p$, or we have $\lfloor n_{i+1} \rfloor \leq -2$. If $r_i \neq p$ then we set $x_i = s_i + \lfloor n_{i+1} \rfloor - e + 1$, while if $r_i = p$ (so that $\lfloor n_{i+1} \rfloor \leq -2$) we set $x_i = s_i + \lfloor n_{i+1} \rfloor - e + 2$. It follows from (2.2.14) that $s_i + \lfloor n_{i+1} \rfloor - e + 1 \equiv t_i + r_i \pmod p$, so we have

$$x_i \not\equiv t_i \pmod p. \tag{2.2.20}$$

We claim that we also have

$$s_i + \lfloor n_{i+1} \rfloor - e + 1 \leq x_i \leq s_i + \lfloor n_{i+1} \rfloor. \tag{2.2.21}$$

Indeed by definition we have $x_i = s_i + \lfloor n_{i+1} \rfloor - e + 1$ or $x_i = s_i + \lfloor n_{i+1} \rfloor - e + 2$, so the lower bound is immediate, and the upper bound is also automatic unless $e = 1$. If $e = 1$, we need to rule out the possibility that we are in the case $x_i = s_i + \lfloor n_{i+1} \rfloor - e + 2$. In this case we assumed that $\lfloor n_{i+1} \rfloor \leq -2$, so in particular $n_{i+1} < -1$; but since we have $s_j - t_j - e \geq -e(p-1)$ for all j, it follows from (2.2.8) that we have $n_j \geq -e$ for all j, and in particular if $e = 1$ we have $n_{i+1} \geq -1$, as required.

We also have $x_i \leq s_i - e$ (because if $\lfloor n_{i+1} \rfloor = -1$ then $x_i = s_i + \lfloor n_{i+1} \rfloor - e + 1 = s_i - e$, and otherwise $\lfloor n_{i+1} \rfloor \leq -2$ and we have $x_i = s_i + \lfloor n_{i+1} \rfloor - e + 2 \leq s_i - e$), i.e.

$$x_i < s_i - e + 1 = s_i - e + \max(n_{i+1}, 1). \tag{2.2.22}$$

(We have written the inequality in this form so that we can apply Lemma 2.2.7.) Set $y'_i = u^{x_i}$ and $y'_j = 0$ for all $j \neq i$. By (2.2.21) and Lemma 2.2.15, it suffices to show that the étale φ-module M arising from taking the y_j in (2.2.16) to be our y'_j is not of the form $\mathfrak{M}[1/u]$ for any Breuil–Kisin module $\mathfrak{M}$ satisfying the constraints of Lemma 2.2.7.

Suppose on the contrary that $\mathfrak{M}$ as in (2.2.5) has $\mathfrak{M}[1/u] \cong M$. This means that there is a change of variables $e'_j = e_j$, $f'_j = f_j + \lambda_j e_j$ with $\lambda_j \in \overline{\mathbf{F}}_p((u))$ having the property that for all j, we have

$$\varphi(f'_{j-1}) = (a)_j u^{s_j} f'_j + y_j e_j.$$

Equivalently, for each j we must have

$$y_j = y'_j + (b)_j u^{t_j} \varphi(\lambda_{j-1}) - (a)_j u^{s_j} \lambda_j. \tag{2.2.23}$$

Recall that we chose $y'_i = u^{x_i}$, where x_i satisfies (2.2.20) and (2.2.22), so the coefficient of u^{x_i} in y_i is zero by Lemma 2.2.7. The coefficient of u^{x_i} in $u^{t_i}\varphi(\lambda_{i-1})$ is also zero (again by (2.2.20)), so it follows from (2.2.23) with $j = i$ that the coefficient of u^{x_i} in $u^{s_i}\lambda_i$ is nonzero. Thus λ_i has a term of degree $x_i - s_i$. By (2.2.22) we have $x_i - s_i \leq -e$.

We claim that this implies that every λ_j has a term of degree at most $-e$. To see this we rewrite (2.2.23) for j replaced by $j+1$ in the form

$$(a)_{j+1} u^{s_{j+1}} \lambda_{j+1} = (y'_{j+1} - y_{j+1}) + (b)_{j+1} u^{t_{j+1}} \varphi(\lambda_j). \tag{2.2.24}$$

If $j + 1 \neq i$ then $y'_{j+1} - y_{j+1} \in \overline{\mathbf{F}}_p[[u]]$, so if λ_j has a term of degree at most $-e$, then $u^{t_{j+1}}\varphi(\lambda_j)$ has a term of degree at most $t_{j+1} - ep$, which must cancel with a term in $u^{s_{j+1}}\lambda_{j+1}$. Thus λ_{j+1} has a term of degree at most $t_{j+1} - ep - s_{j+1} \leq e(p-2) - ep = -2e < -e$, so the claim follows from induction (beginning with the case $j = i$).

Now $v_j \le -e$ denote the u-adic valuation of λ_j. Then $u^{t_{j+1}}\varphi(\lambda_j)$ has a nonzero term of degree $t_{j+1} + pv_j$, which again must cancel with a term in $u^{s_{j+1}}\lambda_{j+1}$. (Indeed, we have $t_{j+1} + pv_j \le e(p-2) - ep < 0$, and the only possible term in any $y'_{j+1} - y_{j+1}$ of negative degree is the term u^{x_i} in y'_i, which cannot cancel with a term of degree $t_i + pv_{i-1}$ by (2.2.20)). We therefore have

$$v_{j+1} \le t_{j+1} - s_{j+1} + pv_j \le t_{j+1} + pv_j \le e(p-2) + pv_j,$$

i.e.

$$pv_j - v_{j+1} \ge -e(p-2). \qquad (2.2.25)$$

Summing these inequalities multiplied by appropriate powers of p, we have

$$\sum_{j=1}^{f} p^{f-j}(pv_{j+i-1} - v_{j+i}) \ge -e(p-2)(p^f-1)/(p-1),$$

so that $v_j \ge -e(p-2)/(p-1) > -e$ for each j. Since we already saw that $v_j \le -e$ for all j, we have a contradiction, and we are done. □

Definition 2.2.26. Let $\overline{\chi}_1, \overline{\chi}_2 : G_K \to \overline{\mathbf{F}}_p^\times$ be two characters. We say that an element of $\mathrm{Ext}^1_{G_K}(\overline{\chi}_2, \overline{\chi}_1)$ is generic if it is not in the image of the restriction map

$$\mathrm{Ext}^1_{\mathrm{BrMod}^{\mathrm{cr}}}(\mathcal{M}(\underline{s};a), \mathcal{M}(\underline{t};b)) \xrightarrow{\mathrm{res}_K} \mathrm{Ext}^1_{G_K}(\overline{\chi}_2, \overline{\chi}_1)$$

for any rank 1 Breuil modules $\mathcal{M}(\underline{s};a)$ and $\mathcal{M}(\underline{t};b))$ with $T(\mathcal{M}(\underline{t};b)) = \overline{\chi}_1$, $T(\mathcal{M}(\underline{s};a)) = \overline{\chi}_2$ and $\sum_{j=1}^{f}(s_j - t_j - e) < 0$.

Remark 2.2.27. Note that by Lemma 2.2.19, the generic extensions in $\mathrm{Ext}^1_{G_K}(\overline{\chi}_2, \overline{\chi}_1)$ are the complement of the union of finitely many proper subspaces.

Remark 2.2.28. Definition 2.2.26 may seem a little ad hoc, but it is closely related to the condition of being a generic $\overline{\mathbf{F}}_p$-point on an irreducible component of the 2-dimensional Emerton–Gee stack (which we recall in Section 2.4). To make this precise, we would need to work simultaneously with arbitrary unramified twists of the characters $\overline{\chi}_1, \overline{\chi}_2$. While it is clear that the arguments above are uniform across such unramified twists, and we could presumably formulate and prove our results in the context of stacks of Breuil modules (and Breuil–Kisin modules), there does not seem to be any benefit in doing so. Indeed, while working with $\overline{\mathbf{F}}_p$-points occasionally leads to slightly clumsy formulations, we view it as a feature of the structural results proved in [EG23] (see e.g. Theorem 2.4.3) that we can prove statements about families of Galois representations (e.g. lifting rings) by only thinking about representations valued in $\overline{\mathbf{F}}_p$.

Remark 2.2.29. While it may be possible to use other integral p-adic Hodge theories (e.g. $(\varphi, \widehat{G})$-modules) to prove a version of Lemma 2.2.19 which could apply to the reductions of crystalline representations in a greater range of Hodge–Tate weights than $[0, p-2]$, it is unlikely that it can be significantly improved. Indeed already for $K = \mathbf{Q}_p$, there are irreducible 2-dimensional crystalline representations of $G_{\mathbf{Q}_p}$ with Hodge–Tate weights $0, p+2$ whose corresponding mod p Breuil–Kisin modules are of the form $\begin{pmatrix} bu^p & x \\ 0 & au^2 \end{pmatrix}$ where $a, b \in \overline{\mathbf{F}}_p^\times$ and $x \in \overline{\mathbf{F}}_p$ are arbitrary, and consequently give all extensions of the corresponding characters of $G_{\mathbf{Q}_p}$ when $a \ne b$. (In addition, it is not clear to us whether the analogue of Lemma 2.2.7 holds for $(\varphi, \widehat{G})$-modules, even in height $[0, p-2]$, although we have not seriously pursued this question.)

2.3. **Generic weight 0 crystalline representations.** In this subsection and the next, in order to be compatible with the notation of [EG23], we work with d-dimensional rather than n-dimensional representations.

Definition 2.3.1. We say that a representation $\overline{\rho} : G_K \to \mathrm{GL}_d(\mathbf{F})$ is generic if it has the form

$$\overline{\rho} \cong \begin{pmatrix} \overline{\chi}_d & * & \dots & * \\ 0 & \overline{\chi}_{d-1} & \dots & * \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & \overline{\chi}_1 \end{pmatrix}$$

and for $i = 1, \dots, d-1$, the off diagonal extension class in $\mathrm{Ext}^1_{G_K}(\overline{\chi}_i, \overline{\chi}_{i+1})$ is generic in the sense of Definition 2.2.26.

Theorem 2.3.2. *Suppose $p > d$ and let $\rho : G_K \to \mathrm{GL}_d(\mathcal{O})$ be a weight 0 crystalline representation such that $\overline{\rho}$ is generic in the sense of Definition 2.3.1. Then*

$$\rho \simeq \begin{pmatrix} \mathrm{ur}_{\lambda_d} & * & \dots & * \\ 0 & \mathrm{ur}_{\lambda_{d-1}}\varepsilon^{-1} & \dots & * \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & \mathrm{ur}_{\lambda_1}\varepsilon^{1-d} \end{pmatrix}$$

for some $\lambda_1, \dots, \lambda_d \in \mathcal{O}^\times$.

Proof. The proof will be by induction on d. The base case $d = 1$ is trivial. For the inductive step, we claim that ρ fits in an exact sequence

$$0 \to \rho' \to \rho \to \mathrm{ur}_{\lambda_1}\varepsilon^{1-d} \to 0.$$

Admitting this for the moment, ρ' is a $d-1$ dimensional crystalline representation of weight 0, and $\overline{\rho}'$ is generic (since $\overline{\rho}$ has a unique $d-1$ dimensional subrepresentation, which is generic). We conclude by induction on d.

We now prove the key claim above. As the Hodge–Tate weights of ρ are contained in the interval $[0, d-1] \subseteq [0, p-2]$, by Theorem 2.1.5 there is a crystalline Breuil module $\mathcal{M}$ of rank d with $\overline{\rho} \cong T(\mathcal{M})$, whose underlying Breuil–Kisin module has height at most $(d-1)$. By Theorem 2.1.4 (2), the unique maximal filtration on $\overline{\rho}$ determines a filtration $0 = \mathcal{M}^0 \subset \mathcal{M}^1 \subset \cdots \subset \mathcal{M}^d = \mathcal{M}$ by crystalline Breuil submodules. Write $\mathcal{M}^i/\mathcal{M}^{i-1} \simeq \mathcal{M}(\underline{s(i)}; a_i)$ in the notation of Lemma 2.2.3.

It follows from Lemma 2.2.19 and the definition of genericity that for each $1 \leq i \leq d-1$ we have

$$\sum_{j=1}^{f} (s(i+1)_j - s(i)_j - e) \geq 0.$$

Summing these inequalities over i, we obtain

$$\sum_{j=1}^{f} (s(d)_j - s(1)_j) \geq ef(d-1).$$

Since the underlying Breuil–Kisin module of $\mathcal{M}$ has height at most $(d-1)$, we have $s(i)_j \leq e(d-1)$ for all i, j, and hence $s(d)_j - s(1)_j \leq e(d-1)$ for all j. Since we also have the reverse inequality summed over f this implies that $s(d)_j - s(1)_j = e(d-1)$ for all j, and hence $s(1)_j = 0$ and $s(d)_j = e(d-1)$ for all j.

Now let $\mathfrak{M}/\mathfrak{S}_{\mathcal{O}}$ be the Breuil–Kisin module associated to ρ, and $\overline{\mathfrak{M}} = \mathfrak{M} \otimes_{\mathcal{O}} \mathbf{F}$. This is the Breuil–Kisin module underlying $\mathcal{M}$ by Theorem 2.1.5. Since $s(d)_j = e(d-1)$ for all j, we have shown that $\overline{\mathfrak{M}}$ has a rank 1 quotient $\overline{\mathfrak{M}} \to \mathfrak{S}_{\mathbf{F}} \cdot v$ where $\varphi(\overline{v}) = \overline{\lambda} u^{e(d-1)} \overline{v}$ for some $\overline{\lambda} \in (k \otimes \mathbf{F})^{\times}$. It follows from [Kis09, Prop. 1.2.11] (or rather its obvious generalization from height 1 to height $(d-1)$ Breuil–Kisin modules) that this lifts to a quotient $\mathfrak{M} \to \mathfrak{S}_{\mathcal{O}} \cdot v$ where $\varphi(v) = \lambda E(u)^{d-1} v$ for some $\lambda \in (W(k) \otimes \mathcal{O})^{\times}$. Indeed, using height $(d-1)$ duality [Liu07, §3.1], we need to lift a rank one 'multiplicative' submodule of $\overline{\mathfrak{M}}^*$ to $\mathfrak{M}^*$, where multiplicative means that the linearization of φ is an isomorphism. As in [Kis09, Prop. 1.2.11], we have a maximal multiplicative submodule $\mathfrak{M}^{*,m}$ of $\mathfrak{M}^*$ which lifts the maximal multiplicative submodule of $\overline{\mathfrak{M}}^*$ and therefore has rank at least one. Since ρ is weight 0 crystalline, its maximal unramified subrepresentation has dimension at most one. It follows that $\mathfrak{M}^{*,m}$ has rank one and is the desired lift.

Finally it follows from the full faithfulness of the functor from lattices in crystalline representations to Breuil–Kisin modules (see [Kis06, Prop. 1.3.15], or for the precise statement we are using here [Kis10, Thm. 1.2.1]) that there is a nonzero map $\rho \to \mathrm{ur}_{\lambda_1} \varepsilon^{1-d}$. □

Corollary 2.3.3. *Let $\overline{\rho} : G_K \to \mathrm{GL}_d(\mathbf{F})$ be a generic representation. Suppose that $\overline{\rho}$ has a crystalline lift of weight 0. Then $\overline{\rho}$ has the form*

$$\overline{\rho} \cong \begin{pmatrix} \mathrm{ur}_{\overline{\lambda}_d} & * & \dots & * \\ 0 & \mathrm{ur}_{\overline{\lambda}_{d-1}} \overline{\varepsilon}^{-1} & \dots & * \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & \mathrm{ur}_{\overline{\lambda}_1} \overline{\varepsilon}^{1-d} \end{pmatrix}$$

and moreover the off-diagonal extensions are peu ramifiée.

Proof. The first statement is immediate from Theorem 2.3.2, while the claim about the extensions follows from the fact that the reduction of a crystalline representation

$$\begin{pmatrix} \mathrm{ur}_{\lambda_2} & * \\ 0 & \mathrm{ur}_{\lambda_1} \varepsilon^{-1} \end{pmatrix}$$

is peu ramifiée (e.g. by [CEGS22b, Lem. A.4]; the reduction of such a representation is finite flat, hence peu ramifiée). □

2.4. **Recollections on Emerton–Gee stacks.** We now recall some of the main results of [EG23], and prove a slight extension of them. We use the notation of [EG23], and in particular we continue to work with d-dimensional rather than n-dimensional representations.

As above, we let $E/\mathbf{Q}_p$ be a finite extension containing the Galois closure of K, with ring of integers $\mathcal{O}$, uniformizer ϖ, and residue field $\mathcal{O}/\varpi = \mathbf{F}$. The stack $\mathcal{X}_d$ over $\mathrm{Spf}\, \mathcal{O}$ is defined in [EG23, Defn. 3.2.1]. It is a stack of (φ, Γ)-modules, but if $\mathbf{F}'/\mathbf{F}$ is a finite extension (or if $\mathbf{F}' = \overline{\mathbf{F}}_p$), then the groupoid of points $x \in \mathcal{X}_d(\mathbf{F}')$ is canonically equivalent to the groupoid of Galois representations $\overline{\rho} : G_K \to \mathrm{GL}_d(\mathbf{F}')$ [EG23, §3.6.1], and we use this identification without comment below. The stack $\mathcal{X}_d$ is a Noetherian formal algebraic stack [EG23, Cor. 5.5.18], and it admits closed substacks cut out by (potentially) crystalline or semistable conditions. In particular there is a closed substack $\mathcal{X}_d^{\mathrm{crys},0}$ of $\mathcal{X}_d$ corresponding to crystalline representations of weight 0, which has the following properties.

Proposition 2.4.1.

(1) $\mathcal{X}_d^{\mathrm{crys},0}$ *is a p-adic formal algebraic stack, which is flat over* $\operatorname{Spf}\mathcal{O}$ *and of finite type. In particular, the special fibre* $\overline{\mathcal{X}}_d^{\mathrm{crys},0} := \mathcal{X}_d^{\mathrm{crys},0} \times_{\operatorname{Spf}\mathcal{O}} \operatorname{Spec}\mathbf{F}$ *is an algebraic stack.*

(2) If A° is a finite flat $\mathcal{O}$-algebra, then $\mathcal{X}_d^{\mathrm{crys},0}(A^\circ)$ is the subgroupoid of $\mathcal{X}_d(A^\circ)$ consisting of G_K-representations which after inverting p are crystalline of weight 0.

(3) The special fibre $\overline{\mathcal{X}}_d^{\mathrm{crys},0} := \mathcal{X}_d^{\mathrm{crys},0} \times_{\operatorname{Spf}\mathcal{O}} \operatorname{Spec}\mathbf{F}$ *is equidimensional of dimension* $[K:\mathbf{Q}_p]d(d-1)/2$.

(4) For any finite extension $\mathbf{F}'$ *of* $\mathbf{F}$ *and any point* $x : \operatorname{Spec}\mathbf{F}' \to \overline{\mathcal{X}}_d^{\mathrm{crys},0}$, *there is a versal morphism* $\operatorname{Spf} R_{\overline{\rho}}^{\mathrm{crys},0,\mathcal{O}'} \to \mathcal{X}_d^{\mathrm{crys},0}$ *at x, where $\overline{\rho} : G_K \to$* $\mathrm{GL}_d(\mathbf{F}')$ *is the representation corresponding to x,* $\mathcal{O}' := W(\mathbf{F}') \otimes_{W(\mathbf{F})} \mathcal{O}$, *and* $R_{\overline{\rho}}^{\mathrm{crys},0,\mathcal{O}'}$ *is the weight* 0 *crystalline lifting ring.*

Proof. We define $\mathcal{X}_d^{\mathrm{crys},0}$ to be the stack $\mathcal{X}_{K,d}^{\mathrm{crys},\underline{\lambda},\tau}$ of [EG23, Defn. 4.8.8], taking $\underline{\lambda}$ to be given by $\lambda_{\sigma,i} = d-i$ for all σ, i, and τ to be trivial. Then the first two claims are [EG23, Thm. 4.8.12], the third is [EG23, Thm. 4.8.14], and the final claim is [EG23, Prop. 4.8.10]. ☐

We now recall some definitions from [EG23, §5.5]. By a *Serre weight* $\underline{k}$ we mean a tuple of integers $\{k_{\overline{\sigma},i}\}_{\overline{\sigma}:k\hookrightarrow\overline{\mathbf{F}}_p, 1\le i\le d}$ with the properties that

- $p-1 \ge k_{\overline{\sigma},i} - k_{\overline{\sigma},i+1} \ge 0$ for each $1 \le i \le d-1$, and
- $p-1 \ge k_{\overline{\sigma},d} \ge 0$, and not every $k_{\overline{\sigma},d}$ is equal to $p-1$.

For each $\overline{\sigma} : k \hookrightarrow \mathbf{F}$, we define the fundamental character $\omega_{\overline{\sigma}}$ to be the composite

$$\omega_{\overline{\sigma}} : I_K \to W_K^{\mathrm{ab}} \xrightarrow{\mathrm{Art}_K^{-1}} \mathcal{O}_K^\times \to k^\times \xrightarrow{\overline{\sigma}} \mathbf{F}^\times.$$

As in [EG23, §5.5], for each Serre weight $\underline{k}$ we choose characters $\omega_{\underline{k},i} : G_K \to \mathbf{F}^\times$ $(i = 1,\dots,d)$ with

$$\omega_{\underline{k},i}|_{I_K} = \prod_{\overline{\sigma}:k\hookrightarrow\mathbf{F}} \omega_{\overline{\sigma}}^{-k_{\overline{\sigma},i}},$$

in such a way that if $k_{\overline{\sigma},i} - k_{\overline{\sigma},i+1} = p-1$ for all $\overline{\sigma}$, then $\omega_{\underline{k},i} = \omega_{\underline{k},i+1}$. (In [EG23, §5.5] it was erroneously claimed that we could impose further constraints on the $\omega_{\underline{k},i}$, but as explained in [EG], these properties are all that we require.) For any $\nu \in \overline{\mathbf{F}}_p$ we write $\mathrm{ur}_\nu : G_K \to \overline{\mathbf{F}}_p^\times$ for the unramified character taking a geometric Frobenius to λ.

We say that a representation $\overline{\rho} : G_K \to \mathrm{GL}_d(\overline{\mathbf{F}}_p)$ is *maximally nonsplit of niveau* 1 if it has a unique filtration by G_K-stable $\overline{\mathbf{F}}_p$-subspaces such that all of the graded pieces are one-dimensional representations of G_K. We assign a unique Serre weight $\underline{k}$ to each such $\overline{\rho}$ in the following way: we say that $\overline{\rho}$ is of weight $\underline{k}$ if and only we can write

$$\overline{\rho} \cong \begin{pmatrix} \mathrm{ur}_{\nu_d}\omega_{\underline{k},d} & * & \dots & * \\ 0 & \mathrm{ur}_{\nu_{d-1}}\overline{\varepsilon}^{-1}\omega_{\underline{k},d-1} & \dots & * \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & \mathrm{ur}_{\nu_1}\overline{\varepsilon}^{1-d}\omega_{\underline{k},1} \end{pmatrix}; \tag{2.4.2}$$

this uniquely determines $\underline{k}$, except that if $\omega_{\underline{k},i} = \omega_{\underline{k},i+1}$ then we need to say whether $k_{\overline{\sigma},i} - k_{\overline{\sigma},i+1} = p-1$ for all $\overline{\sigma}$ or $k_{\overline{\sigma},i} - k_{\overline{\sigma},i+1} = 0$ for all $\overline{\sigma}$. We distinguish these possibilities as follows: if $\omega_{\underline{k},i} = \omega_{\underline{k},i+1}$, then we set $k_{\overline{\sigma},d-i} - k_{\overline{\sigma},d+1-i} = p-1$ for all $\overline{\sigma}$ if and only if $\nu_i = \nu_{i+1}$ and the element of

$$\mathrm{Ext}^1_{G_K}(\mathrm{ur}_{\nu_i}\overline{\varepsilon}^{i-d}\omega_{\underline{k},i}, \mathrm{ur}_{\nu_{i+1}}\overline{\varepsilon}^{i+1-d}\omega_{\underline{k},i+1}) = H^1(G_K, \overline{\varepsilon})$$

determined by $\overline{\rho}$ is très ramifiée.

Let $(\mathbf{G}_m)^d_{\underline{k}}$ denote the closed subgroup scheme of $(\mathbf{G}_m)^d$ parameterizing tuples $(x_1, \dots, x_d)$ for which $x_i = x_{i+1}$ whenever $k_{\overline{\sigma},i} - k_{\overline{\sigma},i+1} = p-1$ for all $\overline{\sigma}$. By the definition that we just made, if $\overline{\rho}$ is maximally nonsplit of niveau 1 and weight $\underline{k}$, then the tuple $(\nu_1, \dots, \nu_d)$ is an $\overline{\mathbf{F}}_p$-point of $(\mathbf{G}_m)^d_{\underline{k}}$ (where the ν_i are as in in (2.4.2)).

We have the following slight variant on [EG23, Thm. 5.5.12].

Theorem 2.4.3.

(1) The Ind-algebraic stack $\mathcal{X}_{d,\mathrm{red}}$ is an algebraic stack, of finite presentation over $\mathbf{F}$.
(2) $\mathcal{X}_{d,\mathrm{red}}$ is equidimensional of dimension $[K : \mathbf{Q}_p]d(d-1)/2$.
(3) The irreducible components of $\mathcal{X}_{d,\mathrm{red}}$ are indexed by the Serre weights $\underline{k}$. More precisely, for each $\underline{k}$ there is an irreducible component $\mathcal{X}^{\underline{k}}_{d,\mathrm{red}}$ containing a dense open substack $\mathcal{U}^{\underline{k}}$, all of whose $\overline{\mathbf{F}}_p$-points are maximally nonsplit of niveau one and weight $\underline{k}$; and the $\mathcal{X}^{\underline{k}}_{d,\mathrm{red}}$ exhaust the irreducible components of $\mathcal{X}_{d,\mathrm{red}}$.
(4) There is an open subscheme T of $(\mathbf{G}_m)^d_{\underline{k}}$ such that for all $(t_1, \dots, t_d) \in T(\overline{\mathbf{F}}_p)$, there is an $\overline{\mathbf{F}}_p$-point of $\mathcal{U}^{\underline{k}}$ corresponding to a representation (2.4.2) *with $\nu_i = t_i$ for all i, and which is generic in the sense of Definition 2.3.1.*

Proof. Everything except for part (4) is part of [EG23, Thm. 5.5.12, Thm. 6.5.1]. Part (4) follows from the version of [EG23, Thm. 5.5.12] proved in [EG], as we explain below.

We begin by taking T to be an open contained in the image of the eigenvalue morphism $\mathcal{U}^{\underline{k}} \to (\mathbf{G}_m)^d_{\underline{k}}$, and then further shrink it so that for any $m < n$ and $(t_1, \dots, t_d) \in T(\overline{\mathbf{F}}_p)$ either:

- we have $k_{\overline{\sigma},i} - k_{\overline{\sigma},i+1} = p-1$ for all $\overline{\sigma}$ and all $m \le i < n$, or
- we have $(\mathrm{ur}_{t_n}\overline{\varepsilon}^{n-d}\omega_{\underline{k},n})/(\mathrm{ur}_{t_m}\overline{\varepsilon}^{m-d}\omega_{\underline{k},m}) \ne \overline{\varepsilon}$.

We then fix $(t_1, \dots, t_d) \in T(\overline{\mathbf{F}}_p)$, and regard each $\mathrm{Ext}^1_{G_K}(\mathrm{ur}_{t_i}\overline{\varepsilon}^{i-d}\omega_{\underline{k},i}, \mathrm{ur}_{t_{i+1}}\overline{\varepsilon}^{i+1-d}\omega_{\underline{k},i+1})$ as an affine space over $\overline{\mathbf{F}}_p$, and as in [EG] we define

$$\mathrm{Ext}^1_{(t_1,\dots,t_d),\underline{k}} \subseteq \prod_{i=1}^{d-1} \mathrm{Ext}^1_{G_K}(\mathrm{ur}_{t_i}\overline{\varepsilon}^{i-d}\omega_{\underline{k},i}, \mathrm{ur}_{t_{i+1}}\overline{\varepsilon}^{i+1-d}\omega_{\underline{k},i+1})$$

to be the closed subvariety of tuples of extension classes $(\psi_1, \dots, \psi_{d-1})$ determined by the condition that for each $i = 1, \dots, d-2$, the cup product $\psi_i \cup \psi_{i+1}$ vanishes.

The version of [EG23, Thm. 5.5.12] proved in [EG] states in particular that for a dense Zariski open subset U of $\mathrm{Ext}^1_{(t_1,\dots,t_d),\underline{k}}$, the corresponding extension classes are realized by some $\overline{\rho} \in \mathcal{U}^{\underline{k}}(\overline{\mathbf{F}}_p)$; so it suffices to show that U contains a point $(\psi_1, \dots, \psi_{d-1})$ with each ψ_i generic. As the locus of generic classes in $\mathrm{Ext}^1_{(t_1,\dots,t_d),\underline{k}}$

is open, and U is dense, it suffices in turn to exhibit a single generic class in $\mathrm{Ext}^1_{(t_1,\dots,t_d),\underline{k}}$.

To do this, first note that $\psi_i \cup \psi_{i+1}$ is an element of the Ext^2 group

$$\mathrm{Ext}^2_{G_K}(\mathrm{ur}_{t_i}\overline{\varepsilon}^{i-d}\omega_{\underline{k},i}, \mathrm{ur}_{t_{i+2}}\overline{\varepsilon}^{i+2-d}\omega_{\underline{k},i+2}).$$

This group vanishes unless $(\mathrm{ur}_{t_{i+2}}\overline{\varepsilon}^{i+2-d}\omega_{\underline{k},i+2})/(\mathrm{ur}_{t_i}\overline{\varepsilon}^{i-d}\omega_{\underline{k},i}) = \overline{\varepsilon}$, which by our choice of T can only occur when $k_{\overline{\sigma},i} - k_{\overline{\sigma},i+1} = k_{\overline{\sigma},i+1} - k_{\overline{\sigma},i+2} = p-1$ for all $\overline{\sigma}$ and $\overline{\varepsilon} = 1$. Thus if $\overline{\varepsilon} \neq 1$, we can just choose each ψ_i to be any generic extension class, and the cup product condition is automatically satisfied.

We assume from now on that $\overline{\varepsilon} = 1$ and fix a maximal interval $m < n$ such that $k_{\overline{\sigma},i} - k_{\overline{\sigma},i+1} = p-1$ for all $\overline{\sigma}$ and all $m \le i < n$. The characters $\mathrm{ur}_{t_i}\overline{\varepsilon}^{i-d}\omega_{\underline{k},i}$ for $m \le i \le n$ are all equal, and we write $\overline{\chi}$ for their common value. The cup product pairing is a perfect pairing

$$\mathrm{Ext}^1_{G_K}(\overline{\chi},\overline{\chi}) \times \mathrm{Ext}^1_{G_K}(\overline{\chi},\overline{\chi}) \to \mathrm{Ext}^2_{G_K}(\overline{\chi},\overline{\chi}) = H^2(G_K,\overline{\varepsilon}) \simeq \mathbf{F}.$$

The generic classes are the complement of the union of finitely many proper subspaces $L_j \subset \mathrm{Ext}^1_{G_K}(\overline{\chi},\overline{\chi})$, with annihilators $L_j^\perp$ under the pairing. Pick a generic class ψ_m which is not in any $L_j^\perp$. Then the annihilator $\langle\psi_m\rangle^\perp$ cannot be contained in $\bigcup_j L_j \cup \bigcup_j L_j^\perp$ (otherwise $\langle\psi_m\rangle^\perp$ is contained in one of the L_j or $L_j^\perp$ which implies that L_j or $L_j^\perp$ is contained in $\langle\psi_m\rangle$). So we can find a generic class $\psi_{m+1} \in \langle\psi_m\rangle^\perp$ which is also not in any $L_j^\perp$. Repeating, we can find a sequence $\psi_m, \psi_{m+1}, \dots \psi_{n-1}$ of generic classes such that $\psi_i \cup \psi_{i+1} = 0$ for $m \le i < n$. □

2.5. **Generic reducedness.** We now compute the underlying cycle of the weight 0 crystalline stack, and deduce our main result on generic reducedness (Theorem 2.5.5).

This underlying cycle is defined as follows. By Theorem 2.4.3, the special fibre $\overline{\mathcal{X}}_d^{\mathrm{crys},0}$ is a closed substack of the special fibre $\overline{\mathcal{X}}_d$, and its irreducible components (with the induced reduced substack structure) are therefore closed substacks of the algebraic stack $\overline{\mathcal{X}}_{d,\mathrm{red}}$ (see [Sta13, Tag 0DR4] for the theory of irreducible components of algebraic stacks and their multiplicities). Furthermore, $\overline{\mathcal{X}}_d^{\mathrm{crys},0}$ and $\overline{\mathcal{X}}_{d,\mathrm{red}}$ are both algebraic stacks over $\mathbf{F}$ which are equidimensional of dimension $[K:\mathbf{Q}_p]d(d-1)/2$. It follows that the irreducible components of $\overline{\mathcal{X}}_d^{\mathrm{crys},0}$ are irreducible components of $\overline{\mathcal{X}}_{d,\mathrm{red}}$, and are therefore of the form $\overline{\mathcal{X}}^{\underline{k}}_{d,\mathrm{red}}$ for some Serre weight $\underline{k}$.

For each $\underline{k}$, we write $\mu_{\underline{k}}(\overline{\mathcal{X}}_d^{\mathrm{crys},0})$ for the multiplicity of $\overline{\mathcal{X}}^{\underline{k}}_{d,\mathrm{red}}$ as a component of $\overline{\mathcal{X}}_d^{\mathrm{crys},0}$. We write $Z_{\mathrm{crys},0} = Z(\overline{\mathcal{X}}_d^{\mathrm{crys},0})$ for the corresponding cycle, i.e. for the formal sum

$$Z_{\mathrm{crys},0} = \sum_{\underline{k}} \mu_{\underline{k}}(\overline{\mathcal{X}}_d^{\mathrm{crys},0}) \cdot \overline{\mathcal{X}}^{\underline{k}}_{d,\mathrm{red}}, \tag{2.5.1}$$

which we regard as an element of the finitely generated free abelian group $\mathbf{Z}[\mathcal{X}_{d,\mathrm{red}}]$ whose generators are the irreducible components $\overline{\mathcal{X}}^{\underline{k}}_{d,\mathrm{red}}$.

Theorem 2.5.2. *Suppose that $p > d$. Then we have an equality of cycles*

$$Z_{\mathrm{crys},0} = \overline{\mathcal{X}}^{\underline{0}}_{d,\mathrm{red}},$$

where $\underline{0}$ is the Serre weight $\underline{k}$ with $k_{\sigma,i} = 0$ for all σ, i. In particular, the special fibre $\overline{\mathcal{X}}_d^{\mathrm{crys},0}$ is generically reduced.

Proof. Suppose (in the notation of Theorem 2.4.3 (3)) that $\overline{\mathcal{X}}^{\underline{k}}_{d,\mathrm{red}}$ is an irreducible component of $\mathcal{X}_{d,\mathrm{red}}$ contained in the $\overline{\mathcal{X}}_d^{\mathrm{crys},0}$. We begin by showing that $\underline{k} = \underline{0}$. By Theorem 2.4.3 (4) after possibly enlarging $\mathbf{F}$, we can pick a point $x : \operatorname{Spec} \mathbf{F} \to \mathcal{U}^{\underline{k}}$ so that the corresponding representation $\overline{\rho} : G_K \to \mathrm{GL}_d(\mathbf{F})$ is generic in the sense of Definition 2.3.1 (it is also maximally nonsplit of niveau one and weight $\underline{k}$, since it comes from a point of $\mathcal{U}^{\underline{k}}$). Since x is in $\overline{\mathcal{X}}_d^{\mathrm{crys},0}$, $\overline{\rho}$ has a crystalline lift of weight 0. We can now apply Corollary 2.3.3 to conclude that $\underline{k} = 0$.

We have now shown that the support of $Z_{\mathrm{crys},0}$ is indeed $\overline{\mathcal{X}}^{\underline{0}}_{d,\mathrm{red}}$, i.e. that the underlying reduced substack of $\overline{\mathcal{X}}_d^{\mathrm{crys},0}$ is equal to $\overline{\mathcal{X}}^{\underline{0}}_{d,\mathrm{red}}$, and it remains to determine the generic multiplicity. To do this, we modify our choice of point x as follows: by definition, we have $(\mathbf{G}_m)^d_{\underline{0}} = (\mathbf{G}_m)^d$, so we can and do choose our point x such that if $i \neq j$, then

$$(\mathrm{ur}_{\nu_i}\overline{\varepsilon}^{i-d})/(\mathrm{ur}_{\nu_j}\overline{\varepsilon}^{j-d}) \neq 1, \overline{\varepsilon}. \tag{2.5.3}$$

We will show that $\overline{\mathcal{X}}_d^{\mathrm{crys},0}$ is reduced in some open neighbourhood of x. Since the reduced locus is open, and $\overline{\mathcal{X}}_d^{\mathrm{crys},0}$ is irreducible, this implies that $\overline{\mathcal{X}}_d^{\mathrm{crys},0}$ is generically reduced.

We claim that the crystalline lifting ring $R_{\overline{\rho}}^{\mathrm{crys},0,\mathcal{O}}$ is formally smooth, where $\overline{\rho}$ corresponds to our chosen point x. Indeed, by Theorem 2.3.2, crystalline lifts of $\overline{\rho}$ of weight 0 are ordinary, and so $R_{\overline{\rho}}^{\mathrm{crys},0,\mathcal{O}}$ is the weight 0 ordinary lifting ring of $\overline{\rho}$. Since $\overline{\rho}$ is maximally nonsplit (i.e. has a unique filtration with rank 1 graded pieces) and satisfies (2.5.3), the deformation problem represented by $R_{\overline{\rho}}^{\mathrm{crys},0,\mathcal{O}}$ coincides with the one considered in [CHT08, 2.4.2] (taking $F_{\tilde{v}}$ there to be our K, n to be our d, and $\chi_{v,i}$ to be ε^{-i}), and the formal smoothness is [CHT08, Lem. 2.4.7].

By Theorem 2.4.1 we have a versal morphism $\operatorname{Spec} R_{\overline{\rho}}^{\mathrm{crys},0,\mathcal{O}}/\varpi \to \overline{\mathcal{X}}_d^{\mathrm{crys},0}$ at x, where $R_{\overline{\rho}}^{\mathrm{crys},0,\mathcal{O}}/\varpi$ is formally smooth and in particular reduced. By [Sta13, Tag 0DR0] we may find a smooth morphism $V \to \overline{\mathcal{X}}_d^{\mathrm{crys},0}$ with source a finite type $\mathcal{O}/\varpi$-scheme, and a point $v \in V$ with residue field $\mathbf{F}$, such that there is an isomorphism $\widehat{\mathcal{O}}_{V,v} \cong R_{\overline{\rho}}^{\mathrm{crys},0,\mathcal{O}}/\varpi$, compatible with the given morphism to $\overline{\mathcal{X}}_d^{\mathrm{crys},0}$. By [Sta13, Tag 00MC] and [Sta13, Tag 033F], the local ring $\mathcal{O}_{V,v}$ is reduced. Since being reduced is an open condition, we see that V is reduced in an open neighbourhood of v; and since it is also a smooth local condition (see [Sta13, Tag 04YH]) it follows that $\overline{\mathcal{X}}_d^{\mathrm{crys},0}$ is reduced in an open neighbourhood of x, and we are done. □

Remark 2.5.4. Since the algebraic representation of GL_d of highest weight 0 is the trivial representation, Theorem 2.5.2 shows that if $p > d$, the cycle $Z_{\underline{0}}$ in the geometric Breuil–Mézard conjecture [EG23, Conj. 8.2.2] is necessarily equal to $\overline{\mathcal{X}}^{\underline{0}}_{d,\mathrm{red}}$. As far as we are aware, this is the only instance in which such a cycle has been computed for $d > 2$ and $K/\mathbf{Q}_p$ arbitrary.

Theorem 2.5.5. *Suppose that $p > d$, that $K/\mathbf{Q}_p$ is a finite extension, and that $E/\mathbf{Q}_p$ is a finite extension containing the Galois closure of K, with ring of integers $\mathcal{O}$ and residue field $\mathbf{F}$.*

Then for any $\overline{\rho} : G_K \to \mathrm{GL}_d(\mathbf{F})$, the special fibre $R_{\overline{\rho}}^{\mathrm{crys},0,\mathcal{O}}/\varpi$ of the weight 0 crystalline lifting ring is generically reduced.

Proof. We follow the proof of [CEGS22a, Thm. 4.6]. By Proposition 2.4.1, we have a versal morphism $\operatorname{Spf} R_{\overline{\rho}}^{\mathrm{crys},0,\mathcal{O}}/\varpi \to \overline{\mathcal{X}}_d^{\mathrm{crys},0}$ at the $\mathbf{F}$-point of $\mathcal{X}_{d,\mathrm{red}}$ corresponding to $\overline{\rho}$. By [Sta13, Tag 0DR0] we may find a smooth morphism $V \to \overline{\mathcal{X}}_d^{\mathrm{crys},0}$ with source a finite type $\mathbf{F}$-scheme, and a point $v \in V$ with residue field $\mathbf{F}$, such that there is an isomorphism $\widehat{\mathcal{O}}_{V,v} \cong R_{\overline{\rho}}^{\mathrm{crys},0,\mathcal{O}}/\varpi$, compatible with the given morphism to $\overline{\mathcal{X}}_d^{\mathrm{crys},0}$.

By Theorem 2.5.2, there is a dense open substack $\mathcal{U}$ of $\overline{\mathcal{X}}_d^{\mathrm{crys},0}$ such that $\mathcal{U}$ is reduced. Since being reduced is a smooth local property, the pullback of $\mathcal{U}$ to V is a reduced open subscheme of V; and this pullback is furthermore dense in V, because the formation of the scheme-theoretic image of $\mathcal{U}$ in $\overline{\mathcal{X}}_d^{\mathrm{crys},0}$ commutes with flat base change [Sta13, Tag 0CMK]. Thus V is generically reduced, and the complete local rings of V at finite type points are generically reduced by [CEGS22a, Lem. 4.5]. In particular $R_{\overline{\rho}}^{\mathrm{crys},0,\mathcal{O}}/\varpi \cong \widehat{\mathcal{O}}_{V,v}$ is generically reduced, as required. □

Remark 2.5.6. The case $d = 2$ of Theorem 2.5.5 is a special case of [CEGS22a, Thm. 4.6]. In both cases the statement is deduced from the corresponding statement for the stack $\overline{\mathcal{X}}_d^{\mathrm{crys},0}$, and indeed in the case $d = 2$, Theorem 2.5.2 is a special case of [CEGS22a, Thm. 7.1, 7.6] (although the generic reducedness statement is proved earlier in [CEGS22a, Prop. 4.1]).

The argument that we use to prove Theorem 2.5.2 is necessarily rather different from the proof of [CEGS22a, Thm. 4.6], which was written before [EG23], and in particular could not use the structure of generic points on the irreducible components of $\mathcal{X}_{2,\mathrm{red}}$. Instead, the proof in [CEGS22a] uses the Kisin resolution of $\mathcal{X}_2^{\mathrm{crys},0}$ (originally defined for lifting rings in [Kis09]). By results on local models for Shimura varieties, this Kisin resolution has reduced special fibre, and the arguments in [CEGS22a] show that the map from the Kisin resolution is an isomorphism on dense open substacks of the source and target. In dimension greater than 2 we do not know of a candidate Kisin resolution for which we could expect to argue in this way.

The result [CEGS22a, Thm. 4.6] is more general than Theorem 2.5.5 (as always, in the special case $d = 2$), because it also proves the analogous statement for the potentially crystalline lifting ring of weight 0 and any tame type. The Breuil–Mézard conjecture implies that the analogous statement necessarily fails for $d \geq 4$ (even if $K = \mathbf{Q}_p$), because the reductions modulo p of the corresponding inertial types contain Serre weights with multiplicities greater than 1, even for generic choices of type (see [LLHLM23, Rem. 8.1.4]). Similarly, Theorem 2.5.2 is best possible in the sense that for any parallel Serre weight $\underline{k}$ greater than 0 (i.e. $k_{\sigma,i}$ is independent of σ, and the $k_{\sigma,i}$ are not all equal), the stack $\mathcal{X}_d^{\mathrm{crys},\underline{k}}$ cannot have generically reduced special fibre once K is sufficiently ramified. (While the Breuil–Mézard conjecture is not known, standard arguments with Taylor–Wiles patching give the expected lower bounds for Breuil–Mézard multiplicities, so it is presumably possible to prove unconditionally that the special fibres of the corresponding stacks are not generically reduced.)

Remark 2.5.7. Despite Remark 2.2.29, it seems plausible to us that Theorem 2.5.2 should also hold if $p \leq d$, but any proof will necessarily be more complicated, and presumably cannot rely only on an analysis of the successive extension classes of characters of the kind that we have made here.

3. An automorphy lifting theorem in weight 0

3.1. **Preliminaries.** Our goal in this section is to state and prove Theorem 3.2.1, which is an automorphy lifting theorem for n-dimensional crystalline weight 0 p-adic representations of G_F, where F is an imaginary CM field in which p is arbitrarily ramified. The key innovations that allow us to prove this theorem are the local-global compatibility result of [CN23] and the generic reducedness result that we proved in Theorem 2.5.5. Given these ingredients, the proof is very close to those of [ACC$^+$23, Theorem 6.1.1] and [MT23, Theorem 1.2], and we refer to those papers for some of the details of the arguments, and for any unfamiliar terminology.

We begin by introducing some terminology and notation we will need for the statement and proof.

3.1.1. *Galois preliminaries.* Fix a continuous irreducible representation $\overline{\rho} : G_F \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$ for a number field F. We fix a coefficient field $E/\mathbf{Q}_p$ such that $\overline{\rho}(G_F) \subset \mathrm{GL}_n(\mathbf{F})$.

We will use the notion of a *decomposed generic* representation $\overline{\rho}$, defined in [ACC$^+$23, Definition 4.3.1]. We will also use the notion of an *adequate subgroup* of $\mathrm{GL}_n(\mathbf{F})$, see for example [MT23, Definition 1.1.1].

Let v be a finite place of F. As in [ACC$^+$23, §6.2.1], a *local deformation problem* is a $\widehat{\mathrm{PGL}}_n$-stable subfunctor of the lifting functor $\mathcal{D}_v^{\square} := \mathcal{D}^{\square,\mathcal{O}}_{\overline{\rho}|_{G_{F_v}}}$, (pro-)representable by a quotient R_v of the lifting ring $R_v^{\square}$. The following local deformation problems will be relevant:

- the lifting functor itself, $\mathcal{D}_v^{\square}$,
- for $v|p$, weight 0 crystalline lifts $\mathcal{D}_v^{\mathrm{crys},\underline{0}}$, represented by $R^{\mathrm{crys},\underline{0},\mathcal{O}}_{\overline{\rho}|_{G_{F_v}}}$,
- the local deformation problem $\mathcal{D}_v^{\chi}$ defined in [ACC$^+$23, §6.2.15]. In this case, we assume that $q_v \equiv 1 \mod p$, that $\overline{\rho}|_{G_{F_v}}$ is trivial, that $p > n$, and we have a tuple $(\chi_i)_{i=1,\dots,n}$ of characters $\chi_i : \mathcal{O}_{F_v}^{\times} \to \mathcal{O}^{\times}$ which are trivial modulo ϖ. Then $\mathcal{D}_v^{\chi}$ classifies lifts $\rho\colon G_{F_v} \to \mathrm{GL}_n(A)$ such that

$$\mathrm{char}_{\rho(\sigma)}(X) = \prod_{i=1}^{n}(X - \chi_i(\mathrm{Art}_{F_v}^{-1}(\sigma)))$$

  for all $\sigma \in I_{F_v}$.

Let S be a finite set of finite places of F containing the p-adic places S_p and all places at which $\overline{\rho}$ is ramified. Then we use the notion of a *global deformation problem* from [ACC$^+$23, Definition 6.2.2]. We will be able to restrict to the case where $\Lambda_v = \mathcal{O}_v$ for all $v \in S$, so our global deformation problems will be tuples $\mathcal{S} = (\overline{\rho}, S, \{\mathcal{O}\}_{v\in S}, \{\mathcal{D}_v\}_{v\in S})$. Each $\mathcal{D}_v$ is a local deformation problem, representable by a quotient R_v of $R_v^{\square}$. There is an associated functor $\mathcal{D}_{\mathcal{S}}$ of deformations of $\overline{\rho}$ satisfying the local condition $\mathcal{D}_v$ for each $v \in S$. It is representable by $R_{\mathcal{S}}$. More generally, if $T \subset S$, we have a functor $\mathcal{D}_{\mathcal{S}}^T$ of T-framed deformations, which is representatble by $R_{\mathcal{S}}^T$. The T-framed global deformation ring $R_{\mathcal{S}}^T$ receives a natural $\mathcal{O}$-algebra map from $R_{\mathcal{S}}^{T,\mathrm{loc}} := \widehat{\otimes}_{v\in T,\mathcal{O}} R_v$.

3.1.2. *Automorphic preliminaries.* Now we assume that F is an imaginary CM number field. On the automorphic side, we will be interested in cuspidal automorphic representations of $\mathrm{GL}_n(\mathbf{A}_F)$ which are *regular algebraic of weight* 0.

This means that the infinitesimal character of π_∞ matches the infinitesimal character of the trivial representation of $\mathrm{GL}_n(F_\infty)$. These automorphic representations contribute to the cohomology groups with trivial coefficients of locally symmetric spaces.

Let $X_\infty = \mathrm{GL}_n(F_\infty)/\mathbf{R}_{>0}K_\infty$ be the symmetric space, with K_∞ a maximal compact subgroup of $\mathrm{GL}_n(F_\infty)$ (since F is totally imaginary, K_∞ is connected). Suppose we have a *good subgroup* $K \subset \mathrm{GL}_n(\mathbf{A}_F^\infty)$. In other words, K is neat, compact, open, and factorizes as $K = \prod_v K_v$ for compact open subgroups $K_v \subset \mathrm{GL}_n(F_v)$. Then we can define a smooth manifold

$$X_K = \mathrm{GL}_n(F)\backslash (X_\infty \times \mathrm{GL}_n(\mathbf{A}_F^\infty)/K)\,.$$

Fix a finite set of finite places S of F containing S_p, with $K_v = \mathrm{GL}_n(\mathcal{O}_v)$ for $v \notin S$. We factorize $K = K_S K^S$. We have an abstract Hecke algebra $\mathcal{H}(\mathrm{GL}_n(\mathbf{A}_F^{\infty,S}), K^S)$ with coefficients in $\mathbf{Z}$, a tensor product of spherical Hecke algebras over finite places $v \notin S$.

Suppose that V is a finite $\mathcal{O}$-module with an action of $G(F) \times K_S$. Then, as explained in [ACC$^+$23, §2.1.2], V descends to a local system of $\mathcal{O}$-modules $\mathcal{V}$ on X_K, and we have a natural Hecke action

$$\mathcal{H}(\mathrm{GL}_n(\mathbf{A}_F^{\infty,S}), K^S) \otimes_{\mathbf{Z}} \mathcal{O} \to \mathrm{End}_{\mathbf{D}(\mathcal{O})}(R\Gamma(X_K, \mathcal{V})).$$

The image of this $\mathcal{O}$-algebra map is a finite $\mathcal{O}$-algebra denoted by $\mathbf{T}^S(K, \mathcal{V})$. If $\mathfrak{m}$ is a maximal ideal of $\mathbf{T}^S(K, \mathcal{V})$, it has an associated semisimple Galois representation

$$\overline{\rho}_{\mathfrak{m}} : G_{F,S'} \to \mathrm{GL}_n(k(\mathfrak{m}))$$

for a suitable set of places S' containing S [ACC$^+$23, Theorem 2.3.5]. For $v \notin S'$, the characteristic polynomial of $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)$ equals the image of

$$\begin{aligned} P_v(X) = X^n - T_{v,1}X^{n-1} + \dots + (-1)^i q_v^{i(i-1)/2} T_{v,i} X^{n-i} + \dots \\ + q_v^{n(n-1)/2} T_{v,n} \in \mathcal{H}(\mathrm{GL}_n(F_v), \mathrm{GL}_n(\mathcal{O}_{F_v}))[X]. \end{aligned}$$

in the residue field $k(\mathfrak{m})$. We write $T_{v,i} \in \mathcal{H}(\mathrm{GL}_n(F_v), \mathrm{GL}_n(\mathcal{O}_{F_v}))$ for the double coset operator

$$T_{v,i} = [\mathrm{GL}_n(\mathcal{O}_{F_v}) \mathrm{diag}(\varpi_v, \dots, \varpi_v, 1, \dots, 1)\, \mathrm{GL}_n(\mathcal{O}_{F_v})],$$

where ϖ_v appears i times on the diagonal.

When $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible, the cohomology groups $H^i(X_K, \mathcal{O})_{\mathfrak{m}} \otimes_{\mathcal{O}} E$ can be described in terms of cuspidal automorphic representations which are regular algebraic of weight 0 [ACC$^+$23, Theorem 2.4.10].

3.2. **An automorphy lifting theorem.** The rest of this section is devoted to the proof of the following theorem, which is a version of [ACC$^+$23, Theorem 6.1.1] and [MT23, Theorem 1.2] allowing arbitrary ramification at primes dividing p, at the price of restricting to weight 0 automorphic representations.

Theorem 3.2.1. *Let F be an imaginary CM or totally real field and let $p > n$ be a prime. Suppose given a continuous representation $\rho : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ satisfying the following conditions:*

(1) ρ is unramified almost everywhere.

(2) For each place $v|p$ of F, the representation $\rho|_{G_{F_v}}$ is crystalline of weight 0, i.e. with Hodge–Tate weights $HT_\tau(\rho) = \{0, 1, 2, \dots, n-1\}$ for each $\tau : F_v \hookrightarrow \overline{\mathbf{Q}}_p$.

(3) *$\overline{\rho}$ is absolutely irreducible and decomposed generic. The image of $\overline{\rho}|_{G_{F(\zeta_p)}}$ is adequate (as a subgroup of $\mathrm{GL}_n(\mathbf{F})$, for sufficiently large $\mathbf{F}$).*
(4) *There exists $\sigma \in G_F - G_{F(\zeta_p)}$ such that $\overline{\rho}(\sigma)$ is a scalar.*
(5) *There exists a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$ satisfying the following conditions:*
    (a) *π is regular algebraic of weight 0.*
    (b) *There exists an isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ such that $\overline{\rho} \cong \overline{r_\iota(\pi)}$.*
    (c) *If $v|p$ is a place of F, then π_v is unramified and $r_\iota(\pi)|_{G_{F_v}} \sim \rho|_{G_{F_v}}$ ("connects to", in the sense of [BLGGT14, §1.4]).*

Then ρ is automorphic: there exists a cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbf{A}_F)$ of weight λ such that $\rho \cong r_\iota(\Pi)$. Moreover, if v is a finite place of F and either $v|p$ or both ρ and π are unramified at v, then Π_v is unramified.

Remark 3.2.2. In assumption (5c), we are using [CN23, Theorem 4.3.1] which shows that $r_\iota(\pi)|_{G_{F_v}}$ is crystalline with the same labelled Hodge–Tate weights as $\rho|_{G_{F_v}}$. Choose a p-adic coefficient field E which contains the Galois closure of F and such that $\overline{\rho}(G_F) \subset \mathrm{GL}_n(\mathbf{F})$. Then assumption (5c) is that $r_\iota(\pi)|_{G_{F_v}}$ and $\rho|_{G_{F_v}}$ define points on the same irreducible component of the weight 0 crystalline lifting ring $R^{\mathrm{crys},\underline{0},\mathcal{O}}_{\overline{\rho}|_{G_{F_v}}} \otimes_{\mathcal{O}} \overline{\mathbf{Q}}_p$.

We begin by imposing some additional assumptions, under which we can use the Calegari–Geraghty version of the Taylor–Wiles–Kisin patching method to prove an automorphy lifting theorem. We then deduce Theorem 3.2.1 by a standard base change argument. We refer the reader to [ACC+23] for any unfamiliar notation.

We let F be an imaginary CM field with maximal totally real subfield F^+ and complex conjugation $c \in \mathrm{Gal}(F/F^+)$. We fix an integer $n \geq 1$, an odd prime $p > n$ and an isomorphism $\iota : \overline{\mathbf{Q}}_p \cong \mathbf{C}$. We let π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$, which is regular algebraic of weight 0. We suppose we have a finite set S of finite places of F, containing the set S_p of places of F above p, and a (possibly empty) subset $R \subset (S \smallsetminus S_p)$.

Then we assume that the following conditions are satisfied:

(1) If l is a prime lying below an element of S, or which is ramified in F, then F contains an imaginary quadratic field in which l splits. In particular, each place of S is split over F^+ and the extension F/F^+ is everywhere unramified.
(2) For each $v \in S_p$, let $\overline{v}$ denote the place of F^+ lying below v. Then there exists a place $\overline{v}' \neq \overline{v}$ of F^+ such that $\overline{v}'|p$ and
$$\sum_{\overline{v}'' \neq \overline{v}, \overline{v}'} [F^+_{\overline{v}''} : \mathbf{Q}_p] > \frac{1}{2}[F^+ : \mathbf{Q}_p].$$
(3) The residual representation $\overline{r_\iota(\pi)}$ is absolutely irreducible and decomposed generic, and $\overline{r_\iota(\pi)}|_{G_{F(\zeta_p)}}$ has adequate image.
(4) If v is a place of F lying above p, then π_v is unramified.
(5) If $v \in R$, then $\pi_v^{\mathrm{Iw}_v} \neq 0$, $q_v \equiv 1 \bmod p$ and $\overline{r_\iota(\pi)}|_{G_{F_v}}$ is trivial.
(6) If $v \in S - (R \cup S_p)$, then π_v is unramified, $v \notin R^c$, and $H^2(F_v, \mathrm{ad}\, \overline{r_\iota(\pi)}) = 0$.
(7) $S - (R \cup S_p)$ contains at least two places with distinct residue characteristics.
(8) If $v \notin S$ is a finite place of F, then π_v is unramified.

We define an open compact subgroup $K = \prod_v K_v$ of $\mathrm{GL}_n(\widehat{\mathcal{O}}_F)$ as follows:

- If $v \notin S$, or $v \in S_p$, then $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$.
- If $v \in R$, then $K_v = \mathrm{Iw}_v$.
- If $v \in S - (R \cup S_p)$, then $K_v = \mathrm{Iw}_{v,1}$ is the pro-v Iwahori subroup of $\mathrm{GL}_n(\mathcal{O}_{F_v})$.

By [ACC+23, Theorem 2.4.10], we can find a coefficient field $E \subset \overline{\mathbf{Q}}_p$ and a maximal ideal $\mathfrak{m} \subset \mathbf{T}^S(K, \mathcal{O})$ such that $\overline{\rho}_{\mathfrak{m}} \cong \overline{r_\iota(\pi)}$. After possibly enlarging E, we can and do assume that the residue field of $\mathfrak{m}$ is equal to $\mathbf{F}$, the residue field of E. For each tuple $(\chi_{v,i})_{v \in R, i=1,\dots,n}$ of characters $\chi_{v,i} : k(v)^\times \to \mathcal{O}^\times$ which are trivial modulo ϖ, we define a global deformation problem

$$\mathcal{S}_\chi = (\overline{\rho}_{\mathfrak{m}}, S, \{\mathcal{O}\}_{v \in S}, \{\mathcal{D}_v^{\mathrm{crys},\underline{0}}\}_{v \in S_p} \cup \{\mathcal{D}_v^{\chi}\}_{v \in R} \cup \{\mathcal{D}_v^{\square}\}_{v \in S-(R \cup S_p)}).$$

We will assume that either $\chi_{v,i} = 1$ for all $v \in R$ and all $1 \le i \le n$, or that for each $v \in R$ the $\chi_{v,i}$ are pairwise distinct.

Extending $\mathcal{O}$ if necessary, we may assume that all irreducible components of our local lifting rings and their special fibres are geometrically irreducible. We fix representatives $\rho_{\mathcal{S}_\chi}$ of the universal deformations which are identified modulo ϖ (via the identifications $R_{\mathcal{S}_\chi}/\varpi \cong R_{\mathcal{S}_1}/\varpi$). We define an $\mathcal{O}[K_S]$-module $\mathcal{O}(\chi^{-1})$, where K_S acts by the composition of χ^{-1} with the projection

$$K_S \to K_R = \prod_{v \in R} \mathrm{Iw}_v \to \prod_{v \in R} (k(v)^\times)^n.$$

Proposition 3.2.3. *There exists an integer $\delta \ge 1$, depending only on n and $[F : \mathbf{Q}]$, an ideal $J \subset \mathbf{T}^S(R\Gamma(X_K, \mathcal{V}_\lambda(\chi^{-1})))_{\mathfrak{m}}$ such that $J^\delta = 0$, and a continuous surjective homomorphism*

$$f_{\mathcal{S}_\chi} : R_{\mathcal{S}_\chi} \to \mathbf{T}^S(R\Gamma(X_K, \mathcal{O}(\chi^{-1})))_{\mathfrak{m}}/J$$

such that for each finite place $v \notin S$ of F, the characteristic polynomial of $f_{\mathcal{S}_\chi} \circ \rho_{\mathcal{S}_\chi}(\mathrm{Frob}_v)$ equals the image of $P_v(X)$ in $\mathbf{T}^S(R\Gamma(X_K, \mathcal{O}(\chi^{-1})))_{\mathfrak{m}}/J$.

Proof. This is a version of [ACC+23, Proposition 6.5.3], using [CN23, Theorem 4.2.15] to verify that we satisfy the crystalline weight 0 condition at $v \in S_p$. □

This proposition means that it makes sense to talk about the support of $H^*(X_K, \mathcal{O})_{\mathfrak{m}}$ over $R_{\mathcal{S}_1}$, since $f_{\mathcal{S}_1}$ realizes $\mathrm{Spec}(\mathbf{T}^S(K, \mathcal{O})_{\mathfrak{m}})$ as a closed subset of $\mathrm{Spec}(R_{\mathcal{S}_1})$.

Here are the essential properties of the (completed tensor products of) local deformation rings in our situation:

Lemma 3.2.4. *Fix a tuple $\chi = (\chi_{v,i})_{v \in R, i=1,\dots,n}$ of characters $\chi_{v,i} : k(v)^\times \to \mathcal{O}^\times$ which are trivial modulo ϖ. We assume that either $\chi_{v,i} = 1$ for all $v \in R$ and all $1 \le i \le n$, or that for each $v \in R$ the $\chi_{v,i}$ are pairwise distinct.*

(1) $R^{S,\mathrm{loc}}_{\mathcal{S}_\chi}$ is equidimensional of dimension $1 + n^2|S| + \frac{n(n-1)}{2}[F : \mathbf{Q}]$ and every generic point has characteristic 0.
(2) Each generic point of $\mathrm{Spec}\, R^{S,\mathrm{loc}}_{\mathcal{S}_\chi}/\varpi$ is the specialization of a unique generic point of $\mathrm{Spec}\, R^{S,\mathrm{loc}}_{\mathcal{S}_\chi}$.
(3) Assume that $\chi_{v,1}, \dots, \chi_{v,n}$ are pairwise distinct for each $v \in R$. Then the natural map $\mathrm{Spec}\, R^{S,\mathrm{loc}}_{\mathcal{S}_\chi} \to \mathrm{Spec}\, R^{S_p,\mathrm{loc}}_{\mathcal{S}_\chi} = \mathrm{Spec}\, R^{S_p,\mathrm{loc}}_{\mathcal{S}_1}$ induces a bijection on irreducible components.
(4) Each characteristic zero point of $\mathrm{Spec}\, R^{S_p,\mathrm{loc}}_{\mathcal{S}_1}$ lies on a unique irreducible component.

(5) *Assume that* $\chi_{v,1},\dots,\chi_{v,n}$ *are pairwise distinct for each* $v\in R$*, and let* C *be an irreducible component of* $\operatorname{Spec} R^{S,\mathrm{loc}}_{\mathcal{S}_1}$*. Write* C_p *for the image of* C *in* $\operatorname{Spec} R^{S_p,\mathrm{loc}}_{\mathcal{S}_1}$ *(so that* C_p *is an irreducible component of* $\operatorname{Spec} R^{S_p,\mathrm{loc}}_{\mathcal{S}_1}$*). Then the generic points of* $C\cap \operatorname{Spec} R^{S,\mathrm{loc}}_{\mathcal{S}_1}/\varpi$ *generalize (via the equality* $\operatorname{Spec} R^{S,\mathrm{loc}}_{\mathcal{S}_1}/\varpi = \operatorname{Spec} R^{S,\mathrm{loc}}_{\mathcal{S}_\chi}/\varpi$*) to the generic point of* $\operatorname{Spec} R^{S,\mathrm{loc}}_{\mathcal{S}_\chi}$ *corresponding to* C_p *via the bijection of part* (3)*. (By part (2), each of these points has a unique generalization.)*

Proof. We begin by noting that [BLGHT11, Lemma 3.3] allows us to describe the set of irreducible components of $\operatorname{Spec}(R^{S,\mathrm{loc}}_{\mathcal{S}_\chi})$ (respectively, its special fibre) as the product over $v\in S$ of the sets of irreducible components of the local deformation rings (respectively, their special fibres). (Here we use that the irreducible components of the local deformation rings that we consider are all in characteristic zero)

The first part follows from [ACC+23, Lemma 6.2.25] (we have a different deformation condition at p, but $R^{\mathrm{crys},\underline{0},\mathcal{O}}_{\overline{\rho}|_{G_{F_v}}}$ is $\mathcal{O}$-flat by definition and equidimensional of dimension $1+n^2+\frac{n(n-1)}{2}[F_v:\mathbf{Q}_p]$ by [Kis08, Theorem 3.3.4]).

For the second part, for each $v\in S$ and local deformation ring R_v we need to check that the generic points of $\operatorname{Spec} R_v/\varpi$ have unique generalizations to $\operatorname{Spec} R_v$. For $v|p$, this follows from Theorem 2.5.5— see [CN23, Lemma 5.3.3] for the argument that generically reduced special fibre implies unique generalizations of its generic points, and note that we are assuming $p>n$. For $v\in R$, the property we need follows from [ACC+23, Props. 6.2.16, 6.2.17]. For $v\in S-(R\cup S_p)$, $R_v=R_v^{\square}$ is formally smooth over $\mathcal{O}$.

The third part follows from the irreducibility of the local deformation rings for $v\in S-S_p$ when the $\chi_{v,i}$ are pairwise distinct [ACC+23, Prop. 6.2.17], and the fourth part from the regularity of $R^{S_p,\mathrm{loc}}_{\mathcal{S}_1}[1/p]$ [Kis08, Theorem 3.3.8].

For the final part, by the third part is enough to note that as we saw above, it follows from Theorem 2.5.5 that the generic points of the special fibre of $\operatorname{Spec} R^{S_p,\mathrm{loc}}_{\mathcal{S}_1}/\varpi = \operatorname{Spec} R^{S_p,\mathrm{loc}}_{\mathcal{S}_\chi}/\varpi$ uniquely generalize to generic points of $\operatorname{Spec} R^{S_p,\mathrm{loc}}_{\mathcal{S}_1} = \operatorname{Spec} R^{S_p,\mathrm{loc}}_{\mathcal{S}_\chi}$. □

Theorem 3.2.5. *Suppose we are given two homomorphisms* $f_1,f_2: R_{\mathcal{S}_1}\to\mathcal{O}$ *with associated liftings* $\rho_1,\rho_2: G_{F,S}\to \mathrm{GL}_n(\mathcal{O})$*. Suppose* $\ker(f_1)\in \operatorname{Supp}_{R_{\mathcal{S}_1}}(H^*(X_K,\mathcal{O})_{\mathfrak{m}})$ *and* $\rho_1|_{G_{F_v}}\sim\rho_2|_{G_{F_v}}$ *for each* $v\in S_p$*. Then* $\ker(f_2)\in \operatorname{Supp}_{R_{\mathcal{S}_1}}(H^*(X_K,\mathcal{O})_{\mathfrak{m}})$*.*

Proof. We patch, as in [ACC+23, §6.5] and [MT23, §8], replacing the Fontaine–Laffaille local condition at $v\in S_p$ with the crystalline weight 0 condition. Once again, we use [CN23, Theorem 4.2.15] to ensure that we have the necessary maps from deformation rings with these local conditions to our Hecke algebras. We record the output of this patching process, complete details of which can be found in [ACC+23, §6.4–6.5]. Fix a tuple $\chi=(\chi_{v,i})_{v\in R, i=1,\dots,n}$ of characters $\chi_{v,i}: k(v)^\times\to\mathcal{O}^\times$ which are trivial modulo ϖ, and with $\chi_{v,1},\dots,\chi_{v,n}$ pairwise distinct for each $v\in R$. Patching will provide us with the following:

(1) A power series ring $S_\infty=\mathcal{O}[\![X_1,\cdots,X_r]\!]$ with augmentation ideal $\mathfrak{a}_\infty=(X_1,\dots,X_r)$.

(2) Perfect complexes C_∞, C'_∞ of S_∞-modules, an isomorphism
$$C_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\varpi \cong C'_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\varpi$$
in $\mathbf{D}(S_\infty/\varpi)$ and an isomorphism
$$C_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\mathfrak{a}_\infty \cong R\operatorname{Hom}_{\mathcal{O}}(R\Gamma(X_K, \mathcal{O})_{\mathfrak{m}}, \mathcal{O})[-d]$$
in $\mathbf{D}(\mathcal{O})$.

(3) Two S_∞-subalgebras
$$T_\infty \subset \operatorname{End}_{\mathbf{D}(S_\infty)}(C_\infty)$$
and
$$T'_\infty \subset \operatorname{End}_{\mathbf{D}(S_\infty)}(C'_\infty),$$
which have the same image in
$$\operatorname{End}_{\mathbf{D}(S_\infty/\varpi)}(C_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\varpi) = \operatorname{End}_{\mathbf{D}(S_\infty/\varpi)}(C'_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\varpi),$$
where these endomorphism algebras are identified using the fixed isomorphism in (2). Call this common image $\overline{T}_\infty$. Note that T_∞ and T'_∞ are finite S_∞-algebras. The map
$$T_\infty \to \operatorname{End}_{\mathbf{D}(\mathcal{O})}(C_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\mathfrak{a}_\infty) = \operatorname{End}_{\mathbf{D}(\mathcal{O})}(R\Gamma(X_K, \mathcal{O})_{\mathfrak{m}})^{op}$$
factors through a map $T_\infty \to \mathbf{T}(K, \mathcal{O})_{\mathfrak{m}}$.

(4) Two Noetherian complete local S_∞-algebras R_∞ and R'_∞, which are power series algebras over $R^{S,\mathrm{loc}}_{\mathcal{S}_1}$ and $R^{S,\mathrm{loc}}_{\mathcal{S}_\chi}$ respectively. We have a surjective $R^{S,\mathrm{loc}}_{\mathcal{S}_1}$-algebra map $R_\infty \to R_{\mathcal{S}_1}$, which factors through an $\mathcal{O}$-algebra map $R_\infty/\mathfrak{a}_\infty \to R_{\mathcal{S}_1}$. We also have surjections $R_\infty \twoheadrightarrow T_\infty/I_\infty$, $R'_\infty \twoheadrightarrow T'_\infty/I'_\infty$, where I_∞ and I'_∞ are nilpotent ideals. We write $\overline{I}_\infty$ and $\overline{I}'_\infty$ for the image of these ideals in $\overline{T}_\infty$. These maps fit into a commutative diagram
$$\begin{array}{ccc} R_\infty & \longrightarrow & R_{\mathcal{S}_1} \\ \big\downarrow & & \big\downarrow \\ T_\infty/I_\infty & \longrightarrow & \mathbf{T}(K, \mathcal{O})^{\mathrm{red}}_{\mathfrak{m}}. \end{array}$$

(5) An isomorphism $R_\infty/\varpi \cong R'_\infty/\varpi$ compatible with the S_∞-algebra structure and the actions (induced from T_∞ and T'_∞) on
$$H^*(C_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\varpi)/(\overline{I}_\infty + \overline{I}'_\infty) = H^*(C'_\infty \otimes^{\mathbf{L}}_{S_\infty} S_\infty/\varpi)/(\overline{I}_\infty + \overline{I}'_\infty),$$
where these cohomology groups are identified using the fixed isomorphism.

(6) Integers $q_0 \in \mathbf{Z}$ and $l_0 \in \mathbf{Z}_{\geq 0}$ such that
$$H^*(X_K, E)_{\mathfrak{m}} \neq 0,$$
and these groups are non-zero only for degrees in the interval $[q_0, q_0 + l_0]$. Moreover, $\dim R_\infty = \dim R'_\infty = \dim S_\infty - l_0$.

With that out of the way, we let $x \in \operatorname{Spec} R_\infty$ be the automorphic point coming from $\ker(f_1)$. By the first part of [CN23, Proposition 5.4.2], there is an irreducible component C_a of $\operatorname{Spec} R_\infty$, containing x, with $C_a \subset \operatorname{Spec} T_\infty$. Let C be any irreducible component of $\operatorname{Spec} R_\infty$ containing $\ker(f_2)$. Since $\rho_1|_{G_{F_v}} \sim \rho_2|_{G_{F_v}}$ for each $v \in S_p$, C and C_a map to the same irreducible component of $\operatorname{Spec} R^{S_p,\mathrm{loc}}_{\mathcal{S}_1}$ (we are using part (4) of Lemma 3.2.4 here, which says that each characteristic 0 point lies in a unique irreducible component of $R^{S_p,\mathrm{loc}}_{\mathcal{S}_1}$). By Lemma 3.2.4, the generic points of $C \cap \operatorname{Spec} R_\infty/\varpi$ and $C_a \cap \operatorname{Spec} R_\infty/\varpi$ all generalize to the same irreducible component of $\operatorname{Spec} R'_\infty$. We can apply the second part of [CN23, Proposition 5.4.2] to deduce that $C \subset \operatorname{Spec} \mathbf{T}_\infty$, and therefore $\ker(f_2)$ is in the support of $H^*(C_\infty)$. It follows as in [ACC+23, Corollary 6.3.9] (see also [CN23, Corollary 5.4.3]) that $\ker(f_2)$ is in the support of $H^*(X_K, \mathcal{O})_{\mathfrak{m}}$, as desired. □

Proof of Theorem 3.2.1. This is immediate from Theorem 3.2.5 via a standard base change argument identical to the one found in [ACC+23, §6.5.12]. □

4. The Dwork family

4.1. **Definitions.** We begin by introducing the Dwork motives we need to consider. For our purposes, we need to consider the non-self dual motives (with coefficients) studied in [Qia21, Qia23] rather than the self-dual (generalized) symplectic motives previously considered in [HSBT10, BLGHT11].

Let $n > 2$ and $N > 100n + 100$ be integers, with N odd and $(N, n) = 1$. Let $\zeta_N \in \overline{\mathbf{Q}}$ be a primitive N^{th} root of unity. Let $R_0 = \mathbf{Z}[\zeta_N, N^{-1}]$, $T_0 = \operatorname{Spec} R_0[t, (1-t^N)^{-1}]$, and let $Z \subset \mathbf{P}^{N-1}_{T_0}$ be the family of smooth hypersurfaces of degree N and dimension $N-2$ defined by the equation

$$X_1^N + \cdots + X_N^N = NtX_1 \ldots X_N.$$

We write $\pi : Z \to T_0$ for the natural projection. Let μ_N denote the group of N^{th} roots of unity in $\mathbf{Z}[\zeta_N]^\times$. Then the group $H = \mu_N^N/\Delta(\mu_N)$ acts on $\mathbf{P}^{N-1}$ by multiplication of coordinates, and the subgroup

$$H_0 = \ker(\prod : H \to \mu_N)$$

preserves Z. The action of H_0 extends to an action of H on the central fibre Z_0 (which is a Fermat hypersurface).

Let $M = \mathbf{Q}(e^{2\pi i/N}) \subset \mathbf{C}$, and set

$$X = \operatorname{Hom}(H, M^\times),$$

$$X_0 = \operatorname{Hom}(H_0, M^\times).$$

A choice of embedding $\tau : \mathbf{Q}(\zeta_N) \to \mathbf{C}$ determines an isomorphism

$$f_\tau : X \cong \ker\left(\sum : (\mathbf{Z}/N\mathbf{Z})^N \to \mathbf{Z}/N\mathbf{Z})\right),$$

but we do not fix a preferred choice. We do choose a character $\underline{\chi} \in (\chi_1, \ldots, \chi_N) \in X$ with the following properties:

- The trivial character of μ_N occurs $n+1$ times among $\chi_1, \ldots, \chi_N$, and each other character appears at most once.
- Let $\rho_1, \ldots, \rho_n$ be the n distinct non-trivial characters $\mu_N \to M^\times$ which do not appear in $\chi_1, \ldots, \chi_N$. Then the stabilizer of the set $\{\rho_1, \ldots, \rho_n\}$ in $\operatorname{Gal}(M/\mathbf{Q})$ is trivial.

The existence of such $\underline{\chi}$ is established in [Qia23, Lem. 3.1], as a consequence of the assumption $N > 100n + 100$. The precise choice is not important.

For any place λ of M of characteristic l, we define $\mathcal{V}_\lambda = (\pi[1/l]_* \mathcal{O}_{M_\lambda})^{H_0 = \underline{\chi}|_{H_0}}$. It is a lisse sheaf of finite free $\mathcal{O}_{M_\lambda}$-modules on $T_0[1/l]$. If k is a perfect field which is an $R_0[1/l]$-algebra, and $t \in T_0(k)$, then we write $V_{t,\lambda} = \mathcal{V}_{\lambda,\bar{t}}$ for the stalk at a geometric point lying above t; it is an $\mathcal{O}_{M_\lambda}[G_k]$-module, finite free as $\mathcal{O}_{M_\lambda}$-module.

Katz [Kat90, Kat09] defines hypergeometric sheaves on $T_1 = \operatorname{Spec} R_0[t, t^{-1}, (1-t)^{-1}]$. We give the definition just in the case of interest. Let $j : T_1 \to \mathbf{G}_{m,R_0}$ be the natural open immersion, and let $f : T_1 \to \mathbf{G}_{m,R_0}$ be the map induced by $t \mapsto 1-t$. Fixing again a place λ of M of characteristic l, let $\mathcal{L}_i$ denote the rank 1 lisse M_λ-sheaf on $\mathbf{G}_{m,R_0}[1/l]$ associated to ρ_i and the μ_N-torsor $\mathbf{G}_m \xrightarrow{(\cdot)^N} \mathbf{G}_m$, and let $\mathcal{F}_i = j[1/l]_! f[1/l]^* \mathcal{L}_i$. We set

$$\mathcal{E}_\lambda = j[1/l]^*(\mathcal{F}_1 *_! \mathcal{F}_2 *_! \cdots *_! \mathcal{F}_n)[n-1],$$

where $*_!$ denotes multiplicative convolution with compact support.

4.2. **Basic properties and good ordinary points.** Associated to specializations of $\mathcal{E}_\lambda$ are compatible systems of Galois representations. In this section, we establish some of their basic properties. Most importantly, we prove (in Proposition 4.2.6) the existence of many specializations which have crystalline ordinary reduction. (In [Qia21], Qian proves that specializations sufficiently close to $t = \infty$ are semistable ordinary, but that is not sufficient for our purposes where we need to work with crystalline representations.)

Theorem 4.2.1.

(1) $\mathcal{E}_\lambda$ is a lisse M_λ-sheaf on $T_1[1/l]$ of rank n. The sheaf $\mathcal{E}_\lambda \otimes_{M_\lambda} \overline{M}_\lambda$ is geometrically irreducible. Moreover, $\mathcal{E}_\lambda$ is pure of weight $n-1$ and there is an isomorphism $\det \mathcal{E}_\lambda \cong M_\lambda(n(1-n)/2)$.

(2) Let k be an $R_0[1/l]$-algebra which is a finite field of cardinality q, and let $x \in T_1(k)$. Then we have

$$\operatorname{tr}(\mathrm{Frob}_k \mid \mathcal{E}_{\lambda,\overline{x}}) = (-1)^{n-1} \sum_{\substack{x_1,\ldots,x_n \in k \\ \prod_{i=1}^n x_i = x}} \prod_{i=1}^n \rho_i((1-x_i)^{(q-1)/N}) \tag{4.2.2}$$

where we identify $\mu_N = k^\times[N]$ and extend ρ_i by $\rho_i(0) = 0$.

(3) There exists a (unique) continuous character

$$\Psi_\lambda : \pi_1(\operatorname{Spec} R_0[1/l]) \to \mathcal{O}_{M_\lambda}^\times$$

with the following property: let $T_0' = T_0[1/t]$, let $j' : T_0' \to T_0$ be the natural open immersion, and let $g : T_0' \to T_1$ be the map induced by $t \mapsto t^{-N}$. Then there is an isomorphism of M_λ-sheaves on $T_0'[1/l]$:

$$(j'[1/l])^* \mathcal{V}_\lambda \otimes_{\mathcal{O}_{M_\lambda}} M_\lambda \cong g^* \mathcal{E}_\lambda \otimes_{M_\lambda} M_\lambda(\Psi_\lambda). \tag{4.2.3}$$

Proof. The construction and properties of $\mathcal{E}_\lambda$ are summarized in [Kat09] (where it is the sheaf denoted $\mathcal{H}^{can}(\mathbf{1}(n \text{ times}), \{\rho_i\})$) and given in detail in [Kat90, Ch. 8]. See also [DK17, §A.1.6]. The computation of the determinant follows from [Kat90, Theorem 8.12.2(1a)], noting that $\prod_{i=1}^n \rho_i = \mathbf{1}$. Property (2) follows from the definition. The existence of Ψ_λ as in (3) follows from [Kat09, Theorem 5.3] (we have also used the relation $[-1]^*\mathcal{H}^{can}(\mathbf{1}(n \text{ times}), \{\rho_i\}) \cong \mathcal{H}^{can}(\{\rho_i^{-1}\}, \mathbf{1}(n \text{ times})))$. The uniqueness follows from geometric irreducibility and Schur's lemma. □

Lemma 4.2.4. *There exists a Hecke character $\Psi : \mathbf{Q}(\zeta_N)^\times \backslash \mathbf{A}_{\mathbf{Q}(\zeta_N)}^\times \to \mathbf{C}^\times$ of type A_0, unramified away from N, and with field of definition contained inside M, such that Ψ_λ is associated to Ψ. In other words, Ψ_λ is 'independent of λ'. Moreover, Ψ is defined over M and we have $\Psi c(\Psi) = |\cdot|^{N-n}$.*

Proof. We may argue as in [Kat09, Question 5.5] to see that almost all Frobenius traces of Ψ_λ lie in M, and are independent of λ. The existence of the character Ψ, of type A_0, follows from the main result of [Hen82]. There are c-linear isomorphisms $\mathcal{V}_{c(\lambda)} \cong \mathcal{V}_\lambda^\vee(1-N)$ and $\mathcal{E}_{c(\lambda)} \cong \mathcal{E}_\lambda^\vee(1-n)$, so the final part is again a consequence of Schur's lemma. □

For any place λ of M of characteristic l, we define $\mathcal{W}_\lambda = \mathcal{V}_\lambda \otimes_{\mathcal{O}_{M_\lambda}} \mathcal{O}_{M_\lambda}(\Psi_\lambda^{-1})$, a lisse sheaf of finite free $\mathcal{O}_{M_\lambda}$-modules on $T_0[1/l]$. Thus $\mathcal{W}_\lambda \otimes_{\mathcal{O}_{M_\lambda}} M_\lambda$ is a lisse M_λ-sheaf of rank n which is pure of weight $n-1$, geometrically irreducible, and of determinant $M_\lambda(n(1-n)/2)$. If k is a perfect field which is an $R_0[1/l]$-algebra, and $t \in T_0(k)$, then we write $W_{t,\lambda} = \mathcal{W}_{\lambda,\bar{t}}$ for the stalk at a geometric point lying above t; it is an $\mathcal{O}_{M_\lambda}[G_k]$-module, finite free as $\mathcal{O}_{M_\lambda}$-module. The local systems $\mathcal{W}_\lambda$ are the ones we will use in building the moduli spaces used in the Moret-Bailly argument in §6.2. In particular, let us write $\overline{\mathcal{W}}_\lambda = \mathcal{W}_\lambda \otimes_{\mathcal{O}_{M_\lambda}} k(\lambda)$ and define $\overline{W}_{t,\lambda}$ similarly.

Proposition 4.2.5. *Let $F/\mathbf{Q}(\zeta_N)$ be a number field.*

(1) Let v be a finite place of F of characteristic l, and let λ be a place of M of characteristic not equal to l. If $l \nmid N$ and $t \in T_0(\mathcal{O}_{F_v})$, then $W_{t,\lambda}$ is unramified, and the polynomial $Q_v(X) = \det(X - \mathrm{Frob}_v \mid W_{t,\lambda})$ has coefficients in $\mathcal{O}_M[X]$ and is independent of λ.

(2) Let v be a finite place of F of characteristic l, and let λ be a place of M of the same characteristic. Let $t \in T_0(F_v)$. Then $W_{t,\lambda}$ is de Rham and for any embedding $\tau : F_v \to \overline{M}_\lambda$, we have $\mathrm{HT}_\tau(W_{t,\lambda}) = \{0, 1, \ldots, n-1\}$. If $l \nmid N$ and $t \in T_0(\mathcal{O}_{F_v})$, then $W_{t,\lambda}$ is crystalline and the characteristic polynomial of Frob_v on $\mathrm{WD}(W_{t,\lambda})$ equals $Q_v(X)$. In particular, $W_{t,\lambda}$ is ordinary if and only if the roots of $Q_v(X)$ in $\overline{M}_\lambda$ have l-adic valuations $0, [k(v) : \mathbf{F}_l], \ldots, (n-1)[k(v) : \mathbf{F}_l]$.

(3) Let $t \in T_0(F)$, and let S be the set of finite places v of F such that either $v|N$, or $v \nmid N$ and $t \notin T_0(\mathcal{O}_{F_v}) \subset T_0(F_v)$. Then

$$(M, S, \{Q_v(X)\}_{v \notin S}, \{W_{t,\lambda}^{ss}\}_\lambda, \{\{0, 1, \ldots, n-1\}\}_\tau)$$

is a weakly compatible system of l-adic representations of G_F over M of rank n, pure of weight $n-1$, in the sense of [BLGGT14, §5.1].

Proof. For the first part, we note that Z_t is smooth and proper over $\mathcal{O}_{F_v}$, so by smooth proper base change $H^*_{\text{ét}}(\overline{Z}_{t,F_v}, \mathcal{O}_{M_\lambda})$ is unramified and there is an isomorphism

$$H^{N-2}_{\text{ét}}(Z_{t,\overline{F}_v}, \mathcal{O}_{M_\lambda}) \cong H^{N-2}_{\text{ét}}(Z_{\bar{t},\overline{k(v)}}, \mathcal{O}_{M_\lambda}),$$

where $\bar{t}$ denotes the image of t in $T_0(k(v))$. [KM74, Theorem 2(2)] shows that for any $h \in H$ the characteristic polynomial of $h \cdot \mathrm{Frob}_v$ on this group has coefficients in $\mathcal{O}_M$ and is independent of the choice of $\lambda \nmid l$, and this implies that $Q_v(X)$ also has coefficients in $\mathcal{O}_M[X]$ and is independent of λ.

For the second part, note that $W_{t,\lambda}$ is de Rham because it is a subquotient of $H^{N-2}_{\text{ét}}(Z_t, M_\lambda) \otimes M_\lambda(\Psi_\lambda^{-1})$, which is de Rham. To compute the Hodge–Tate weights, we use [Qia23, Lemma 3.10], which implies that there is an integer M_τ such that $\mathrm{HT}_\tau(V_{t,\lambda}) = \{M_\tau, M_\tau + 1, \ldots, M_\tau + (n-1)\}$. Since $W_{t,\lambda}$ is a twist of $V_{t,\lambda}$, there is an integer M'_τ such that $\mathrm{HT}_\tau(W_{t,\lambda}) = \{M'_\tau, M'_\tau + 1, \ldots, M'_\tau + (n-1)\}$. Looking at determinants shows that $nM'_\tau + n(n-1)/2 = n(n-1)/2$, hence $M'_\tau = 0$.

If further $l \nmid N$ and $t \in T_0(\mathcal{O}_{F_v})$ then again Z_t is smooth and proper, so $H^{N-2}_{\text{ét}}(Z_t, M_\lambda)$ is crystalline, and also $M_\lambda(\Psi_\lambda^{-1})$ is crystalline, hence $W_{t,\lambda}$ is crystalline. The crystalline comparison theorem implies that there is an isomorphism

$$D_{cris}(H^{N-2}_{\text{ét}}(Z_{t,\overline{F}_v}, \mathbf{Q}_l)) \cong H^{N-2}_{cris}(Z_{\overline{t}}/F_{v,0}),$$

respecting the action of Frobenius ϕ_v and H_0 on each side. Here $F_{v,0}$ denotes the maximal absolutely unramified subfield of F_v. Choosing an embedding $\sigma_0 : F_{v,0} \to \overline{M}_\lambda$, there is an isomorphism

$$D_{cris}(H^{N-2}_{\text{ét}}(Z_{t,\overline{F}_v}, \mathbf{Q}_l)) \otimes_{F_{v,0},\sigma_0} \overline{M}_\lambda \cong H^{N-2}_{cris}(Z_{\overline{t}}/F_{v,0}) \otimes_{F_{v,0},\sigma_0} \overline{M}_\lambda,$$

equivariant for the $\overline{M}_\lambda$-linear action of $\phi_v^{[k(v):\mathbf{F}_l]}$. By definition, $\mathrm{WD}(W_{t,\lambda})$ is the unramified representation of W_{F_v} over $\overline{M}_\lambda$ afforded by the Ψ_λ^{-1}-twist of the $\underline{\chi}|_{H_0}$-isotypic subspace of the left-hand side. We therefore need to check that the characteristic polynomial of $\phi_v^{[k(v):\mathbf{F}_l]}$ on the Ψ_λ^{-1}-twist of the $\underline{\chi}|_{H_0}$-isotypic subspace of the right-hand side equals $Q_v(X)$. This follows again from [KM74, Theorem 2(2)] (applicable here by the main result of [GM87]). The characterization of ordinary representations follows from [Ger19, Lemma 2.32].

The third part follows from the first two parts and the definition of a weakly compatible system. □

We now apply the results of Drinfeld–Kedlaya [DK17] to deduce that the $W_{t,\lambda}$ are ordinary for generic choices of t.

Proposition 4.2.6. *Let v be a place of $\mathbf{Q}(\zeta_N)$ of characteristic $l \nmid N$, and let λ be a place of M of the same characteristic. Then there exists a non-empty Zariski open subset $U(v;\lambda) \subset T_{0,k(v)}$ with the following property: for any finite extension $F_w/\mathbf{Q}(\zeta_N)_v$ and any $t \in T_0(\mathcal{O}_{F_w})$ such that $\overline{t} = t \bmod (\varpi_w) \in U(v;\lambda)(k(w))$, $W_{t,\lambda}$ is a crystalline ordinary representation of G_{F_w}.*

Proof. Fix an auxiliary place μ of M of characteristic not l. If $k/k(v)$ is a finite extension of cardinality q and $x \in T_0(k)$, we write $Q_x(X) \in \mathcal{O}_M[X]$, for the characteristic polynomial of Frob_x on $W_{x,\mu}$. Let $s_1(x) \geq s_2(x) \geq \cdots \geq s_n(x)$ denote $[k : \mathbf{F}_l]^{-1}$ times the l-adic valuations of the roots of $Q_x(X)$ in $\overline{M}_\lambda$. Observe that these normalized slopes $s_i(x)$ do not change if k is replaced by a larger extension (leaving the point x unchanged). By Proposition 4.2.5, it suffices to show the existence of a non-empty Zariski open subset $U \subset T_{0,k(v)}$ such that if $k/k(v)$ is a finite extension and $x \in U(k)$, then $s_i(x) = n - i$ for each $i = 1, \ldots, n$.

By [DK17, Theorem 1.3.3], we can find a non-empty Zariski open subset $V \subset T_{0,k(v)}$ such that the numbers $s_i(x)$ are constant for $x \in V(k)$, and moreover such that $s_i(x) \leq s_{i+1}(x) + 1$ for each $i = 1, \ldots, n-1$. To complete the proof, it suffices to show that there is a non-empty Zariski open subset $U \subset V$ such that if $x \in U(k)$, then $s_n(x) = 0$. Indeed, consideration of determinants shows that $s_1(x)+s_2(x)+\cdots+s_n(x) = n(n-1)/2$ for all $x \in T_0(k)$. If $x \in V(k)$ and $s_n(x) = 0$ then $s_i(x) \leq n - i$ for each $i = 1, \ldots, n$, hence $s_1(x) + \cdots + s_n(x) \leq n(n-1)/2$, with equality if and only if $s_i(x) = n - i$ for each $i = 1, \ldots, n$.

Finally, by (4.2.3), it is enough to show the analogous statement for the pullback of $\mathcal{E}_\mu$ to $T_{1,k(v)}$. We will prove this by showing that there is a non-empty Zariski open subset $U_1 \subset T_{1,k(v)}$ such that if $x \in T_{1,k(v)}(k)$, then $\mathrm{tr}(\mathrm{Frob}_x \mid \mathcal{E}_{\mu,\overline{x}}) \not\equiv 0 \bmod \lambda$, or in other words (using (4.2.2)) that

$$\sum_{\substack{x_1,\dots,x_n\in k\\ \prod_{i=1}^n x_i=x}}\prod_{i=1}^n \rho_i((1-x_i)^{(q-1)/N})\not\equiv 0 \bmod \lambda.$$

We will produce this set U_1 by a computation following [DK17, §A.3]. Let $\tau : \mathbf{Q}(\zeta_N)\to M$ be an isomorphism identifying the place v with the place λ. The character $k(v)^\times\to k(v)^\times$, $z\mapsto \tau^{-1}\rho_i(z^{(q_v-1)/N})$, is given by the formula $z\mapsto z^{c_i}$ for some integer c_i with $1\le c_i\le q_v-2$. If $q=q_v^d$ then we find that the pre-image under τ of the left-hand side of the displayed equation is given by

$$\sum_{\substack{x_1,\dots,x_n\in k\\ \prod_{i=1}^n x_i=x}}\prod_{i=1}^n \mathbf{N}_{k/k(v)}(1-x_i)^{c_i}=\sum_{\substack{x_1,\dots,x_n\in k\\ \prod_{i=1}^n x_i=x}}\prod_{i=1}^n(1-x_i)^{\widetilde{c}_i},$$

where $\widetilde{c}_i=c_i\cdot(q-1)/(q_v-1)<q-1$. This we can in turn compute as

$$\begin{aligned}\sum_{\substack{x_1,\dots,x_n\in k\\ \prod_{i=1}^n x_i=x}}\sum_{\substack{0\le r_i\le\widetilde{c}_i\\ i=1,\dots,n}}\prod_{i=1}^n\binom{\widetilde{c}_i}{r_i}(-x_i)^{r_i}\\ =\sum_{\substack{0\le r_i\le\widetilde{c}_i\\ i=1,\dots,n}}(-1)^{r_1+\cdots+r_n}\left(\prod_{i=1}^n\binom{\widetilde{c}_i}{r_i}\right)\sum_{x_1,\dots,x_{n-1}\in k^\times}\left(\prod_{i=1}^{n-1}x_i^{r_i-r_n}\right)x^{r_n}.\end{aligned}$$

We now use that if $r\in\mathbf{Z}$ and $r\not\equiv 0 \bmod (q-1)$, then $\sum_{z\in k^\times}z^r=0$. This implies that the inner sum vanishes except if $r_i=r_n$ for each $i=1,\dots,n-1$. We obtain (noting that there are only finitely many non-zero terms in the sum on the right-hand side):

$$\sum_{0\le r\le\min(\widetilde{c}_i)}(-1)^{nr}\prod_{i=1}^n\binom{\widetilde{c}_i}{r}(-1)^{n-1}x^r=(-1)^{n-1}\sum_{r\ge0}(-1)^{nr}\prod_{i=1}^n\binom{\widetilde{c}_i}{r}x^r.$$

Define a polynomial $u(T)\in k(v)[T]$:

$$u(T)=\sum_{r\ge0}(-1)^{nr}\prod_{i=1}^n\binom{c_i}{r}T^r.$$

We claim that there is an equality

$$\sum_{r\ge0}(-1)^{nr}\prod_{i=1}^n\binom{\widetilde{c}_i}{r}x^r=\mathbf{N}_{k/k(v)}(u(x)).$$

This will complete the proof: indeed, it will imply that we can take $U_1=T_{1,k(v)}[1/u]$ (noting that u is non-zero, since its constant term is 1, so U_1 is indeed non-empty, and also u is independent of the field extension $k/k(v)$). To show this, we expand

$$\begin{aligned}\mathbf{N}_{k/k(v)}(u(x)) &= \left(\sum_{r\geq 0}(-1)^{nr}\prod_{i=1}^{n}\binom{c_i}{r}x^r\right)^{1+q_v+\cdots+q_v^{d-1}} \\ &= \sum_{r_0,\ldots,r_{d-1}\geq 0}(-1)^{n(r_0+\cdots+r_{d-1}q_v^{d-1})}\prod_{i=1}^{n}\prod_{j=0}^{d-1}\binom{c_i}{r_j}x^{r_0+r_1q_v+\cdots+r_dq_v^{d-1}}.\end{aligned}$$

We now observe that a given tuple $(r_0,\ldots,r_{d-1})$ can contribute a non-zero summand only if $r_j < q_v - 1$ for each j, so each value of $r = r_0 + r_1q_v + \cdots + r_{d-1}q_v^{d-1}$ is represented at most once. Furthermore, since $(1+X)^{\widetilde{c}_i} = \prod_{j=0}^{d-1}(1+X^{q_v^j})^{c_i}$ in $\mathbf{F}_l[X]$, we have in this case a congruence

$$\binom{\widetilde{c}_i}{r} \equiv \prod_{j=0}^{d-1}\binom{c_i}{r_j} \bmod l,$$

showing that $\mathbf{N}_{k/k(v)}(u(x))$ indeed equals

$$\sum_{r\geq 0}(-1)^{nr}\prod_{i=1}^{n}\binom{\widetilde{c}_i}{r}x^r,$$

as desired. □

4.3. **Basics on unitary groups over finite fields.** In order to discuss the possible (residual) images of the Galois representations associated to our Dwork family, we recall here some basic facts about unitary groups over finite fields which will be used in the sequel. (Nothing here is original but we include it for convenience of exposition.)

Let l/k be a quadratic extension of finite fields. Let p be a prime and let $M \in M_n(l)$. Let M^t denote the transpose of M and M^c the conjugate of M by the generator of $\mathrm{Gal}(l/k)$. We define the adjoint $M^\dagger$ of M to be $M^\dagger := (M^c)^t = (M^t)^c$. Note that $(AB)^\dagger = B^\dagger A^\dagger$. We recall:

Definition 4.3.1. The unitary group $\mathrm{GU}_n(l)$ is the subgroup of matrices $M \in \mathrm{GL}_n(l)$ satisfying $M^\dagger M = \lambda \in k^\times$. Let ν be the multiplier character $\nu : \mathrm{GU}_n(l) \to k^\times$ sending M to $M^\dagger M$ and let $\mathrm{SU}_n(l)$ denote the kernel of ν.

If V is a representation of a finite group G over l, let $V^c := V \otimes_{l,c} l$ denote the representation obtained by conjugating the coefficients by the generator of $\mathrm{Gal}(l/k)$. If $x \in V$, we set $x^c := x \otimes 1 \in V^c$. If $x \in l$, we write either cx or x^c for the conjugate of x by $c \in \mathrm{Gal}(l/k)$.

Definition 4.3.2. If l/k is a quadratic extension of finite fields and $\mathrm{Gal}(l/k) \simeq \langle c\rangle$, then a Hermitian form on a vector space V over l is an l-valued pairing on V which is k-bilinear, satisfies $\langle ax, y\rangle = a\langle x, y\rangle$ and $\langle x, ay\rangle = ca\langle x, y\rangle$ for $a \in l$, and moreover satisfies $\langle y, x\rangle = \langle x, y\rangle^c = c\langle x, y\rangle$.

Remark 4.3.3. Scaling the pairing by an element $\eta \in l$ such that $c\eta = -\eta$, all the conditions remain true except that now $\langle x, y\rangle = -c\langle x, y\rangle$.

The basic fact concerning unitary groups over finite fields is that there is essentially only one non-degenerate Hermitian form. In practice, it will be useful to formulate this in the following lemma.

Lemma 4.3.4. *Let V be a vector space over l with an absolutely irreducible representation of a group G. Suppose that there is an l-linear isomorphism*

$$V^\vee \simeq V^c \otimes \chi^{-1} \tag{4.3.4}$$

for some multiplier character $\chi : G \to k^\times$. Then, after a suitable choice of basis for V, the corresponding map $G \to \mathrm{GL}_n(l)$ has image in $\mathrm{GU}_n(l)$.

Proof. An isomorphism (4.3.4) is equivalent to the existence of a G-equivariant non-degenerate bilinear pairing:

$$\psi : V \times V^c \to \chi,$$

where by abuse of notation we consider χ as a 1-dimensional vector space over l. By Schur's Lemma, the isomorphism in (4.3.4) is unique up to scaling and thus ψ is also unique up to scaling. If we define ψ' to be the map:

$$\psi'(x, y^c) = \psi(y, x^c)^c,$$

then ψ' is also a G-equivariant bilinear map from $V \times V^c$ to χ and hence ψ' is equal to $\psi' = \lambda\psi$ for some $\lambda \in l^\times$, that is,

$$\psi(y, x^c)^c = \psi(x, y^c) \cdot \lambda.$$

Applying this twice, we get $\psi(x, y^c) = \lambda \cdot \lambda^c \cdot \psi(x, y^c)$, and thus $N_{l/k}(\lambda) = 1$. By Hilbert Theorem 90, it follows that $\lambda = c\eta/\eta$ for some $\eta \in l$. Replacing $\psi(x, y)$ by $\psi(x, y)/\eta$, we deduce that $\psi(x, y^c) = \psi(y, x^c)^c$. It follows that

$$\langle x, y \rangle := \psi(x, y^c)$$

defines a non-degenerate Hermitian form on V in the sense of Definition 4.3.2. Let A denote the matrix associated to this Hermitian form, so that $A^\dagger = A$. Then $G \subset \mathrm{GU}(V, A)$, that is, matrices M such that $M^\dagger A M = \lambda \cdot A$ for some $\lambda \in k^\times$. But now we use the fact that there is a unique non-degenerate equivalence class of Hermitian forms associated to l/k, namely, they are all equivalent to $A = I$ and so $G \subset \mathrm{GU}_n(l)$. (See, for example, [Lew82, §4].) □

4.4. **Moduli spaces and monodromy.** We shall now discuss a number of moduli spaces related to finding Dwork motives with fixed residual representations, and compute the corresponding monodromy groups. Since it will be important to find such motives whose p-adic representations are related to symmetric powers of "niveau two" representations, for our applications we will have to take $p \equiv -1 \bmod N$, and thus be in cases excluded by [Qia23].

Definition 4.4.1. Let $l_1, l_2 \nmid 2N$ be distinct primes and let λ_1, λ_2 be places of $\mathbf{Q}(\zeta_N)$ of these characteristics. Suppose we are given the following data:

(1) A field $F/\mathbf{Q}(\zeta_N)$ and for each $i = 1, 2$ an étale sheaf $\overline{U}_{\lambda_i}$ on $\operatorname{Spec} F$ of $k(\lambda_i)$-modules of rank n.
(2) For each $i = 1, 2$ an isomorphism $\eta_i : \wedge^n \overline{\mathcal{W}}_{\lambda_i} \to \wedge^n \overline{U}_{\lambda_i, T_{0,F}}$ of sheaves of $k(\lambda_i)$-modules.
(3) For each $i = 1, 2$, if $-1 \bmod N \in \langle l_i \rangle \leq (\mathbf{Z}/N\mathbf{Z})^\times$ (equivalently: if $c \in \mathrm{Gal}(k(\lambda_i)/\mathbf{F}_{l_i})$), then we fix in addition a perfect $\mathbf{F}_l$-bilinear morphism $\langle \cdot, \cdot \rangle_{\overline{U}_{\lambda_i}} : \overline{U}_{\lambda_i} \times \overline{U}_{\lambda_i} \to k(\lambda_i)(1-n)$ satisfying the following conditions:
   (a) For all $x, y \in \overline{U}_{\lambda_i}$, $a \in k(\lambda_i)$, we have $\langle ax, y \rangle = a\langle x, y \rangle$, $\langle x, ay \rangle = c(a)\langle x, y \rangle$.
   (b) For all $x, y \in \overline{U}_{\lambda_i}$, we have $\langle y, x \rangle = -c\langle x, y \rangle$.

That is, the pairing is Hermitian in the sense of Definition 4.3.2 up to a scalar η with $c\eta = -\eta$, see Remark 4.3.3.

Note that if $-1 \bmod N \in \langle l_i \rangle$ then there is also a perfect $\mathbf{F}_l$-bilinear morphism $\langle \cdot, \cdot \rangle_{\overline{\mathcal{W}}_{\lambda_i}} : \overline{\mathcal{W}}_{\lambda_i} \times \overline{\mathcal{W}}_{\lambda_i} \to k(\lambda_i)(1-n)$ satisfying the same two conditions, induced by taking Poincaré duality on the relative cohomology with $\mathcal{O}_{M_\lambda^+}$-coefficients of the hypersurface $Z \to T_0$ and extending sesquilinearly to $\mathcal{O}_{M_\lambda}$-coefficients. If k/F is a field extension and $t \in T_0(k)$, then we write $\langle \cdot, \cdot \rangle_{\overline{W}_{t,\lambda_i}}$ for the induced perfect pairing on $\overline{W}_{t,\lambda_i}$.

Given such data, let us write $\mathcal{F}(\{\overline{U}_{\lambda_i}\})$ for the functor which sends a scheme $S \to T_{0,F}$ to the set of pairs of isomorphisms $\phi_i : \overline{\mathcal{W}}_{\lambda_i,S} \to \overline{U}_{\lambda_i,S}$ $(i = 1, 2)$ satisfying the following conditions:

- For each $i = 1, 2$, $\wedge^n \phi_i = \eta_i$.
- For each $i = 1, 2$, if $-1 \bmod N \in \langle l_i \rangle$, then ϕ_i intertwines $\langle \cdot, \cdot \rangle_{\overline{\mathcal{W}}_{\lambda_i},S}$ and $\langle \cdot, \cdot \rangle_{\overline{U}_{\lambda_i},S}$.

Then $\mathcal{F}(\{\overline{U}_{\lambda_i}\})$ is represented by a finite étale $T_{0,F}$-scheme $T(\{\overline{U}_{\lambda_i}\})$.

We need the following variant of [Qia23, Proposition 3.8].

Proposition 4.4.2. *With notation as above, $T(\{\overline{U}_{\lambda_i}\})$ is a geometrically irreducible smooth F-scheme.*

Proof. We need to show that the geometric monodromy group $\pi_1(T_{0,\overline{\mathbf{Q}}})$ acts transitively on the fibres of $T(\{\overline{U}_{\lambda_i}\})$ over T_0. The existence of the pairing $\langle \cdot, \cdot \rangle_{\overline{\mathcal{W}}_{\lambda_i}}$ shows that if $-1 \bmod N \in \langle l_i \rangle$, then the image of the geometric monodromy group acting on the geometric generic fibre of $\overline{\mathcal{W}}_{\lambda_i}$ may be identified with a subgroup of $\mathrm{SU}_n(k(\lambda_i))$, and otherwise it may be identified with a subgroup of $\mathrm{SL}_n(k(\lambda_i))$.

We claim that it is enough to show that equality holds in either of these cases. Indeed, let H_i denote the image at each prime l_i (which would then be either $\mathrm{SU}_n(k(\lambda_i))$ or $\mathrm{SL}_n(k(\lambda_i))$). Since we are assuming $l_1, l_2 \nmid 2$ and $n > 2$, it follows that the H_i are perfect and their associated projective groups (i.e. the H_i modulo their subgroups of scalar matrices) are simple (Lemma 5.2.3), and moreover $H_1 \ncong H_2$ (also by Lemma 5.2.3). Goursat's lemma implies that the image of geometric monodromy acting on $\overline{\mathcal{W}}_{\lambda_1} \times \overline{\mathcal{W}}_{\lambda_2}$ must be $H_1 \times H_2$, completing the proof.

If $-1 \bmod N \notin \langle l_i \rangle$, then the required statement follows from [Qia23, Lemma 3.7]. Now suppose that $-1 \bmod N \in \langle l_i \rangle$. In this case, we can follow the proof of [Qia23, Lemma 3.7] (now allowing the case $-1 \bmod N \in \langle l_i \rangle$, which is used there to exclude the possibility of image a special unitary group) to conclude that H_i is isomorphic to a subgroup of $\mathrm{SU}_n(k(\lambda_i))$ which maps to $\mathrm{SU}_n(k(\lambda_i))$ or $\mathrm{SL}_n(k(\lambda_i))$ with image a normal subgroup of index dividing N. Since N is coprime to n by assumption, the only possibility is that this map is in fact an isomorphism and that $H_i \cong \mathrm{SU}_n(k(\lambda_i))$, as required. □

Remark 4.4.3. If $l \equiv 1 \bmod N$, so that l splits completely in $\mathbf{Q}(\zeta_N)$, and $\lambda | l$, then the data of an étale sheaf $\overline{U}_\lambda$ on $\operatorname{Spec} F$ satisfying conditions (1), (2), and (3) of Definition 4.4.1 is nothing more than a representation

$$\overline{r}_l : G_F \to \mathrm{GL}_n(\mathbf{F}_l)$$

with determinant $\overline{\varepsilon}^{-n(n-1)/2}$. If $l \equiv -1 \bmod N$, however, then (in light of Lemma 4.3.4) these conditions correspond to a representation

$$\overline{r}_l : G_F \to \mathrm{GU}_n(\mathbf{F}_{l^2})$$

with multiplier character $\overline{\varepsilon}^{1-n}$ (and determinant $\overline{\varepsilon}^{-n(n-1)/2}$). In practice, we shall only consider the moduli spaces T with primes l_1, l_2 that are either $\pm 1 \bmod N$. in which case we sometimes write $T(\{\overline{U}_{\lambda_i}\})$ as $T(\overline{r}_{\lambda_1}, \overline{r}_{\lambda_2})$, replacing the étale sheaf with the corresponding representations, which are always assumed to satisfy conditions (1), (2), and (3) of Definition 4.4.1. We will also use the simpler variant $T(\overline{r}_\lambda)$ where there is a single prime $l \equiv 1 \bmod N$, a choice of place $\lambda | l$, and a representation $\overline{r}_\lambda : G_F \to \mathrm{GL}_n(\mathbf{F}_l)$ of determinant $\overline{\varepsilon}^{-n(n-1)/2}$. The F-scheme $T(\overline{r}_\lambda)$ is geometrically irreducible.

The following lemma will be used to prove the existence of local points on $T(\{\overline{U}_{\lambda_i}\})$ in certain cases.

Lemma 4.4.4. *Let $l > n$ be a prime such that $l \equiv -1$ mod N, and let v, λ be places of $\mathbf{Q}(\zeta_N)$, M, respectively, of residue characteristic l. Let $\tau, c\tau : k(v) \to k(\lambda)$ be the two distinct isomorphisms, and let $\omega_\tau : I_{\mathbf{Q}(\zeta_N)_v} \to k(\lambda)^\times$ be the character $\tau \circ \mathrm{Art}^{-1}_{\mathbf{Q}(\zeta_N)_v}$. Then there is an isomorphism*

$$\overline{W}_{0,\lambda} \cong \bigoplus_{i=1}^{n} \omega_\tau^{i-1} \omega_{c\tau}^{n-i}.$$

Proof. The action of H_0 on Z_0 extends to an action of H, leading to a decomposition

$$\overline{W}_{0,\lambda} = \bigoplus_{j=1}^{n} \overline{W}_{0,\lambda,j},$$

where $W_{0,\lambda,j}$ is the Ψ_λ^{-1}-twist of the H-eigenspace in $H^{N-1}(Z_{0,\overline{\mathbf{Q}}}, \mathcal{O}_{M_\lambda})$ for the character $(\chi_1 \rho_j^{-1}, \ldots, \chi_N \rho_j^{-1})$, and $\overline{W}_{0,\lambda,j} := W_{0,\lambda,j} \otimes_{\mathcal{O}_{M_\lambda}} k(\lambda)$. Here we have used the computation of [DMOS82, I.7.4], which moreover shows that each summand here has rank 1 over $k(\lambda)$. Moreover, this decomposition is orthogonal with respect to $\langle \cdot, \cdot \rangle_{\overline{W}_{0,\lambda}}$, showing that $\overline{W}_{0,\lambda,j} \otimes_{k(\lambda),c} k(\lambda) \cong \overline{W}^\vee_{0,\lambda,j} \varepsilon^{1-n}$ as $k(\lambda)[G_{\mathbf{Q}(\zeta_N)}]$-modules.

After permuting $\rho_1, \ldots, \rho_n$, we can assume that $\mathrm{HT}_\tau(W_{0,\lambda,j}) = j-1$. Then we have $\overline{W}_{0,\lambda,j} \cong k(\lambda)(\omega_\tau^{j-1}\omega_{c\tau}^{a_j})$ for some integers a_j with $\{a_1, \ldots, a_n\} = \{0, \ldots, n-1\}$. The last sentence of the previous paragraph shows that we must in fact have $j - 1 + a_j = n - 1$, completing the proof. □

4.5. **A result of Moret-Bailly.** We will use the following variant of the extensions [Cal12, Theorem 3.1], [BLGGT14, Proposition 3.1.1] of the main result of [MB89].

Proposition 4.5.1. *Let F be an imaginary CM field, Galois over $\mathbf{Q}$, and let T/F be a smooth, geometrically irreducible variety. Suppose given the following data:*

1. *A finite extension F^{avoid}/F and disjoint finite sets S_0 of rational primes.*
2. *For each $l \in S_0$ and each place $v | l$ of F, a Galois extension L_v/F_v. These have the property that if $\sigma \in G_{\mathbf{Q}_l}$ then $\sigma(L_v) = L_{\sigma(v)}$.*
3. *For each $l \in S_0$ and each place $v | l$ of F, a non-empty open subset $\Omega_v \subset T(L_v)$, invariant under the action of $\mathrm{Gal}(L_v/F_v)$.*

Then we can find a finite CM extension F'/F and a point $P \in T(F')$ with the following properties:

(1) *$F'/\mathbf{Q}$ is Galois and F'/F is linearly disjoint from F^{avoid}/F.*
(2) *For each $l \in S_0$ and each place $v|l$ of F and $w|v$ of F', there is an isomorphism $F'_w \cong L_v$ of F_v-algebras such that $P \in \Omega_v \subset T(F'_w) \cong T(L_v)$.*

Suppose given further a finite group G and a surjective homomorphism $f : \pi_1^{\acute{e}t}(T) \to G$. Then we can further choose P so that the image of $f \circ P_* : G_{F'} \to G$ is surjective.

Proof. Without the last sentence, this is a special case of [BLGGT14, Proposition 3.1.1] (taking $K_0 = \mathbf{Q}$ in the notation there), noting that (as in [Cal12, Theorem 3.1]) we can choose F' to be of the form $F' = FE$ for a Galois, totally real extension $E/\mathbf{Q}$, and therefore in particular to be CM.

To get the last sentence, it suffices to add further local conditions at places of sufficiently large norm, ensuring using a Chebotarev density theorem for schemes of finite type over $\mathbf{Z}$ that the image of $f \circ P_*$ meets every conjugacy class of G (in close analogy with the argument of [Cal12, Proposition 3.2] – the surjectivity is then a consequence of Jordan's theorem). To define the necessary local conditions, we can spread T out to a geometrically irreducible scheme $\mathcal{T}$, smooth and of finite type over $\mathcal{O}_F$, such that f factors through $\pi_1^{\text{ét}}(\mathcal{T})$. Then [Ser12, Corollary 9.12] shows that for any $X > 0$ and any conjugacy class $C \subset G$, we can find a finite place v of F of norm $q_v > X$ and a point $x \in \mathcal{T}(k(v))$ such that the image of (arithmetic) Frobenius under $f \circ x_*$ lies in C. For each conjugacy class C of G, we choose one such place v_C and point x_C for each conjugacy class of G and take Ω_{v_C} to be the pre-image of x_C in $\mathcal{T}(\mathcal{O}_{F_{v_C}}) \subset T(F_{v_C})$. We may assume that if $C \neq C'$ then v_C and $v_{C'}$ have distinct residue characteristics $l_C \neq l_{C'}$, and then replace S_0 by $S_0 \cup \{l_C \mid C \subset G\}$. Finally, if $v|l_C$ and $v \neq v_C$, we take $\Omega_v = T(F_v)$. Provided the norm q_{v_C} is sufficiently large, these sets Ω_v will also be non-empty, as required. □

5. Preliminaries on deformation rings and Galois theory

5.1. **Lemmas on components of Galois deformation rings.** We begin by defining a certain local representation which shall appear repeatedly in the sequel.

Definition 5.1.1. For $n, m \in \mathbf{Z}_{\geq 1}$, let $\varepsilon_2, \varepsilon_2' : G_{\mathbf{Q}_{p^2}} \to \overline{\mathbf{Z}}_p^\times$ be the two Lubin–Tate characters trivial on $\mathrm{Art}_{\mathbf{Q}_{p^2}}(p)$, and let $\rho_{n,m,0}$ denote the representation

$$\rho_{n,m,0} = \bigoplus_{i=1}^{n} \varepsilon_2^{m(n-i)} (\varepsilon_2')^{m(i-1)} : G_{\mathbf{Q}_{p^2}} \to \mathrm{GL}_n(\overline{\mathbf{Z}}_p). \tag{5.1.2}$$

We assume that $p > nm$, so the representation $\rho_{n,m,0}$ is Fontaine–Laffaille. If the value of n is implicit, we often simply write ρ_0 for $\rho_{n,1,0}$.

Lemma 5.1.3. *Let $K_0/\mathbf{Q}_{p^2}$ be an unramified extension and let $\rho : G_{K_0} \to \mathrm{GL}_n(\overline{\mathbf{Z}}_p)$ be any crystalline representation of Hodge–Tate weights $\{0, m, 2m, \ldots, (n-1)m\}$ (with respect to any embedding $K_0 \to \overline{\mathbf{Q}}_p$) such that $\overline{\rho}|_{I_{K_0}} = \overline{\rho}_{n,m,0}|_{I_{K_0}}$. Then:*

(1) *There is a finite unramified extension K_1/K_0 such that $\overline{\rho}|_{G_{K_1}} = \overline{\rho}_{n,m,0}|_{G_{K_1}}$.*
(2) *For any finite extension K/K_0 such that $\overline{\rho}|_{G_K} = \overline{\rho}_{n,m,0}|_{G_K}$, we have $\rho|_{G_K} \sim \rho_{n,m,0}|_{G_K}$ ("connects to", in the sense of* [BLGGT14, §1.4]*).*

Proof. The first claim is clear. For the second, choose K_1/K_0 minimal such that $\overline{\rho}|_{G_{K_1}} = \overline{\rho}_{n,m,0}|_{G_{K_1}}$. Since $p > nm$ and K_1 is unramified, the lifting ring $R_{\overline{\rho}|_{G_{K_1}}}^{\mathrm{crys},\{0,\ldots,(n-1)m\},\mathcal{O}}$ is formally smooth by Fontaine–Laffaille theory. It follows that $\rho|_{G_{K_1}} \sim \rho_0|_{G_{K_1}}$, and then these representations are still connected after passing to any further finite extension. ☐

Lemma 5.1.4. *Let K be a finite extension of $\mathbf{Q}_p$. Let ρ_1, ρ_2 be ordinary, crystalline weight 0 representations of G_K with $\overline{\rho}_1 = \overline{\rho}_2$ the trivial representation. Then $\rho_1 \sim \rho_2$.*

Proof. This follows immediately from [Ger19, Lemma 3.14] — the ordinary weight 0 crystalline lifting ring of the trivial representation is irreducible. ☐

Lemma 5.1.5. *Let K be a finite extension of $\mathbf{Q}_p$, and let $\rho : G_K \to \mathrm{GL}_n(\overline{\mathbf{Z}}_p)$ be crystalline of weight 0. Then there exists a constant $c = c(K, \rho, n)$ with the following property:*

- *if $t : G_K \to \mathrm{GL}_n(\overline{\mathbf{Z}}_p)$ is crystalline of weight 0, and $t \equiv \rho|_{G_K} \bmod p^c$, then $t \sim \rho|_{G_K}$.*

Proof. Up to conjugation, the image of ρ lands in $\mathrm{GL}_n(\mathcal{O}_E)$ for some finite extension $E/\mathbf{Q}_p$ with residue field k. Let $R = R^{\mathrm{crys},\underline{0}}_{\overline{\rho}|_{G_K}} \otimes_{W(k)} \mathcal{O}_E$ denote the weight 0 crystalline lifting ring of $\overline{\rho}|_{G_K} : G_K \to \mathrm{GL}_n(k)$. By assumption, R has specializations corresponding to ρ and to t. We may choose a finite set of elements $\{g_k : 1 \leq k \leq d\}$ of G_K such that

$$R = \mathcal{O}_E[\![X_{ijk} : 1 \leq i, j \leq n, 1 \leq k \leq d]\!]/I$$

for an ideal I, and the universal lifting $\rho^{\mathrm{univ}} : G_K \to \mathrm{GL}_n(R)$ of $\overline{\rho}$ satisfies

$$\rho^{\mathrm{univ}}(g_k) = \rho(g_k) + [X_{ijk}]_{i,j=1}^n,$$

so that $\mathfrak{p} = (X_{ijk})$ is the dimension one prime associated to ρ. The condition that $t \equiv \rho|_{G_K} \bmod p^c$ is then equivalent to the condition that the corresponding homomorphism $t : R \to \overline{\mathbf{Z}}_p$ satisfies $v_p(t(X_{ijk})) \geq c$ for all i, j, k.

The generic fibre of R is formally smooth at $\mathfrak{p}$ by [Kis08, Theorem 3.3.8], and so in particular there is a unique minimal prime $\mathfrak{P}$ of $R[1/p]$ contained in the prime $\mathfrak{p}$. Suppose that $\mathfrak{Q}$ is any minimal prime ideal of $R[1/p]$ which is not contained in $\mathfrak{p}$. Then $\mathfrak{Q}$ contains an element $P(X_{ijk}) \in R[1/p]$ which doesn't vanish at $X_{ijk} = 0$ and hence has a non-zero constant term. After scaling if necessary, we may assume that $P \in R$. But now any specialization of P with $v_p(X_{ijk}) > v_p(P(0, 0, \ldots, 0))$ for every (i, j, k) will be non-zero, and hence, if $c > v_p(P(0, 0, \ldots, 0))$, then t cannot lie on the irreducible component corresponding to $\mathfrak{Q}$. Since R has only finitely many minimal prime ideals (it is Noetherian), there exists a choice of c which guarantees that t lies on the component corresponding to $\mathfrak{P}$. ☐

5.2. **Lemmas on big image conditions.** In order to apply Theorem 3.2.1 to a p-adic representation of G_F, one needs first to establish that the image of the residual representation (and its restriction to $G_{F(\zeta_p)}$) satisfies certain technical hypotheses, in particular conditions (3) and (4). In this section, we prove some lemmas showing that a number of representations of a form we shall encounter later have these properties. We first combine these conditions into the following definition:

Definition 5.2.1. Say that a representation $\overline{s} : G_F \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$ satisfies the Taylor–Wiles big image conditions if the following hold:

(1) The representation $\overline{s}$ is decomposed generic.
(2) The representation $\overline{s}|_{G_{F(\zeta_p)}}$ has adequate image.

(3) There exists $\sigma \in G_F - G_{F(\zeta_p)}$ such that $\overline{s}(\sigma)$ is scalar.

We have:

Lemma 5.2.2. *Suppose that $\overline{s} : G_F \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$ satisfies the Taylor–Wiles big image conditions. Suppose that $F/\mathbf{Q}$ is Galois. Let H/F be a finite extension whose Galois closure over $\mathbf{Q}$ is linearly disjoint over F from the composite of $F(\zeta_p)$ and the Galois closure over $\mathbf{Q}$ of the fixed field of $\ker(\overline{s})$. Then $\overline{s}|_{G_H}$ satisfies the Taylor–Wiles big image conditions.*

Proof. Let $\widetilde{H}$ be the Galois closure of H over $\mathbf{Q}$. Since $\overline{s}|_{G_H}$ satisfies the Taylor–Wiles conditions if $\overline{s}|_{G_{\widetilde{H}}}$ does, we assume that $H = \widetilde{H}$ is Galois over $\mathbf{Q}$. The conditions ensure that the images of $\overline{s}$ and $\overline{s}|_{G_H}$ coincide, and also the images of $\overline{s}|_{G_{F(\zeta_p)}}$ and $\overline{s}|_{G_{H(\zeta_p)}}$ coincide. Thus condition (2) of Definition 5.2.1 holds. Let M be the Galois closure of the fixed field of $\ker(\overline{s})$. Then we have an isomorphism

$$\mathrm{Gal}(H \cdot M(\zeta_p)/F) \simeq \mathrm{Gal}(M(\zeta_p)/F) \times \mathrm{Gal}(H/F),$$

and so $\mathrm{Gal}(M(\zeta_p)/F) \simeq \mathrm{Gal}(H \cdot M(\zeta_p)/H)$ via the map $\sigma \to (\sigma, 1)$. Moreover,

$$\mathrm{Gal}(H \cdot M(\zeta_p)/\mathbf{Q}) \subset \mathrm{Gal}(M(\zeta_p)/\mathbf{Q}) \times \mathrm{Gal}(H/\mathbf{Q})$$

is the subgroup of elements whose projection to $\mathrm{Gal}(F/\mathbf{Q}) \times \mathrm{Gal}(F/\mathbf{Q})$ is the diagonal. There exists a conjugacy class $\langle\sigma\rangle \in \mathrm{Gal}(M(\zeta_p)/F)$ such that any rational prime unramified in $H \cdot M(\zeta_p)$ whose Frobenius element corresponds to σ is decomposed generic for $\overline{s}$. Then $(\sigma, 1)$ will be decomposed generic for $\overline{s}|_{G_H}$. Similarly, if $\sigma \in \mathrm{Gal}(M(\zeta_p)/F) - \mathrm{Gal}(M(\zeta_p)/F(\zeta_p))$ is an element such that $\overline{s}(\sigma)$ is scalar, then the same is true of $(\sigma, 1) \in \mathrm{Gal}(H \cdot M(\zeta_p)/H)$. □

We shall also use the following group-theoretic fact.

Lemma 5.2.3. *Consider the collection of groups G either of the form $\mathrm{PSL}_n(\mathbf{F}_{p^k})$ or of the form $\mathrm{PSU}_n(\mathbf{F}_{p^{2k}})$ for all primes p and integers $k \geq 1$, $n \geq 2$. Then G is simple unless $(n, p) \in \{(2, 2), (2, 3), (3, 2)\}$. These groups are all pairwise mutually non-isomorphic as n and p both vary except for the following isomorphisms:*

(1) $\mathrm{PSL}_2(\mathbf{F}_{p^k}) \simeq \mathrm{PSU}_2(\mathbf{F}_{p^{2k}})$,
(2) $\mathrm{PSL}_2(\mathbf{F}_5) \simeq \mathrm{PSL}_2(\mathbf{F}_4)$,
(3) $\mathrm{PSL}_2(\mathbf{F}_7) \simeq \mathrm{PSL}_3(\mathbf{F}_2)$.

If we restrict G to be of the form $G = \mathrm{PSL}_2(\mathbf{F}_{p^k})$ or $\mathrm{PSU}_n(\mathbf{F}_{p^{2k}})$, and $A \in G$ is the image of any matrix with eigenvalues in $\mathbf{F}_p$, then any automorphism of G preserves these eigenvalues up to scalar.

Proof. If $n = 2$, then there is an isomorphism $\mathrm{PSU}_2(\mathbf{F}_{p^{2k}}) \simeq \mathrm{PSL}_2(\mathbf{F}_{p^k})$. Otherwise, $\mathrm{PSL}_n(\mathbf{F}_{p^k}) \simeq A_{n-1}(p^k)$ and $\mathrm{PSU}_n(\mathbf{F}_{p^{2k}}) \simeq {}^2A_{n-1}(p^{2k})$ is the Steinberg group. These groups are (twisted in the second case) Chevalley groups. The simplicity statement follows from [Ste68, Thm 37(b)] (see also [Car72]). The list of exceptional isomorphisms between (possibly twisted) simple Chevelley groups was determined in [Ste68, Thm 37(a)].

By a theorem of Steinberg [Ste60] (and [Car72, Thm 12.5.1]), the outer automorphism group of either $\mathrm{PSL}_n(\mathbf{F}_{p^k})$ or $\mathrm{PSU}_n(\mathbf{F}_{p^{2k}})$ is generated by diagonal automorphisms (conjugation by diagonal elements), by field automorphisms (acting on $\mathbf{F}_{p^k}$ or $\mathbf{F}_{p^{2k}}$ respectively), and the graph automorphism coming from the automorphism of the Dynkin diagram A_{n-1} if $n > 2$ (associated to the inverse transpose map). Certainly diagonal automorphisms preserve eigenvalues and field automorphisms act on the eigenvalues and so preserve eigenvalues in $\mathbf{F}_p$. There are no graph automorphisms for $n = 2$. For $n > 2$, the graph automorphism $M \mapsto (M^t)^{-1} = \sigma(M^\dagger)^{-1} = \sigma M$ in the unitary group coincides with the field automorphism, and so also preserves rational eigenvalues. □

Lemma 5.2.4. *Let $F/\mathbf{Q}$ be Galois. Consider representations:*

$$\overline{r}_A : G_F \to \mathrm{GL}_2(\mathbf{F}_p),$$
$$\overline{r}_B : G_F \to \mathrm{GU}_m(\mathbf{F}_{p^2}) \to \mathrm{GL}_m(\mathbf{F}_{p^2}),$$

such that the images $\overline{r}_A(G_{F(\zeta_p)})$ and $\overline{r}_B(G_{F(\zeta_p)})$ equal $\mathrm{SL}_2(\mathbf{F}_p)$ and $\mathrm{SU}_m(\mathbf{F}_{p^2})$ respectively. Consider the representation:

$$\overline{s} = \mathrm{Sym}^{n-1}\,\overline{r}_A \otimes \overline{r}_B : G_F \to \mathrm{GL}_{mn}(\mathbf{F}_{p^2})$$

Assume that $p > 2mn + 1$, and if $m = 2$ assume that the fixed fields of the kernels of the projective representations associated to $\overline{r}_A$ and $\overline{r}_B$ are linearly disjoint over $F(\zeta_p)$.

1. *The representation $\overline{s}$ satisfies conditions (1) and (2) of Definition 5.2.1.*
2. *If $\det(\overline{r}_A) = \overline{\varepsilon}^{-m}$ and $\overline{r}_B$ has multiplier character $\overline{\varepsilon}^{1-m}$, then $\overline{s}$ has image in $\mathrm{GU}_{mn}(\mathbf{F}_{p^2})$ with multiplier character $\overline{\varepsilon}^{1-mn}$.*
3. *If in addition to the assumptions in (2), one additionally assumes that $\overline{\varepsilon}(G_F) = \mathbf{F}_p^\times$, then $\overline{s}$ also satisfies condition (3) of Definition 5.2.1 and thus satisfies the Taylor–Wiles big image conditions.*

Proof. Let H_A and H_B denote the extensions of $F(\zeta_p)$ corresponding to the fixed fields of the kernels of the projective representations associated to $\overline{r}_A$ and $\overline{r}_B$. Our assumption on the images of $\overline{r}_A$ and $\overline{r}_B$ imply that

$$\mathrm{Gal}(H_A/F(\zeta_p)) \simeq \mathrm{PSL}_2(\mathbf{F}_p), \quad \mathrm{Gal}(H_B/F(\zeta_p)) \simeq \mathrm{PSU}_m(\mathbf{F}_{p^2}).$$

Let $\widetilde{H}_A$ and $\widetilde{H}_B$ denote the Galois closures of H_A and H_B over $\mathbf{Q}$, and let $\widetilde{H}$ denote the compositum of $\widetilde{H}_A$ and $\widetilde{H}_B$. Since $F/\mathbf{Q}$ is Galois and $\mathrm{PSL}_2(\mathbf{F}_p)$ and $\mathrm{PSU}_n(\mathbf{F}_{p^2})$ are simple (as $p \geq 5$), we have isomorphisms $\mathrm{Gal}(\widetilde{H}_A/F(\zeta_p)) \sim \mathrm{PSL}_2(\mathbf{F}_p)^r$ and $\mathrm{Gal}(\widetilde{H}_B/F(\zeta_p)) \simeq \mathrm{PSU}_m(\mathbf{F}_{p^2})^s$ respectively for some positive integers r and s. Thus

$$\mathrm{Gal}(\widetilde{H}/F(\zeta_p)) \simeq \mathrm{Gal}(\widetilde{H}_A/F(\zeta_p)) \times \mathrm{Gal}(\widetilde{H}_B/F(\zeta_p)),$$

since either $m > 2$ and the groups have no common quotients, or $m = 2$ and the fields are linearly disjoint by assumption.

If $G = \mathrm{PSL}_2(\mathbf{F}_p)$ or $G = \mathrm{PSU}_m(\mathbf{F}_{p^2})$, then G is simple by Lemma 5.2.3. Moreover, by the same lemma, for any equivalence class of matrices A with eigenvalues in $\mathbf{F}_p$, any automorphism of G preserves the (unordered set of) eigenvalues of any member of A up to scalars. We also have $\mathrm{Aut}(G^m) \simeq \mathrm{Aut}(G) \rtimes S_m$.

Let $\alpha = \beta^2 \in \mathbf{F}_p^{\times 2}$ be an element such that $1, \alpha, \ldots, \alpha^{mn-1}$ are all distinct; such an α exists because $p - 1 \geq 2mn$. Let A be a matrix in $\mathrm{PSL}_2(\mathbf{F}_p)$ with eigenvalues (up to scalars) 1 and α, and let $B \in \mathrm{PSU}_m(\mathbf{F}_{p^2})$ have eigenvalues $1, \alpha^n, \ldots, \alpha^{n(m-1)}$. Explicitly, let $A = \mathrm{diag}(\beta, \beta^{-1}) \in \mathrm{SL}_2(\mathbf{F}_p)$, and then construct B as follows. Certainly $A^n \in \mathrm{SL}_2(\mathbf{F}_p)$. The image of A^n under the $m-1$th symmetric power map lands in $\mathrm{Sp}_m(\mathbf{F}_p)$ or $\mathrm{SO}_m(\mathbf{F}_p)$ depending on the parity of m (here we mean the symplectic or orthogonal groups defined using the bilinear form induced by the standard symplectic form on $\mathbf{F}_p^2$). These groups are conjugate to subgroups of $\mathrm{SU}_m(\mathbf{F}_{p^2})$ by Lemma 4.3.4. By Chebotarev, we find a prime q unramified in $\widetilde{H}$ and such that for a fixed choice of $\mathfrak{q}|q$ in $\widetilde{H}$, $\mathrm{Frob}_{\mathfrak{q}} \in \mathrm{Gal}(\widetilde{H}/F(\zeta_p)) \subset \mathrm{Gal}(\widetilde{H}/\mathbf{Q})$ has the form

$$(A, A, \dots, A) \times (B, B, \dots, B) \in \mathrm{PSL}_2(\mathbf{F}_p)^r \times \mathrm{PSU}_m(\mathbf{F}_{p^2})^s.$$

The eigenvalues (up to scalar) of A and B are preserved by the action of $\mathrm{Gal}(F/\mathbf{Q})$; this follows from our description of the automorphism group of each factor.

The images of these elements in $\mathrm{Gal}(F(\zeta_p)/\mathbf{Q})$ are trivial, so such a prime q will split completely in $F(\zeta_p)$ and so satisfy $q \equiv 1 \bmod p$. Moreover, the Frobenius elements at all other primes above q will be conjugate inside $\mathrm{Gal}(\widetilde{H}/\mathbf{Q})$. Hence the image of (any conjugate of) $\mathrm{Frob}_{\mathfrak{q}}$ under $\overline{s}$ has eigenvalues (up to scalar) given by $1, \alpha, \dots, \alpha^{mn-1}$. In particular, they are all distinct. Since $q \equiv 1 \bmod p$, this implies that $\overline{s}$ is decomposed generic, which is property (1) of Definition 5.2.1.

To see that $\overline{s}|_{G_{F(\zeta_p)}}$ has adequate image, it suffices to show that the image is absolutely irreducible and thus is also adequate by [Tho12, Theorem A.9] (using the assumption $p > 2mn+1$). The irreducibility follows from the fact that the $\mathrm{SL}_2(\mathbf{F}_p)$-representation $\mathrm{Sym}^{n-1} \overline{\mathbf{F}}_p^2$ and the standard representation of $\mathrm{SU}_m(\mathbf{F}_{p^2})$ are both irreducible as long as $p > n$, since the image of $\overline{s}|_{G_{F(\zeta_p)}}$ is $\mathrm{SL}_2(\mathbf{F}_p) \times \mathrm{SU}_m(\mathbf{F}_{p^2})$. This proves property (2)of Definition 5.2.1.

Assume that $\det(\overline{r}_A) = \overline{\varepsilon}^{-m}$ and that $\overline{r}_B$ has multiplier character $\overline{\varepsilon}^{1-m}$. Then $\overline{r}_A \otimes \overline{r}_B$ is absolutely irreducible and self-dual (i.e. there is an isomorphism of the form (4.3.4)) with multiplier character

$$\overline{\varepsilon}^{-m(n-1)} \cdot \overline{\varepsilon}^{1-m} = \overline{\varepsilon}^{1-mn},$$

and so the image lies in $\mathrm{GU}_{mn}(\mathbf{F}_{p^2})$ with this multiplier character by Lemma 4.3.4. This establishes condition (2).

Assume that $\overline{\varepsilon}(G_F) = \mathbf{F}_p^\times$. Let M_A and M_B denote the fixed fields of the kernels of $\overline{r}_A$ and $\overline{r}_B$, and let M be the compositum of M_A and M_B. By our assumption on linear disjointness of H_A and H_B, $\mathrm{Gal}(M/F(\zeta_p))$ is the direct product $\mathrm{SL}_2(\mathbf{F}_p) \times \mathrm{SU}_m(\mathbf{F}_{p^2})$, and $\mathrm{Gal}(M/F)$ is the subgroup of matrices (A, B) of $\mathrm{GL}_2(\mathbf{F}_p) \times \mathrm{GU}_m(\mathbf{F}_{p^2})$ with $\det(A) = \eta^{-m}$ and $\nu(B) = \eta^{1-m}$ for some $\eta \in \mathbf{F}_p^\times$. Hence the image certainly contains $(\beta^m I_2, \beta^{m-1} I_m)$, where I_n denotes the trivial matrix in $\mathrm{SL}_n(\mathbf{F}_p)$ and $\beta \in \mathbf{F}_p^\times$ is a primitive root. Then, by Chebotarev, there exists $\sigma \in G_F$ whose image in $\mathrm{Gal}(M/F)$ is this element. Since $p > 2mn+1 \geq 2m+1$, we have $\beta^{2m} \neq 1$. Since $\overline{\varepsilon}^{-m}(\sigma) = \beta^{2m}$, the element σ is not contained in $G_{F(\zeta_p)}$. On the other hand, we see that $\overline{s}(\sigma)$ is also scalar, and we are done. □

We shall need the following well-known property of induced representations (specialized to the context in which we shall apply it in the proof of the following lemma).

Lemma 5.2.5. *Let $E/\mathbf{Q}$ be a cyclic Galois extension of degree m linearly disjoint from F, and let $L = E \cdot F$. Let $\overline{\psi} : G_L \to \mathbf{F}_p^\times$ be a character and let $\overline{r}_B = \mathrm{Ind}_{G_L}^{G_F} \overline{\psi} : G_F \to \mathrm{GL}_m(\overline{\mathbf{F}}_p)$. Let q be a prime of $\mathbf{Q}$ such that $\overline{r}_B$ is unramified at all $v|q$, q splits completely in F, and* Frob_q *generates* $\mathrm{Gal}(E/\mathbf{Q})$*. Then, for $v|q$ in F, the eigenvalues of $\overline{r}_B(\mathrm{Frob}_v)$ are of the form $\lambda, \zeta\lambda, \dots, \zeta^{m-1}\lambda$ for some λ where ζ is a primitive mth root of unity.*

Proof. The assumption that E is linearly disjoint from F ensures that $\mathrm{Gal}(L/F) \simeq \mathrm{Gal}(E/\mathbf{Q})$ is cyclic of order m. There is an isomorphism $\overline{r}_B \simeq \overline{r}_B \otimes \chi$ where χ is a character (factoring through $\mathrm{Gal}(L/F)$) of order m.

Thus $\overline{r}_B(\mathrm{Frob}_v)$ is conjugate to $\chi(\mathrm{Frob}_v)\overline{r}_B(\mathrm{Frob}_v) = \zeta \cdot \overline{r}_B(\mathrm{Frob}_v)$, where ζ is a primitive mth root of unity and the result follows. □

We shall also need the following variant of Lemma 5.2.4:

Lemma 5.2.6. *Let $F/\mathbf{Q}$ be Galois. Consider representations:*

$$\overline{r}_A : G_F \to \mathrm{GL}_2(\mathbf{F}_p),$$
$$\overline{r}_B : G_F \to \mathrm{GL}_m(\mathbf{F}_p).$$

Assume that:

1. *The image of $\overline{r}_A(G_{F(\zeta_p)})$ equals $\mathrm{SL}_2(\mathbf{F}_p)$.*
2. *There is a cyclic Galois extension $E/\mathbf{Q}$ of degree m and linearly disjoint from F, such that, setting $L = E \cdot F$, there is a character $\overline{\psi} : G_L \to \mathbf{F}_p^\times$ with $\overline{r}_B \cong \mathrm{Ind}_{G_L}^{G_F} \overline{\psi}$ and $\overline{r}_B|_{G_{F(\zeta_p)}}$ irreducible.*

Consider the representation:

$$\overline{s} = \mathrm{Sym}^{n-1} \overline{r}_A \otimes \overline{r}_B : G_F \to \mathrm{GL}_{mn}(\mathbf{F}_p).$$

Assume that $p > 2mn + 1$. Then:

1. *The representation $\overline{s}$ satisfies conditions (1) and (2) of Definition 5.2.1.*
2. *If $\det(\overline{r}_A) = \overline{\varepsilon}^{-m}$ and $\det(\overline{r}_B) = \overline{\varepsilon}^{-m(m-1)/2}$ and the image of $\overline{\varepsilon}(G_L) = \mathbf{F}_p^\times$, then $\overline{s}$ also satisfies condition (3) of Definition 5.2.1 and thus satisfies the Taylor–Wiles big image conditions.*

Proof. The representation $\overline{r}_B$ has solvable image. The assumption that $\overline{r}_B|_{G_{F(\zeta_p)}}$ is irreducible implies that $F(\zeta_p)$ and E are linearly disjoint. As in the proof of Lemma 5.2.4, let H_A and H_B denote the extensions of $F(\zeta_p)$ corresponding to the fixed fields of the kernels of the projective representations associated to $\overline{r}_A$ and $\overline{r}_B$ and $\widetilde{H}_A$, $\widetilde{H}_B$ their Galois closures over $\mathbf{Q}$. Let $\widetilde{H}$ be the compositum of $\widetilde{H}_A$ and $\widetilde{H}_B$. We deduce once more that

$$\mathrm{Gal}(\widetilde{H}_A/F(\zeta_p)) \simeq \mathrm{PSL}_2(\mathbf{F}_p)^r$$

and, since $\mathrm{PSL}_2(\mathbf{F}_p)$ has no solvable quotients,

$$\mathrm{Gal}(\widetilde{H}/F(\zeta_p)) \simeq \mathrm{Gal}(\widetilde{H}_A/F(\zeta_p)) \times \mathrm{Gal}(\widetilde{H}_B/F(\zeta_p)).$$

We have $E \subset \widetilde{H}_B$ and, since $\overline{r}|_{G_{F(\zeta_p)}}$ is irreducible, $\mathrm{Gal}(\widetilde{H}_B/F(\zeta_p)) \to \mathrm{Gal}(E/\mathbf{Q})$ is surjective.

Let $\alpha \in \mathbf{F}_p^\times$ be an element such that $1, \alpha, \ldots, \alpha^{mn-1}$ are all distinct; such an α exists because $p-1 \geq mn$. By Chebotarev, we find a prime q unramified in $\widetilde{H}$, split in $F(\zeta_p)$, and such that for a fixed choice of $\mathfrak{q}|q$ in $\widetilde{H}$, $\mathrm{Frob}_\mathfrak{q} \in \mathrm{Gal}(\widetilde{H}/F(\zeta_p)) \subset \mathrm{Gal}(\widetilde{H}/\mathbf{Q})$ has the form

$$(A, A, \ldots, A) \times \sigma \in \mathrm{PSL}_2(\mathbf{F}_p)^r \times \mathrm{Gal}(\widetilde{H}_B/F(\zeta_p)),$$

where A has eigenvalues with ratio α and σ projects to a generator of $\mathrm{Gal}(E/\mathbf{Q})$. Hence, by Lemma 5.2.5, the image of (any conjugate of) $\mathrm{Frob}_\mathfrak{q}$ under $\overline{s}$ has eigenvalues (up to scalar) given by:

$$\alpha^i \zeta^j, i = 0, \ldots, n-1, j = 0, \ldots, m-1,$$

and in particular all eigenvalues are distinct since otherwise $\alpha^{km} = 1$ for some $k < n$. Since $q \equiv 1 \bmod p$, this implies that $\overline{s}$ is decomposed generic, which is property (1).

Property (2) of Definition 5.2.1 follows exactly as in the proof of Lemma 5.2.4.

Now suppose that $\det(\overline{r}_A) = \overline{\varepsilon}^{-m}$ and $\det(\overline{r}_B) = \overline{\varepsilon}^{-m(m-1)/2}$ and that $\overline{\varepsilon}(G_L) = \mathbf{F}_p^\times$. We now show that property (3) of Definition 5.2.1 holds. Let M_A and M_B denote the fixed fields of the kernels of $\overline{r}_A|_{G_{F(\zeta_p)}}$ and $\overline{r}_B|_{G_{F(\zeta_p)}}$ respectively. Since $\mathrm{SL}_2(\mathbf{F}_p)$ has no solvable quotients, the map $G_{M_B} \to \mathrm{Gal}(M_A/F(\zeta_p))$ is surjective. Let $\beta \in \mathbf{F}_p^\times$ be a primitive root. We claim that we can find $\sigma \in G_F$ such that $\overline{r}_A(\sigma) = \beta^{-m^2} \cdot I_2$ and $\overline{r}_B(\sigma) = \beta^{m(1-m)} \cdot I_m$. To see this, first choose $g \in G_L$ such that $\overline{\varepsilon}(g) = \beta^2$, and let $h = \prod_{\tau \in \mathrm{Gal}(L/F)} {}^\tau g$. Then $\overline{r}_B(h) = \beta^{m(1-m)} I_m$, as $\prod_{\tau \in \mathrm{Gal}(L/F)} {}^\tau \overline{\psi} = \det \overline{r}_B|_{G_L} = \overline{\varepsilon}^{-m(m-1)/2}$. We now choose σ of the form $h\gamma$ where $\gamma \in G_{M_B}$; since $\overline{r}_B(\gamma)$ is trivial, this means that $\overline{r}_B(\sigma) = \overline{r}_B(h)$ is of the correct form. On the other hand, we have $\det \overline{r}_A(h) = \overline{\varepsilon}^{-m}(g)^m = \beta^{-2m^2}$. Since $G_{M_B} \to \mathrm{Gal}(M_A/F(\zeta_p)) \simeq \mathrm{SL}_2(\mathbf{F}_p)$ is surjective, we choose $\gamma \in G_{M_B}$ so that $\overline{r}_A(\gamma) = \beta^{-m^2} \cdot \overline{r}_A(h)^{-1}$, and then $\overline{r}_A(\sigma) = \beta^{-m^2} \cdot I_2$.

By construction, $\overline{s}(\sigma)$ is scalar. On the other hand, $\overline{\varepsilon}(\sigma) = \beta^{2m} \neq 1$, as $p - 1 > 2m$ because $p > 2nm + 1$, so $\sigma \in G_F - G_{F(\zeta_p)}$, as required. □

5.3. **Character building lemmas.** In this section, we construct some induced extensions with certain desirable local properties. We begin with the following well-known lemma:

Lemma 5.3.1 (Globalizing local characters)**.** *Let F be a number field, and let S be a finite set of places of F. Let $\psi_v : G_{F_v} \to \mathbf{Z}/n\mathbf{Z}$ be a collection of characters for all $v \in S$. Assume that S does not contain any places $v|2$. Then there exists a global character $\chi : G_F \to \mathbf{Z}/n\mathbf{Z}$ such that $\chi|_{G_{F_v}} = \psi_v$ for all $v \in S$.*

Proof. This is a consequence of [AT09, §X Thm. 5] (see also [Con11, Appendix A]). More precisely, the claim holds (without the hypothesis on S) if n is odd. If n is even, there exists an explicitly defined element

$$a_{F,n} \in (F^\times)^{n/2}$$

which is a perfect nth power for all but a finite set (possibly empty) places $S_{F,n}$ of primes $v|2$. Then the ψ_v come from a global character χ of order n if and only if either $S_F \not\subset S$, or $S_F \subset S$ and

$$\prod_{v \in S_F} \psi_v(a_{F,n}) = 1.$$

Since we have assumed that S contains no places above 2, either S_F is empty or $S_F \not\subset S$, so the result follows. □

Remark 5.3.2. One cannot drop the hypothesis on S in general because of the Grunwald–Wang phenomenon (e.g. $F = \mathbf{Q}$, $v = 2$, $n = 8$, and ψ_2 unramified with order 8). If one considers general F, one cannot even globalize a local character ψ_v up to a character ϕ_v which is unramified at v. Let $F = \mathbf{Q}(\sqrt{-5})$, and consider a character

$$\chi : F^\times \backslash \mathbf{A}_F^\times \to \mathbf{Z}/8\mathbf{Z}.$$

Since $16 \in F^\times$ is a perfect 8th power for all $v|F$ of odd residue characteristic (this is true even for $F = \mathbf{Q}$) and also in $F_\infty^\times \simeq \mathbf{C}^\times$, it follows that the restriction:

$$\chi_2 : F_2^\times \to \mathbf{Z}/8\mathbf{Z}$$

satisfies $\chi_2(16) = 1$. Since $F_2/\mathbf{Q}_2$ is ramified of degree 2, we find that 16 has valuation 8 and hence $\phi_2(16) = 1$ for any unramified character $F_2^\times \to \mathbf{Z}/8\mathbf{Z}$. Since neither 2 nor -1 is a square in $F_2 \simeq \mathbf{Q}_2(\sqrt{-5})$, it follows that 16 is not a perfect 8th power in $F_2^\times$, and thus there exists a local character $\psi_2 : F_2^\times \to \mathbf{Z}/8\mathbf{Z}$ such that $\psi_2(16) = -1$. But from the above, we see that there is no global character χ such that $\chi_2 = \psi_2\phi_2$ for an unramified character ϕ_2.

We now show that any character over a CM field can be written as an mth power of another character over some finite CM extension satisfying certain properties. The argument is essentially the same as in the proof of [ACC+23, Theorem 7.1.11].

Lemma 5.3.3. *Let η be a finite order character of G_F for a CM field F, let m be an integer, and let F^{avoid}/F be a finite extension. Then there exists a totally real Galois extension $M/\mathbf{Q}$ linearly disjoint from F^{avoid} and a character ψ of $G_{M\cdot F}$ such that $\eta|_{G_{MF}} = \psi^m$. Moreover, if η is unramified at all v dividing some finite set of primes T of $\mathbf{Q}$ not including 2, then we may take M to be totally split at all primes dividing those in T, and ψ to be unramified at primes dividing those in T.*

Proof. By induction, it suffices to consider the case when m is prime. Assume that η has order n. There is an exact sequence:

$$0 \to \mathbf{Z}/m\mathbf{Z} \to \mathbf{Z}/mn\mathbf{Z} \to \mathbf{Z}/n\mathbf{Z} \to 0.$$

The character η gives a class in $H^1(F, \mathbf{Z}/n\mathbf{Z})$ which we want to write as an mth power, which amounts to lifting this class to $H^1(F, \mathbf{Z}/mn\mathbf{Z})$. The obstruction to this is an element $\partial\eta$ lying in

$$H^2(F, \mathbf{Z}/m\mathbf{Z}) \hookrightarrow H^2(F(\zeta_m), \mu_m)) \simeq \mathrm{Br}(F(\zeta_m))[m],$$

where the injectivity of the first map follows from Hochschild–Serre and the fact that $[F(\zeta_m) : F]$ is prime to m (since m is prime). From the Albert–Brauer–Hasse–Noether theorem, there is an injection

$$\mathrm{Br}(F(\zeta_m))[m] \hookrightarrow \bigoplus_v \mathrm{Br}(F(\zeta_m)_v)[m].$$

The image of the class $\partial\eta$ is zero for all v not dividing a finite set of places S of $\mathbf{Q}$ (the places where η is ramified), and is zero for v dividing places in T (since η is unramified there and there is no obstruction to lifting an unramified local character). Since F is totally imaginary and $\mathrm{Br}(\mathbf{C}) = 0$, we may assume S consists only of finite primes and we may also assume that $\infty \in T$. If K is a local field and L/K has degree m then the map $\mathrm{Br}(K)[m] \to \mathrm{Br}(L)$ is trivial [CF86, §VI, Thm. 3]. Hence any class in $\mathrm{Br}(F(\zeta_m))[m]$ is trivial in $F(\zeta_m)\cdot M$ whenever $[M_v : \mathbf{Q}_v]$ is divisible by $m[F(\zeta_m) : \mathbf{Q}]$ for any prime v in S. Hence it suffices to find such a Galois extension $M/\mathbf{Q}$ disjoint from F^{avoid}, in which the places in T are totally split (since $\infty \in T$ this implies that M is totally real). This is essentially done in [AT09, §X Thm. 6] and we can appeal to [CHT08, Lemma 4.1.2] for the precise statement we need. Since η is unramified at primes in T, for each $v|T$ the image of $\psi|_{I_{(M\cdot F)_v}}$ has order dividing m. Thus by Lemma 5.3.1 we may twist ψ by another character of order m (which doesn't change ψ^m) so that it is unramified at $v|T$. □

6. Automorphy of compatible systems

6.1. **Compatible systems and purity.** We recall the following definition from [ACC+23, §7].

Definition 6.1.1. Let F be a number field. A very weakly compatible system $\mathcal{R}$ (of rank n representations of G_F, with coefficients in M) is by definition a tuple

$$(M, S, \{Q_v(X)\}, \{r_\lambda\}, \{H_\tau\}),$$

where:

(1) M is a number field;
(2) S is a finite set of finite places of F;
(3) for each finite place $v \notin S$ of F, $Q_v(X) \in M[X]$ is a monic degree n polynomial;
(4) for each $\tau : F \hookrightarrow \overline{M}$, H_τ is a multiset of integers;
(5) for each finite place λ of M (say of residue characteristic l),

$$r_\lambda : G_F \to \mathrm{GL}_n(\overline{M}_\lambda)$$

is a continuous, semi-simple representation satisfying the following conditions:
   (a) If $v \notin S$ and $v \nmid l$, then $r_\lambda|_{G_{F_v}}$ is unramified and the characteristic polynomial of Frob_v equals $Q_v(X)$.
   (b) For l outside a set of primes of Dirichlet density 0, r_λ is crystalline and $\mathrm{HT}_\tau(r_\lambda) = H_\tau$.
   (c) For every l, we have $\mathrm{HT}_\tau(\det r_\lambda) = \sum_{h \in H_\tau} h$.

If F'/F is a finite extension then we may define the restricted very weakly compatible system

$$\mathcal{R}|_{G_{F'}} = (M, S_{F'}, \{Q_w(X)\}, \{r_\lambda|_{G_{F'}}\}, \{H'_\tau\}),$$

where $S_{F'}$ is the set of places of F' lying above S, $Q_w(X) = \det r_\lambda(X - \mathrm{Frob}_w)$ (thus independent of λ), and $H'_\tau = H_{\tau|_F}$. If

$$\mathcal{R}_1 = (M, S_1, \{Q_{1,v}(X)\}, \{r_{1,\lambda}\}, \{H_{1,\tau}\}),\ \mathcal{R}_2 = (M, S_2, \{Q_{2,v}(X)\}, \{r_{2,\lambda}\}, \{H_{2,\tau}\})$$

are very weakly compatible systems with a common coefficient field M, then we can define the tensor product

$$\mathcal{R}_1 \otimes \mathcal{R}_2 = (M, S_1 \cup S_2, \{Q_v(X)\}, \{r_{1,\lambda} \otimes r_{2,\lambda}\}, \{H_\tau\}),$$

where we take $Q_v(X) = \det(r_{1,\lambda} \otimes r_{2,\lambda})(X - \mathrm{Frob}_v)$ (thus independent of λ) and $H_\tau = \{k + l \,|\, k \in H_{1,\tau}, l \in H_{2,\tau}\}$ (sums taken with multiplicity).

The following definition summarizes some possible properties of very weakly compatible systems. These were all defined in [ACC+23], with the exception of (3) ('weakly automorphic'). This condition arises for us because we consider tensor products of compatible systems, one of which has poorly controlled ramification. Lemma 6.1.4 gives conditions under which 'weakly automorphic' can be upgraded to 'automorphic'.

Definition 6.1.2. Let

$$\mathcal{R} = (M, S, \{Q_v(X)\}, \{r_\lambda\}, \{H_\tau\})$$

be a very weakly compatible system. We say that $\mathcal{R}$ is:

(1) pure of weight $m \in \mathbf{Z}$, if it satisfies the following conditions:
   (a) for each $v \notin S$, each root α of $Q_v(X)$ in $\overline{M}$, and each $\iota : \overline{M} \hookrightarrow \mathbf{C}$ we have

$$|\iota\alpha|^2 = q_v^m;$$

(b) for each $\tau : F \hookrightarrow \overline{M}$ and each complex conjugation c in $\mathrm{Gal}(\overline{M}/\mathbf{Q})$ we have

$$H_{c\tau} = \{m - h : \ h \in H_\tau\}.$$

(2) automorphic, if there is a regular algebraic, cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$ and an embedding $\iota : M \hookrightarrow \mathbf{C}$ such that for every finite place $v \notin S$ of F, π_v is unramified and $\mathrm{rec}^T_{F_v}(\pi_v)(\mathrm{Frob}_v)$ has characteristic polynomial $\iota(Q_v(X))$.

(3) weakly automorphic of level prime to T, if T is a finite set of finite places of F, disjoint from S, and there is a regular algebraic, cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$ and an embedding $\iota : M \hookrightarrow \mathbf{C}$ such that for all but finitely many finite places $v \notin S$ of F, and for every $v \in T$, π_v is unramified and $\mathrm{rec}^T_{F_v}(\pi_v)(\mathrm{Frob}_v)$ has characteristic polynomial $\iota(Q_v(X))$. We will say that $\mathcal{R}$ is simply 'weakly automorphic' if it is weakly automorphic of level prime to the empty set.

(4) irreducible, if for l outside a set of primes of Dirichlet density 0, and for all $\lambda | l$ of M, r_λ is irreducible.

(5) strongly irreducible, if for every finite extension F'/F, the compatible system $\mathcal{R}|_{G_{F'}}$ is irreducible.

For a CM number field F, and a regular algebraic weight λ, cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbf{A}_F)$, there is an associated automorphic very weakly compatible system

$$\mathcal{R} = (M, S, \{Q_v(X)\}, r_{\pi,\lambda}, H_\tau),$$

where $H_\tau = \{\lambda_{\tau,1} + 1, \lambda_{\tau,2}\}$ (see [ACC+23, Lemma 7.1.10]).

We now recall that the potential automorphy of symmetric powers of $\mathcal{R}$ is enough to imply purity.

Lemma 6.1.3. *Let $\mathcal{R} = (M, S, \{Q_v(X)\}, \{r_\lambda\}, \{H_\tau\})$ be a very weakly compatible system of rank 2 representations of G_F such that $H_\tau = \{0, m\}$ for a fixed $m \in \mathbf{N}$ for all τ. Fix a finite place v_0 of F which is not in S, and let $X_0 = \{v_0\}$. Suppose that for infinitely many $n \geq 1$, we can find a finite Galois extension F_n/F such that the very weakly compatible system $\mathrm{Sym}^{n-1}\mathcal{R}|_{F_n}$ is weakly automorphic of level prime to $X_{0,F_n} = \{v | v_0\}$. Then the roots α_1, α_2 of $Q_{v_0}(X)$ in $\overline{M}$ satisfy*

$$|\iota\alpha|^2 = q_{v_0}^m$$

for each $\iota : \overline{M} \hookrightarrow \mathbf{C}$.

Proof. Choose a place $v_n | v_0$ in F_n and fix $\iota : \overline{M} \hookrightarrow \mathbf{C}$. We are assuming that $\mathrm{Sym}^{n-1}\mathcal{R}|_{F_n}$ is associated to a cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbf{A}_{F_n})$ and Π_{v_n} is unramified. Up to a finite order character, the determinant of our rank n automorphic compatible system is given by the $-mn(n-1)/2$th power of the cyclotomic character, so the central character of Π is (again, up to a finite order Hecke character) $|\cdot|^{n(m-1)(1-n)/2}$, and in particular $\Pi|\cdot|^{(m-1)(n-1)/2}$ is unitary. Since we know that $|\iota(\alpha_1\alpha_2)| = q_{v_0}^m$, it suffices to prove that $|\iota\alpha_i| \leq q_{v_0}^{m/2}$ for $i = 1, 2$. Let $q_{v_n} = q_{v_0}^f$. As in the proof of [ACC+23, Cor. 7.1.13], we can apply the Jacquet–Shalika bound [JS81, Cor. 2.5] to deduce that that $|\iota\left(\alpha_i^{f(n-1)}\right)| \leq q_{v_n}^{((m-1)(n-1)+n)/2}$, so $|\iota\alpha_i| \leq q_{v_0}^{m/2+1/2(n-1)}$. Letting n tend to ∞ gives the desired bound on $|\iota\alpha_i|$. □

Lemma 6.1.4. *Let F be a CM number field, and let*

$$\mathcal{R} = (M, S, \{Q_v(X)\}, \{r_\lambda\}, \{H_\tau\})$$

be a very weakly compatible system of rank n representations of G_F which is weakly automorphic, corresponding to a regular algebraic, cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$, and pure of weight $m \in \mathbf{Z}$. Then $\mathcal{R}$ is automorphic.

Proof. Choose some embedding of M in $\mathbf{C}$. By assumption, there is a finite set $S' \supseteq S$ of finite places of F such that for each $v \notin S'$, π_v is unramified and $\mathrm{rec}^T_{F_v}(\pi_v)(\mathrm{Frob}_v)$ has characteristic polynomial $Q_v(X)$. We must show that this holds for all $v \notin S$. Choose $v \in S' - S$, a rational prime p not lying under v, and an isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$. Let λ denote the place of M induced by ι^{-1}. Then the Chebotarev density theorem implies that there is an isomorphism $r_\lambda \cong r_\iota(\pi)$. By assumption, $r_\lambda|_{G_{F_v}}$ is unramified and pure of weight m. By [Var24, Theorem 1], there is an isomorphism $r_\iota(\pi)|^{\mathrm{ss}}_{W_{F_v}} \cong \iota^{-1} \mathrm{rec}^T_{F_v}(\pi_v)^{\mathrm{ss}}$. We deduce that π_v is a subquotient of an unramified principal series, namely the one with Satake parameter determined by $Q_v(X)$. Since $r_\lambda|_{G_{F_v}}$ is pure, this principal series representation is irreducible and π_v is unramified, as desired. □

Lemma 6.1.5. *Let F be a number field and let*

$$\mathcal{R} = (M, S, \{Q_v(X)\}, \{r_\lambda\}, \{H_\tau\})$$

be a very weakly compatible system of rank 2 representations of G_F which is strongly irreducible. Let $\mathcal{L}(\mathcal{R})$ denote the set of primes l satisfying the following conditions:

1. *$l \notin S$ and for each place $\lambda | l$ of M, r_λ is crystalline of Hodge–Tate weights H_τ.*
2. *For each place $\lambda | l$ of M, $\overline{r}_\lambda(G_{\widetilde{F}})$ contains a conjugate of $\mathrm{SL}_2(\mathbf{F}_l)$, where $\widetilde{F}$ is the Galois closure of $F/\mathbf{Q}$.*

Then $\mathcal{L}(\mathcal{R})$ has Dirichlet density 1.

Proof. The set of primes l having property (1) has Dirichlet density 1, by definition of a very weakly compatible system. The lemma therefore follows from [ACC+23, Lemma 7.1.3]. □

6.2. **Potential automorphy theorems.** Our goal in this section is to prove Theorem 6.2.1. The proof will occupy the whole section, but to keep the presentation organized and somewhat motivated, we deduce it from Theorem 6.2.4 below, which we will in turn deduce from Proposition 6.2.3.

Theorem 6.2.1. *Let F be an imaginary CM number field, and let*

$$\mathcal{R} = (M, S, \{Q_v(X)\}, \{r_\lambda\}, \{H_\tau\})$$

be a very weakly compatible system of rank 2 representations of G_F. Let $m \geq 1$ be an integer, and suppose that the following conditions are satisfied:

1. *For each τ, $H_\tau = \{0, m\}$.*
2. *$\det r_\lambda = \varepsilon^{-m}$.*
3. *$\mathcal{R}$ is strongly irreducible.*

Then $\mathcal{R}$ is pure of weight m, and for each $n \geq 1$, there exists a finite CM extension F_n/F such that $F_n/\mathbf{Q}$ is Galois and $\mathrm{Sym}^{n-1} \mathcal{R}|_{G_{F_n}}$ is automorphic.

Proof. We may assume that $m \geq 2$, since otherwise the result follows from [ACC$^+$23, Cor 7.1.12]. Let $v_0 \notin S$ be a place of F. Theorem 6.2.4 states that we can find, for each $n \geq 1$, a CM extension F_n/F, Galois over $\mathbf{Q}$, such that $\mathrm{Sym}^{n-1} \mathcal{R}|_{G_{F_n}}$ is weakly automorphic of level prime to $\{v|v_0\}$. By Lemma 6.1.3, the roots of $Q_{v_0}(X)$ are q_{v_0}-Weil numbers of weight m. Since $v_0 \notin S$ is arbitrary, this shows that the compatible system $\mathcal{R}$ is pure of weight m. We can then apply Lemma 6.1.4 to conclude that $\mathrm{Sym}^{n-1} \mathcal{R}|_{G_{F_n}}$ is automorphic, as required. □

Remark 6.2.2. We assume in our arguments below that $m > 1$. Our argument certainly applies in principle to the case $m = 1$, but certain statements we make through the proof assume that $m \geq 2$, and so this assumption avoids having to make the necessary extra remarks to cover the case $m = 1$. Moreover, our argument in the case $m = 1$ would involve tensoring $\mathcal{R}$ with auxiliary 1-dimensional representations, and not so surprisingly can be simplified to the point where it becomes very similar to the proof of [ACC$^+$23, Cor 7.1.12].

Before giving our first technical result towards the proof of Theorem 6.2.4 (and hence Theorem 6.2.1 above), we sketch the idea of the proof. We begin with the strongly irreducible, very weakly compatible system $\mathcal{R}$ of rank 2 and parallel Hodge–Tate weights $\{0, m\}$, and wish to show that $\mathrm{Sym}^{n-1} \mathcal{R}$ is potentially (weakly) automorphic. This presents difficulties since the compatible system $\mathrm{Sym}^{n-1} \mathcal{R}$ has parallel Hodge–Tate weights $\{0, m, 2m, \ldots, (n-1)m\}$, while the auxiliary motives that we can construct to show potential automorphy have consecutive (and parallel) Hodge–Tate weights (and moreover, our automorphy lifting theorem Theorem 3.2.1 applies only to Galois representations with consecutive Hodge–Tate weights). To get around this, we construct auxiliary compatible systems as follows:

- An auxiliary compatible system $\mathcal{R}_{\mathrm{aux}}$ of rank m and with consecutive (and parallel) Hodge–Tate weights $\{0, 1, \ldots, m-1\}$. Then $(\mathrm{Sym}^{n-1} \mathcal{R}) \otimes \mathcal{R}_{\mathrm{aux}}$ has rank nm and consecutive (and parallel) Hodge–Tate weights $\{0, 1, \ldots, nm-1\}$.
- A second auxiliary compatible system $\mathcal{R}_{\mathrm{CM}}$ of rank m and with consecutive (and parallel) Hodge–Tate weights $\{0, 1, \ldots, m-1\}$ which is moreover induced from a character.
- A third auxiliary compatible system $\mathcal{S}_{\mathrm{UA}}$ of rank nm with consecutive (and parallel) Hodge–Tate weights $\{0, 1, \ldots, mn-1\}$, and which is moreover automorphic. We will construct $\mathcal{S}_{\mathrm{UA}}$ (and $\mathcal{R}_{\mathrm{aux}}$) as a member of the families of motives considered in §4. (The subscript 'UA' stands for 'universally automorphic'.)

These are chosen to behave well with respect to distinct primes p, r as follows:

- There is a congruence modulo p linking $\mathcal{S}_{\mathrm{aux}} := (\mathrm{Sym}^{n-1} \mathcal{R}) \otimes \mathcal{R}_{\mathrm{aux}}$ and $\mathcal{S}_{\mathrm{UA}}$. We will apply Theorem 3.2.1 to conclude that $\mathcal{S}_{\mathrm{aux}}$ is automorphic.
- There is a congruence modulo r linking $\mathcal{R}_{\mathrm{aux}}$ and $\mathcal{R}_{\mathrm{CM}}$, and therefore also linking $\mathcal{S}_{\mathrm{aux}}$ and $\mathcal{S}_{\mathrm{CM}} := (\mathrm{Sym}^{n-1} \mathcal{R}) \otimes \mathcal{R}_{\mathrm{CM}}$. We will apply Theorem 3.2.1 a second time to conclude that $\mathcal{S}_{\mathrm{CM}}$ is automorphic.
- Since $\mathcal{R}_{\mathrm{CM}}$ is induced from a Hecke character, $\mathcal{S}_{\mathrm{CM}}$ is also induced (from an n-dimensional compatible system). We will then be able to apply the description of the image of automorphic induction given in [AC89] to conclude that $\mathrm{Sym}^{n-1} \mathcal{R}$ is itself automorphic.

The most significant conditions that must be satisfied to apply Theorem 3.2.1 in each case are the non-degeneracy of the residual images and the 'connects' relation locally at the p-adic (resp. r-adic) places of F. The non-degeneracy of the residual images will be easy to arrange by careful choice of data. It is the 'connects' relation that is more serious and imposes the circuitous route followed here to prove the theorem.

The statement of Proposition 6.2.3 below is long but is merely a precise formulation of the properties required of the various auxiliary compatible systems needed to carry out the above sketch. The main point in the proof of Theorem 6.2.4 will be to show how to construct auxiliary compatible systems with these properties.

Proposition 6.2.3. *Let F be an imaginary CM number field, let $m \geq 2$ and $n \geq 1$ be integers, and let X_0 be a finite set of finite places of F. Let*

$$\mathcal{R} = (M, S, \{Q_v(X)\}, \{r_\lambda\}, \{H_\tau\})$$

be a very weakly compatible system of rank 2 representations of G_F satisfying the following conditions:

(1) *For each τ, $H_\tau = \{0, m\}$.*
(2) $\det r_\lambda = \varepsilon^{-m}$.
(3) *$\mathcal{R}$ is strongly irreducible.*
(4) $X_0 \cap S = \emptyset$.
(5) *$F/\mathbf{Q}$ is Galois and contains an imaginary quadratic field F_0.*

We fix an embedding $M \hookrightarrow \mathbf{C}$, and regard M as a subfield of $\mathbf{C}$. Suppose we can find the following additional data:

(6) *A cyclic totally real extension $E/\mathbf{Q}$ of degree m, linearly disjoint from F, and a character $\Psi : \mathbf{A}_L^\times \to M^\times$, where $L = E \cdot F$, satisfying the following conditions:*
   (a) *There is an embedding $\tau_0 : F_0 \to \mathbf{C}$, and a labelling $\tau_1, \dots, \tau_m : E \cdot F_0 \to \mathbf{C}$ of the embeddings $E \cdot F_0 \to \mathbf{C}$ which extend τ_0 such that for each $\alpha \in L^\times$, we have*

   $$\Psi(\alpha) = \prod_{i=1}^{m} \tau_i(\mathbf{N}_{L/E\cdot F_0}(\alpha))^{m-i} c\tau_i(\mathbf{N}_{L/E\cdot F_0}(\alpha))^{i-1}.$$

   We let $\{\Psi_\lambda\}$ denote the weakly compatible system associated to Ψ, and let

   $$\mathcal{R}_{\mathrm{CM}} = \{\operatorname{Ind}_{G_L}^{G_F} \Psi_\lambda\} = (M, S_{\mathrm{CM}}, \{Q_{\mathrm{CM},v}(X)\}, \{r_{\mathrm{CM},\lambda}\}, \{H_{\mathrm{CM},\tau}\})$$

   denote the induced weakly compatible system. (Then $H_{\mathrm{CM},\tau} = \{0, 1, \dots, m-1\}$ for all τ, and we take S_{CM} to be the set of places of F ramified in L or above which Ψ is ramified.)
   (b) *For all λ, $\det r_{\mathrm{CM},\lambda} = \varepsilon^{-m(m-1)/2}$.*

(7) *Distinct primes p, r, not dividing any place of S, and places $\mathfrak{p}, \mathfrak{r}$ of M lying above them.*
(8) *A weakly compatible system of rank m representations of G_F*

   $$\mathcal{R}_{\mathrm{aux}} = (M, S_{\mathrm{aux}}, \{Q_{\mathrm{aux},v}(X)\}, \{r_{\mathrm{aux},\lambda}\}, \{H_{\mathrm{aux},\tau}\}),$$

   satisfying the following conditions:
   (a) *$\mathcal{R}_{\mathrm{aux}}$ is pure of weight $m-1$ and S_{aux} does not intersect $X_0 \cup \{v|pr\}$.*
   (b) *For all λ, $\det r_{\mathrm{aux},\lambda} = \varepsilon^{-m(m-1)/2}$. For all τ, $H_{\mathrm{aux},\tau} = \{0, 1, \dots, m-1\}$.*

(9) A weakly compatible system

$$\mathcal{S}_{\mathrm{UA}} = (M, S_{\mathrm{UA}}, \{Q_{\mathrm{UA},v}(X)\}, \{s_{\mathrm{UA},\lambda}\}, \{H_{\mathrm{UA},\tau}\})$$

of rank nm representations of G_F, satisfying the following conditions:

(a) $\mathcal{S}_{\mathrm{UA}}$ is pure of weight $nm-1$ and S_{UA} does not intersect $X_0 \cup \{v|pr\}$.

(b) For all λ, $\det s_{\mathrm{UA},\lambda} = \varepsilon^{-nm(nm-1)/2}$. For all τ, $H_{\mathrm{UA},\tau} = \{0, 1, \ldots, nm-1\}$.

(c) $\mathcal{S}_{\mathrm{UA}}$ is weakly automorphic of level prime to $X_0 \cup \{v|pr\}$.

Let $\mathcal{S}_{\mathrm{aux}} = \{s_{\mathrm{aux},\lambda}\} = (\mathrm{Sym}^{n-1}\mathcal{R}) \otimes \mathcal{R}_{\mathrm{aux}}$ and $\mathcal{S}_{\mathrm{CM}} = \{s_{\mathrm{CM},\lambda}\} = (\mathrm{Sym}^{n-1}\mathcal{R}) \otimes \mathcal{R}_{\mathrm{CM}}$. These compatible systems of rank nm have coefficients in the number field M. Suppose that these data satisfy the following additional conditions:

(10) L/F is unramified at $X_0 \cup \{v|pr\}$, and Ψ is unramified at the places of L lying above $X_0 \cup \{v|pr\}$. (Then $S_{\mathrm{CM}} \cap (X_0 \cup \{v|pr\}) = \emptyset$.)

(11) $p > 2nm+1$, and $[F(\zeta_p) : F] = p-1$.

(12) $r > 2nm+1$, r splits completely in $E \cdot F_0$, and $[L(\zeta_r) : L] = r-1$.

(13) Up to conjugation, there are sandwiches

$$\mathrm{SL}_2(\mathbf{F}_p) \leq \overline{r}_{\mathfrak{p}}(G_F) \leq \mathrm{GL}_2(\mathbf{F}_p)$$

and

$$\mathrm{SL}_2(\mathbf{F}_r) \leq \overline{r}_{\mathfrak{r}}(G_F) \leq \mathrm{GL}_2(\mathbf{F}_r).$$

If $m > 2$ then the image $\overline{r}_{\mathrm{aux},\mathfrak{p}}(G_F)$ is a conjugate of $\mathrm{GU}_m(\mathbf{F}_{p^2})$ and $\overline{r}_{\mathrm{aux},\mathfrak{p}}$ has multiplier character $\overline{\varepsilon}^{1-m}$. If $m = 2$ then the image $\overline{r}_{\mathrm{aux},\mathfrak{p}}(G_F)$ is a conjugate of $\mathrm{GL}_2(\mathbf{F}_p)$. The representation $\overline{r}_{\mathrm{CM},\mathfrak{r}}|_{G_{F(\zeta_r)}}$ is irreducible. If $m = 2$, then the extensions of $F(\zeta_p)$ cut out by the projective representations associated to $\overline{r}_{\mathfrak{p}}|_{G_{F(\zeta_p)}}$ and $\overline{r}_{\mathrm{aux},\mathfrak{p}}|_{G_{F(\zeta_p)}}$ are linearly disjoint.

(14) There are isomorphisms $\overline{s}_{\mathrm{UA},\mathfrak{p}} \cong \overline{s}_{\mathrm{aux},\mathfrak{p}}$ and $\overline{r}_{\mathrm{aux},\mathfrak{r}} \cong \overline{r}_{\mathrm{CM},\mathfrak{r}}$.

(15) There is a decomposition $S_p = \Sigma^{ord} \sqcup \Sigma^{ss}$ of the set S_p of p-adic places of F such that for each place $v|p$ of F, F_v contains $\mathbf{Q}_{p^2}$, $\overline{r}_{\mathfrak{p}}|_{G_{F_v}}$ and $\overline{\rho}_{2,m,0}|_{G_{F_v}}$ (cf. Definition 5.1.1) are trivial, and:

(a) if $v \in \Sigma^{ord}$, then $r_{\mathfrak{p}}|_{G_{F_v}}$ is crystalline ordinary;

(b) if $v \in \Sigma^{ss}$, then $r_{\mathfrak{p}}|_{G_{F_v}} \sim \rho_{2,m,0}|_{G_{F_v}}$.

(16) If $v \in \Sigma^{ord}$, then $\overline{r}_{\mathrm{aux},\mathfrak{p}}|_{G_{F_v}}$ is trivial and $r_{\mathrm{aux},\mathfrak{p}}|_{G_{F_v}}$ and $s_{\mathrm{UA},\mathfrak{p}}|_{G_{F_v}}$ are both crystalline ordinary. If $v \in \Sigma^{ss}$, then $\overline{r}_{\mathrm{aux},\mathfrak{p}}|_{G_{F_v}}$ is trivial, $r_{\mathrm{aux},\mathfrak{p}}|_{G_{F_v}}$ and $s_{\mathrm{UA},\mathfrak{p}}|_{G_{F_v}}$ are both crystalline, and $r_{\mathrm{aux},\mathfrak{p}}|_{G_{F_v}} \sim \rho_{m,1,0}|_{G_{F_v}}$ and $s_{\mathrm{UA},\mathfrak{p}}|_{G_{F_v}} \sim \rho_{nm,1,0}|_{G_{F_v}}$.

(17) For each place $v|r$ of F, $\overline{r}_{\mathrm{aux},\mathfrak{r}}|_{G_{F_v}} \cong \overline{r}_{\mathrm{CM},\mathfrak{r}}|_{G_{F_v}}$ is trivial and $r_{\mathrm{aux},\mathfrak{r}}|_{G_{F_v}}$ is crystalline ordinary.

Then $\mathrm{Sym}^{n-1}\mathcal{R}$ is weakly automorphic of level prime to X_0.

Proof. We first show that $\mathcal{S}_{\mathrm{aux}}$ is weakly automorphic of level prime to $X_0 \cup \{v|r\}$ by applying Theorem 3.2.1 to $s_{\mathrm{aux},\mathfrak{p}}$. To justify this, we need to check that $\overline{s}_{\mathrm{aux},\mathfrak{p}} \cong \overline{s}_{\mathrm{UA},\mathfrak{p}}$ satisfies the Taylor–Wiles conditions (as formulated in Definition 5.2.1) and that for each place $v|p$ of F, we have $s_{\mathrm{aux},\mathfrak{p}}|_{G_{F_v}} \sim s_{\mathrm{UA},\mathfrak{p}}|_{G_{F_v}}$. The Taylor–Wiles conditions hold by assumption (13) and Lemma 5.2.4. If $v \in \Sigma^{\mathrm{ord}}$, then $\overline{s}_{\mathrm{aux},\mathfrak{p}}|_{G_{F_v}}$ is trivial, and both $s_{\mathrm{aux},\mathfrak{p}} \cong (\mathrm{Sym}^{n-1} r_{\mathfrak{p}}) \otimes r_{\mathrm{aux},\mathfrak{p}}|_{G_{F_v}}$ and $s_{\mathrm{UA},\mathfrak{p}}|_{G_{F_v}}$ are crystalline ordinary, so Lemma 5.1.4 implies that $s_{\mathrm{aux},\mathfrak{p}}|_{G_{F_v}} \sim s_{\mathrm{UA},\mathfrak{p}}|_{G_{F_v}}$. If $v \in \Sigma^{\mathrm{ss}}$, then $\overline{s}_{\mathrm{aux},\mathfrak{p}}|_{G_{F_v}}$ is trivial and our assumptions imply that $\mathrm{Sym}^{n-1} r_{\mathfrak{p}}|_{G_{F_v}} \sim \rho_{n,m,0}|_{G_{F_v}}$,

$r_{\mathrm{aux},\mathfrak{p}}|_{G_{F_v}} \sim \rho_{m,1,0}|_{G_{F_v}}$ and $s_{\mathrm{UA},\mathfrak{p}}|_{G_{F_v}} \sim \rho_{nm,1,0}|_{G_{F_v}}$, hence

$$s_{\mathrm{aux},\mathfrak{p}}|_{G_{F_v}} \sim \rho_{n,m,0}|_{G_{F_v}} \otimes \rho_{m,1,0}|_{G_{F_v}} \cong \rho_{nm,1,0}|_{G_{F_v}} \sim s_{\mathrm{UA},\mathfrak{p}}|_{G_{F_v}}.$$

Therefore $\mathcal{S}_{\mathrm{aux}}$ is weakly automorphic of level prime to $X_0 \cup \{v|r\}$.

We next show that $\mathcal{S}_{\mathrm{CM}}$ is weakly automorphic of level prime to X_0 by applying Theorem 3.2.1 to $s_{\mathrm{CM},\mathfrak{r}}$. The Taylor–Wiles conditions for $\overline{s}_{\mathrm{CM},\mathfrak{r}} \cong \overline{s}_{\mathrm{aux},\mathfrak{r}}$ hold by assumption (13) and Lemma 5.2.6. To check the connectedness conditions, let $v|r$ be a place of F. Then $\overline{r}_{\mathrm{CM},\mathfrak{r}}|_{G_{F_v}} \cong \overline{r}_{\mathrm{aux},\mathfrak{r}}|_{G_{F_v}}$ is trivial and $r_{\mathrm{aux},\mathfrak{r}}|_{G_{F_v}}$ is crystalline ordinary, by assumption (17). Since r splits completely in E by assumption (12), v splits completely in L, and we can label the places $w_i|v$ so that $w_i|_E$ is the place induced by the embedding τ_i. There is an isomorphism

$$r_{\mathrm{CM},\mathfrak{r}}|_{G_{F_v}} \cong \oplus_{w|v}\alpha_i,$$

where for each $i = 1, \ldots, m$, $\alpha_i : G_{F_v} \to \overline{M}_{\mathfrak{r}}^{\times}$ is a continuous character with the property that for any $u \in \mathcal{O}_{F_v}^{\times}$, we have

$$\alpha_i(\mathrm{Art}_{F_v}(u)) = \prod_{\substack{\tau \in \mathrm{Hom}(L_{w_i}, \overline{M}_{\mathfrak{r}}) \\ \tau|_{F_0} = \tau_0}} \tau(u)^{-(m-i)}$$

if v lies above the place of F_0 induced by τ_0, and

$$\alpha_{w,i}(\mathrm{Art}_{F_v}(u)) = \prod_{\substack{\tau \in \mathrm{Hom}(L_{w_i}, \overline{M}_{\mathfrak{r}}) \\ \tau|_{F_0} = c\tau_0}} \tau(u)^{-(i-1)}$$

otherwise. It follows that $r_{\mathrm{CM},\mathfrak{r}}|_{G_{F_v}}$ is also crystalline ordinary, with Hodge–Tate weights $\{0, \ldots, m-1\}$ matching those of $r_{\mathrm{aux},\mathfrak{r}}|_{G_{F_v}}$. By Lemma 5.1.4, we have $r_{\mathrm{CM},\mathfrak{r}}|_{G_{F_v}} \sim r_{\mathrm{aux},\mathfrak{r}}|_{G_{F_v}}$, and using [BLGGT14, p. 530, (5)] it follows that

$$s_{\mathrm{CM},\mathfrak{r}}|_{G_{F_v}} = \mathrm{Sym}^{n-1} r_{\mathfrak{r}}|_{G_{F_v}} \otimes r_{\mathrm{CM},\mathfrak{r}}|_{G_{F_v}} \sim \mathrm{Sym}^{n-1} r_{\mathfrak{r}}|_{G_{F_v}} \otimes r_{\mathrm{aux},\mathfrak{r}}|_{G_{F_v}} = s_{\mathrm{aux},\mathfrak{r}}|_{G_{F_v}}.$$

We can now show that $\mathrm{Sym}^{n-1}\mathcal{R}$ is weakly automorphic of level prime to X_0. Let π be the regular algebraic, cuspidal automorphic representation of $\mathrm{GL}_{nm}(\mathbf{A}_F)$ which is associated to the compatible system $\mathcal{S}_{\mathrm{CM}}$. By construction, π is unramified at X_0. Let $\eta : F^{\times}\backslash\mathbf{A}_F^{\times} \to \mathbf{C}^{\times}$ be the character of order m associated to the inducing field L/F of $\mathcal{R}_{\mathrm{CM}}$. Then $\pi \cong \pi \otimes (\eta \circ \det)$, so by cyclic base change [AC89, Ch. 3, Thm 4.2], we deduce that π is the induction of a cuspidal automorphic representation Π for $\mathrm{GL}_n(\mathbf{A}_L)$, which by consideration of the infinity type of π must also be regular algebraic. More precisely, for any place w of L lying above a place v of F, we have

$$\mathrm{rec}_{F_v}(\pi)|_{W_{L_w}} = \oplus_{i=0}^{m-1} \mathrm{rec}_{L_w}(\Pi^{\sigma^i}),$$

where σ is a generator for $\mathrm{Gal}(L/F)$. Since L is CM and Π is regular algebraic, Π has an associated compatible system of l-adic Galois representations. If l is a prime and $\iota : \overline{\mathbf{Q}}_l \to \mathbf{C}$ is an isomorphism, with ι^{-1} inducing the place λ of M, then we find

$$r_\iota(\pi)|_{G_L} = \oplus_{i=0}^{m-1} \mathrm{Sym}^{n-1} r_\lambda|_{G_L} \otimes \Psi_\lambda^{\sigma^i} \cong \oplus_{i=0}^{m-1} r_\iota(\Pi^{\sigma^i}).$$

Choosing λ so that $\mathrm{Sym}^{n-1} r_\lambda$ is irreducible (e.g. $\lambda = \mathfrak{p}$), we find that $\mathrm{Sym}^{n-1} r_\lambda|_{G_L}$ is a character twist of $r_\iota(\Pi)$. Undoing the twist and making cyclic descent (using the irreducibility of $\mathrm{Sym}^{n-1} r_\lambda|_{G_L}$, as in [ACC+23, Proposition 6.5.13]) shows that $\mathrm{Sym}^{n-1}\mathcal{R}$ is weakly automorphic over F of level prime to X_0, as desired. □

The next theorem is proved by constructing the data required by Proposition 6.2.3 (after possibly extending the base field F).

Theorem 6.2.4. *Let F be an imaginary CM number field, and let*

$$\mathcal{R} = (M, S, \{Q_v(X)\}, \{r_\lambda\}, \{H_\tau\})$$

be a very weakly compatible system of rank 2 representations of G_F. Let $m \geq 2$ be an integer, and suppose that the following conditions are satisfied:

1. *For each τ, $H_\tau = \{0, m\}$.*
2. $\det r_\lambda = \varepsilon^{-m}$.
3. *$\mathcal{R}$ is strongly irreducible.*

Let $v_0 \notin S$ be a place of F. Then for each $n \geq 1$, there is a CM extension F_n/F, Galois over $\mathbf{Q}$, such that $\mathrm{Sym}^{n-1}\mathcal{R}|_{G_{F_n}}$ is weakly automorphic of level prime to $\{v|v_0\}$.

Proof. We can fix $n \geq 1$. Let p_0 denote the residue characteristic of v_0, let F_0 be an imaginary quadratic field, and let F_1 denote the Galois closure of $F \cdot F_0$ over $\mathbf{Q}$. Embed M in $\mathbf{C}$ arbitrarily, and let X_1 denote the set of places of F_1 lying above v_0. It suffices to prove the following statement:

- There exists a CM extension F'/F_1, Galois over $\mathbf{Q}$, such that (after possibly enlarging M) $\mathcal{R}|_{G_{F'}}$ satisfies the hypotheses of Proposition 6.2.3 with X_0 taken to be the set of places of F' lying above X_1.

Indeed, Proposition 6.2.3 will then imply that $\mathrm{Sym}^{n-1}\mathcal{R}|_{G_{F'}}$ is weakly automorphic of level prime to $X_0 = \{v|v_0\}$, which is what we need to prove. To prove this statement, we will consider a series of CM extensions F_{j+1}/F_j ($j = 1, 2, \dots$), each Galois over $\mathbf{Q}$. For any such extension F_j/F_1, $\mathcal{R}|_{G_{F_j}}$ satisfies Assumptions (1)–(5) of Proposition 6.2.3 with respect to X_j, the set of places of F_j lying above v_0. The extensions F_{j+1}/F_j will be chosen in order to satisfy the remaining assumptions.

Let $E/\mathbf{Q}$ be any totally real cyclic extension of degree m linearly disjoint from F_1, in which p_0 is unramified. (We can find such E by taking the degree m subfield of $\mathbf{Q}(\zeta_{p'})$, where p' is any sufficiently large prime $\equiv 1 \bmod 2m$.) Let $L_1 = E \cdot F_1$. For any extension F_j/F_1, we will set $L_j = E \cdot F_j$. Choose an odd prime $q_1 \nmid X_0$ which splits completely in L_1 and a place $v_1|q_1$ of F_1 which splits completely as $v_1 = w_1 \dots w_m$ in L_1. Fix an embedding $\tau_0 : F_0 \to \mathbf{C}$, and a labelling $\tau_1, \dots, \tau_m : E \cdot F_0 \to \mathbf{C}$ of the embeddings $E \cdot F_0 \to \mathbf{C}$ which extend τ_0. After enlarging M, using [HSBT10, Lemma 2.2], we can find a character $\Psi_0 : \mathbf{A}_{L_1}^\times \to M^\times$, unramified at the places above X_1, such that for each $\alpha \in L_1^\times$, we have

$$\Psi_0(\alpha) = \prod_{i=1}^{m} \tau_i(\mathbf{N}_{L_1/E\cdot F_0}(\alpha))^{m-i} c\tau_i(\mathbf{N}_{L_1/E\cdot F_0}(\alpha))^{i-1},$$

and moreover such that the characters $\Psi_0|_{\mathcal{O}_{L_{w_i}}^\times}$ ($i = 1, \dots, m$) are wildly ramified, pairwise distinct, and satisfy $\prod_{i=1}^m \Psi_0|_{\mathcal{O}_{L_{w_i}}^\times} = 1$ (where we identify $F_{v_1} = L_{w_i}$ for each i). If λ is a place of M, then $\varepsilon^{m(m-1)/2}\det \mathrm{Ind}_{G_{L_1}}^{G_{F_1}} \Psi_{0,\lambda}$ is a character of finite order which is unramified at v_0 and v_1. Using Lemma 5.3.3, and possibly enlarging M further, we can find a CM extension F_2/F_1, linearly disjoint from $E/\mathbf{Q}$ and Galois over $\mathbf{Q}$, and a twist $\Psi_2 : \mathbf{A}_{L_2}^\times \to M^\times$ of $\Psi_0 \circ \mathbf{N}_{L_2/L_1}$ by a character of $L_2^\times \backslash \mathbf{A}_{L_2}^\times$ of finite order, unramified above v_0 and v_1, such that for any place

λ of M, $\det \operatorname{Ind}_{G_{L_2}}^{G_{F_2}} \Psi_{2,\lambda} = \varepsilon^{-m(m-1)/2}$. (If $v_0|2$, then the twist whose existence is guaranteed from Lemma 5.3.3 may be ramified above X_2; if so, it is certainly ramified of finite order, and we enlarge F_2 further so that $\operatorname{Ind}_{G_{L_2}}^{G_{F_2}} \Psi_{2,\lambda}$ is unramified at all places above X_2 and L_2/F_2 is unramified at all places above X_2.) If F_j/F_2 is a finite extension, then we set $\Psi_j = \Psi_2 \circ \mathbf{N}_{F_j/F_2}$. Let $\mathcal{R}_{\mathrm{CM}} = \{\operatorname{Ind}_{G_{L_2}}^{G_{F_2}} \Psi_{2,\lambda}\} = \{r_{\mathrm{CM},\lambda}\}$.

We now choose any primes N, $p \in \mathcal{L}(\mathcal{R}|_{G_{F_2}})$, $r \in \mathcal{L}(\mathcal{R}|_{G_{F_2}})$ (cf. Lemma 6.1.5) not dividing v_0v_1 and satisfying the following conditions:

- $N > 100nm + 100$ and N is unramified in L_2 and M.
- $p \equiv -1 \bmod N$ and $p > 2nm + 1$.
- $r \equiv 1 \bmod N$ and $r > 2nm + 1$.
- p splits completely in L_2 and M and r splits completely in $L_2(\zeta_p)$ and M.
- The character Ψ_2 is unramified at the places of L above p and r.

Choosing $\mathfrak{p}|p$ and $\mathfrak{r}|r$ arbitrarily, there will be sandwiches up to conjugation

$$\mathrm{SL}_2(\mathbf{F}_p) \leq \overline{r}_{\mathfrak{p}}(G_{F_2}) \leq \mathrm{GL}_2(\mathbf{F}_p)$$

and

$$\mathrm{SL}_2(\mathbf{F}_r) \leq \overline{r}_{\mathfrak{r}}(G_{F_2}) \leq \mathrm{GL}_2(\mathbf{F}_r),$$

and for each p-adic (resp. r-adic) place v of F, $r_{\mathfrak{p}}|_{G_{F_v}}$ (resp. $r_{\mathfrak{r}}|_{G_{F_v}}$) is crystalline. (Here we are using the definition of $\mathcal{L}(\mathcal{R}|_{G_{F_2}})$ and the fact that p, r split in M.) The representation $\overline{r}_{\mathrm{CM},\mathfrak{r}}$ can be chosen to take values in $\mathrm{GL}_m(\mathbf{F}_r)$. Since the prime N is unramified in $L_2(\zeta_r)$, $E/\mathbf{Q}$ is linearly disjoint from $F_2(\zeta_N, \zeta_r)/\mathbf{Q}$. The different inertial behaviour of Ψ_0 at places dividing v_1 implies that $\overline{r}_{\mathrm{CM},\mathfrak{r}}|_{G_{F_2(\zeta_N,\zeta_r)}}$ is absolutely irreducible.

Let v be a p-adic place of F. Then $F_{2,v} = M_{\mathfrak{p}} = \mathbf{Q}_p$. By [Ber10, Théorème 3.2.1], either $r_{\mathfrak{p}}|_{G_{F_{2,v}}}$ is (crystalline) ordinary, or there is an isomorphism $\overline{r}_{\mathfrak{p}}|_{G_{\mathbf{Q}_{p^2}}} \cong \overline{\rho}_{2,m,0}$ (notation as in Definition 5.1.1). In the latter case, Lemma 5.1.3 shows that for any finite extension $K/\mathbf{Q}_{p^2}$, we have $r_{\mathfrak{p}}|_{G_K} \sim \rho_{2,m,0}|_{G_K}$. We write Σ_2^{ord} (resp. Σ_2^{ss}) for the set of p-adic places of F_2 such that $r_{\mathfrak{p}}|_{G_{F_{2,v}}}$ is (resp. is not) ordinary. If F_j/F_2 is a finite extension, then we write Σ_j^{ord} for the set of places of F_j lying above a place of Σ_2^{ord} (and define Σ_j^{ss} similarly).

Let $B/F_2(\zeta_N, \zeta_p, \zeta_r)$ be the extension cut out by $\overline{r}_{\mathfrak{p}} \times \overline{r}_{\mathfrak{r}} \times \overline{r}_{\mathrm{CM},\mathfrak{r}}$. We now choose a solvable CM extension $F_3/F_2(\zeta_N)$, Galois over $\mathbf{Q}$ and linearly disjoint from $B \cdot F_2/F_2(\zeta_N)$. Since $p \equiv -1 \bmod N$, for each place $v|p$ of F_3, F_v contains $\mathbf{Q}_{p^2}$. We moreover adjoin $e^{2\pi i/N}$ to M and extend $\mathfrak{p}$, $\mathfrak{r}$ arbitrarily to places of this enlarged M.

At this point we choose (for later use) a semistable elliptic curve $A/\mathbf{Q}$ with good reduction at p, r, and p_0. We choose a prime q with the following properties:

- $q > 2nm + 1$ and q splits in M. In particular, $q \equiv 1 \bmod N$. We choose a place $\mathfrak{q}|q$ of M.
- $\overline{\rho}_{A,q}(G_{F_3}) = \mathrm{GL}_2(\mathbf{F}_q)$ and A has good ordinary reduction at q.

Let B' denote the composite of B with the extension of F_3 cut out by $\overline{\rho}_{A,q}$.

Having chosen an integer N and extension $F_3/\mathbf{Q}(\zeta_N)$, we have access to the families of motives over $T_0 = \mathbf{P}^1_{F_3} - \{\mu_N, \infty\}$ constructed in §4. We will use the families of motives both of rank m and of rank nm. We write ${}_mW_{t,\lambda}$, ${}_{nm}W_{t,\lambda}$ for the $\mathcal{O}_{M_\lambda}[G_K]$-modules of ranks m, nm constructed in §4 associated to an extension

K/F_3 and point $t \in T_0(K)$. We claim that we can find a CM extension F_4/F_3, Galois over $\mathbf{Q}$ and linearly disjoint from $B' \cdot F_3/F_3$ such that for any place $v|prp_0q$ of F_4, the representations $\overline{r}_{\mathfrak{p}}|_{G_{F_{4,v}}}$, $\overline{r}_{\mathfrak{r}}|_{G_{F_{4,v}}}$, $\overline{r}_{\mathrm{CM},\mathfrak{r}}|_{G_{F_{4,v}}}$ and $\overline{\rho}_{A,q}|_{G_{F_{4,v}}}$ are all trivial, and the following additional data exists for $k \in \{m, nm\}$:

(i) If $v \in \Sigma_4^{\mathrm{ord}}$, then there is a non-empty open subset ${}_k\Omega_v \subset T_0(\mathcal{O}_{F_{4,v}})$ such that if $t \in {}_k\Omega_v$, then ${}_k\overline{W}_{t,\mathfrak{p}}$ is trivial and ${}_kW_{t,\mathfrak{p}}$ is crystalline ordinary. Moreover, ${}_k\overline{W}_{t,\mathfrak{r}}$ and ${}_k\overline{W}_{t,\mathfrak{q}}$ are both trivial.
(ii) If $v \in \Sigma_4^{\mathrm{ss}}$, then there is a non-empty open subset ${}_k\Omega_v \subset T_0(\mathcal{O}_{F_{4,v}})$ such that if $t \in {}_k\Omega_v$, then ${}_k\overline{W}_{t,\mathfrak{p}}$ and $\overline{\rho}_{k,1,0}|_{G_{F_{4,v}}}$ are trivial, ${}_kW_{t,\mathfrak{p}}$ is crystalline, and ${}_kW_{t,\mathfrak{p}} \sim \rho_{k,1,0}|_{G_{F_{4,v}}}$. Moreover, ${}_k\overline{W}_{t,\mathfrak{r}}$ and ${}_k\overline{W}_{t,\mathfrak{q}}$ are both trivial.
(iii) If $v|r$ is a place of F_4, then there is a non-empty open subset ${}_k\Omega_v \subset T_0(\mathcal{O}_{F_{4,v}})$ such that if $t \in \Omega_v$, then ${}_k\overline{W}_{t,\mathfrak{r}}$ is trivial and $W_{t,\mathfrak{r}}$ is crystalline ordinary. Moreover, ${}_k\overline{W}_{t,\mathfrak{q}}$ and ${}_k\overline{W}_{t,\mathfrak{p}}$ are both trivial.
(iv) If $v|p_0$ is a place of F_4, then there is a non-empty open subset ${}_k\Omega_v \subset T_0(\mathcal{O}_{F_{4,v}})$ such that if $t \in {}_k\Omega_v$, then ${}_k\overline{W}_{t,\mathfrak{r}}$, ${}_k\overline{W}_{t,\mathfrak{q}}$ and ${}_k\overline{W}_{t,\mathfrak{p}}$ are all trivial.
(v) If $v|q$ is a place of F_4, then there is a non-empty open subset ${}_k\Omega_v \subset T_0(\mathcal{O}_{F_{4,v}})$ such that if $t \in {}_k\Omega_v$, then ${}_k\overline{W}_{t,\mathfrak{q}}$ is trivial and ${}_kW_{t,\mathfrak{q}}$ is crystalline ordinary. Moreover, ${}_k\overline{W}_{t,\mathfrak{p}}$ and ${}_k\overline{W}_{t,\mathfrak{r}}$ are both trivial.

Indeed, we can take $F_4 = K^+ \cdot F_3$, where $K^+/\mathbf{Q}$ is a Galois, totally real extension with K_v^+ large enough for each place $v|prp_0q$, as we now explain, dropping the subscript k which is fixed for the next two paragraphs. For (i), we claim that it is enough to show that once $F_{4,v}$ is large enough, we can find a single point of $t \in T_0(F_{4,v})$ such that $\overline{W}_{t,\mathfrak{p}}$, $\overline{W}_{t,\mathfrak{r}}$, and $\overline{W}_{t,\mathfrak{q}}$ are all trivial and $W_{t,\mathfrak{p}}$ is crystalline ordinary. Indeed, by a version of Krasner's Lemma due to Kisin [Kis99, Theorem 5.1], for any $c > 0$ there exists an open ball U_t around t in $T_0(\mathcal{O}_{F_{4,v}})$, such that for any $t' \in U_t$, the pairs of representations $W_{\mathfrak{p},t}/(p^c)$, $W_{\mathfrak{p},t'}/(p^c)$ and $W_{\mathfrak{r},t}/(r^c)$, $W_{\mathfrak{r},t'}/(r^c)$ and $W_{\mathfrak{q},t}/(q^c)$, $W_{\mathfrak{q},t'}/(q^c)$ are isomorphic. By Lemma 5.1.5, we can choose $c > 1$ so that this forces $W_{\mathfrak{p},t} \sim W_{\mathfrak{p},t'}$, hence (by Lemma 5.1.4) that $W_{\mathfrak{p},t'}$ is crystalline ordinary. The existence of a crystalline ordinary point t follows from Proposition 4.2.6 and Proposition 4.2.5(2), after which we enlarge $F_{4,v}$ further if necessary to force the residual representations to be trivial. Then we take $\Omega_v = U_t$.

For (iii) and (v), the argument is essentially the same as case (i), while for (iv), it is even simpler. For (ii), we enlarge $F_{4,v}$ so that $\overline{\rho}_{k,1,0}|_{G_{F_{4,v}}}$ and $\overline{W}_{\mathfrak{p},0}|_{G_{F_{4,v}}}$ are trivial. By Lemma 5.1.3 and Lemma 4.4.4, we have $W_{\mathfrak{p},0}|_{G_{F_{4,v}}} \sim \rho_{k,1,0}|_{G_{F_{4,v}}}$. Employing the same argument as in the previous paragraph, using [Kis99, Theorem 5.1] and Lemma 5.1.5, we can find a non-empty open neighbourhood $\Omega_v \subset T_0(\mathcal{O}_{F_{4,v}})$ of $0 \in T_0(\mathcal{O}_{F_{4,v}})$ such that if $t \in \Omega_v$, then $W_{\mathfrak{p},t}$ is crystalline and $W_{\mathfrak{p},t} \sim W_{\mathfrak{p},0}|_{G_{F_{4,v}}}$. Since $\sim$ is a transitive relation, this leads to a choice of Ω_v with the desired property.

To construct the compatible system $\mathcal{R}_{\mathrm{aux}}$, we will apply Proposition 4.5.1. If $m = 2$ we can use a modular curve with level r-structure, and since the argument in this case is a straightforward (and considerably simpler) variant on the argument that we use if $m > 2$, we leave this case to the reader. In the case $m > 2$ we use the moduli space $T = T(\overline{r}_{\mathrm{CM},\mathfrak{r}}|_{G_{F_4}})$ defined in Remark 4.4.3, which is defined since $r \equiv 1 \bmod N$ and $\overline{r}_{\mathrm{CM},\mathfrak{r}}$ takes values in $\mathrm{GL}_m(\mathbf{F}_r)$, with determinant $\overline{\varepsilon}^{-m(m-1)/2}$. We take $F^{\mathrm{avoid}} = B' \cdot F_4$. We take the homomorphism $\pi_1^{\text{ét}}(T_{F_4}) \to \mathrm{GU}_m(\mathbf{F}_{p^2})$ to be the one associated to the local system $\overline{\mathcal{W}}_{\mathfrak{p}}$. We take $S_0 = \{p, r, p_0, q\}$. If v is a place lying above a prime in S_0, we take $L_v = F_{4,v}$ and Ω_v to be the pre-image in $T(F_{4,v})$ of the set ${}_m\Omega_v$. Note that Ω_v is certainly open, and it is non-empty because we have arranged that for each place $v|S_0$ of F_4, and for each $t \in {}_m\Omega_v$, $\overline{r}_{\mathrm{CM},\mathfrak{r}}|_{G_{F_{4,v}}}$ and $\overline{W}_{t,\mathfrak{r}}$ are both trivial (hence isomorphic!).

Proposition 4.5.1 now yields an imaginary CM extension F_5/F_4, Galois over $\mathbf{Q}$ and in which the places above S_0 all split completely, and a weakly compatible system $\{W_{t,\lambda}\}$ of representations of G_{F_5} with coefficients in $\mathbf{Q}(e^{2\pi i/N}) \subset M$. We take $\mathcal{R}_{\mathrm{aux}} = \{r_{\mathrm{aux},\lambda}\} = \{W_{t,\lambda}\}$ and note that the statement of Proposition 4.5.1 and the definition of the sets Ω_v imply that $\mathcal{R}_{\mathrm{aux}}$ has the following properties:

- $\overline{r}_{\mathrm{aux},\mathfrak{p}}(G_{F_5}) = \mathrm{GU}_m(\mathbf{F}_{p^2})$ (note we are assuming that $m > 2$).
- If $v \in \Sigma_5^{\mathrm{ord}}$, then $\overline{r}_{\mathrm{aux},\mathfrak{p}}|_{G_{F_{5,v}}}$ is trivial and $r_{\mathrm{aux},\mathfrak{p}}|_{G_{F_{5,v}}}$ is crystalline ordinary.
- If $v \in \Sigma_5^{\mathrm{ss}}$, then $\overline{r}_{\mathrm{aux},\mathfrak{p}}|_{G_{F_{5,v}}}$ is trivial and $r_{\mathrm{aux},\mathfrak{p}}|_{G_{F_{5,v}}} \sim \rho_{m,1,0}|_{G_{F_{5,v}}}$.
- S_{aux} is disjoint from $X_5 \cup \{v|pr\}$. (Use Proposition 4.2.5.)
- There is an isomorphism $\overline{r}_{\mathrm{aux},\mathfrak{r}} \cong \overline{r}_{\mathrm{CM},\mathfrak{r}}|_{G_{F_5}}$. For each place $v|r$ of F_5, $\overline{r}_{\mathrm{aux},\mathfrak{r}}|_{G_{F_{5,v}}}$ is trivial and $r_{\mathrm{aux},\mathfrak{r}}|_{G_{F_{5,v}}}$ is crystalline ordinary.

We set $\mathcal{S}_{\mathrm{aux}} = (\mathrm{Sym}^{n-1}\mathcal{R}|_{G_{F_5}}) \otimes \mathcal{R}_{\mathrm{aux}}$, and now construct $\mathcal{S}_{\mathrm{UA}}$. The places $v|prp_0q$ split in F_5/F_4, so if v is a place of F_5 dividing prp_0q we may define ${}_k\Omega_v = {}_k\Omega_{v|_{F_4}}$ to keep in hand the data (i)–(v) defined above. We will apply Proposition 4.5.1 to the moduli space

$$T = T(\overline{s}_{\mathrm{aux},\mathfrak{p}}, \mathrm{Sym}^{nm-1}\overline{\rho}_{A,q}|_{G_{F_5}}).$$

We take $F^{\mathrm{avoid}} = B' \cdot F_5$. We do not specify a homomorphism f. We take $S_0 = \{p, r, p_0, q\}$. If v is a place lying above a prime in S_0, we take $L_v = F_{5,v}$ and Ω_v to be the pre-image in $T(F_{5,v})$ of the set ${}_{nm}\Omega_v$. Once again, this pre-image is non-empty because we have trivialized all of the relevant local residual representations. (Since $p \equiv -1 \bmod N$, the definition of T involves a choice of Hermitian structure. We are therefore invoking the fact here that over a finite field, any two Hermitian spaces of the same dimension are isomorphic.) Proposition 4.5.1 then yields a CM extension F_6/F_5, Galois over $\mathbf{Q}$, and a point $t \in T(F_6)$ corresponding to a weakly compatible system $\mathcal{S}_{\mathrm{UA}} = \{s_{\mathrm{UA},\lambda}\} = \{W_{t,\lambda}\}$ of rank nm representations of G_{F_6} with the following properties:

- There are isomorphisms $\overline{s}_{\mathrm{UA},\mathfrak{p}} \cong \overline{s}_{\mathrm{aux},\mathfrak{p}}|_{G_{F_6}}$ and $\overline{s}_{\mathrm{UA},\mathfrak{q}} \cong \mathrm{Sym}^{nm-1}\overline{\rho}_{A,q}|_{G_{F_6}}$.
- If $v \in \Sigma_6^{\mathrm{ord}}$, then $\overline{s}_{\mathrm{UA},\mathfrak{p}}|_{G_{F_{6,v}}}$ is trivial and $s_{\mathrm{UA},\mathfrak{p}}|_{G_{F_{6,v}}}$ is crystalline ordinary.
- If $v \in \Sigma_6^{\mathrm{ss}}$, then $\overline{s}_{\mathrm{UA},\mathfrak{p}}|_{G_{F_{6,v}}}$ is trivial and $s_{\mathrm{UA},\mathfrak{p}}|_{G_{F_{6,v}}} \sim \rho_{nm,1,0}|_{G_{F_{6,v}}}$.
- For each place $v|q$ of F_6, $\overline{s}_{\mathrm{UA},\mathfrak{p}}|_{G_{F_{6,v}}}$ is trivial and $s_{\mathrm{UA},q}|_{G_{F_{6,v}}}$ is crystalline ordinary.
- S_{UA} is disjoint from $X_6 \cup \{v|pr\}$.

We now claim that Assumptions (1)–(17) of Proposition 6.2.3 are satisfied for the compatible system $\mathcal{R}|_{G_{F_6}}$, set $X_0 = X_6$ of places of F_6, and auxiliary compatible systems $\mathcal{R}_{\mathrm{CM}}|_{G_{F_6}}$, $\mathcal{R}_{\mathrm{aux}}|_{G_{F_6}}$, and $\mathcal{S}_{\mathrm{UA}}$ (defined over F_6 by construction). Let us verify these assumptions in turn.

- As already observed, (1)–(5) are automatically satisfied.
- We take $\Psi = \Psi_6$. The extension $E/\mathbf{Q}$ is linearly disjoint from F_6 because $E \leq B$, while Ψ has the given infinity type, so (6) is satisfied.
- The primes p, r are prime to S by construction, so (7) is satisfied.
- $\mathcal{R}_{\mathrm{aux}}|_{G_{F_6}}$ has the claimed properties by construction, so (8) is satisfied. The same is true for $\mathcal{S}_{\mathrm{UA}}$, except we need to justify the fact that $\mathcal{S}_{\mathrm{UA}}$

is weakly automorphic of level prime to $X_6 \cup \{v|pr\}$. Note that the q-adic representation $\mathrm{Sym}^{nm-1} \rho_{A,q}|_{G_{F_6}}$ is automorphic by the combination of the main results of [BCDT01, Die15, NT22, AC89] (or alternately by [CNT23]), associated to a regular algebraic, cuspidal automorphic representation of $\mathrm{GL}_{nm}(\mathbf{A}_{F_6})$ which is ι-ordinary with respect to any isomorphism $\iota : \overline{\mathbf{Q}}_q \to \mathbf{C}$. (We could also verify the automorphy, at the cost of further extending the field F_6, by a further application of Proposition 4.5.1 as is done in [ACC+23].) We would now like to apply [MT23, Theorem 1.3] to conclude that $\mathcal{S}_{\mathrm{UA}}$ is weakly automorphic of level prime to $X_6 \cup \{v|pr\}$ (noting that the cited result includes the conclusion that the automorphic representation witnessing the weak automorphy of $\mathcal{S}_{\mathrm{UA}}$ is unramified at any place where both $\rho_{A,q}$ and $s_{\mathrm{UA},\mathfrak{q}}$ are unramified). We must verify that $\mathrm{Sym}^{nm-1} \overline{\rho}_{A,q}|_{G_{F_6}}$ satisfies the Taylor–Wiles conditions (as formulated in Definition 5.2.1). By Lemma 5.2.2, it suffices to check that $\mathrm{Sym}^{nm-1} \overline{\rho}_{A,q}$ satisfies these conditions (as a representation of $G_{\mathbf{Q}}$), and this follows from the definitions, together with an application of [GHTT12, Theorem A.9] (using our assumption $q > 2nm + 1$).

- L/F and Ψ are unramified above $X_0 \cup \{v|pr\}$ by construction, so (10) is satisfied.
- We have chosen the primes p, r so that $p > 2nm + 1$ and $r > 2nm + 1$. At each step the extension F_{j+1}/F_j has been chosen linearly disjoint from $L_j(\zeta_p, \zeta_r)$, so (11) and (12) are satisfied.
- The images $\overline{r}_{\mathfrak{p}}(G_{F_2})$, $\overline{r}_{\mathfrak{r}}(G_{F_2})$ and $\overline{r}_{\mathrm{aux},\mathfrak{p}}(G_{F_5})$ are large by construction, and at each step the extension F_{j+1}/F_j has been chosen so that the image does not change on restriction to the smaller Galois group. Moreover, $\overline{r}_{\mathrm{CM},\mathfrak{r}}|_{G_{F_2(\zeta_r)}}$ is irreducible, and again the analogous property holds over F_6 by construction. Therefore (13) is satisfied.
- Assumptions (14)–(17) hold by construction.

This completes the proof. □

7. Applications

7.1. **The Ramanujan Conjecture.** We are now in a position to prove the (more general verisons of the) main theorems of the introduction as a consequence of Theorem 6.2.1. Let F be an imaginary CM field, and let π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$. We write $(a_\tau \geq b_\tau)_{\tau : F \hookrightarrow \mathbf{C}}$ for the weight of π. Recall that we say that π is *of parallel weight* if $a_\tau - b_\tau$ is independent of τ.

Theorem 7.1.1. *Let F be an imaginary CM field, and let π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$ of parallel weight. Then, for all primes v of F, the representation π_v is (essentially) tempered.*

Proof. Since π is assumed to have parallel weight, there is an integer $m \geq 1$ such that $a_\tau - b_\tau = m - 1$ for all $\tau : F \hookrightarrow \mathbf{C}$. By Clozel's purity lemma [Clo90, Lemma 4.9], there is an integer w with $a_\tau + b_{c\tau} = w$ for all τ. It follows that $b_\tau + b_{c\tau} = w - m + 1$ is independent of τ. In particular, there exists an algebraic Hecke character of $\mathbf{A}_F^\times$ with weight $(b_\tau)_{\tau : F \hookrightarrow \mathbf{C}}$, so after twisting we may assume that $(a_\tau, b_\tau) = (m - 1, 0)$ for all τ. The central character of π is then of the form $\psi |\cdot|^{1-m}$ for a finite order Hecke character ψ.

Exactly as in the proof of [ACC$^+$23, Theorem 7.1.1], we can find a quadratic CM extension F'/F for which the character $\psi \circ N_{F'/F}$ is a square. We can check temperedness after base change to F'. Twisting by a finite order Hecke character, we may then assume that π has central character $|\cdot|^{1-m}$. Exactly as in the proof of [ACC$^+$23, Cor. 7.1.15], we can make a further solvable base change to reduce to checking temperedness of the unramified π_v. Hence it suffices to show that the associated very weakly compatible system $\mathcal{R}$ (cf. [ACC$^+$23, Lemma 7.1.10]) is pure. By [ACC$^+$23, Lemma 7.1.2], either $\mathcal{R}$ is strongly irreducible, Artin up to twist, or induced from a quadratic extension. If $\mathcal{R}$ is induced, then purity follows from the purity of rank one (very weakly) compatible systems. The compatible family $\mathcal{R}$ cannot be Artin up to twist because that is incompatible with having distinct Hodge–Tate weights. Thus $\mathcal{R}$ is strongly irreducible, and the result follows from Theorem 6.2.1. □

7.2. **The potential automorphy of compatible systems and the Sato–Tate conjecture.**

Theorem 7.2.1. *Let F be a CM field, and let $\mathcal{R} = (M, S, \{Q_v(X)\}, r_\lambda, H_\tau)$ be a very weakly compatible system of rank 2 representations of G_F that is strongly irreducible. Suppose there exists an integer $m \geq 1$ such that $H_\tau = \{0, m\}$ for each embedding $\tau : F \to \overline{M}$. Then $\mathcal{R}$ is pure of weight m, and for each $n \geq 1$, there exists a finite CM extension F'/F, Galois over $\mathbf{Q}$, such that $\operatorname{Sym}^{n-1} \mathcal{R}|_{G_{F'}}$ is automorphic.*

If one alternatively assumes that $\mathcal{R}$ is irreducible but not strongly irreducible, then $\mathcal{R}$ is pure of weight m, and for each $n \geq 1$, $\operatorname{Sym}^{n-1} \mathcal{R}$ decomposes as a direct sum of compatible systems of dimension at most 2 which are automorphic.

Proof. Assume that $\mathcal{R}$ is strongly irreducible. As in the proof of Theorem 7.1.1, we can reduce to the case where $\mathcal{R}$ has determinant ε^{-m}. But now Theorem 7.2.1 follows directly from Theorem 6.2.1.

If $\mathcal{R}$ is not strongly irreducible, then from [ACC$^+$23, Lemma 7.1.2] it follows that $\mathcal{R}$ is induced from a compatible system of algebraic Hecke characters for some quadratic extension F'/F (the condition on the Hodge–Tate weights ensures that $\mathcal{R}$ is not Artin up to twist). Then the symmetric powers $\operatorname{Sym}^{n-1} \mathcal{R}$ decompose as a sum of two-dimensional induced compatible systems and (when n is odd) a one-dimensional compatible system. In particular for any n, $\operatorname{Sym}^{n-1} \mathcal{R}$ decomposes as a direct sum of automorphic compatible systems, and the purity statement follows from the purity of (the Galois representations associated to) algebraic Hecke characters. □

We next give a statement of the Sato–Tate conjecture, including Theorem B as a special case, before giving the proof when π has parallel weight. Let F be an imaginary CM field, and let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$ which is regular algebraic of weight λ and not CM (i.e. not automorphically induced). Thus there is an integer w such that $\lambda_{\tau,1} + \lambda_{\tau c,2} = w$ for all $\tau \in \operatorname{Hom}(F, \mathbf{C})$. The central character of π has the form $\omega_\pi = |\cdot|^{-w}\psi$, where $\psi : F^\times \backslash \mathbf{A}_F^\times \to \mathbf{C}^\times$ is a unitary Hecke character of type A_0. We define the Sato–Tate group of π, $\mathrm{ST}(\pi)$, as follows:

- If ψ has finite order $a \geq 1$, then $\mathrm{ST}(\pi) = \mathrm{U}_2(\mathbf{R})_a := \{g \in \mathrm{U}_2(\mathbf{R}) \mid \det(g)^a = 1\}$.

- If ψ has infinite order, then $\mathrm{ST}(\pi) = \mathrm{U}_2(\mathbf{R})$.

Lemma 7.2.2. *$\mathrm{ST}(\pi)$ is a compact subgroup of $\mathrm{GL}_2(\mathbf{C})$. If v is a finite place of F such that π_v is unramified and essentially tempered, then the $\mathrm{GL}_2(\mathbf{C})$-conjugacy class of $q_v^{-w/2}\operatorname{rec}_{F_v}(\pi_v)(\mathrm{Frob}_v)$ intersects $\mathrm{ST}(\pi)$ in a unique conjugacy class of $\mathrm{ST}(\pi)$.*

Proof. The group $\mathrm{ST}(\pi)$ is compact since $\mathrm{U}_2(\mathbf{R})$ is. It is well-known that two elements of $\mathrm{U}_2(\mathbf{R})$ which become conjugate in $\mathrm{GL}_2(\mathbf{C})$ are conjugate by an element of $\mathrm{SU}_2(\mathbf{R})$. All we need to show then is that if v is a finite place of F such that π_v is unramified, and $\operatorname{rec}_{F_v}(\pi_v)(\mathrm{Frob}_v) = \mathrm{diag}(\alpha_v, \beta_v)$, then α_v, β_v are complex numbers of absolute value $q_v^{w/2}$, and further if ψ has finite order a then $(q_v^{-w}\alpha_v\beta_v)^a = 1$.

Since π_v is essentially tempered, we have $|\alpha_v| = |\beta_v|$. On the other hand, we have $\alpha_v\beta_v = \psi(\varpi_v)q_v^w$, hence $|\alpha_v\beta_v| = q_v^w$ (as ψ is unitary), and if ψ has finite order a then $(q_v^{-w}\alpha_v\beta_v)^a = 1$. □

If v is a place such that π_v is unramified and essentially tempered, then we write $[\pi_v] \in \mathrm{ST}(\pi)$ for a representative of the conjugacy class of $q_v^{-w/2}\operatorname{rec}_{F_v}(\pi_v)(\mathrm{Frob}_v) \in \mathrm{GL}_2(\mathbf{C})$.

Theorem 7.2.3. *Suppose that π has parallel weight. Let S_π denote the set of finite places of F at which π is unramified. With notation as above, the classes of elements $[\pi_v] \in \mathrm{ST}(\pi)$ ($v \notin S_\pi$) are equidistributed with respect to the Haar probability measure μ_{ST} of $\mathrm{ST}(\pi)$. More precisely, for any continuous, conjugation-invariant function $f : \mathrm{ST}(\pi) \to \mathbf{C}$, we have*

$$\lim_{X\to\infty} \frac{\sum_{v\notin S_\pi, q_v<X} f([\pi_v])}{\#\{v \notin S_\pi, q_v < X\}} = \int_{g\in\mathrm{ST}(\pi)} f(g)\, d\mu_{\mathrm{ST}(\pi)}.$$

Proof. If ρ is a finite-dimensional irreducible representation of $\mathrm{ST}(\pi)$, let us define

$$L^{S_\pi}(\pi, \rho, s) = \prod_{v\notin S_\pi} \det(1 - q_v^{-s}\rho([\pi_v]))^{-1},$$

an Euler product which converges absolutely in the right half-plane $\mathrm{Re}(s) > 1$. According to the criterion of Serre [Ser98, Ch. I, Appendix], the theorem will be proved if we can show that for each non-trivial such ρ, $L^{S_\pi}(\pi, \rho, s)$ admits a meromorphic continuation to $\mathbf{C}$ which is holomorphic and non-vanishing on the line $\mathrm{Re}(s) = 1$. This may be deduced from the potential automorphy of the symmetric powers $\mathrm{Sym}^{n-1}\mathcal{R}$ of the compatible system associated to π, exactly as in e.g. [Gee09, §7] and [BLGHT11, §8], after noting that $\mathcal{R}$ is strongly irreducible (again invoking [ACC+23, Lemma 7.1.2] and the assumption that π is not CM). Note also that the list of non-trivial one-dimensional representations of $\mathrm{ST}(\pi)$ depends on the order of the character ψ. □

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