Solutions to Exercises II

1. X = number of detectives in sample of size 10.

   X is binomial with $n = 10$ and $p = 0.04$.

$$\begin{aligned}
\mathrm{P}(X \geq 2) &= 1 - [\mathrm{P}(X=0) + \mathrm{P}(X=1)] \\
&= 1 - \left[0.96^{10} + 10 \times 0.04^{1} \times 0.96^{9}\right] = \boxed{0.0582}. \\
\mathrm{P}(X \geq 6) &= \mathrm{P}(X=6) + \mathrm{P}(X=7) + \mathrm{P}(X=8) + \mathrm{P}(X=9) + \mathrm{P}(X=10) \\
&= \binom{10}{6} 0.04^{6} \times 0.96^{4} + \ldots \\
&= 7 \times 10^{-7} + \text{ smaller terms}
\end{aligned}$$

   Therefore, with $p = 0.04$, the probability of obtaining 6 or more defectives is very small. If we were to obtain such a sample, we might doubt whether p was indeed as small as 0.04.

2. X = number of cored samples that have the required density.

   X is binomial with $n = 5$ and $p = 0.8$.

$$\begin{aligned}
\mathrm{P}(\text{subgrade acceptable}) &= \mathrm{P}(\geq 4) = \mathrm{P}(X=4) + \mathrm{P}(X=5) \\
&= \binom{5}{4} 0.8^{4} \times 0.2^{1} + \binom{5}{5} 0.8^{5} \times 0.2^{0} \\
&= 0.4096 + 0.3277 = \boxed{0.7373}
\end{aligned}$$

3. Rate is 1 in every 200m.

   i. Let X = number of imperfections in a 200m. length.
      X is Poisson with mean 1:

$$\mathrm{P}(X = x) = \begin{cases} \frac{e^{-1}1^{x}}{x!} & x = 0, 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

      $\mathrm{P}(X = 0) = e^{-1} = \boxed{0.3679}$.

   ii. Let Y = number of imperfections in a 400m. length.
      Y is Poisson with mean 2:

$$\mathrm{P}(Y = y) = \begin{cases} \frac{e^{-2}2^{y}}{y!} & y = 0, 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

      $\mathrm{P}(Y = 0) = e^{-2} = \boxed{0.1353}$.

   Let Z= number of lengths that are free from imperfections.

   Z is binomial with $n = 4$ and $p = 0.1353$.

$$\begin{aligned}
&\mathrm{P}(\text{only one length is free from imperfections}) = \mathrm{P}(Z = 1) \\
&= \binom{4}{1} 0.1353^{1}(1 - 0.1353)^{3} = \boxed{0.3499}.
\end{aligned}$$

4. X is exponential with parameter $\lambda = \frac{1}{3}$.

$$\mathrm{F}_X(x_0) = \mathrm{P}(X < x_0) = \int_0^{x_0} \frac{1}{3} e^{-\frac{x}{3}}\, \mathrm{d}x = \left[-e^{-\frac{x}{3}}\right]_0^{x_0} = 1 - e^{-\frac{x_0}{3}}$$

(a) $\mathrm{P}(X < 2) = \mathrm{F}_X(2) = 1 - e^{-\frac{2}{3}} = \boxed{0.4866}$.

(b)

$$\begin{aligned} \mathrm{P}(4 < X < 7) &= \mathrm{F}_X(7) - \mathrm{F}_X(4) \\ &= \left(1 - e^{-\frac{7}{3}}\right) - \left(1 - e^{-\frac{4}{3}}\right) = \boxed{0.1666}. \end{aligned}$$

(c) $\mathrm{P}(X > 4) = 1 - \mathrm{F}_X(4) = e^{-\frac{4}{3}} = \boxed{0.2636}$.

(d)

$$\begin{aligned} \mathrm{P}(X < 7 | X > 4) &= \frac{\mathrm{P}(X < 7 \cap X > 4)}{\mathrm{P}(X > 4)} \\ &= \frac{\mathrm{P}(4 < X < 7)}{\mathrm{P}(X > 4)} \\ &= \frac{0.1666}{0.2636} = \boxed{0.6320}. \end{aligned}$$

Note: due to the memoryless property of the exponential distribution, this probability is equal to

$$\mathrm{P}(X < 3) = F_X(3) = 1 - e^{-\frac{3}{3}} = 0.6321.$$

(e) Mean of exponential $= \mathrm{E}(X) = \lambda^{-1} = 3$ minutes.

5. Let $X=$ size of outer diameter.

$$X \sim \mathrm{N}(2.30, 0.06) \quad \text{so} \quad Y = \frac{X - 2.30}{0.06} \sim \mathrm{N}(0, 1).$$

Lower specification = 2.31 - 0.10 = 2.21cm. (SCRAP).

Upper specification = 2.31 + 0.10 = 2.41cm. (REWORK).

(a) Percentage of scrap:

$$\begin{aligned} \mathrm{P}(X < 2.21) &= \mathrm{P}\left(Y < \frac{2.21 - 2.30}{0.06}\right) = \mathrm{P}(Y < -1.50) \\ &= \Phi(-1.50) = 1 - \Phi(1.50) = 1 - 0.9332 \\ &= 0.0668. \quad \text{i.e. percentage is } \boxed{6.68\%}. \end{aligned}$$

(b) Percentage reworkable:

$$\begin{aligned} \mathrm{P}(X > 2.41) &= \mathrm{P}\left(Y > \frac{2.41 - 2.30}{0.06}\right) = \mathrm{P}(Y > 1.83) \\ &= 1 - \Phi(1.83) = 1 - 0.9664 \\ &= 0.0336. \quad \text{i.e. percentage is } \boxed{3.36\%}. \end{aligned}$$

(c) If the mean of X were $\mu = 2.31$cm., then we want

$$\mathrm{P}(2.21 \leq X \leq 2.41).$$

Let

$$Y = \frac{X - 2.31}{0.06} \quad \text{then} \quad Y \sim \mathrm{N}(0,1).$$

$$\begin{aligned}
\mathrm{P}(2.21 \leq X \leq 2.41) &= \mathrm{P}\left(\frac{2.21 - 2.31}{0.06} \leq Y \leq \frac{2.41 - 2.31}{0.06}\right) \\
&= \mathrm{P}(-1.67 \leq Y \leq 1.67) \\
&= \Phi(1.67) - \Phi(-1.67) = \Phi(1.67) - (1 - \Phi(1.67)) \\
&= 0.9525 - (1 - 0.9525) = 0.905.
\end{aligned}$$

i.e. percentage is $\boxed{9.05\%}$, compared with 6.86+3.36 = 10.22% previously.

6. X = diameter of piston, $X \sim \mathrm{N}(10.42, 0.03^2)$, $\mu_X = 10.42, \sigma_X = 0.03$.
Y = diameter of cylinder, $Y \sim \mathrm{N}(10.52, 0.04^2)$, $\mu_Y = 10.52, \sigma_X = 0.04$.

X and Y are independent, so $Z = Y - X$ is also normal with mean
$\mu_Z = E(Y - X) = \mu_Y - \mu_X = 0.1$ and variance,
$\sigma_Z^2 = \mathrm{var}(Y - X) = \sigma_X^2 + \sigma_Y^2 = 0.03^2 + 0.04^2 = 0.0025 \Rightarrow \sigma_Z = 0.05$.

Proportion of pistons that will not fit into cylinders is given by

$$\begin{aligned}
\mathrm{P}(Z < 0) &= \mathrm{P}\left(\frac{Z - 0.1}{0.05} < \frac{0 - 0.1}{0.05}\right) = \mathrm{P}\left(\frac{Z - 0.1}{0.05} < -2\right) \\
&= \Phi(-2) = 1 - \Phi(2) = \boxed{0.0228}.
\end{aligned}$$

We now have a binomial experiment with $n = 100$ and probability of a pair fitting $p = 1 - 0.0228 = 0.9772$.

Let X = number of pairs that fit.

(a) $\mathrm{P}(X = 100) = \binom{100}{100}(0.9772)^{100}(0.0228)^0 = \boxed{0.0996}$.

(b)

$$\begin{aligned}
\mathrm{P}(\text{less than 2 will fail to fit}) &= \mathrm{P}(X \geq 99) = \mathrm{P}(X = 99) + \mathrm{P}(X = 100) \\
&= \binom{100}{99}(0.9772)^{99}(0.0228)^1 + 0.0996 \\
&= 0.2324 + 0.0996 = \boxed{0.3320}.
\end{aligned}$$

7. **Hints:**

(a) For a Poisson distribution $\mathrm{P}(X = 0) = e^{-\mu}$, we can estimate $\mathrm{P}(X = 0)$ from the data as $\frac{58}{100}$, so one estimate of μ could be taken as $\boxed{-\log_e(0.58)}$.

(b) Could also estimate μ using the mean of the data $= (33 \times 1 + 7 \times 2 + 2 \times 3)/100 = \boxed{0.53}$.

(c) Then use the estimate of μ to calculate $\mathrm{P}(X = i)$ for $i = 0, 1, 2, 3, > 3$, from the definition of the Poisson distribution.