Solutions to Exercises I

1. (a) $A \cap B' \cap C'$
   (b) $A \cap B \cap C'$
   (c) $A \cap B \cap C$
   (d) $A \cup B \cup C$
   (e) $(A \cap B \cap C') \cup (A \cap B' \cap C) \cup (A' \cap B \cap C) \cup (A \cap B \cap C)$
   (f) $(A \cap B' \cap C') \cup (A' \cap B \cap C') \cup (A' \cap B' \cap C)$
   (g) $(A \cap B \cap C') \cup (A \cap B' \cap C) \cup (A' \cap B \cap C)$
   (h) $A' \cap B' \cap C'$
   (i) $(A \cap B \cap C)'$

2. (a) $1 - \left(\frac{5}{6}\right)^4 = 0.5177.$
   (b) $1 - \left(\frac{35}{36}\right)^{24} = 0.4914.$
   $\rightarrow$ (a) is more probable.

3. (a) $S = \{(i,j): \quad i,j = 1,2,\ldots,6\}.$
   (b)

$$B = \{(i,j): \quad i = 1,2,3 \ \text{ and } \ j = 1,2,\ldots,6\}.$$

$$C = \{(i,j): \quad i+j = 6 \ \text{ and } \ i,j = 1,2,\ldots,6\}.$$

$$\mathrm{P}((i,j)) = \frac{1}{36} \ \text{ for } \ i,j = 1,2,\ldots,6.$$

$$\mathrm{P}(B) = \frac{18}{36} = \boxed{\tfrac{1}{2}} \quad \mathrm{P}(C) = \boxed{\tfrac{5}{36}}.$$

$$\begin{aligned} B \cap C &= \{(1,5),(2,4),(3,3)\} \\ \mathrm{P}(B \cap C) &= \frac{3}{36} = \frac{1}{12} \end{aligned}$$

$$\begin{aligned} \mathrm{P}(C|B) &= \frac{\mathrm{P}(B \cap C)}{\mathrm{P}(B)} = \frac{\frac{1}{12}}{\frac{1}{2}} = \boxed{\tfrac{1}{6}} \\ \mathrm{P}(B|C) &= \frac{\mathrm{P}(B \cap C)}{\mathrm{P}(C)} = \frac{\frac{1}{12}}{\frac{5}{36}} = \boxed{\tfrac{3}{5}} \end{aligned}$$

4. (a) $S = \{1,2,\ldots,n-r+1\}$
   (b) $S = \{r, r+1, \ldots, n\}$

5. Let S = event that it snows tomorrow,
   R = event that it rains tomorrow,
   L = event that I am late.

$$\mathrm{P}(S) = \frac{3}{5}, \quad \mathrm{P}(R) = \frac{2}{5}, \quad \mathrm{P}(L|R) = \frac{1}{5}, \quad \mathrm{P}(L|S) = \frac{3}{5}.$$

$$\begin{aligned} \mathrm{P}(L) &= \mathrm{P}(L|R)\mathrm{P}(R) + \mathrm{P}(L|S)\mathrm{P}(S) \\ &= \frac{1}{5} \times \frac{2}{5} + \frac{3}{5} \times \frac{3}{5} = \boxed{\tfrac{11}{25}} \end{aligned}$$

6. (a)

$$\begin{aligned}
\mathrm{P}(\text{car makes a turn}) &= \mathrm{P}(L \cup R) = \mathrm{P}(L) + \mathrm{P}(R) \quad \text{(since disjoint)} \\
&= \tfrac{1}{3} + \tfrac{1}{3} = \boxed{\tfrac{2}{3}}.
\end{aligned}$$

$$\begin{aligned}
\mathrm{P}(L|\text{makes a turn}) &= \frac{\mathrm{P}(\text{makes a turn}|L)\mathrm{P}(L)}{\mathrm{P}(\text{makes a turn})} \\
&= \frac{1\times\frac{1}{3}}{\frac{2}{3}} = \boxed{\tfrac{1}{2}}.
\end{aligned}$$

(b)

$$\begin{aligned}
\mathrm{P}(\text{at least one car turns left}) &= 1 - \mathrm{P}(\text{neither car turns left}) \\
&= 1 - \left(\tfrac{2}{3}\right)^2 = \boxed{\tfrac{5}{9}}
\end{aligned}$$

Let A = event that at least one car makes a turn
Let B = event that at least one car turns left

$$\begin{aligned}
\mathrm{P}(B|A) &= \frac{\mathrm{P}(A|B)\mathrm{P}(B)}{\mathrm{P}(A)} \\
&= \frac{1 \times \frac{5}{9}}{\mathrm{P}(A)}
\end{aligned}$$

Let C_i = event that car i makes a turn, $i = 1, 2$.

$$\begin{aligned}
\mathrm{P}(A) = \mathrm{P}(\text{at least one car makes a turn}) &= 1 - \mathrm{P}(\text{neither car makes a turn}) \\
&= 1 - \mathrm{P}(C_1' \cap C_2') \\
&= 1 - \mathrm{P}(C_1')\mathrm{P}(C_2') \quad \text{(independent)} \\
&= 1 - \tfrac{1}{3} \times \tfrac{1}{3} = \tfrac{8}{9}.
\end{aligned}$$

Alternatively,

$$\begin{aligned}
\mathrm{P}(\text{at least one car makes a turn}) &= \mathrm{P}(\text{car 1 makes a turn}) + \\
&\quad \mathrm{P}(\text{car 2 makes a turn}) - \mathrm{P}(\text{both cars make a turn}) \\
&= \mathrm{P}(C_1) + \mathrm{P}(C_2) - \mathrm{P}(C_1 \cap C_2) \\
&= \mathrm{P}(C_1) + \mathrm{P}(C_2) - \mathrm{P}(C_1) \times \mathrm{P}(C_2) \quad \text{(independent)} \\
&= \tfrac{2}{3} + \tfrac{2}{3} - \left(\tfrac{2}{3}\right)^2 = \tfrac{8}{9}.
\end{aligned}$$

So,

$$\begin{aligned}
\mathrm{P}(B|A) &= \frac{\frac{5}{9}}{\mathrm{P}(A)} \\
&= \frac{5}{9} \times \frac{9}{8} = \boxed{\tfrac{5}{8}}
\end{aligned}$$

7. Let O_i = output of resistor i, $i = 1, 2$.

$$\mathrm{P}(O_i < 14) = 0.05, \quad \mathrm{P}(O_i > 16) = 0.1, \quad i = 1, 2$$

(a)

$$\mathrm{P}((14 < O_1 < 16) \cap (14 < O_2 < 16)) = \mathrm{P}(14 < O_1 < 16)\mathrm{P}(14 < O_2 < 16).$$

Now,

$$\begin{aligned}\mathrm{P}(14 < O_i < 16) &= 1 - \mathrm{P}((O_i < 14) \cup (O_i > 16)) \\ &= 1 - [\mathrm{P}(O_i < 14) + \mathrm{P}(O_i > 16)] \\ &= 1 - [0.05 + 0.1] = 0.85.\end{aligned}$$

So,

$$\mathrm{P}((14 < O_1 < 16) \cap (14 < O_2 < 16)) = (0.85)^2 = \boxed{0.7225}.$$

(b)

$$\begin{aligned}\mathrm{P}(\text{at least one is above 16}) &= 1 - \mathrm{P}(\text{both are less than 16}) \\ &= 1 - \mathrm{P}((O_1 < 16) \cap (O_2 < 16)) \\ &= 1 - \mathrm{P}(O_1 < 16)\mathrm{P}(O_2 < 16) \\ &= 1 - 0.9^2 = \boxed{0.19}.\end{aligned}$$

8. D = disease $\quad$ D' = no disease
T = test positive $\quad$ T' = test negative

$$\begin{array}{llllll}\mathrm{P}(T|D) & = & 0.9 & \mathrm{P}(T'|D) & = & 0.1 \\ \mathrm{P}(T'|D) & = & 0.1 & \mathrm{P}(T'|D') & = & 0.9 \\ \mathrm{P}(D) & = & 0.01 & \mathrm{P}(D') & = & 0.99\end{array}$$

$$\begin{aligned}\mathrm{P}(D|T) &= \frac{\mathrm{P}(T|D)\mathrm{P}(D)}{\mathrm{P}(T)} \\ &= \frac{\mathrm{P}(T|D)\mathrm{P}(D)}{\mathrm{P}(T|D)\mathrm{P}(D) + \mathrm{P}(T|D')\mathrm{P}(D')} \\ &= \frac{\frac{9}{10} \times \frac{1}{100}}{\frac{9}{10} \times \frac{1}{100} + \frac{1}{10} \times \frac{99}{100}} \\ &= \frac{\frac{9}{1000}}{\frac{108}{1000}} = \frac{9}{108} = \boxed{\tfrac{1}{12}}.\end{aligned}$$

If we obtain a positive test result there is only a 1 in 12 chance that the person actually has the disease.

9. Let S = event system survives.
A_i= event that component A_i survives, $i = 1, 2, 3$.
B_i= event that component B_i survives, $i = 1, 2$.
C_i= event that component C_i survives, $i = 1, 2, 3$.

$\mathrm{P}(A_i') = 0.1, \quad \mathrm{P}(B_i') = 0.2, \quad \mathrm{P}(C_i') = p.$

$S = A_1 \cap A_2 \cap A_3 \cap (B_1 \cup B_2) \cap (C_1 \cup C_2 \cup C_3)$.
$\mathrm{P}(S) = \mathrm{P}(A_1)\mathrm{P}(A_2)\mathrm{P}(A_3)\mathrm{P}(B_1 \cup B_2)\mathrm{P}(C_1 \cup C_2 \cup C_3)$ (independence).

$\mathrm{P}(B_1 \cup B_2) = 1 - \mathrm{P}(B_1' \cap B_2')$
$\mathrm{P}(C_1 \cup C_2 \cup C_3) = 1 - \mathrm{P}(C_1' \cap C_2' \cap C_3')$.

So $\mathrm{P}(S) = \mathrm{P}(A_1)\mathrm{P}(A_2)\mathrm{P}(A_3)(1 - \mathrm{P}(B_1' \cap B_2'))(1 - \mathrm{P}(C_1' \cap C_2' \cap C_3'))$.

(a) $\mathrm{P}(S) = 0.9^3(1 - 0.2^2)(1 - 0.5^3) = \boxed{0.61236}$.

(b)

$$\begin{aligned}
\mathrm{P}(S) < 0.5 &\Rightarrow 0.5 > 0.9^3(1-0.2^2)(1-p^3)\\
&\Rightarrow 1 - p^3 < \tfrac{0.5}{0.9^3(1-0.2^2)}\\
&\Rightarrow p^3 > \left(1 - \tfrac{0.5}{0.9^3(1-0.2^2)}\right)\\
&\Rightarrow 3\log(p) > \log\left(1 - \tfrac{0.5}{0.9^3(1-0.2^2)}\right)\\
&\Rightarrow p > \exp\left[\tfrac{1}{3}\log\left(1 - \tfrac{0.5}{0.9^3(1-0.2^2)}\right)\right] = \boxed{0.6585}.
\end{aligned}$$

10. (a) $\mathrm{P}(\text{I survive}) = \boxed{\tfrac{6}{9}}$.

(b) $\mathrm{P}(\text{sister survives}|\text{I survived}) = \boxed{\tfrac{5}{8}}$.

(c) $\mathrm{P}(\text{sister survives}|\text{I died}) = \boxed{\tfrac{6}{8}}$.

(d) If I choose first the probability of survival is $\frac{6}{9}$. We must work out the probability of my survival if my sister chooses first. The events

A = my sister chooses first and dies, and
B = my sister chooses first and survives form a partition.

$$\begin{aligned}
\mathrm{P}(\text{I survive on the } 2^{\text{nd}}) &= \mathrm{P}(A)\mathrm{P}(\text{I survive on the } 2^{\text{nd}}|A) +\\
&\qquad \mathrm{P}(B)\mathrm{P}(\text{I survive on the } 2^{\text{nd}}|B)\\
&= \tfrac{3}{9} \times \tfrac{6}{8} + \tfrac{6}{9} \times \tfrac{5}{8} = \tfrac{6}{9}
\end{aligned}$$

so, the probability is unchanged and it makes no difference which of us goes first.

(e)

$$\begin{aligned}
&\mathrm{P}(\text{I survive on the } 1^{\text{st}}|\text{my sister survived on } 2^{\text{nd}})\\
&= \frac{\mathrm{P}\left(\text{my sister survived on } 2^{\text{nd}}|\text{I survive on the } 1^{\text{st}}\right)\mathrm{P}\left(\text{I survive on the } 1^{\text{st}}\right)}{\mathrm{P}\left(\text{my sister survived on } 2^{\text{nd}}\right)}\\
&= \frac{\frac{5}{8} \times \frac{6}{9}}{\frac{6}{9}} \quad \text{as } \mathrm{P}(\text{my sister survived on } 2^{\text{nd}}) = \mathrm{P}(\text{I survived on } 2^{\text{nd}})\\
&= \boxed{\tfrac{5}{8}}.
\end{aligned}$$