Example 2.1

Suppose we toss three coins. The possible outcomes can be described as follows:

1^{st}	2^{nd}	3^{rd}
H	H	H
H	H	T
H	T	H
T	H	H
H	T	T
T	H	T
T	T	H
T	T	T

i.e. 8 possible outcomes.

Let A = event of 2 Heads and 1 Tail occurring.

So $A = \{\text{HHT,HTH,THH}\}$. i.e. 3 of the outcomes are contained in the event A. If we obtain the outcome THH we say the event A has occurred.

Example 2.2

Suppose we are interested in describing the state of supply of concrete and steel to a construction site;

E_1 = event that there is a shortage of concrete,

E_2 = event that there is a shortage of steel.

$E_1 \cup E_2$ = short of *either* concrete or steel or *both*.

$E_1 \cap E_2$ = short of *both* concrete and steel.

Example 2.3

Toss of 3 coins.

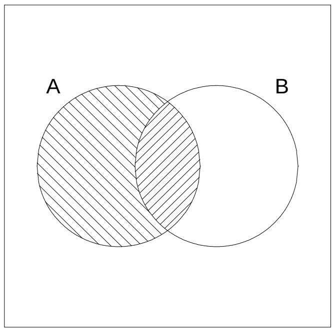
B = event that the first two tosses produce different outcomes.

$B = \{\text{HTH, THH, HTT, THT}\}$ $B' = \{\text{HHH, HHT, TTH, TTT}\}$

Example 2.4

We can write $A = (A \cap B) \cup (A \cap B')$

– $(A \cap B)$ and $(A \cap B')$ are disjoint (Important Trick).



Examples of De Morgan's law:

Example 2.5

Recall, $(A \cup B)' = A' \cap B'$.

Consider a chain made up of two links. Chain breaks if either link fails.

Let A = breakage of link 1,

B = breakage of link 2.

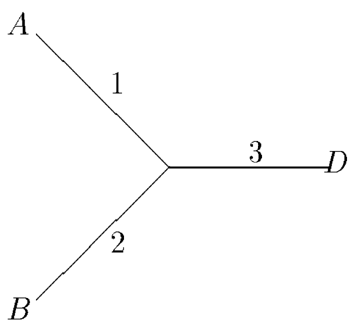
Failure of chain = $A \cup B \quad \Rightarrow$ no failure =$(A \cup B)'$.

Also, no failure means both links do not fail i.e. $A' \cap B'$.

Therefore, $(A \cup B)' = A' \cap B'$.

Example 2.6

Water for a city comes from two sources, A and B:



Let E_1 = event that branch 1 fails.

E_2 = event that branch 2 fails.

E_3 = event that branch 3 fails.

$\Rightarrow$ No water in the city $= (E_1 \cap E_2) \cup E_3$.

Alternatively

Water in city $= (E_1' \cap E_3') \cup (E_2' \cap E_3')$,

therefore, no water in city

$$= [(E_1' \cap E_3') \cup (E_2' \cap E_3')]'$$

$$= (E_1' \cap E_3')' \cap (E_2' \cap E_3')' \quad \text{by De Morgan's law}$$

$$= (E_1 \cup E_3) \cap (E_2 \cup E_3) \quad \text{by De Morgan's law}$$

$$= (E_1 \cap E_2) \cup E_3.$$

Example 2.7

Selecting a number from the real line: $S = \{x : -\infty < x < \infty\}$.

Let

$$A = \{x : 1 \le x \le 5\}$$

$$B = \{x : 3 < x \le 7\}$$

$$C = \{x : x \le 0\}$$

Describe the following:

(a) $A' = \{x : x < 1 \text{ or } x > 5\}$.

(b) $A \cup B = \{x : 1 \le x \le 7\}$.

(c) $B \cap C' = B$.

(d) $A' \cap B' \cap C' = \{x : 0 < x < 1 \text{ or } x > 7\}$.

(e) $(A \cup B) \cap C = \phi$.

Example 2.8

Let E_i = event that student i fails the stats exam, $i = 1, 2$.

$$\mathrm{P}(E_1) = 0.7, \quad \mathrm{P}(E_2) = 0.8, \quad \mathrm{P}(E_1 \cap E_2) = 0.6$$

What is the probability that at least one will fail?

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) = 0.7 + 0.8 - 0.6 = 0.9$$

Note,

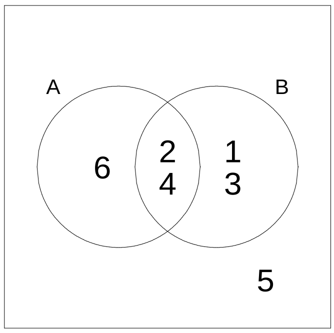
$$P(E_1|E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} = \frac{0.6}{0.8} = 0.75 \neq P(E_1) \quad \text{NOT independent}$$

Example 2.9

Roll of a dice.

Let event A = even = {2,4,6}, and event B = {1,2,3,4}.

We show that A and B are independent.



$$P(A) = \frac{1}{2}, \quad P(B) = \frac{2}{3}, \quad P(A \cap B) = \frac{1}{3}.$$

So, $P(A \cap B) = P(A)P(B)$.

i.e.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/3}{2/3} = \frac{1}{2} = P(A).$$

half of the events in B are even.

Similarly,

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1/3}{1/2} = \frac{2}{3} = P(B).$$

two-thirds of the events in A are in B.

Example 2.10

Recall Example 2.6, E_i = failure of branch i.

Failures occur independently with the following probabilities:

$$P(E_1) = 0.3, \quad P(E_2) = 0.2, \quad P(E_3) = 0.1.$$

$$
\begin{aligned}
\mathrm{P(failure)} &= \mathrm{P}((E_1 \cap E_2) \cup E_3) \\
&= \mathrm{P}(E_1 \cap E_2) + \mathrm{P}(E_3) - \mathrm{P}(E_1 \cap E_2 \cap E_3) \ \text{ addition law of probability} \\
&= \mathrm{P}(E_1)\mathrm{P}(E_2) + \mathrm{P}(E_3) - \mathrm{P}(E_1)\mathrm{P}(E_2)\mathrm{P}(E_3) \ \text{ since } E_1, E_2, E_3 \text{ independent} \\
&= 0.3 \times 0.2 + 0.1 - 0.3 \times 0.2 \times 0.1 \\
&= 0.06 + 0.1 - 0.006 \\
&= 0.154
\end{aligned}
$$

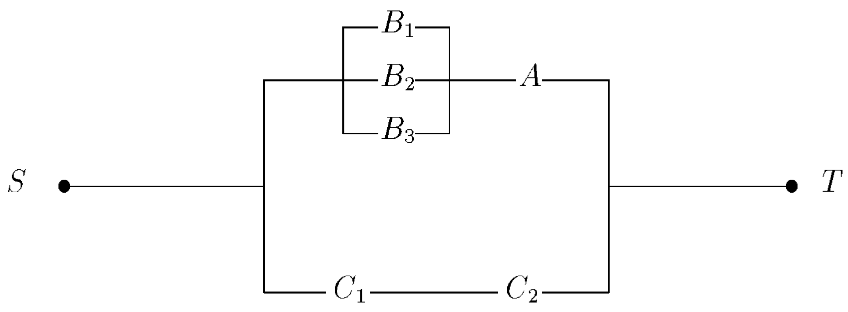
Suppose we wish to improve branch 3 so that the overall probability of failure is 0.1.

What probability do we need to have for branch 3?

Let $\mathrm{P}(E_3) = p$, then

$$
\begin{aligned}
\mathrm{P(failure)} &= \mathrm{P}((E_1 \cap E_2) \cup E_3) \\
&= \mathrm{P}(E_1)\mathrm{P}(E_2) + \mathrm{P}(E_3) - \mathrm{P}(E_1)\mathrm{P}(E_2)\mathrm{P}(E_3) \\
&= 0.3 \times 0.2 + p - 0.3 \times 0.2 \times p \\
&= 0.06 + p - 0.06p \\
&= 0.06 + 0.94p = 0.1 \\
&\Rightarrow 0.94p = 0.04 \Rightarrow p = 0.0426
\end{aligned}
$$

Mixed Systems



Components mutually independent,

A = event that component of type A functions, $\mathrm{P}(A) = \frac{1}{2}$.

B_i =event that component of type B functions, $\mathrm{P}(B_i) = \frac{1}{3}$, $i = 1, 2, 3$.

C_i =event that component of type C functions, $\mathrm{P}(C_i) = \frac{3}{4}$, $i = 1, 2$.

Let N = event that the network functions.

U = event that upper path functions.

L = event that the lower path functions.

$$N = (U \cap L) \cup (U \cap L') \cup (L \cap U')$$

i.e. $\mathrm{P}(N) = \mathrm{P}(U \cap L) + \mathrm{P}(U \cap L') + \mathrm{P}(L \cap U')$ since events are disjoint.

Alternatively:

$$\begin{aligned} \mathrm{P}(N) &= 1 - \mathrm{P}(N') \\ &= 1 - \mathrm{P}(\text{network fails}) \end{aligned}$$

$$N' = U' \cap L',$$

i.e. $\mathrm{P}(N') = \mathrm{P}(U' \cap L') = \mathrm{P}(U')\mathrm{P}(L')$ since independent.

U functions if A functions and if the B system functions.

i.e. $U = A \cap B$ where

B = event that B system functions $= B_1 \cup B_2 \cup B_3$.

Now, $B_1 \cup B_2 \cup B_3 = (B_1' \cap B_2' \cap B_3')'$.

So,

$$\begin{aligned}
\mathrm{P}(B) &= 1 - \mathrm{P}(B_1' \cap B_2' \cap B_3') = 1 - \mathrm{P}(B_1')\mathrm{P}(B_2')\mathrm{P}(B_3') \\
&= 1 - \left(\frac{2}{3}\right)^3 = 1 - \frac{8}{27} = \frac{19}{27} \\
\mathrm{P}(U) &= \mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B) \\
&= \frac{1}{2} \times \frac{19}{27} = \frac{19}{54} \\
\text{so, } \mathrm{P}(U') &= 1 - \frac{19}{54} = \frac{35}{54} \\
\mathrm{P}(L) &= \mathrm{P}(C_1 \cap C_2) = \mathrm{P}(C_1)\mathrm{P}(C_2) \\
&= \left(\frac{3}{4}\right)^2 = \frac{9}{16} \\
\text{so, } \mathrm{P}(L') &= 1 - \mathrm{P}(L) = 1 - \frac{9}{16} = \frac{7}{16}. \\
\Rightarrow \mathrm{P}(N) &= \mathrm{P}(U')\mathrm{P}(L') = \frac{35}{54} \times \frac{7}{16} = 0.284 \\
\text{so, } \mathrm{P}(N) &= \mathrm{P}(\text{system function}) = 1 - \mathrm{P}(N') \\
&= 1 - 0.284 = 0.716
\end{aligned}$$

Example 2.11

Two boxes containing long bolts and short bolts.

Box 1 contains 60 long and 40 short.

Box 2 contains 10 long and 20 short.

We select a box at random and randomly select a bolt.

What is the probability that the bolt is long?

Let A_i = event that Box i is selected, $i = 1, 2$. $\mathrm{P}(A_i) = \frac{1}{2}$.

Let B = event that a long bolt is selected.

$$\mathrm{P}(B|A_1) = \frac{60}{100}, \quad \mathrm{P}(B|A_2) = \frac{10}{30}.$$

A_1 and A_2 form a partition:

$$\begin{aligned}
\mathrm{P}(B) &= \mathrm{P}(A_1)\mathrm{P}(B|A_1) + \mathrm{P}(A_2)\mathrm{P}(B|A_2) \\
&= \tfrac{1}{2} \times \tfrac{6}{10} + \tfrac{1}{2} \times \tfrac{1}{3} \\
&= \tfrac{3}{10} + \tfrac{1}{6} = \tfrac{18+10}{60} = \tfrac{28}{60} = \tfrac{7}{15}
\end{aligned}$$

Given that a long bolt is selected, what is the probability that Box 1 was selected?

$$
\begin{aligned}
\mathrm{P}(A_1|B) &= \frac{\mathrm{P}(B|A_1)\mathrm{P}(A_1)}{\mathrm{P}(B)} \\
&= \frac{\frac{6}{10}\times\frac{1}{2}}{\frac{7}{15}} \\
&= \frac{3}{10}\times\frac{15}{7} = \frac{9}{14}
\end{aligned}
$$

Note: in general

$$\mathrm{P}(A_i|B) = \frac{\mathrm{P}(B|A_i)\mathrm{P}(A_i)}{\sum_{j=1}^{k}\mathrm{P}(A_j)\mathrm{P}(B|A_j)}$$