6 Markov Chains

A stochastic process $\{X_n; n = 0, 1, \ldots\}$ in discrete time with finite or infinite state space S is a Markov Chain with stationary transition probabilities if it satisfies:

(1) For each $n \geq 1$, if A is an event depending only on any subset of $\{X_{n-1}, X_{n-2}, \ldots, 0\}$, then, for any states i and j in S,

$$\mathrm{P}(X_{n+1} = j \mid X_n = i \text{ and } A) = \mathrm{P}(X_{n+1} = j \mid X_n = i).$$

(2) For any given states i and j

$$\mathrm{P}(X_{n+1} = j \mid X_n = i) \quad \text{is same } \forall\, n \geq 0.$$

(1) is the MARKOV PROPERTY,
More generally:
For each $n \geq 1$ and $m \geq 1$, if A is as in (1), then for any states i and j in S:

$$\mathrm{P}(X_{n+m} = j \mid X_n = i \text{ and } A) = \mathrm{P}(X_{n+m} = j \mid X_n = i).$$

Denote transition probabilities in (2) by

$$p_{ij} = \mathrm{P}(X_{n+1} = j \mid X_n = i).$$

Key consequences:

$$\begin{aligned}
\mathrm{P}(X_{n+1} = j \cap X_{n+2} = k \mid X_n = i) &= \mathrm{P}(X_{n+2} = k \mid X_{n+1} = j \cap X_n = i)\mathrm{P}(X_{n+1} = j \mid X_n = i) \\
&= \mathrm{P}(X_{n+2} = k \mid X_{n+1} = j)\mathrm{P}(X_{n+1} = j \mid X_n = i) \\
&= p_{jk}p_{ij}.
\end{aligned}$$

6.1 Transition Matrix: $P = \{p_{ij}\}$

e.g. Gambler's ruin with $a = 4$ and $p + q = 1$

$$P = \begin{array}{c|ccccc|} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & q & 0 & p & 0 & 0 \\ 2 & 0 & q & 0 & p & 0 \\ 3 & 0 & 0 & q & 0 & p \\ 4 & 0 & 0 & 0 & 0 & 1 \\ \hline \end{array}$$

NOTE: $\sum_j p_{ij} = 1$.

e.g. Bernoulli process: success probability p.

$$X_n = \text{length of current run of successes.}$$

S for $\{X_n;\ n = 0, 1, 2, \ldots\}$ is $\{0, 1, 2, \ldots\}$.

At time n: if in state i

then at time $n + 1$: in state $i + 1$ with probability p.

in state 0 with probability q.

$$P = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & \ldots \\ \hline 0 & q & p & 0 & 0 & 0 & \ldots \\ 1 & q & 0 & p & 0 & 0 & \ldots \\ 2 & q & 0 & 0 & p & 0 & \ldots \\ 3 & q & 0 & 0 & 0 & p & \ldots \end{array}$$

Note: for some important Markov chains it is difficult to find explicit form for transition probabilities.

e.g. Galton-Watson branching process

$$\begin{aligned} Z_n &= \text{size of } n\text{th generation.} \\ X_k &= \#\text{ of offspring of } k\text{th member of } n\text{th generation.} \end{aligned}$$

So,

$$p_{ij} = \mathrm{P}(Z_{n+1} = j \mid Z_n = i) = \mathrm{P}(X_1 + X_2 + \ldots + X_i = j).$$

If each X_k has pgf $\Pi(s)$ then $X_1 + X_2 + \ldots X_i$ has pgf $[\Pi(s)]^i$

$\Rightarrow p_{ij}$ is coefficient of s^j in $[\Pi(s)]^i$.

The <u>n-step transition probability</u> of a Markov chain is the probability that it goes from state i to state j in n transitions:

$$p_{ij}^{(n)} = \mathrm{P}(X_{n+m} = j \,|\, X_m = i)$$

and the associated n-step transition matrix is

$$P^{(n)} = \{p_{ij}^{(n)}\} \qquad (P^{(1)} = P).$$

Now,

$$\mathrm{P}(i \text{ to } j \text{ in } n \text{ steps}) = \text{sum of probs of all paths } i \text{ to } j \text{ in } n \text{ steps.}$$

We have

$$p_{ij}^{(n+m)} = \sum_k p_{ik}^{(m)} p_{kj}^{(n)} \qquad \text{CHAPMAN-KOLMOGOROV EQUATIONS.}$$

So, in terms of transition matrices

$$P^{(m+n)} = P^{(m)} P^{(n)}$$

and in particular

$$P^{(n)} = P^{(n-1)} P$$

so that,

$$P^{(n)} = P^n \quad \text{for } \; n \geq 1.$$

e.g. In Melbourne, during the first 3 months of 1983:

		Weather on next day		
		Dry	Wet	Total
Weather	Dry	57	12	69
on one day	Wet	12	8	20

$$\mathrm{P} = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0.826 & 0.174 \\ 1 & 0.600 & 0.400 \end{array} \quad = \quad \begin{array}{|cc} & \\ \hline p_{00} & p_{01} \\ p_{00} & p_{01} \end{array}$$

To calculate the probability that it will be dry two days after a wet day:

$$\mathrm{P}(X_2 = 0 \mid X_0 = 1) = p_{10}p_{00} + p_{11}p_{10} = 0.736.$$

If we are interested in $\mathrm{P}(X_7 = 0 \mid X_0 = 1)$, the calculations become unwieldy $\rightarrow$ use matrices:

$$P^{(7)} = P^7 = \begin{pmatrix} 0.826 & 0.174 \\ 0.600 & 0.400 \end{pmatrix}^7 = \begin{pmatrix} 0.775 & 0.225 \\ 0.775 & 0.225 \end{pmatrix}.$$

Transition matrix P is useful if we know the initial state. But what if we have a large population or have a probability distribution for initial states?

$$\begin{aligned} \text{Let} \quad a_i^{(0)} &= \mathrm{P}(X_0 = i) \quad i \in S \\ \boldsymbol{a}^{(0)} &= (a_0, a_1, \ldots) \\ \text{Let} \quad a_j^{(n)} &= \mathrm{P}(X_n = j) \\ \boldsymbol{a}^{(n)} &= (a_0^{(n)}, a_1^{(n)}, \ldots) \end{aligned}$$

How are $\boldsymbol{a}^{(n)}$ and $\boldsymbol{a}^{(0)}$ related?

$$\begin{aligned} \mathrm{P}(X_n = j) &= \sum_i \mathrm{P}(X_n = j \mid X_0 = i)\mathrm{P}(X_0 = i) \\ \Rightarrow \boldsymbol{a}^{(n)} &= \boldsymbol{a}^{(0)} P^n \quad n \geq 1. \end{aligned}$$

e.g. A student has two was of getting to college: A and B. Each day he picks one, and his choice is influenced only by his previous days choice:

– if A, then $\mathrm{P}(A)$ next day $= \frac{1}{2}$

– if B, then $\mathrm{P}(A)$ next day $= \frac{3}{4}$

When he first arrived at college (day 0), he had no preference.

1. The transition matrix is given by:

$$\begin{array}{c} A \\ B \end{array} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

2. Probabilities of choosing A and B on day 1:

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{5}{8} & \frac{3}{8} \end{pmatrix} = (0.625, 0.375).$$

3. Probabilities of choosing A and B on day 2:

$$\begin{pmatrix} \frac{5}{8} & \frac{3}{8} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{19}{32} & \frac{13}{32} \end{pmatrix} = (0.594, 0.406).$$

Extending this example:

	0	1	2	3	4
$a_0^{(n)} = \mathrm{P}(X_n = A)$	0.500	0.625	0.594	0.602	0.600
$a_1^{(n)} = \mathrm{P}(X_n = B)$	0.500	0.375	0.406	0.398	0.400

Also, P^n starts at

$$P^1 = \begin{pmatrix} 0.500 & 0.500 \\ 0.750 & 0.250 \end{pmatrix}$$

and seems to converge to

$$\begin{pmatrix} 0.600 & 0.400 \\ 0.600 & 0.400 \end{pmatrix}$$

(c.f. the previous weather example).

1. The $p_{ij}^{(n)}$ have settled to a limiting value.

2. This value is independent of initial state.

3. The $a_j^{(n)}$ also approach this limiting value.

If a Markov chain displays such equilibrium behaviour it is

in <u>probabilistic equilibrium</u>

or <u>stochastic equilibrium</u>

The limiting value is $\boldsymbol{\pi}$.

Not all Markov chains behave in this way.

For a Markov chain which does achieve stochastic equilibrium:

$$\begin{aligned} p_{ij}^{(n)} &\to \pi_j \quad \text{as } n \to \infty \\ a_j^{(n)} &\to \pi_j \end{aligned}$$

π_j is the <u>limiting probability</u> of state j.

Two interpretations of $\boldsymbol{\pi}$:

1. The probability that the process will be in state j after running for a long time.

2. The proportion of time it spends in state j after running for a long time.

To see 2.: Define

$$I_n = \begin{cases} 1 & \text{if } X_n = j \\ 0 & \text{if } X_n \neq j \end{cases}$$

Number of visits to j in first N transitions $= \sum_{n=1}^{N} I_n$.

So, starting from state i

$$\begin{aligned}
\mathrm{E}\left(\sum_{n=1}^{N} I_n \,|\, X_0 = i\right) &= \sum_{n=1}^{N} \mathrm{E}(I_n \,|\, X_0 = i) \\
&= \sum_{n=1}^{N} [1 \times \mathrm{P}(I_n = 1 \,|\, X_0 = i) + 0 \times \mathrm{P}(I_n = 0 \,|\, X_0 = i)] \\
&= \sum_{n=1}^{N} \mathrm{P}(X_n = j \,|\, X_0 = i) \\
&= \sum_{n=1}^{N} p_{ij}^{(n)}
\end{aligned}$$

$$\Rightarrow \frac{\sum_{i=1}^{N} p_{ij}^{(n)}}{N} = \text{expected proportion time in } j.$$

So

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} p_{ij}^{(n)} = \pi_j$$

(recall, if $b_1, b_2, b_3, \ldots$ converges to b, then so does $b_1, (b_1+b_2)/2, (b_2+b_2+b_3)/3, \ldots$)

6.2 Finding the limiting distribution

We have

$$\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j$$

and,

$$p_{ij}^{(n+1)} = \sum_k p_{ik}^{(n)} p_{kj}.$$

So, letting $n \to \infty$,

$$\begin{aligned}
\pi_j &= \sum_k \pi_k p_{kj} \\
\boldsymbol{\pi} &= \boldsymbol{\pi} P \qquad (4)
\end{aligned}$$

and,

$$\pi_j \geq 0 \; \forall \; j \in S \tag{5}$$

and,

$$\sum_j p_{ij}^{(n)} = 1 \quad \forall n,$$

so (take limit)

$$\sum_j \pi_j = 1 \tag{6}$$

We can find the limiting distribibition by solving (4) subject to (5) and (6).

e.g.

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

Solve

$$(\pi_0 \; \pi_1) = (\pi_0 \; \pi_1) \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

s.t. $\pi_0, \pi_1 \geq 0, \pi_0 + \pi_1 = 1$:

$$\begin{cases} \pi_0 = \frac{1}{2}\pi_0 + \frac{3}{4}\pi_1 \\ \pi_1 = \frac{1}{2}\pi_0 + \frac{1}{4}\pi_1 \end{cases}$$

(two equations, but linearly dependent, so discard either one).

So, find non-negative solutions of

$$\begin{aligned} \pi_0 &= \frac{1}{2}\pi_0 + \frac{3}{4}\pi_1 \\ \pi_0 + \pi_1 &= 1 \end{aligned}$$

$$\Rightarrow \pi_0 = 0.6 \qquad \pi_1 = 0.4.$$

Cautionary notes:

1. Let

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\pi} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Now P and $\boldsymbol{\pi}$ satisfy the conditions

$$\boldsymbol{\pi} = \boldsymbol{\pi} P, \quad \boldsymbol{\pi} \geq 0, \quad \sum \pi_j = 1$$

(convince yourselves of this).

But $p_{ij}^{(n)}$ does not tend to a limit.

2. Let

$$P = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix}$$

Both,

$$\begin{aligned} \boldsymbol{\pi} &= \begin{pmatrix} \frac{3}{4} & 0 & \frac{1}{4} \end{pmatrix} \\ \boldsymbol{\pi} &= \begin{pmatrix} \frac{1}{3} & 0 & \frac{2}{3} \end{pmatrix} \end{aligned}$$

both satisfy $\boldsymbol{\pi} = \boldsymbol{\pi} P$, so the solution is not necessarily unique.

Limiting distributions must satisfy

$$\boldsymbol{\pi} = \boldsymbol{\pi} P, \quad \boldsymbol{\pi} \geq 0, \quad \sum \pi_j = 1.$$

But: satisfying these, does not mean there's a limiting distribution.

And: sometimes, there's more that one solution.

Distributions satisfying the 3 conditions are **stationary distributions**. So limiting $\Rightarrow$ stationary.

"Stationary because", if initial distribution is $\boldsymbol{a}^{(0)} = \boldsymbol{\pi}$ then $\boldsymbol{a}^{(\boldsymbol{n})} = \boldsymbol{\pi}$ always.

Questions: when is a stationary distribution limiting? next section...

Results also hold for infinite sample space. But more care is needed.

$$\sum_j p_{ij}^{(n)} = 1 \quad \not\Rightarrow \quad \sum_j \pi_j = 1$$

$$\left(\lim_n \sum_j p_{ij}^{(n)} \quad \text{may not equal} \quad \sum_j \lim_n p_{ij}^{(n)} \right)$$

e.g. unrestricted simple random walk

$$p_{ij}^{(n)} \to 0 \quad \text{as} \quad n \to \infty \quad \forall \ i, j$$

so limiting probabilities are all 0 so $\sum \pi_j \neq 1$.

6.3 Communicating classes

- when does a Markov chain have a unique stationary distribution?
- is this roughly the same as the distribution obtained when the chain's been running for a long time?

(These are extremely important in the context of MCMC).

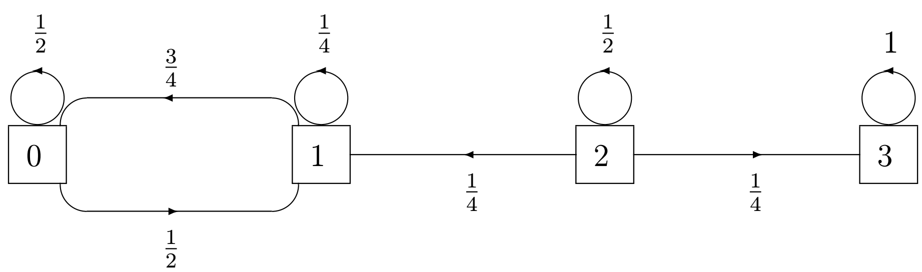
Definition: State j is accessible from state i if $p_{ij}^{(n)} > 0$ for some $n \geq 0$.

(Note: $p_{ii}^{(0)} = 1 \Rightarrow$ every state accessible from itself. If two states are each accessible from the other, they communicate: $i \leftrightarrow j$.

Often easier to see what's going on from the transition diagram than from transition matrix.

e.g. $S = \{0, 1, 2, 3\}$,

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{3}{4} & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



The pairs of communicating states are given by

$$0 \leftrightarrow 0 \quad 0 \leftrightarrow 1 \quad 1 \leftrightarrow 1 \quad 2 \leftrightarrow 2 \quad 3 \leftrightarrow 3$$

So $S = \{0, 1, 2, 3\}$ may be divided into non-overlappng sets:

$$\{0, 1\} \quad \{2\} \quad \{3\}$$

The state space of any Markov chain may be divided into non-overlapping subsets of states such that two states are in the same subset if and only if they communicate.

These subsets are communicating classes (or just "classes").

A Markov chain is irreducible if all the states communicate.

A "closed" class is one that is impossible to leave, so $p_{ij} = 0 \quad \text{if } i \in C, j \notin C$. $\Rightarrow$ an irreducible MC has only one class, which is necessarily closed.

MCs with more than one class, may consist of both closed and non-closed classes: for the previous example

$$\{0,1\} \quad \text{is closed,} \qquad \{3\} \quad \text{is closed,} \qquad \{2\} \quad \text{is not closed.}$$

Theorem: Every Markov Chain with a finite state space has a unique stationary distribution unless the chain has two or more closed communicating classes.

Note: two or more communicating classes but only one closed $\rightarrow$ unique stationary distribution.

We know that:

$$\{\text{limiting distributions}\} \subset \{\text{stationary distributions}\}.$$

So, if a MC satisfying the conditions above (finite state space, 0 or 1 closed classes) has a limiting distribution, then this is the unique stationary distribution of the chain.

But, a finite MC with a unique stationary distribution may not have a limiting distribution unless we satisfy one more condition...

6.4 Periodicity of states

Recall

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

1 closed communicating class $\Rightarrow$ a unique stationary distribution.

But,

$$P^{2m} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad P^{2m+1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

So

$$p_{00}^{(n)} = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \qquad p_{11}^{(n)} = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

no limiting distribution – return to 0 or 1 can only occur in either an even or odd number of steps.

These states are periodic (with period 2).

A periodicity of state i is defined as

$$d_i = \gcd\{n \geq 1 \,:\, p_{ii}^{(n)} > 0\}.$$

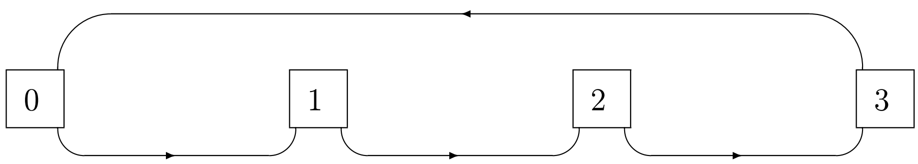
If d_i=1 then state i is called aperiodic.

e.g.:

1. unrestricted random walk with $p + q = 1$ state 0 has period 2.
2. unresticted random walk with $p + q + r = 1$ and $r > 0$ is aperiodic.
3.

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

transition diagram:



periodic – with return only after $4, 8, 12, \ldots$ transitions,

$\Rightarrow$ period 4.

Note: all states in a communicating class are either aperiodic or periodic with the same period.

$\Rightarrow$ so classes can be described as aperiodic or periodic.

$\Rightarrow$ an irreducible MC is either periodic or aperiodic.

Theorem: If an irreducible MC with finite state space $\{0, 1, 2, \ldots m\}$ is aperiodic, then for all states i and j

$$p_{ij}^{(n)} \to \pi_j \quad \text{as} \quad n \to \infty$$

where $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots, \pi_m)$ is the unique stationary distribution of the chain (i.e. the limiting distribution is the stationary distribution).

When we studied random walks we found that all states were recurrent ($p = q$) or transient ($p \neq q$).

This is not true for MCs in general.

e.g.

$$P = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

state 0 is recurrent and state 1 is transient.

- Every class which is not closed is transient.
- Every closed finite class is recurrent ($\Rightarrow$ every finite irreducible MC is recurrent).
- Infinite closed classes can be either transient or recurrent.

  e.g. SRW $p + q = 1$:

  - states form a single closed class.
  - but transient if $p \neq q$, recurrent if $p = q$.

	Finite	Infinite
Closed	Recurrent	Recurrent or Transient
Not Closed	Transient	Transient

6.5 Return probabilities and return times

Recall,

$$
\begin{aligned}
p_{ij}^{(n)} &= \mathrm{P}(X_n = j \mid X_0 = i) \\
p_{ii}^{(n)} &= \text{prob of return to } i \text{ in } n \text{ transitions (not necessarily first return)} \\
p_{ii}^{(0)} &= 1 \\
f_{ii}^{(n)} &= \text{prob. of return to } i \text{ for the first time after } n \text{ transitions} \\
f_{ii}^{(0)} &= 0 \quad \text{by definition} \\
f_{ii} &= \text{prob that a return to } i \text{ eventually occurs} \\
f_{ii} &= f_{ii}^{(1)} + f_{ii}^{(2)} + \ldots + f_{ii}^{(n)} + \ldots
\end{aligned}
$$

State i is recurrent if $f_{ii} = 1$

transient if $f_{ii} < 1$

e.g.

$$
P = \begin{array}{c} 0 \\ 1 \end{array} \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}
$$

$$
\begin{aligned}
f_{00}^{(1)} &= 1-\alpha \\
f_{00}^{(2)} &= \alpha\beta
\end{aligned}
$$

For $n \geq 3$ the first return to state 0 occurs after n transitions if

(a). 1st transition is from 0 to 1, and

(b). next $(n-2)$ transitions are from 1 to 1, and

(c). nth transition is from 1 to 0.

$$
\begin{aligned}
\Rightarrow f_{00}^{(n)} &= \alpha(1-\beta)^{n-2}\beta \\
\Rightarrow f_{00} &= \sum_{n=0}^{\infty} f_{ii}^{(n)} \\
&= 1-\alpha + \sum_{n=2}^{\infty} \alpha(1-\beta)^{n-2}\beta \\
&= 1-\alpha + \alpha\beta \sum_{m=0}^{\infty} (1-\beta)^m \\
&= 1-\alpha + \frac{\alpha\beta}{1-(1-\beta)} \\
&= 1-\alpha+\alpha = 1
\end{aligned}
$$

$\Rightarrow$ state 0 is recurrent.

For random walks we denoted the general return probability by u_n and the first return probability by f_n.

We found

$$u_n = f_0 u_n + f_1 u_{n-1} + f_2 u_{n-2} + \dots f_{n-1} u_1 + f_n u_0$$

and

$$U(s) = 1 + F(s)U(s)$$

and that a random walk is recurrent iff $\sum u_n = \infty$.

The same is true for Markov Chains

$$\begin{aligned} P_{ii}(s) &= \sum_{n=0}^{\infty} p_{ii}^{(n)} s^n \\ F_{ii}(s) &= \sum_{n=0}^{\infty} f_{ii}^{(n)} s^n \\ P_{ii}(s) &= 1 + F_{ii}(s) P_{ii}(s) \end{aligned}$$

state i is recurrent iff $\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$.

Theorem: Two states in the same communicating class are either both recurrent or both transient.

6.5.1 Mean recurrence times

$\{f_{ii}^{(n)};\ n \geq 1\}$ is the distribution of T_i, the time of first return to state i.

$$\begin{aligned} \mu_i = \mathrm{E}(T_i) &= \text{mean (first) return time of state } i \\ &= \text{mean recurrence time of state } i \end{aligned}$$

e.g. unrestricted simple random walk with $p + q = 1$, we have

$$F(s) = 1 - (1 - 4pqs^2)^{\frac{1}{2}}$$

When $p = q = 1/2$ the random walk is recurrent, and

$$F(s) = 1 - (1 - s^2)^{\frac{1}{2}}.$$

The mean is given by $F'(1)$

$$F'(s) = s(1 - s^2)^{-\frac{1}{2}} \Rightarrow F'(1) = \infty.$$

i.e. the mean time to return to the origin (or any other state) is infinite for a SRW with $p = q = 1/2$ even though eventual return is certain.

Recurrent states which have an infinite mean recurrence time are null recurrent.

Recurrent states which have a finite mean recurrence time are positive recurrent.

Positive or null recurrence is shared by all the states in a class $\rightarrow$ positive or null recurrent classes.

If a chain is irreducible then we have positive or null recurrent chains.

Every finite closed class is positive recurrent.

Every finite irreducible MC is positive recurrent.

Type of class	Finite	Infinite
Closed	positive recurrent	positive recurrent null recurrent transient
Not closed	transient	transient

Theorem: The basic limit theorem for Markov Chains

For each state i of a recurrent irreducible aperiodic Markov Chain

$$\lim_{n\to\infty} p_{ii}^{(n)} = \frac{1}{\sum_{n=0}^{\infty} n f_{ii}^{(n)}} = \frac{1}{\mu_i}.$$

μ_i = mean return times.

Recall, every finite irreducible MC has a unique stationary distribution and if the chain is aperiodic then

$$\lim_{n\to\infty} p_{ii}^{(n)} = \pi_i \quad \text{for each } i$$

So, for a finite irreducible aperiodic MC

$$\pi_i = \frac{1}{\mu_i},$$

so we can find the mean return times from the stationary distribution.

e.g.

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

(with $\alpha, \beta \neq 0$ or 1).

Stationary distribution is

$$\boldsymbol{\pi} = (\pi_0 \quad \pi_1) = \left(\frac{\beta}{\alpha+\beta} \quad \frac{\alpha}{\alpha+\beta}\right)$$

$$\Rightarrow \mu_0 = \frac{\alpha+\beta}{\beta} \qquad \mu_1 = \frac{\alpha+\beta}{\alpha}.$$

Summary

There are three kinds of irreducible Markov chains

1. Positive Persistent

    (a) Stationary distribution $\boldsymbol{\pi}$ exists.

    (b) $\boldsymbol{\pi}$ is unique.

    (c) All mean recurrence times are finite: $F'_{ii}(1) = 1/\pi_i$.

    (d)
    $$\frac{N_i(n)}{n} \to \pi_i \ \ \forall i \quad \text{as } n \to \infty$$
    where $N_i(n)$ is the number of visits to state i after n transitions.

    (e) If aperiodic then
    $$\mathrm{P}(X_n = i) \to \pi_i \ \ \forall i \quad \text{as } n \to \infty$$

2. Null Persistent

    (a) Persistent, but all mean recurrence times are infinite.

    (b) No stationary distribution exists.

    (c)
    $$\frac{N_i(n)}{n} \to 0 \ \ \forall i \quad \text{as } n \to \infty$$

    (d)
    $$\mathrm{P}(X_n = i) \to 0 \ \ \forall i \quad \text{as } n \to \infty$$

3. Transient

    (a) Any particular state is eventually never visited.

    (b) No stationary distribution exists.

    (c)
    $$\frac{N_i(n)}{n} \to 0 \ \ \forall i \quad \text{as } n \to \infty$$

    (d)
    $$\mathrm{P}(X_n = i) \to 0 \ \ \forall i \quad \text{as } n \to \infty$$

Diagram (for irreducible chains)