5. (i) (a) 5 (1 of 2)

$$\begin{aligned} \mathrm{P}(Y=0) &= \mathrm{P}(Y=0 \mid X=0)\mathrm{P}(X=0) + \mathrm{P}(Y=0 \mid X=1)\mathrm{P}(X=1) \\ &= 0.9 \times 0.3 + 0.1 \times 0.7 = 0.34 \end{aligned}$$

2

(b)

$$\mathrm{P}(X=0 \mid Y=0) = \frac{\mathrm{P}(Y=0 \mid X=0)\mathrm{P}(X=0)}{\mathrm{P}(Y=0)} = \frac{0.9 \times 0.3}{0.34} = 0.7941$$

3

(c) Let $N =$ number of digits in error, then $N \sim \mathit{Binomial}(5, 0.1)$ and $\mathrm{P}(N=n) = \binom{5}{n}(0.1)^n(0.9)^{5-n}$. 2

$$\mathrm{P}(N \leq 1) = \mathrm{P}(N=0) + \mathrm{P}(N=1) = (0.9)^5 + 5(0.1)(0.9)^4 = 0.9185$$

2

(ii) (a)

$$\begin{aligned} \mathrm{P}(S \mid A_1) &= \frac{\mathrm{P}(A_1 \mid S)\mathrm{P}(S)}{\mathrm{P}(A_1)} = \frac{\mathrm{P}(A_1 \mid S)\mathrm{P}(S)}{\mathrm{P}(A_1 \mid S)\mathrm{P}(S) + \mathrm{P}(A_1 \mid \overline{S})\mathrm{P}(\overline{S})} \\ &= \frac{0.2 \times 0.2}{0.2 \times 0.2 + 0.05 \times 0.8} = 0.5 \end{aligned}$$

2

(b)

$$\begin{aligned} \mathrm{P}(S \mid A_1 \cap A_2) &= \frac{\mathrm{P}(A_1 \cap A_2 \mid S)\mathrm{P}(S)}{\mathrm{P}(A_1 \cap A_2)} \\ &= \frac{\mathrm{P}(A_1 \mid S)\mathrm{P}(A_2 \mid S)\mathrm{P}(S)}{\mathrm{P}(A_1 \cap A_2 \mid S)\mathrm{P}(S) + \mathrm{P}(A_1 \cap A_2 \mid \overline{S})\mathrm{P}(\overline{S})} \\ &= \frac{\mathrm{P}(A_1 \mid S)\mathrm{P}(A_2 \mid S)\mathrm{P}(S)}{\mathrm{P}(A_1 \mid S)\mathrm{P}(A_2 \mid S)\mathrm{P}(S) + \mathrm{P}(A_1 \mid \overline{S})\mathrm{P}(A_2 \mid \overline{S})\mathrm{P}(\overline{S})} \\ &= \frac{0.2 \times 0.4 \times 0.2}{0.2 \times 0.4 \times 0.2 + 0.05 \times 0.1 \times 0.8} = 0.8 \end{aligned}$$

3

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5. (ii) (c) 5 (2 of 2)

$$
\begin{aligned}
P(S \mid A_1 \cap A_2 \cap A_3) &= \frac{P(A_1 \cap A_2 \cap A_3 \mid S)P(S)}{P(A_1 \cap A_2 \cap A_3)} \\
&= \frac{P(A_1 \mid S)P(A_2 \mid S)P(A_3 \mid S)P(S)}{P(A_1 \cap A_2 \cap A_3 \mid S)P(S) + P(A_1 \cap A_2 \cap A_3 \mid \overline{S})P(\overline{S})} \\
&= \frac{P(A_1 \mid S)P(A_2 \mid S)P(A_3 \mid S)P(S)}{P(A_1 \mid S)P(A_2 \mid S)P(A_3 \mid S)P(S) + P(A_1 \mid \overline{S})P(A_2 \mid \overline{S})P(A_3 \mid \overline{S})P(\overline{S})} \\
&= \frac{0.2 \times 0.4 \times 0.2 \times 0.2}{0.2 \times 0.4 \times 0.2 \times 0.2 + 0.05 \times 0.1 \times 0.01 \times 0.8} = 0.9877
\end{aligned}
$$

4

(d) The probabilities increase as we have $P(A_i \mid S) > P(A_i \mid \overline{S})$, so as we include more terms, we introduce more information about whether the email is SPAM or not. 2

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6. (i) 6

$$\begin{aligned}
T &\sim N(6, 0.25) \\
\frac{T-6}{\sqrt{0.25}} &\sim N(0,1) \\
R(t) = \mathrm{P}(T > t) &= \mathrm{P}\left(\frac{T-6}{0.5} > \frac{t-6}{0.5}\right) \\
&= 1 - \Phi\left(\frac{t-6}{0.5}\right) \\
R(7) &= 1 - \Phi(2) = 1 - 0.977 = 0.023.
\end{aligned}$$

4

(ii) (a)

$$\int_{-\infty}^{\infty} f(t)\,dt = \int_0^{\infty} \lambda e^{-\lambda t}\,dt = \left[-e^{-\lambda t}\right]_0^{\infty} = 1.$$

Also, $f(t) \geq 0 \ \forall t$, so $f(t)$ is a valid pdf. 3

(b)

$$\begin{aligned}
R(t) &= \mathrm{P}(T > t) = \int_t^{\infty} f(u)\,du \\
&= \int_t^{\infty} \lambda e^{-\lambda u}\,du = \left[-e^{-\lambda u}\right]_t^{\infty} \\
&= e^{-\lambda t}
\end{aligned}$$

For component A: $R_A(t) = e^{-0.1t}$
For component B: $R_B(t) = e^{-0.5t}$ 3

$$h(t) = \frac{f(t)}{R(t)} = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$$

For component A: $h_A(t) = 0.1$
For component B: $h_B(t) = 0.5$ 3

(c) when $t = 1.5$ (90 minutes):
For component A: $R_A(1.5) = e^{-0.1\times 1.5} = e^{-0.15} = 0.8607$
For component B: $R_B(1.5) = e^{-0.5\times 1.5} = e^{-0.75} = 0.4724$ 2

(d) Let T = lifetime of system.
Let A_i = event that component A_i, $i = 1, 2$ is functioning at 90 minutes.
B_i = event that component B_i, $i = 1, 2, 3, 4$ is functioning at 90 minutes.

$$\begin{aligned}
R(1.5) &= \mathrm{P}(T > 1.5) = \mathrm{P}(A_1 \cap A_2 \cap ((B_1 \cap B_2) \cup (B_3 \cap B_4))) \\
&= \mathrm{P}(A_1)\mathrm{P}(A_2)[\mathrm{P}(B_1 \cap B_2) + \mathrm{P}(B_3 \cap B_4) - \mathrm{P}(B_1 \cap B_2 \cap B_3 \cap B_4)] \\
&= \mathrm{P}(A_1)\mathrm{P}(A_2)[\mathrm{P}(B_1)\mathrm{P}(B_2) + \mathrm{P}(B_3)\mathrm{P}(B_4) - \mathrm{P}(B_1)\mathrm{P}(B_2)\mathrm{P}(B_3)\mathrm{P}(B_4)] \\
&= e^{-0.15}e^{-0.15}[e^{-0.75}e^{-0.75} + e^{-0.75}e^{-0.75} - e^{-0.75}e^{-0.75}e^{-0.75}e^{-0.75}] \\
&= e^{-0.3}(2e^{-1.5} - e^{-3}) = 0.2937.
\end{aligned}$$

5

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