On bifurcations of a homoclinic "figure eight" of a multi-dimensional saddle

D.V. Turaev

We consider a two-parameter family of C^3-smooth dynamical systems X_μ on an $(m+k)$-dimensional $(m \geqslant 2, k \geqslant 2)$ C^3-smooth manifold that depend smoothly on $\mu = (\mu_1, \mu_2)$. It will be assumed that X_μ has a saddle equilibrium state O, and the roots $\lambda_i(\mu)$ and $\gamma_j(\mu)$ of the characteristic equation at O satisfy the relations

$$\operatorname{Re} \lambda_i(\mu) < \lambda_1(\mu) < 0 < \gamma_1(\mu) < \operatorname{Re} \gamma_j(\mu) \qquad (2 \leqslant i \leqslant m,\ 2 \leqslant j \leqslant k)$$

and $\lambda_1(\mu)+\gamma_1(\mu) > 0$. The following assumptions are made for $\mu = 0$: 1) W_0^s and W_0^u intersect in two trajectories Γ_1 and Γ_2 that are homoclinic to O; 2) Γ_1 and Γ_2 do not lie in the non-leading submanifolds of the manifolds W_0^s and W_0^u and are tangent to each other both as $t \to +\infty$ and as $t \to -\infty$; 3) the separatrix quantities A_1 and A_2 (see [1], [4]) are non-zero. Assume that the family X_μ is transversal to the membrane of codimension two singled out by the conditions 1)–3). We choose the parameters in such a way that, for $\mu_i = 0$ $(i = 1, 2)$, X_μ has a trajectory homoclinic to O that is homotopic to Γ_i in a small neighbourhood V of the contour $\Gamma_1 \cup \Gamma_2 \cup O$, and a cycle [1] is created upon passing into the domain $\mu_i < 0$.

On the (μ_1, μ_2)-plane there are curves $L_1 : \mu_1 = h_1(\mu_2)$ and $L_2 : \mu_2 = h_2(\mu_1)$, $L_i \subset \{\mu_i A_i > 0,\ \mu_{3-i} < 0\}$, $\lim\limits_{\mu_{3-i}\to 0} h_i = \lim\limits_{\mu_{3-i}\to 0} \dfrac{dh_i}{d\mu_{3-i}} = 0$, that, together with the coordinate axes, separate the plane into six domains: $\mathscr{D}_0 = \{\mu_1 > 0,\ \mu_2 > 0\}$,

$$\mathscr{D}_1 = \{\mu_2 < 0,\ A_1 h_1(\mu_2) > A_1\mu_1 > 0\},$$

$$\mathscr{D}_2 = \{\mu_1 < 0,\ A_2 h_2(\mu_1) > A_2\mu_2 > 0\},$$

$$\mathscr{D}_3 = \{\mu_2 > 0,\ \mu_1 < 0\}\setminus\mathscr{D}_2, \qquad \mathscr{D}_4 = \{\mu_2 < 0,\ \mu_1 > 0\}\setminus\mathscr{D}_1,$$

$$\mathscr{D}_5 = \{\mu_1 < 0,\ \mu_2 < 0\}\setminus(\mathscr{D}_1 \cup \mathscr{D}_2).$$

The set Ω_μ of trajectories of the system X_μ that lie entirely in V consists of: the single point O in the domain $\mathscr{D}_0$; the point O, a saddle cycle homotopic to Γ_1, and a heteroclinic trajectory with O as ω-limit and a cycle as α-limit in the domain $\mathscr{D}_3$; the point O, a saddle cycle homotopic to Γ_2, and a heteroclinic trajectory in the domain $\mathscr{D}_4$; the point O, a set B on which X_μ is equivalent to a suspension over the Bernoulli scheme of two symbols, and trajectories with trajectories in B as α-limits and O as ω-limit in the domain $\mathscr{D}_5$. In the case when $A_1 > 0$ and $A_2 > 0$ the bifurcation set in the domains $\mathscr{D}_1$ and $\mathscr{D}_2$ has a Cantor structure. In the cases when $A_1 < 0$ and $A_2 > 0$ and when $A_1 < 0$ and $A_2 < 0$, the bifurcation set contains a Cantor pencil of curves separating the domains $\mathscr{D}_1$ and $\mathscr{D}_2$ into countably many domains, of which each contains a countable set of isolated bifurcation curves. For a complete description of the passage from the domains $\mathscr{D}_3$ and $\mathscr{D}_4$ to the domain $\mathscr{D}_5$ we need some definitions.

We denote by S_a, S_a^+, and S_a^- the sets of two-sided infinite, right-side infinite, and left-side infinite sequences of symbols in the alphabet a ($a = \{1, 2\}$ or $a = \{0, 1, 2\}$). Following [3], we define three order relations $>_1$, $>_2$, and $>_3$ on $S^+_{\{0,1,2\}}$ according to the rule: if $x = \{x_i\}_{i=0}^{+\infty} \in S^+_{\{0,1,2\}}$, $y = \{y_i\}_{i=0}^{+\infty} \in S^+_{\{0,1,2\}}$, $x_i = y_i$ for $i < j$, and $y_j > x_j$ for some j (let $2 > 0 > 1$), then 1) $y >_1 x$; 2) if in addition the word $\{x_i\}_{i=0}^{j-1}$ contains an even number of 1's, then $y >_2 x$, otherwise $x >_2 y$; 3) if in addition to the first assumptions j is even, then $y >_3 x$, otherwise $x >_3 y$. A sequence $x = \{x_i\} \in S_a$ (or S_a^+) is said to be (s, l)-admissible (where $s \in S^+_{\{0,1,2\}}$ and $l \in \{1, 2, 3\}$) if, for any j, $\{x_i\}_{i=j}^{+\infty} = 0^\omega$,(1) or $s \geqslant_l \{x_i\}_{i=j}^{+\infty}$ when s begins with a 2 and $x_j = 2$, or $\{x_i\}_{i=j}^{+\infty} \geqslant_l s$ when s

(1) By p^ω $(p^{-\omega})$ we mean the right- (left-) infinite sequence consisting of the blocks p.

begins with a 1 and $x_j = 1$. A sequence $s \in S^+_{\{0,1,2\}}$, $s \neq 0^\omega$, is said to be l-selfadmissible if it is (s, l)-admissible. Suppose that $s \in S^+_{\{0,1,2\}}$ is l-selfadmissible. A *kneading* system* $K(s, l)$ is defined to be a set, equipped with a shift mapping, that consists of sequences x such that x is an (s, l)-admissible sequence in $S_{\{1,2\}}$, or $x = y0^\omega$ with $y \in S^-_{\{1,2\}}$ and ys an (s, l)-admissible sequence, or $x = 0^{-\omega}s$ in the case when $s \in S^+_{\{1,2\}}$.

Theorem. *For each μ in the domains $\mathscr{D}_1$ and $\mathscr{D}_2$ there is an l-selfadmissible $s_\mu \in S^+_{\{0,1,2\}}$ such that $X_\mu|_{\Omega_\mu}$ is topologically equivalent to a suspension*(2) *over $K(s_\mu, l)$ ($l = 1$ for $A_1 > 0$ and $A_2 > 0$, $l = 2$ for $A_1 < 0$ and $A_2 > 0$, and $l = 3$ for $A_1 < 0$ and $A_2 < 0$).*

As μ_1 (μ_2) varies in the domain $\mathscr{D}_1$ ($\mathscr{D}_2$) for each fixed μ_2 (μ_1), s_μ varies monotonically and runs through all l-selfadmissible values that begin with 1 (2). For each l-selfadmissible sequence $p0^\omega$ the set of μ such that $s_\mu = p0^\omega$ forms a domain $\mathscr{D}$. As follows from [3], $K(p0^\omega, l)$ is topologically conjugate to a topological Markov chain with finitely many states. For each l-selfadmissible $q \neq p0^\omega$ the set of μ such that $s_\mu = q$ forms a curve of the form $\mu_2 = h(\mu_1)$ or $\mu_1 = h(\mu_2)$, where

$$\lim_{\mu_i \to 0} h = \lim_{\mu_i \to 0} \frac{dh}{d\mu_i} = 0.$$

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Scientific Research Institute of Applied Mathematics and Cybernetics

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*The translator and editor are uncertain about this word.

(2)The saddle O corresponds to the trajectory $0^{-\omega}0^\omega$ in the suspension (see [2] for suspensions that include equilibrium states).