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Hyperbolic Sets near Homoclinic Loops to a Saddle for Systems with a First Integral

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Abstract—A complete description of dynamics in a neighborhood of a finite bunch of homoclinic loops to a saddle equilibrium state of a Hamiltonian system is given.

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This paper is dedicated to Segey Bolotin and Dmitry Treschev on occasion of their birthdays

In [1] the structure of a small neighborhood of a bunch of homoclinic loops to a saddle equilibrium of a Hamiltonian system was described. It was shown that if there are three or more homoclinic loops in general position, then the dynamics is chaotic, and the same is true when there are two homoclinic loops which are tangent to each other upon leaving and entering the saddle. The chaos manifests itself in the presence of hyperbolic sets in a neighborhood of the loops; Theorem 1 of [1] gave a full description of these sets. Shilnikov's original idea was that this and other results of [1] can provide simple criteria for nonintegrability and chaos in Hamiltonian systems, along with the homoclinic to a periodic orbit [2] and a homoclinic to a saddle-focus [3–6]. Recently, it has become clear that the homoclinic bunches of the type considered in [1] appear naturally at resonance crossings in near-integrable systems. The hyperbolic sets described in [1] contain heteroclinic orbits that connect saddle periodic orbits which, in the near-integrable setting, correspond to different resonances, so the homoclinic bunches occur to be essential for the switching between resonances at the Arnold diffusion [7–10].1)

The short note [1] contained no proofs. Variational versions of the result of [1] were proved in [11, 12]. Due to the recent interest in Arnold diffusion, different proofs were also proposed in [8, 9].2) In the present paper the proof of the original version is given in the hope it can be useful in Arnold diffusion studies and other applications, e.g., scattering problems [12]. Since the dynamics we discuss here is hyperbolic, we do not require any high regularity from the system and use only C^1-tools (various arguments based on partial hyperbolicity) in the computations. The other results from [1] can be proved in the same way.

Consider a $2n$-dimensional ($n \geqslant 2$) dynamical system

$$\dot{z} = X(z)$$

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1)See also preprints J.-P. Marco "Generic Hyperbolic Properties of Classical Systems on the Torus T^2" (2011) and "Generic Hyperbolic Properties of Nearly Integrable Systems on A^3" (2014).

2)See also the preprint of J.-P. Marco (2011) mentioned above.

of class C^r ($r \geqslant 1$). Let the system have at least C^2-smooth first integral H, i.e.,

$$H'(z)X(z) = 0. \tag{1}$$

A Hamiltonian system with n degrees of freedom is a natural example, but the symplectic structure is not important here. Suppose that X has a saddle equilibrium state O at the origin (i.e., $X(0) = 0$, and the eigenvalues of the matrix $X'(0)$ do not lie on the imaginary axis). By (1), $H'(0)X'(0) = 0$. Since $X'(0)$ is nondegenerate by assumption, it follows that the linear part of H at zero vanishes. Assume that the quadratic part of H at O is nondegenerate. It is easy to check that this nondegeneracy assumption and condition (1) imply that the system near O can be brought to the following form by a linear transformation:

$$\dot{x} = -Ax + o(x, y), \qquad \dot{y} = A^\top y + o(x, y), \tag{2}$$

where $x \in R^n$, $y \in R^n$, and all the eigenvalues of the $(n \times n)$-matrix A have positive real parts. Moreover, the first integral takes the following form in these coordinates:

$$H = (y, Ax) + o(x^2 + y^2). \tag{3}$$

Indeed, a general system near a saddle O can be written in the form

$$\dot{x} = -Ax + o(x, y), \qquad \dot{y} = C^\top y + o(x, y),$$

where the spectra of matrices A and C lie strictly to the right of the imaginary axis. The stable manifold of O is tangent to $y = 0$ at O and the unstable manifold is tangent to $x = 0$. As H must stay zero on the stable and unstable manifolds (we assume $H(0) = 0$), it follows that

$$H = (y, Qx) + o(x^2 + y^2)$$

for some matrix Q. The condition of the nondegeneracy of the quadratic part of H at O implies that Q is a square matrix (i.e., x and y have the same dimension n) and reads as $det(Q) \neq 0$. The invariance condition (1) gives

$$-(y, QAx) + (y, CQx) + o(x^2 + y^2) = 0 \qquad \text{for all} \qquad (x, y),$$

hence $C = QAQ^{-1}$. Now, by introducing new variables $y_{new} = Q^\top y$ and $x_{new} = A^{-1}x$, we bring the system to the form (2) and the integral H to the form (3).

We further assume that that the eigenvalues of A satisfy

$$0 \ < \ \lambda_1 \ < \mathrm{Re}\, \lambda_i$$

for all $i = 2, \ldots, n$, so the matrix A is brought to the form

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & B \end{pmatrix},$$

where the eigenvalues of the $(n-1) \times (n-1)$ matrix B are $\lambda_2, \ldots, \lambda_n$. Thus, system (2) is written as

$$\begin{aligned} \dot{x}_1 &= -\lambda_1 x_1 + o(x, y), \qquad & \dot{u} &= -Bu + o(x, y), \\ \dot{y}_1 &= \lambda_1 y_1 + o(x, y), \qquad & \dot{v} &= B^\top v + o(x, v), \end{aligned} \tag{4}$$

where we denote $u = (x_2, \ldots, x_n)$ and $v = (y_2, \ldots, y_n)$. The expression (3) for the first integral recasts as

$$H = \lambda_1 x_1 y_1 + (v, Bu) + o(x^2 + y^2). \tag{5}$$

The unstable manifold $W^u(O)$ is tangent at O to the space $(x_1, u) = 0$, and the stable manifold $W^s(O)$ is tangent to $(y_1, v) = 0$. We denote by $W^{ss} \subset W^s$ and $W^{uu} \subset W^u$ the strong stable and strong unstable $(n-1)$-dimensional manifolds, which are tangent at O to $x_1 = 0$ and $y_1 = 0$, respectively. These are smooth invariant manifolds; W^{ss} divides W^s into two parts, W^s_+ and W^s_-.

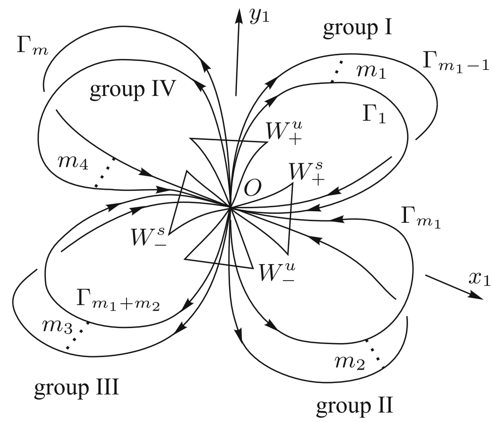

Fig. 1. Schematic figure of the bunch of homoclinic loops.

Similarly, we have $W^u = W^{uu} \cup W^u_+ \cup W^u_-$. We assume that W^u_+ adjoins W^{uu} from the side of positive y_1, and W^{s+} adjoins W^{ss} from the side of positive x_1.

Our main assumption is that *W^u and W^s intersect transversely along m homoclinic orbits* $\Gamma_1, \dots, \Gamma_m$. Note that the n-dimensional manifolds W^u and W^s both lie in the $(2n-1)$-dimensional zero level set of the smooth integral H. Therefore, they can intersect transversely along the one-dimensional orbits Γ_j, moreover, this condition is of general position. Another genericity assumption we make is that *none of the homoclinic loops $\Gamma_1, \dots \Gamma_m$ lie in W^{uu} or W^{ss}*. This means that the orbits Γ_j enter and leave the saddle O along the leading directions, the axes x_1 and y_1, respectively. Let us enumerate the loops in such a way that

$$\Gamma_j \subset W^u_+ \cap W^s_+ \quad \text{for} \ \ j \leqslant m_1 \qquad \text{(group I)},$$

$$\Gamma_j \subset W^u_- \cap W^s_+ \quad \text{for} \ \ m_1 < j \leqslant m_1 + m_2 \qquad \text{(group II)},$$

$$\Gamma_j \subset W^u_- \cap W^s_- \quad \text{for} \ \ m_1 + m_2 < j \leqslant m_1 + m_2 + m_3 \qquad \text{(group III)},$$

$$\Gamma_j \subset W^u_+ \cap W^s_- \quad \text{for} \ \ m_1 + m_2 + m_3 < j \leqslant m_1 + m_2 + m_3 + m_4 \qquad \text{(group IV)}.$$

Thus, m_1 loops leave and enter O from the positive side, m_2 loops leave O towards negative y_1 and enter from the side $x_1 > 0$, m_3 loops leave and enter O from the negative side, and m_4 loops leave O towards positive y_1 and enter from the side $x_1 < 0$ (see Fig. 1). Some of m_k can be zero, but altogether they give $m_1 + m_2 + m_3 + m_4 = m$.

Denote by X_h the restriction of the system to the level set $H = h$. Let $\mathcal{V}$ be a small neighborhood of the set $\Gamma_1 \cup \dots \cup \Gamma_m \cup O$. Denote by Ω_h the set of orbits of X_h, which lie in $\mathcal{V}$ for all times $t \in (-\infty, +\infty)$.

Theorem 1. *For a sufficiently small $\mathcal{V}$ and a sufficiently small $h_0 > 0$ that depends on $\mathcal{V}$, the following holds:*

1. $\Omega_0 = \Gamma_1 \cup \dots \cup \Gamma_m \cup O$.
2. *If $h \neq 0$, $|h| < h_0$, then the set Ω_h is hyperbolic. The restriction of X_h to Ω_h is topologically equivalent to a suspension over a topological Markov chain, which is defined by the transition matrix* $\mathcal{M}^+ = \begin{pmatrix} m_1 & m_2 \\ m_4 & m_3 \end{pmatrix}$ *at $h \in (0, h_0)$ and* $\mathcal{M}^- = \begin{pmatrix} m_2 & m_1 \\ m_3 & m_4 \end{pmatrix}$ *at $h \in (-h_0, 0)$.*

The transition matrices in item 2 of the theorem define oriented graphs with 2 vertices (see Fig. 2). The entry at the intersection of row s and column s' of the transition matrix is the number of edges that go from the vertex s to the vertex s'. The corresponding topological Markov chain

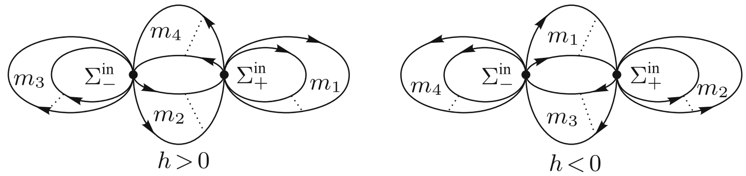

Fig. 2. Graphs of topological Markov chains $\mathcal{M}^+$ $(h > 0)$ and $\mathcal{M}^-$ $(h < 0)$.

is the set of all infinite (in both directions) paths $\{\Gamma_{j_i}\}_{i=-\infty}^{+\infty}$ over the edges of the graph, with the shift map $\sigma : \{\Gamma_{j_i}\} \mapsto \{\Gamma_{j_{i+1}}\}$ acting on this set. The edges of the graphs correspond to homoclinic loops Γ_j in the following way: the entry m_1 in the transition matrix corresponds to the first m_1 loops (those in $W^u_+ \cap W^s_+$), the entry m_2 corresponds to the next m_2 loops (those in $W^u_- \cap W^s_+$), and so on. Note that the neighborhood $\mathcal{V}$ consists of a small ball $\mathcal{V}_0$ around the saddle O and m handles, $\mathcal{V}_1, \ldots, \mathcal{V}_m$, such that $\mathcal{V}_j$ surrounds $\Gamma_j \backslash \mathcal{V}_0$. Each orbit in Ω_h at $h \neq 0$ enters $\mathcal{V}_0$, then gets into one of the handles $\mathcal{V}_j$, returns to $\mathcal{V}_0$, enters a new handle, and so on. This means that each orbit is coded by a bi-infinite sequence of symbols Γ_j, according to the above-described itinerary. Thus, item 2 of the theorem provides a complete description of all possible codings and also establishes that the coding determines the corresponding orbit uniquely (for each given h).

By uniqueness of the orbit with a given coding, periodic codings correspond to periodic orbits. These orbits are saddle (as elements of the hyperbolic set Ω_h), so they smoothly depend on h. Thus, given a coding compatible with the transition matrix $\mathcal{M}^+$ or $\mathcal{M}^-$, the family of corresponding periodic orbits parameterized by $h > 0$ or $h < 0$, respectively, forms a smooth two-dimensional manifold bounded by homoclinic loops at $h = 0$ (the neighborhood $\mathcal{V}$ can be taken as small as possible, so the limit of Ω_h as $h \to 0$ must be contained in Ω_0).

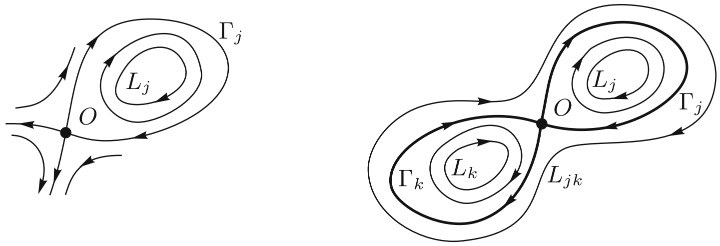

Fig. 3. The behavior on the invariant manifold $\mathcal{L}_j$ (left) and invariant manifold $\mathcal{L}_{jk}$ (right).

In particular, for each of the $(m_1 + m_3)$ homoclinic loops Γ_j that belong to group I or group III we have a unique saddle periodic orbit $L_j(h)$ with the code Γ_j^ω at every small $h > 0$. This is a single-round orbit, i.e., it goes through the handle around Γ only once and closes up. As $h \to 0$, the limit of $L_j(h)$ is the loop Γ_j (see Fig. 3). We will show in the proof of the theorem (in the beginning of the proof of Lemma 1) that our loops Γ_j satisfy conditions of the *nonlocal center manifold theorem for homoclinic loops* (see [13–17] or Theorem 6.3 in [18]). This theorem establishes that there exists a smooth, normally hyperbolic, two-dimensional invariant manifold $\mathcal{L}_j$, which is tangent to the axes x_1 and y_1 at O and contains all the orbits that never leave the small neighborhood $\mathcal{V}_0 \cup \mathcal{V}_j$ of $\Gamma_j \cup O$. The restriction of the system on the surface $\mathcal{L}_j$ has a smooth first integral $H = \lambda_1 x_1 y_1 + o(x_1^2 + y_1^2)$ (see (5)). Thus, $\mathcal{L}_j \cap \{h > 0\}$ is filled by the single-round periodic orbits $L_j(h)$. The same conclusion holds at $h < 0$ for the homoclinic loops Γ_j and their corresponding families of single-round periodic orbits $L_j(h)$ from groups II and IV. By item 2 of our theorem, every two surfaces $\mathcal{L}_j$ from the same group have a heteroclinic connection on every positive (groups I and III) or every negative (groups II and IV) level set of H. These connections correspond to transverse (on the given level set of H) intersections of the stable and unstable manifolds of the periodic orbits L_j — the transversality is a part of the hyperbolicity property of the sets Ω_h. When the groups are different, heteroclinic connections between them are present when a corresponding entry in the transition matrix is nonzero.

Now consider a pair (a figure eight) of homoclinic loops Γ_j and Γ_k which leave and enter O from the opposite sides, i.e., Γ_j belongs to group I while Γ_k belongs to group III, or Γ_j belongs to group II while Γ_k belongs to group IV. As the loops are not tangent to each other at O, the nonlocal center manifold theorem holds for this pair of loops as well [16]. Namely, there exists a smooth, normally hyperbolic, two-dimensional invariant manifold $\mathcal{L}_{jk}$, which is tangent to the axes x_1 and y_1 at O and contains all the orbits that never leave the small neighborhood $\mathcal{V}_0 \cup \mathcal{V}_j \cup \mathcal{V}_k$ of $\Gamma_j \cup \Gamma_k \cup O$ (see Theorem 6.5 of [18]). In other words, the manifolds $\mathcal{L}_j$ and $\mathcal{L}_k$ near the loops Γ_j and Γ_k can be chosen in such a way that their union is the smooth invariant manifold $\mathcal{L}_{jk}$. The restriction of the system on the two-dimensional surface $\mathcal{L}_j$ has a smooth first integral H whose quadratic part at O is nondegenerate. Thus, $\mathcal{L}_{jk}$ is filled by the single-round periodic orbits $L_j(h)$ and $L_k(h)$ at $h > 0$ if Γ_j and Γ_k belong to groups I and III, and $h < 0$ if the loops belong to groups II and IV, while at the opposite sign of h the invariant surface is filled by the double-round periodic orbits $L_{jk}(h)$, which correspond to the coding $(\Gamma_j\Gamma_k)^\omega$ (see Fig. 3). It follows from item 2 of our theorem that every two different surfaces $\mathcal{L}_{jk}$ have a transverse heteroclinic connection at all small $h \neq 0$.

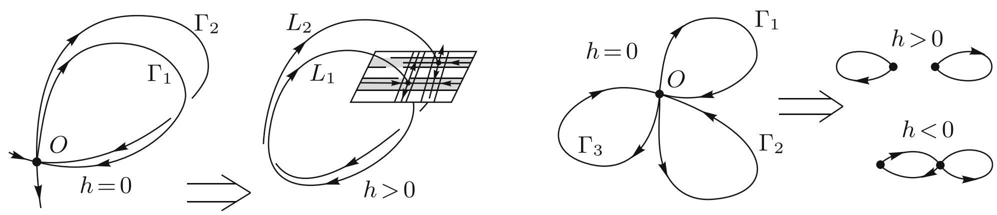

Fig. 4. The simplest cases with positive entropy. Left: When two loops are tangent to each other at O, the saddle periodic orbits $L_1(h)$ and $L_2(h)$ at $h > 0$ come sufficiently close to each other, so their stable and unstable manifolds intersect and a Smale horseshoe exists. Right: A case of three homoclinic loops is shown, with the graphs of corresponding Markov chains at $h > 0$ (no chaos) and $h < 0$ (chaos).

If $m = 1$, then our theorem shows that the only orbits that never leave $\mathcal{V}$ are the single-round periodic orbits $L(h)$, the loop Γ, and the saddle O. If $m = 2$, and the loops leave and enter O from the opposite directions (i.e., $m_1 = m_3 = 1$, $m_2 = m_4 = 0$, or $m_2 = m_4 = 1$, $m_1 = m_3 = 0$), then only the double-round cycles are added to the list. These cases are analogous to systems on a plane. Another case of simple dynamics, which is different from the two-dimensional case, corresponds to two loops which are tangent to each other when they leave O, but they enter O from opposite directions (or they leave O from the opposite directions and are tangent when they enter O). In this case we have only single-round orbits $L(h)$ for both signs of h. Other cases, i.e., $m \geqslant 3$, or $m = 2$ and the loops are tangent to each other upon leaving and entering O, produce chaos, i.e., the restriction of X to Ω_h has positive entropy for all small h of appropriate sign.

Let us now prove the theorem. Let the equation of the local unstable manifold $W^u(O)$ be $\{x_1 = x^u(y_1, v), u = u^u(y_1, v)\}$ and the equation of the local stable manifold $W^s(O)$ be $\{y_1 = y^s(x_1, u), v = v^s(x_1, u)\}$, where x^u, u^u, y^s, and v^s are smooth functions that vanish at zero along with the first derivatives. Take a sufficiently small $d > 0$ and consider two cross-sections $\Sigma_+^{in} : \{x_1 = d\}$, $\Sigma_-^{in} : \{x_1 = -d\}$ to W^s and two cross-sections $\Sigma_+^{out} : \{y_1 = d\}$, $\Sigma_-^{out} : \{y_1 = -d\}$ to W^u. Let $M_j^{out} = (x_j^{out} = x^u(y_j^{out}, v_j^{out}), u_j^{out} = u^u(y_j^{out}, v_j^{out}), y_j^{out}, v_j^{out})$ be the points of intersection of the loop Γ_j with $\Sigma_+^{out} \cup \Sigma_-^{out}$. For the loops Γ_j from groups I and IV, we have $y_j^{out} = d$, while we have $y_j^{out} = -d$ for the loops from groups II and III. As all the loops leave O tangent to the y_1-axis, it follows that $(v_j^{out}, x_j^{out}, u_j^{out}) = o(d)$. Similarly, denote as $M_j^{in} = (x_j^{in}, u_j^{in}, y_j^{in} = y^s(x_j^{in}, u_j^{in}), v_j^{in} = v^s(y_j^{in}, v_j^{in}))$ the points of intersection of the loop Γ_j with $\Sigma_+^{in} \cup \Sigma_-^{in}$. We have $x_j^{in} = d$ for the loops Γ_j from groups I and II, and $x_j^{in} = -d$ for the loops from groups III and IV; we also have $(u_j^{in}, y_j^{in}, v_j^{in}) = o(d)$. Let Π_j^{out} and Π_j^{in} be sufficiently small neighborhoods (within the cross-sections $\Sigma_\pm^{out}$ and $\Sigma_\pm^{in}$) of the points M_j^{out} and M_j^{in}, respectively. Denote $\Pi_j^{in}(h) = \Pi_j^{in} \cap \{H = h\}$ and $\Pi_j^{out}(h) = \Pi_j^{out} \cap \{H = h\}$ (see Fig. 5). If d is taken sufficiently small, then $\frac{\partial H}{\partial y_1} \neq 0$ at $|x_1| = d$

(see (5)). Therefore, y_1 is a well-defined smooth function of (u, v) everywhere on $\Pi_j^{in}(h)$, so the coordinates (u, v) define the points of $\Pi_j^{in}(h)$ uniquely for every given small h. Similarly, x_1 is uniquely defined by the values of u and v for every point of $\Pi_j^{out}(h)$ for every given small h.

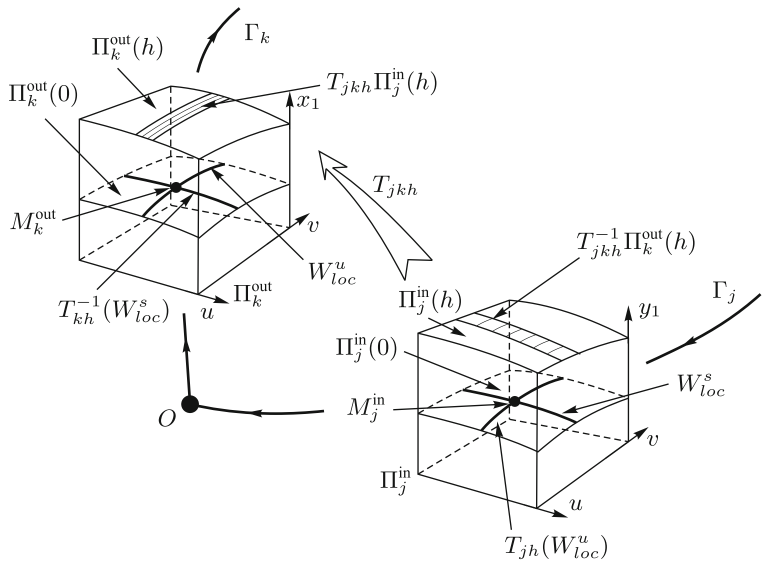

Fig. 5. Illustration to the proof of the theorem.

The orbits from the set Ω_h must enter the d-neighborhood of O by intersecting $\Sigma^{in} = \Sigma^{in}_- \cup \Sigma^{in}_+$ at one of the neighborhoods $\Pi_j^{in}(h)$ and leave the d-neighborhood of O by intersecting $\Sigma^{out} = \Sigma^{out}_- \cup \Sigma^{out}_+$ at one of the neighborhoods $\Pi_j^{out}(h)$. If $\{\Gamma_{j_i}\}_{i=-\infty}^{+\infty}$ is the coding of such an orbit, then it intersects Σ^{in} at a point of $\Pi_{j_{s-1}}^{in}(h)$, then intersects Σ^{out} at a point of $\Pi_{j_s}^{out}(h)$, then intersects Σ^{in} at a point of $\Pi_{j_s}^{in}(h)$, and so on. The following lemma shows that these intersections can only follow a path in the graph defined by the transition matrix $\mathcal{M}^+$ at $h > 0$ and $\mathcal{M}^-$ at $h < 0$.

Lemma 1. *Let d be sufficiently small, and the size of the neighborhoods $\Pi_j^{in,out}$ be sufficiently small ($\ll d$).*

1. *No orbit thats starts at $\Pi_j^{in}(0)$ can leave the d-neighborhood of O by crossing $\Pi_k^{out}(0)$, for any $j, k = 1, \ldots, m$.*
2. *If $h \neq 0$, then an orbit that starts at $\Pi_j^{in}(h)$ can get to $\Pi_k^{out}(h)$ at the moment it leaves the d-neighborhood of O only for such j and k that*

$$\operatorname{sgn}(x_j^{in} y_k^{out}) = \operatorname{sgn}(h). \tag{6}$$

Proof. Let us recall the theorem on a nonlocal invariant manifold in a neighborhood of a homoclinic loop (see, e.g., [18, Chapter 6]). It says that if the loop Γ exits the saddle tangent to the leading axis (the axis y_1 in our case), and if the unstable manifold $W^u(O)$ is transverse to the extended stable manifold W^{se} at any point of the loop, then the system has a smooth, repelling, invariant manifold $W^{cs}(\Gamma)$ such that $\dim(W^{cs}(\Gamma)) = \dim(W^s(O)) + 1 = n + 1$, the manifold $W^{cs}(\Gamma)$ contains O (along with its local stable manifold) and the loop Γ, it is tangent at O to the space $v = 0$, and it contains all the orbits that stay in a small neighborhood of Γ for all positive times t. In our situation, the only condition we need to check in order to guarantee the existence of such a manifold $W^{cs}(\Gamma)$ is the transversality of $W^u(O)$ to $W^{se}(O)$ at the points of Γ.

The $(n+1)$-dimensional manifold W^{se} contains $W^s_{loc}(O)$ and is tangent to $v = 0$ at O [18]. Let us show that our condition that W^u and W^s intersect along the orbit Γ transversely within the level set $H = 0$ is equivalent to the transversality of W^u to W^{se} at the points of Γ. Indeed, as all the manifolds involved are invariant, it is enough to verify the transversality at any point of Γ. At the point $M^{in} = \Gamma \cap \Sigma^{in}$ we have $|x_1| = d$, and $(y_1, u, v) = o(d)$, so the level set $\{H = 0\}$ near this point is given by the equation $y_1 = F(u, v)$ where the first derivatives of F can be made as small as we want by taking d sufficiently small (see (5)). Therefore, the level set $\{H = 0\}$ intersects W^{se} at the point M^{in} transversely, since M^{in} is close to O and W^{se} is tangent to $v = 0$ at O. Hence, at the point M^{in} the transversality of W^u to W^{se} is equivalent to the transversality of W^u to $W^{se} \cap \{H = 0\}$ within the level set $H = 0$. It remains to note that $W^{se} \cap \{H = 0\}$ coincides with W^s_{loc} near the point M^{in} (by the transversality of W^{se} to $\{H = 0\}$, the $(2n-1)$-dimensional level set $\{H = 0\}$ must intersect W^{se} by an n-dimensional smooth manifold near M^{in}, and the manifold W^s_{loc} is n-dimensional and belongs to both W^{se} and $\{H = 0\}$).

Thus, when the conditions of our theorem hold, the nonlocal invariant manifold theorem is applicable to each of the loops Γ_j, so each such loop belongs to an $(n+1)$-dimensional smooth, repelling, invariant manifold $W^{cs}(\Gamma_j)$ which is tangent to $v = 0$ at O and contains all orbits that stay in a small neighborhood of Γ_j for all positive times. We do not assume our system is reversible, but the problem itself is still symmetric with respect to time reversal, so we also have, for each of the loops Γ_j, the existence of an $(n+1)$-dimensional smooth, attracting, invariant manifold $W^{cu}(\Gamma_j)$ which is tangent to $u = 0$ at O and contains all orbits that stay in a small neighborhood of Γ_j for all negative times. The intersection $W^c(\Gamma_j) = W^{cu}(\Gamma_j) \cap W^{cs}(\Gamma_j)$ is a smooth two-dimensional invariant manifold which is tangent to the (x_1, y_1)-plane at O and contains all entire orbits that never leave a small neighborhood of Γ_j.

We will use the following trick in order to simplify the computations. Let us take some $\delta > 0$. Note that given any two homoclinic loops Γ_j and Γ_k, we can, by an obvious use of the flow-box argument, modify our system outside the δ-neighborhood of O such that the new system will have a homoclinic loop Γ which would coincide with Γ_k when it leaves O and with Γ_j when it enters O. This can be done in such a way that the modified system still has the first integral H and the transversality condition holds. Then the modified system will have a smooth invariant manifold W^{cs} which contains the newly created loop Γ. The intersection of this manifold with the δ-neighborhood of O will be a local invariant manifold of the original system. Thus, we find that our original system has a smooth invariant manifold W^{cs}_{jk} in the neighborhood of O, which contains W^s_{loc} and, for every choice of $d < \delta$, it contains the points $M^{in}_j = \Gamma_j \cap \{|x_1| = d|\}$ and $M^{out}_k = \Gamma_k \cap \{|y_1| = d\}$. This manifold is tangent to $v = 0$ at O, so it is given by

$$v = v^{cs}_{jk}(x_1, u, y_1), \tag{7}$$

where the derivative of v^{cs}_{jk} in the δ-neighborhood of O is bounded by a constant ε, which can be taken as small as we want by taking δ sufficiently small. This also implies

$$\|v^{cs}_{jk}\| \leqslant \varepsilon \|x_1, u, y_1\|. \tag{8}$$

For the restriction of the system on the manifold W^{cs}_{jk}, the unstable manifold of O is one-dimensional. It is a curve divided by O into two halves (separatrices), one of which passes through the point M^{out}_k. The orbits that start close enough to the stable manifold of O must leave its neighborhood close enough to the unstable manifold. So, the orbits, which start in W^{cs}_{jk} above W^s_{loc} when y^{out}_k (the y-coordinate of M^{out}_k) is positive or below W^s_{loc} when $y^{out}_k < 0$, get close to M^{out}_k and reach the cross-section Π^{out}_k. On the contrary, as W^s_{loc} divides W^{cs}_{jk}, the orbits that start in W^{cs}_{jk} on the opposite side of W^s_{loc} leave the neighborhood of O close to the separatrix that does not contain M^{out}_k, so they do not get to Π^{out}_k. The orbits that start at W^s_{loc} do not get to Π^{out}_k either (they tend to O).

On the manifold W^{cs}_{jk} we have (see (5), (7), (8))

$$\frac{\partial H}{\partial y_1} = x_1 + O(u)\frac{\partial v^{cs}_{jk}}{\partial y_1} + o(x, y).$$

As $|x_1| = d$ and $(u, y_1, v) = o(d)$ in Π_j^{in}, it follows that

$$\operatorname{sgn}\left(\frac{\partial H}{\partial y_1}\right) = \operatorname{sgn}\left(x_j^{in}\right)$$

everywhere in $\Pi_j^{in} \cap W_{jk}^{cs}$. Note that the integral H vanishes everywhere on W_{loc}^s. Therefore, $H > 0$ above W_{loc}^s when $x_j^{in} = d$, and $H < 0$ above W_{loc}^s when $x_j^{in} = -d$. Thus, the above arguments show that the orbits that start in $\Pi_j^{in}(h) \cap W_{jk}^{cs}$ get to $\Pi_k^{out}(h)$ if and only if (6) holds, in agreement with the statement of the lemma.

In order to show that the same is true for all the other orbits that start in $\Pi_j^{in}(h)$, we recall that the point O is partially hyperbolic in the following sense: the tangent space at O is a direct sum of two invariant subspaces of the linearized system (the v-space and the (x_1, u, y_1)-space) such that the expansion along the directions from the first subspace is stronger than any possible expansion in the second subspace (the expansion in the (x_1, u, y_1)-space is bounded by $e^{\lambda_1 t}$, while the expansion in v is larger than $e^{\bar{\lambda} t}$ where $\lambda_1 < \bar{\lambda} < \min_{i \geqslant 2} \mathrm{Re}\lambda_i$, see (4)). It follows that given any $\varepsilon > 0$, the forward flow of the linearized system takes the cone

$$\|x_1, u, y_1\| \leqslant \varepsilon \|v\|$$

strictly inside itself. Moreover, the linearized flow restricted to this cone expands in the v-directions with the rate $e^{\bar{\lambda} t}$ at least.

For any fixed ε, this *cone property* is obviously inherited by the system linearized along any orbit that lies in a sufficiently small δ-neighborhood of O. This also implies that for any two orbits $(x^{(1)}, y^{(1)})$ and $(x^{(2)}, y^{(2)})$ in the δ-neighborhood of O, if

$$\|x_1^{(2)}(t) - x_1^{(1)}(t), u^{(2)}(t) - u^{(1)}(t), y_1^{(2)}(t) - y_1^{(1)}(t)\| \leqslant \varepsilon \|v^{(2)}(t) - v^{(1)}(t)\| \tag{9}$$

at $t = 0$, then the same holds true for all positive times t such that the orbits stay in the δ-neighborhood of O. Moreover,

$$\|v^{(2)}(t) - v^{(1)}(t)\| \geqslant e^{\bar{\lambda} t} \|v^{(2)}(0) - v^{(1)}(0)\|. \tag{10}$$

Now take any point $P_1 = (x_1^{(1)}, u^{(1)}, y_1^{(1)}, v^{(1)}) \in \Pi_j^{in}(h)$. We have $x_1^{(1)} = x_j^{in} = \pm d$ and $u^{(1)} = o(d)$. By (5), the intersection $S(u^{(1)}, h) = \Pi_j^{in}(h) \cap \{u = u^{(1)} = \text{const}\}$ satisfies

$$h = x_j^{in} y_1 + (v, Bu^{(1)}) + o((x^{(1)})^2 + (y^{(1)})^2).$$

This gives $\dfrac{\partial y_1}{\partial v} = \dfrac{1}{d}\left[O(u^{(1)}) + o(x^{(1)}, y^{(1)})\right] = o(1)_{d \to 0}$. Thus, when d is taken sufficiently small, the manifold $S(u^{(1)}, h)$ is given by an equation $(x_1, u, y_1) = G(v)$ where G is an ε-Lipshitz function. If ε was taken small enough, this guarantees the existence of a unique point of intersection with the manifold W_{jk}^{cs} given by (7). Denote this point $P_2 = (x_1^{(2)}, u^{(2)}, y_1^{(2)}, v^{(2)})$. By construction, the orbits of points $P_{1,2}$ satisfy (9) at $t = 0$, so they will satisfy the same inequality at all positive t, as long as they stay in the δ-neighborhood of O. The orbit of point P_2 stays in the invariant manifold W_{jk}^{cs}, so the norm of $v^{(2)}(t)$ is much smaller than the norm of $(x_1^{(2)}(t), u^{(2)}(t), y_1^{(2)}(t))$ (see (8)). Then, it follows from (9) that as long as the orbit of P_1 stays in the d-neighborhood of O, the orbit of P_2 stays in, say, $2d$-neighborhood of O, hence $\|v^{(2)}\| \leqslant 2\varepsilon d < d$.

Now, again by (9), we conclude that if the orbit of the arbitrarily chosen point $P_1 \in \Pi_j^{in}(h)$ leaves the d-neighborhood of O by crossing Π_k^{out} (which corresponds to $y_1^{(1)}(t) = y_k^{out} = \pm d$), then $|y_1^{(2)}(t) - y_k^{out}| \leqslant 2\varepsilon d \ll d$ at this moment of time. This means that the orbit of the corresponding point $P_2 \in W_{jk}^{cs} \cap \Pi_j^{in}(h)$ is, at this t, above W_{loc}^s if $y_k^{out} > 0$ and below W_{loc}^s if $y_k^{out} < 0$, i.e., it must also hit the cross-section Π_k^{out} at some time. As we have shown above, this is possible only if $h \neq 0$ and condition (6) is satisfied, which proves the lemma. □

Item 1 of the lemma immediately implies item 1 of the theorem. Item 2 implies that the coding of every orbit of Ω_h must give a path in the graph defined by the transition matrix $\mathcal{M}^\sigma$ (where $\sigma = \mathrm{sgn}(h)$) defined in item 2 of the theorem. In order to finish the proof of the theorem, it remains to show that given any bi-infinite path in this graph, there exists a unique orbit in Ω_h with the corresponding coding. We prove this fact by showing that the Poincaré maps $\hat{T}_{jkh} : \Pi_j^{in}(h) \to \Pi_k^{in}(h)$ defined by the orbits, which pass from Π_j^{in} to Π_k^{out} near O and then follow the homoclinic loop Γ_k, are hyperbolic.

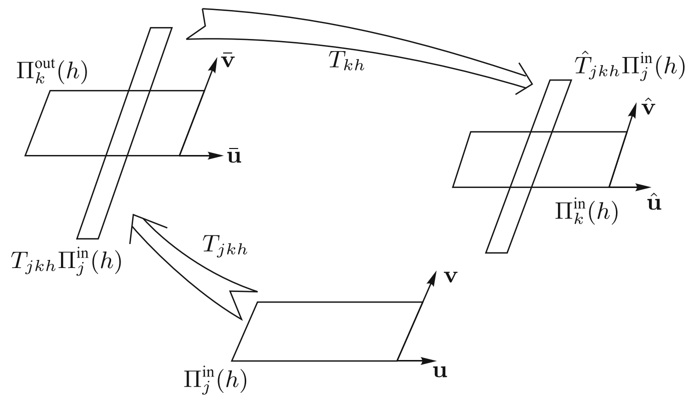

Fig. 6. Illustration to the proof of the theorem. The maps T_{jkh} and $\hat{T}_{jkh} = T_{kh} \circ T_{jkh}$ contract in u and expand in v.

Recall that for any fixed h the coordinates (u, v) define the points in Π_j^{in} (and in Π_k^{out} as well) uniquely. The hyperbolicity of the maps $\hat{T}_{jkh}$ essentially means contraction in u and expansion in v (see Fig. 6). The computations employed for the proof of the hyperbolicity are done by decomposing $\hat{T}_{jkh}$ into the product of two Poincaré maps, the *local map* $T_{jkh} : \Pi_j^{in}(h) \to \Pi_k^{out}(h)$ by the orbits that lie in the d-neighborhood of O and the *global map* $T_{kh} : \Pi_k^{out}(h) \to \Pi_k^{in}(h)$ by the orbits close to $\Gamma_k \cap \mathcal{V}_k$. We start by showing a very strong hyperbolicity of the local maps T_{jkh} (see Lemma 3).

Let us call a smooth manifold of the type $(x_1, u, y_1) = G(v)$, where $\|G'(v)\| \leqslant \varepsilon$, a v-graph. While proving Lemma 1, we noticed that the positive time shift of a v-graph remains a v-graph all the time it stays in the δ-neighborhood of O (see (9)). Moreover, the derivative of the time-t map restricted to the v-graph is strongly expanding:

$$\left\| \frac{\partial v(0)}{\partial v(t)} \right\| \leqslant e^{-\bar{\lambda} t} \tag{11}$$

(see (10)). As we showed in the proof of Lemma 1, the intersection $S(u, h)$ of the set $u = \mathrm{const}$ with $\Pi_j^{in}(h)$ is a v-graph, which has a unique point P of the intersection with the manifold W_{jk}^{cs} for any chosen k. We also showed that, by construction of W_{jk}^{cs}, if condition (6) holds for j, k, and h, then the forward orbit $(x_1^*(t), u^*(t), y_1^*(t), v^*(t))$ of the point P leaves the d-neighborhood of O by intersecting the subset $\Pi_k^{out}(h)$ of the cross-section Σ, i.e., $y_1^*(t) - y_k^{out}$ changes sign while $v^*(t)$ remains small, $\|v^*(t)\| \ll d = |y_k^{out}|$. When the point P is close to W_{loc}^s, i.e., at h small enough, the time t needed to get to $\Pi_k^{out}(h)$ is large, so the flow map restricted to the v-graph $S(u, h)$ is strongly expanding (see (11)). Therefore, in $S(u, h)$ there is a small neighborhood of P such that the image of this neighborhood by the time-t map covers the whole d-neighborhood of $v^*(t)$ in the v-space, for all t from a certain interval when the orbit of P passes from one side from Σ to the other one. This means that for any $\bar{v}$ from a neighborhood of v_k^{out} there exists a point in $S(u, h)$ whose orbit intersects $\Pi_k^{out}(h)$ at a point with the v-coordinate equal to $\bar{v}$.

As the value of u in this argument can be taken arbitrary in a small neighborhood of u_j^{in}, we arrive at the following conclusion.

Lemma 2. *For all sufficiently small h and the indices j, k satisfying condition (6), and for all u and $\bar v$ such that $\|u - u_j^{in}\|$ and $\|v - v_k^{out}\|$ are small, there exist $\bar u$ and v such that $(u, v) \in \Pi_j^{in}(h)$ and the orbit of the point (u, v) leaves the d-neighborhood of O by intersecting the cross-section Σ at the point $(\bar u, \bar v) \in \Pi_k^{out}(h)$, i.e., $(\bar u, \bar v) = T_{jkh}(u, v)$.*

The lemma below considerably strengthens this statement. Recall that the point M_j^{in} lies in the manifold $W^s_{loc} \cap \Pi_j^{in}$, which has the equation $v = v^s(x_j^{in}, u)$, and M_k^{out} lies in the manifold $W^u_{loc} \cap \Pi_j^{in}$, which has the equation $u = u^u(y_k^{out}, v)$, where u^u and v^s are smooth functions. We make a change of the coordinate v in Π_j^{in} by the rule $v_\circ = v - v^s(x_j^{in}, u)$, and a change of the coordinate u by the rule $u_\circ = u - u^u(y_k^{out}, v)$ in Π_k^{out} (note that we do not change the coordinate v in Π_k^{out} and u in Π_j^{in}).

Lemma 3. *The map $(u, \bar v) \mapsto (\bar u, v)$ is single-valued, smooth, and strongly contracting:*

$$\left\| \frac{\partial(\bar u_\circ, v_\circ)}{\partial(u, \bar v)} \right\| \to 0 \tag{12}$$

as $h \to 0$.

Proof. The map $T_{jkh} : (u, v) \mapsto (\bar u, \bar v)$ is smooth (it is a Poincaré map of a smooth system of differential equations). Therefore, we will prove that $(u, \bar v) \mapsto (\bar u, v)$ is a well-defined smooth map if we show that $\frac{\partial \bar v}{\partial v}$ is invertible. For this it is enough to check that the map $v \mapsto \bar v$ at every constant u is uniformly expanding. In order to do this, take some small h and denote as $S(u, h)$ the section of $\Pi_j^{in}(h)$ that corresponds to a constant u close to u_j^{in}. Let $P \in S(u, h)$ be a point defined by a certain value of $v = v(0)$ such that the orbit $(x_1(t), u(t), y_1(t), v(t))$ of P hits Π_k^{out} at a point Q at some time t. As $h \to 0$, the point P tends to W^s_{loc}, the point Q tends to W^u_{loc}, and the flight time t tends to infinity. If we give an infinitesimal increment Δv to the initial condition $v(0)$ on $S(u, h)$, the time of arrival at the cross-section Π_k^{out} (which corresponds to a constant $y_1 = y_k^{out}$) will get the increment

$$\Delta t = -\frac{1}{\dot y_1} \frac{\partial y_1(t)}{\partial v(0)} \Delta v,$$

where $\dot y_1$ is taken at the end point Q. Therefore, the derivatives of the Poincaré map T_{jkh} are related to the derivatives of the time-t map by the rule

$$\frac{\partial \bar v}{\partial v} = \frac{\partial v(t)}{\partial v(0)} - \frac{\dot v}{\dot y_1} \frac{\partial y_1(t)}{\partial v(0)}, \qquad \frac{\partial \bar u}{\partial v} = \frac{\partial u(t)}{\partial v(0)} - \frac{\dot u}{\dot y_1} \frac{\partial y_1(t)}{\partial v(0)}, \tag{13}$$

where $\dot y_1$, $\dot u$, and $\dot v$ are taken at the point Q. By multiplying by Δv we get

$$\Delta \bar v = \Delta v(t) - \frac{\dot v}{\dot y_1} \Delta y_1(t). \tag{14}$$

Since the increment in the initial conditions is tangent to the v-graph $S(u, h)$, the increment $(\Delta x_1(t), \Delta u(t), \Delta y_1(t), \Delta v(t))$ of the time-t image is also tangent to a v-graph, so

$$|\Delta y_1(t)| \leqslant \varepsilon \|\Delta v(t)\|. \tag{15}$$

We also have that $\dot v = o(d) \ll \dot y_1$ at the points of $\Pi_k^{out}(h)$. Thus, $\Delta \bar v = (1 + o(1)_{d \to 0}) \Delta v(t)$ in (14). Hence, if the constant d was taken sufficiently small in advance, then the strong expansion in v by the flow map (see (11)) gives us the strong expansion in v by the Poincaré map, as required.

Now, by (11), we have

$$\frac{\partial v_\circ}{\partial \bar v} = \frac{\partial v}{\partial \bar v} = O(e^{-\bar\lambda t}).$$

Since the flight time t tends to infinity as $h \to 0$, this shows that $\dfrac{\partial v}{\partial \bar{v}} \to 0$ as $h \to 0$. To complete the lemma, it remains to show that $\dfrac{\partial \bar{u}_\circ}{\partial \bar{v}} \to 0$ as $h \to 0$ (the estimates for derivatives with respect to u follow by the symmetry of the problem). By (13),

$$\frac{\partial \bar{u}_\circ}{\partial v} = \frac{\partial u_\circ(t)}{\partial v(0)} - \frac{\dot{u}_\circ}{\dot{y}_1}\frac{\partial y_1(t)}{\partial v(0)},$$

$$\frac{\partial \bar{u}_\circ}{\partial \bar{v}} = \left[\frac{\partial u_\circ(t)}{\partial v(t)} - \frac{\dot{u}_\circ}{\dot{y}_1}\frac{\partial y_1(t)}{\partial v(t)}\right] \cdot \left(1 - \frac{\dot{v}}{\dot{y}_1}\frac{\partial y_1(t)}{\partial v(t)}\right)^{-1},$$

where $u_\circ(t) = u(t) - u^u(y(t), v(t))$. By (15), we have that $\dfrac{\partial y_1(t)}{\partial v(t)} = \dfrac{\partial y_1(t)}{\partial v(0)}\left(\dfrac{\partial v(t)}{\partial v(0)}\right)^{-1}$ is uniformly bounded. Thus, we will prove that $\dfrac{\partial \bar{u}_\circ}{\partial \bar{v}} \to 0$ if we show that $\dot{u}_\circ$ and $\dfrac{\partial u_\circ(t)}{\partial v(t)}$ tend to zero.

Recall that $u_\circ = 0$ at the points of W^u_{loc}, therefore $\dot{u}_\circ \to 0$ as $h \to 0$, because the point $Q \in \Pi_k^{out}(h)$ where $\dot{u}_\circ$ is computed tends to W^u_{loc} as $h \to 0$. The last thing to prove, namely that

$$\frac{\partial u_\circ(t)}{\partial v(t)} \to 0 \tag{16}$$

as $h \to 0$, follows directly from *lambda-lemma* [19]. Indeed, note that the manifold $S = \{x_1 = x_j^{in} = \mathrm{const}, u = \mathrm{const}\} = \cup_h S(u,h)$ is transverse to the local stable manifold, therefore, by lambda-lemma, its time-t shifts S_t converge, in the C^1-topology, to W^u_{loc} as $t \to +\infty$. This means, in particular, that the manifolds S_t in a small neighborhood of M_k^{out} are given by the equation $\{x_1 = f_t(y_1, v), u = g_t(y_1, v)\}$ where the functions f_t and g_t are such that $f_t \to x^u$ and $g_t \to u^u$ as $t \to +\infty$, along with their derivatives with respect to y_1 and v. Our starting point P belongs to S, hence its time-t shift always belongs to S_t, so $u(t) = g_t(y_1(t), v(t))$ and $u_\circ(t) = g_t(y_1(t), v(t)) - u^u(y_1(t), v(t))$. Now (16) follows from $\dfrac{\partial(g_t - u^u)}{\partial v} \to 0$.

Lemma 3 shows that the local Poincaré map T_{jkh} can be written in the so-called cross-form, where the values of the coordinate u at the image and v at the preimage are uniquely determined by the values of u at the preimage and v at the image. Moreover, the fact that the cross-form is contracting means that the map itself is hyperbolic, i.e., it contracts in the u-directions and expands in the v-directions (see [18, Chapter 3] for details). Let us show that the global map $T_{kh} : \Pi_k^{out}(h) \to \Pi_k^{in}(h)$ defined by the orbits close to $\Gamma_k \cap \mathcal{V}_k$ can also be written in the cross-form.

Indeed, the flight time from the cross-section Π_k^{out} to the cross-section Π_k^{in} is bounded, therefore the map T_{kh} is a diffeomorphism that depends continuously on h. So we can write a Taylor expansion

$$\begin{aligned} \hat{u} - u_k^{in} &= \alpha_0(h) + \alpha_1(h)\bar{u} + \alpha_2(h)(\bar{v} - v_k^{out}) + o(\bar{u}, \bar{v} - v_k^{out}), \\ \hat{v} &= \beta_0(h) + \beta_1(h)\bar{u} + \beta_2(h)(\bar{v} - v_k^{out}) + o(\bar{u}, \bar{v} - v_k^{out}), \end{aligned} \tag{17}$$

where $(\bar{u}, \bar{v})$ are the coordinates of a point in $\Pi_k^{out}(h)$, and $(\hat{u}, \hat{v})$ are the coordinates of its image in $\Pi_k^{in}(h)$; the coefficients α and β depend continuously on h. From now on we use the coordinates $u_\circ$ and $v_\circ$ and omit the index $_\circ$. In these coordinates the equation of $W^u_{loc} \cap \Pi_k^{out}(0)$ is $u = 0$ and the equation of $W^s_{loc} \cap \Pi_k^{in}(0)$ is $v = 0$. The map T_{k0} takes the point $M_k^{out}(0, v_k^{out}) = \Gamma_k \cap \Pi_k^{out}$ to $M_k^{in}(u_k^{in}, 0) = \Gamma_k \cap \Pi_k^{in}$, so $\alpha_0(0) = 0$, $\beta_0(0) = 0$. Recall that we assume that the intersection of W^u and W^s along the homoclinic loop Γ is transverse (within the level set $H = 0$). This means that $T_{k0}W^u_{loc}$ is transverse to W^s_{loc} within the set $\Pi_k^{in}(0)$. By (17), this means that $\beta_2(0)$ is invertible. Hence, we can express $\bar{v} - v_k^{out}$ from the second equation of (17) as a function of $\bar{u}$ and $\hat{v}$ for all small h, which means that the map $\hat{T}_{kh}$ can be written in a cross-form:

$$\hat{u} - u_k^{in} = p(\bar{u}, \hat{v}, h), \qquad \bar{v} - v_k^{out} = q(\bar{u}, \hat{v}, h), \tag{18}$$

where p and q are smooth functions vanishing at zero.

By an elementary computation, one obtains the following

Lemma 4. *Let U, V and $\bar U$, $\bar V$ be convex subsets of a Banach space. Let two maps $T_1 : (u,v) \mapsto (\bar u, \bar v) \in \bar U \times \bar V$ and $T_2 : (\bar u, \bar v) \mapsto (\hat u, \hat v) \in U \times V$ be written in a cross-form:*

$$T_1(u,v) = (\bar u, \bar v) \qquad \text{if and only if} \qquad \bar u = p_1(u, \bar v), \qquad v = q_1(u, \bar v),$$

and

$$T_2(\bar u,\bar v) = (\hat u, \hat v) \qquad \text{if and only if} \qquad \hat u = p_2(\bar u, \hat v), \qquad \bar v = q_2(\bar u, \hat v),$$

for some smooth functions $p_1 : U \times \bar V \to \bar U$, $q_1 : U \times \bar V \to V$, $p_2 : \bar U \times V \to U$, and $q_2 : \bar U \times V \to \bar V$. Let

$$\max\left\{\left\|\frac{\partial p_1}{\partial u}\right\|, \left\|\frac{\partial p_1}{\partial \bar v}\right\|, \left\|\frac{\partial q_1}{\partial u}\right\|, \left\|\frac{\partial q_1}{\partial \bar v}\right\|\right\} \leqslant K_1,$$

$$\max\left\{\left\|\frac{\partial p_2}{\partial \bar u}\right\|, \left\|\frac{\partial p_2}{\partial \hat v}\right\|, \left\|\frac{\partial q_2}{\partial \bar u}\right\|, \left\|\frac{\partial q_2}{\partial \hat v}\right\|\right\} \leqslant K_2,$$

for some constants $K_{1,2}$. If $K_1K_2 < 1$, then the map $T_3 = T_2 \circ T_1$ is also written in a cross-form:

$$T_3(u,v) = (\hat u, \hat v) \qquad \text{if and only if} \qquad \hat u = p_3(u, \hat v), \qquad v = q_3(u, \hat v),$$

where the smooth functions p_3, q_3 are defined everywhere on $U \times V$ and, if we define the norm in $U \times V$ as $\|u, v\| = \max\{\sqrt{K_1}\|u\|, \sqrt{K_2}\|v\|\}$, then

$$\left\|\frac{\partial(p_3, q_3)}{\partial(u, \hat v)}\right\| \leqslant \frac{\sqrt{K_1K_2}}{1 - \sqrt{K_1K_2}}.$$

If we take T_{jkh} as the map T_1 and T_{kh} as the map T_2, then Lemma 4 tells us that the Poincaré map $\hat T_{jkh} = T_{kh} \circ T_{jkh} : \Pi^{in}_j(h) \to \Pi^{in}_k(h)$ can be written in the cross-form. Moreover, the cross-map is contracting, which means the map $\hat T_{jkh}$ is hyperbolic. Namely, the constant K_1 can be taken as small as we want if h is sufficiently small (see (12)), so the condition $K_1K_2 < 1$ of Lemma 4 is fulfilled. Then we have

Lemma 5. *There exists a small $\nu > 0$ such that for all sufficiently small $h \neq 0$ and the indices j, k satisfying condition* (6)*, there exist smooth functions p_{jkh}, q_{jkh} that take the ν-neighborhood of $(u = u^{in}_j, v = v_\circ = 0)$ into the ν-neighborhood of $(u = u^{in}_k, v = v_\circ = 0)$ such that*

$$\hat T_{jkh}(u,v) = (\hat u, \hat v) \qquad \text{if and only if} \qquad \hat u = p_{jkh}(u, \hat v), \qquad v = q_{jkh}(u, \hat v).$$

Moreover,

$$\left\|\frac{\partial p_{jkh}}{\partial u}\right\| + \left\|\frac{\partial p_{jkh}}{\partial \hat v}\right\| < 1, \qquad \left\|\frac{\partial q_{jkh}}{\partial u}\right\| + \left\|\frac{\partial q_{jkh}}{\partial \hat v}\right\| < 1$$

for an appropriate choice of norms in the u- and v-spaces.

Now Shilnikov's lemma on the *fixed point of a saddle map in the infinite product of Banach spaces* [2] immediately implies that given any coding sequence $\{\Gamma_{j_i}\}_{i=-\infty}^{+\infty}$ such that $\mathrm{sgn}(x^{in}_{j_j} y^{out}_{j_{i+1}}) = \mathrm{sgn}(h)$ for all i, there exists a unique sequence of points $P_i \in \Pi^{in}_{j_j}(h)$ such that $\hat T_{j_i j_{i+1} h} P_i = P_{i+1}$ for all i. By construction, this sequence consists of points where a certain orbit entirely lying in $\mathcal{V}$ intersects Σ^{in}, and this orbit has the code $\{\Gamma_{j_i}\}_{i=-\infty}^{+\infty}$. Thus, we have established a one-to-one correspondence between the orbits in Ω_h and the bi-infinite codings defined by the transition matrices $\mathcal{M}^{\pm}$, which completes the proof of the theorem.

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