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On Global Bifurcations in Three-Dimensional Diffeomorphisms Leading to Wild Lorenz-Like Attractors

S. V. Gonchenko[1*], L. P. Shilnikov[1], and D. V. Turaev[2***]**

[1]*Research Institute of Applied Mathematics and Cybernetics, 10, Ulyanova Str. 603005 Nizhny Novgorod, Russia*

[2]*Ben Gurion University, Beer-Sheva, Israel*

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Abstract—We study dynamics and bifurcations of three-dimensional diffeomorphisms with nontransverse heteroclinic cycles. We show that bifurcations under consideration lead to the birth of wild-hyperbolic Lorenz attractors. These attractors can be viewed as periodically perturbed classical Lorenz attractors, however, they allow for the existence of homoclinic tangencies and, hence, wild hyperbolic sets.

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1. INTRODUCTION

It is well-known that bifurcations of homoclinic tangencies can create stable periodic orbits, see e.g. [1–6]. This property of being "pregnant" by stable dynamics is one of the main characteristics of area-contracting maps with homoclinic tangencies, [4, 6]. However, as it was shown in many papers, see e.g. [4, 6–9], bifurcations of homoclinic tangencies in maps that do not contract two-dimensional areas can lead to attractors of a non-trivial structure, e.g. to stable invariant tori. Moreover, it was announced in [4] that genuine strange attractors can be born at homoclinic bifurcations.

We do not speak here about Hénon-like strange attractors that are born at homoclinic bifurcations of two-dimensional diffeomorphisms [10], as they do not seem to be robust, i.e. an arbitrarily small perturbation may lead to the birth of stable periodic orbits. The same holds true for many "physical" attractors observed in numerical experiments, where an observed chaotic behavior can easily correspond to some periodic orbit with a very large period (plus inevitable noise); see more discussion in [2, 11].

Genuine strange attractors are free of these problems; an example of such attractors is given by hyperbolic attractors and the Lorenz attractor. Though the latter is not structurally stable, [12, 13], it contains no stable periodic orbits, and every orbit in it has a positive maximal Lyapunov exponent. Moreover, these properties are robust [14]. The reason is that Lorenz attractor possesses a pseudo-hyperbolic structure. In terms of Refs. [15] and [16] this means that the following properties hold:

1) there is a direction in which the flow is strongly contracting ("strongly" means that any possible contraction in transverse directions is always strictly weaker); and

2) transverse to this direction the flow expands areas.

*E-mail: gosv100@uic.nnov.ru
**E-mail: lpshilnikov@mail.ru
***E-mail: turaev@math.bgu.ac.il

The robustness of pseudo-hyperbolicity ensures the robustness of the chaotic behavior of orbits in the Lorenz attractor. Importantly, pseudo-hyperbolicity is also maintained for small time-periodic perturbations. Therefore, a periodically forced Lorenz attractor provides one more example of a genuine strange attractor (for more discussion see [17] and [16]). The difference with the Lorenz attractor itself is that homoclinic tangencies appear within the periodically perturbed attractor [16]. It follows that systems with periodically forced Lorenz attractors fall into Newhouse regions and demonstrate an extremely rich dynamics, see [4, 18, 19] and [20]. Such attractors were called *wild-hyperbolic* in [15].[1)]

Generalizing these ideas, we say that wild-hyperbolic attractors possess the following two main properties that distinguish them from other attractors: (1) wild-hyperbolic attractors allow homoclinic tangencies (hence, they belong to Newhouse regions); (2) every such attractor and all nearby attractors (in the C^r-topology with $r \geqslant 2$) have no stable periodic orbits.

Importantly, wild-hyperbolic attractors can be created by local bifurcations of periodic orbits; in particular, when the periodic orbit has three or more Floquet multipliers on the unit circle [22]. Therefore, wild-hyperbolic attractors easily appear in concrete models when there are sufficiently many parameters to permit such a degeneracy. In particular, this is the case for the three-dimensional map $(x, y, z) \mapsto (\bar{x}, \bar{y}, \bar{z})$ defined by

$$\bar{x} = y, \quad \bar{y} = z, \quad \bar{z} = M_1 + Bx + M_2 y - z^2, \tag{1.1}$$

with three parameters, M_1, M_2 and B. It was shown in [17] and [23] that map (1.1), called the *3D Hénon map*[2)], possesses a wild-hyperbolic attractor for some region of parameter values.

The proof notices first that at $(M_1, M_2, B) = (-1/4, 1, 1)$ the map (1.1) has a fixed point with the multipliers $(-1, -1, +1)$. Then the main idea is to apply the results of Ref. [22] where it was shown that a normal form for a map near a fixed point with the triplet $(-1, -1, +1)$ of multipliers coincides, in certain cases, with the Shimizu–Morioka system

$$\dot{X} = Y, \quad \dot{Y} = X(1 - Z) - \lambda Y, \quad \dot{Z} = -\alpha Z + X^2. \tag{1.2}$$

Thus, it was checked in [17] and [23] that near the bifurcation point $(M_1, M_2, B) = (-1/4, 1, 1)$ the second iteration of map (1.1) is $o(\tau)$-close to the time-τ map of the flow (1.2), with a small τ, and with α, λ taking arbitrary positive values as (M_1, M_2, B) vary.

It was shown numerically in [25, 26] (recently, a free of computer assistance, analytical proof was obtained in [27]) that system (1.2) possess a Lorenz attractor for some parameter domain. Therefore, for a certain open region in the space of (M_1, M_2, B), the 3D Hénon map has an attractor which can be viewed as a τ-periodic perturbation of the Lorenz attractor. A justification of the fact that such attractor is wild-hyperbolic is given in papers [17] and [16]. In fact, numerical simulations show [17] that this attractor persists in a sufficiently big domain of parameter values and has a significate size.

In Figure 1, numerically obtained 3D pictures are shown to be plots of iterations of a single point by (1.1) for $M_1 = 0, M_2 = 0.85, B = 0.7$ (on top) and for $M_1 = 0, M_2 = 0.815, B = 0.7$ (from below). The resemblance to the traditional picture of the Lorenz attractor is astonish. Recall that this is a discrete trajectory of just one point. Still it mimics very well a motion along a continue trajectory of the classical Lorenz attractor. Note that the attractor in the bottom part of Fig. 1 is very similar to the Lorenz attractor with a lacuna from Shimizu–Marioka system, see [25, 26].

The special role of the 3D Hénon map (1.1) is due to the fact that this map appears to be a rescaled normal form for the first return map near certain types of a homoclinic tangency. It is well-known that one- and two-dimensional quadratic maps, like the logistic (parabola) map $\bar{y} = M_1 - y^2$,

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[1)] The term "wild" goes back to Newhouse [21] who introduced the term "wild hyperbolic set" for uniformly hyperbolic basic sets whose stable and unstable manifolds have a tangency — as Newhouse proved, the latter property is persistent, consequently systems with wild hyperbolic sets comprise open sets in the space of dynamical systems.

[2)] The map (1.1) is a natural generalization of the famous two-dimensional Hénon map [24] that can be written as $\bar{y} = z, \;\; \bar{z} = M_1 + M_2 y - z^2$; indeed, like the latter, the map (1.1) is quadratic, has constant Jacobian $\det(Df) \equiv B$ and, moreover, when $B = 0$, reduces to the two-dimensional Hénon map

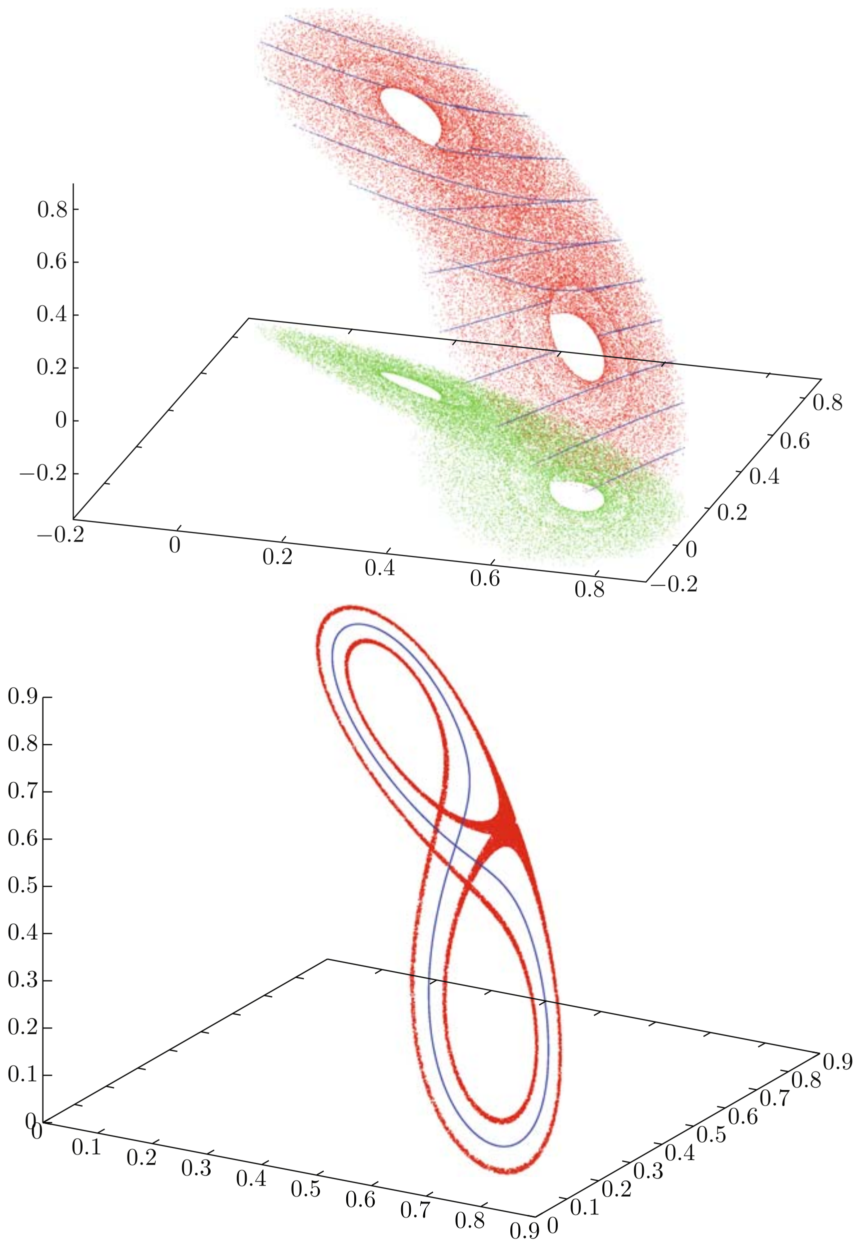

Fig. 1. Plots of attractors observed numerically for $M_1 = 0, B = 0.7$ and $M_2 = 0.85$ (on the top) or $M_2 = 0.815$ (below). In the top part, projections on the (x, y) variables and slices near planes of constant z are also displayed. In the bottom part, it is shown the closed line (looking as a "figure eight") which is a saddle closed invariant curve inside the lacuna. Note the similarity to the Lorenz attractors of system (1.2).

the standard Hénon map $(\bar{x}, \bar{y}) = (y, M_1 - Bx - y^2)$, Mira map $(\bar{x}, \bar{y}) = (y, M_1 + M_2 y - x^2)$, emerge naturally at the study of bifurcations of quadratic homoclinic tangencies, see e.g. [4, 6, 28]. In the paper [23], three-dimensional maps were considered which have a homoclinic tangency to a saddle-focus fixed point that satisfies an additional condition: the Jacobian of the map at the saddle-focus equals to 1 at the bifurcation moment. It was shown that the first return map near such homoclinic tangency coincides with the 3D Hénon map in certain rescaled coordinates, and the birth of a wild-hyperbolic Lorenz-like attractor at this bifurcation was established in this way.

In the present paper we obtain a stronger result: we consider a class of codimension-1 heteroclinic cycles for which the normal form of rescaled first return maps coincides with the 3D Hénon map (1.1), and prove the existence of Newhouse regions where three-dimensional diffeomorphisms with *infinitely many coexisting wild-hyperbolic Lorenz attractors* are dense (and form a residual subset).

2. STATEMENT OF RESULTS

Consider an orientation-preserving three-dimensional diffeomorphism $F_0 \in C^r$, $r \geqslant 4$, which satisfies the following conditions (see Fig. 2):

(A) F_0 has two fixed points: a saddle-focus O_1 with multipliers $(\lambda e^{i\varphi}, \lambda e^{-i\varphi}, \gamma_1)$ where $0 < \lambda < 1 < \gamma_1$, $\varphi \in (0, \pi)$, and a saddle O_2 with real multipliers $\lambda_1, \lambda_2, \gamma_2$ such that $0 < |\lambda_2| < |\lambda_1| < 1 < |\gamma_2|$;

(B) At one of the fixed points the Jacobian of the map is less than 1 in the absolute value, and it is greater than 1 at the other fixed point.

(C) The invariant manifolds $W^u(O_1)$ and $W^s(O_2)$ intersect transversely at the points of a heteroclinic orbit Γ_{12}; the invariant manifolds $W^u(O_2)$ and $W^s(O_1)$ have a quadratic tangency at the points of a heteroclinic orbit Γ_{21}.

(D) The extended unstable manifold W^{uext} (see [29][3)]) of O_2 intersects the stable manifold of O_1 transversely at the points of the heteroclinic orbit Γ_{21}.

We stress that according to condition B the diffeomorphism contracts volume near one of the fixed points and expands volume near the other point. We will call such diffeomorphisms *contracting-expanding*. Accordingly, we will call the heteroclinic cycle $C = \{O_1, O_2, \Gamma_{12}, \Gamma_{21}\}$ *contracting-expanding nontransverse heteroclinic cycle*.

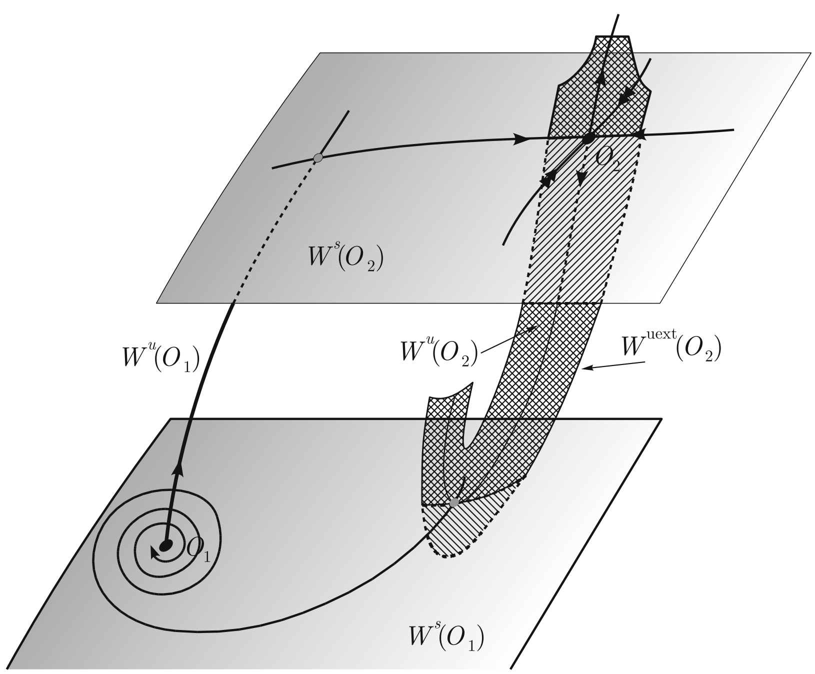

Fig. 2. Heteroclinic cycle in the map F_0 under assumptions A-D.

Diffeomorphisms that are close to F_0 and satisfy condition C comprise, in the space of C^r-diffeomorphisms, a codimension-one bifurcation surface $\mathcal{H}_{het}$. Let us include F_0 in a three-parameter family F_ϵ which is transverse to this surface. As the first governing parameter ε_1 we take the splitting parameter μ for the quadratic heteroclinic tangency. The choice of the second parameter $\epsilon_2 = \varphi(\epsilon) - \varphi(0)$ is stipulated by the fact [30] that diffeomorphisms on $\mathcal{H}_{het}$ possess continuous invariants of the topological conjugacy on the set of non-wandering orbits, the so-called Ω-moduli, and the most important modulus is the angular argument φ of the complex multipliers of the saddle-focus. Also, as we have the "contracting-expanding" situation, we must include among the

3) in given case, a local extended manifold $W^{uext}_{loc}(O_2)$ is an F_0-invariant two-dimensional surface containing $W^u_{loc}(O_2)$ and touching in O_2 to the eigenvectors corresponding to λ_1 and γ_2

governing parameters some ϵ_3 which will control the Jacobians $J_1 := \lambda^2\gamma_1$ and $J_2 := \lambda_1\lambda_2\gamma_2$. To this aim, we introduce the following functional: $S(F) = -\frac{\ln J_1}{\ln J_2}$, and take $\epsilon_3 = S(F_\epsilon) - S(F_0)$.

Let U be a sufficiently small and fixed neighborhood of the heteroclinic cycle $\{O_1, O_2, \Gamma_{12}, \Gamma_{21}\}$. It can be represented as a union of small neighborhoods U_1 of O_1 and U_2 of O_2 with a number of small neighborhoods of those points of the heteroclinic orbits Γ_{12} and Γ_{21} which do not belong to $U_1 \cup U_2$. Take any of the latter neighborhoods and consider the first return maps in it. In Sec. 3, we will prove the following result about the form of the first return maps for some domains of parameter values.

Theorem 1. *Let F_ε be the three-parameter family defined above. Then the point $\varepsilon = 0$ in the space of parameters is a limit of an infinite sequence of open domains, Δ_{ij}, such that at $\varepsilon \in \Delta_{ij}$ the first return map to a certain open region V_{ij} takes the form*

$$\begin{aligned} \bar{X} &= Y + o(1)_{i,j\to+\infty}, \qquad \bar{Y} = Z + o(1)_{i,j\to+\infty}, \\ \bar{Z} &= M_1 + BX + M_2Y - Z^2 + o(1)_{i,j\to+\infty}, \end{aligned} \tag{2.1}$$

where (X, Y, Z) are coordinates in V_{ij} whose range covers all finite values as $i, j \to +\infty$, parameters (M_1, M_2, B) are related to ε by a diffeomorphism, and as ε run Δ_{ij} the values of (M_1, M_2, B) run regions that grow to cover all finite values of (M_1, M_2) and all finite positive values of B as $i, j \to +\infty$. The $o(1)$-terms are functions of (X, Y, Z, M_1, M_2, B) that tend to zero as $i, j \to +\infty$, along with all derivatives up to the order $(r - 2)$.

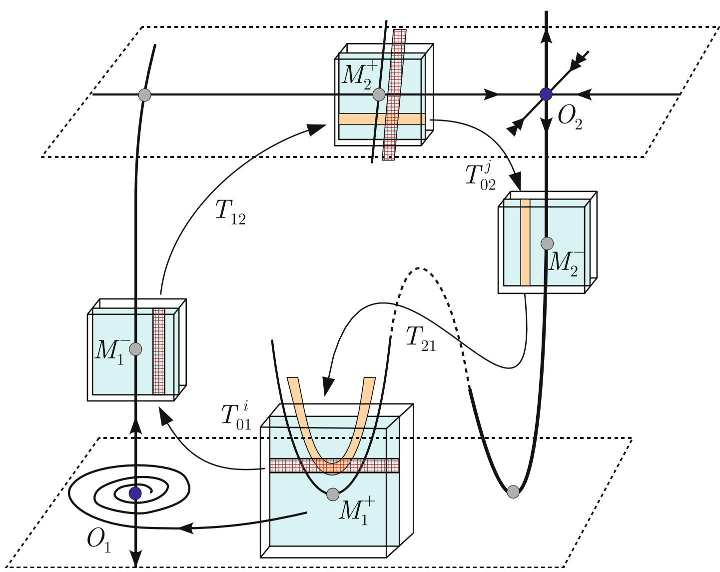

Fig. 3. Local and global maps T_{01}, T_{02} and T_{12}, T_{21}, respectively, for the diffeomorphism F_ε.

Theorem 1 shows that the 3D Hénon map (1.1) is a normal form for the first return maps near the heteroclinic cycle under consideration (since it emerges when one omits the $o(1)$-terms in (2.1)). If, at $\epsilon = 0$, we choose two pairs of heteroclinic points $M_1^- \in W^u_{loc}(O_1)$, $M_2^+ \in W^s_{loc}(O_2)$ and $M_2^- \in W^u_{loc}(O_2)$, $M_1^+ \in W^s_{loc}(O_1)$ (see Fig. 3) such that M_1^- is the last point of the heteroclinic orbit Γ_{12} that lies in U_1 and M_2^+ is the first point of Γ_{12} that lies in U_2, while M_2^- is the last point of the heteroclinic orbit Γ_{21} that lies in U_2 and M_1^+ is the first point of Γ_{21} that lies in U_1, then there exist some integers $q_1 > 0$ and $q_2 > 0$ such that $M_2^+ = F_0^{q_1}(M_1^-)$ and $M_1^+ = F_0^{q_2}(M_2^-)$; the regions V_{ij} in Theorem 1 all lie in a small neighborhood of M_1^+ — they shrink to this point as $i, j \to +\infty$, their images $F_\varepsilon^i V_{ij}$ lie in a small neighborhood of M_1^- (hence the regions $F_\varepsilon^{i+q_1} V_{ij}$ lie in a small neighborhood of M_2^+) and the regions $F_\varepsilon^{i+q_1+j} V_{ij}$ lie in a small neighborhood of M_2^-, so

$F_\varepsilon^{i+q_1+j+q_2}V_{ij}$ lie in a small neighborhood of M_1^+ again: the maps $F_\varepsilon^{i+q_1+j+q_2}|_{V_{ij}}$ are the first return maps described by Theorem 1. The rescaled parameters (M_1, M_2, B) satisfy the following relations (see exact formulas (3.16),(3.11),(3.17)):

$$M_1 \sim \gamma_1^{2i}\gamma_2^{2j}(\mu + O(|\lambda|^j + |\gamma_1|^{-i}))\ ,\ M_2 \sim (\lambda\gamma_1)^i(\lambda_1\gamma_2)^j \cos(i\varphi + \beta)\ ,$$
$$B \sim (\lambda^2\gamma_1)^i(\lambda_1\lambda_2\gamma_2)^j \equiv J_1^i J_2^j,$$

where β is some constant coefficient. It is clear that we can obtain any finite values of M_1 and M_2 by taking i and j sufficiently large and by arbitrary small tuning of the parameters μ and φ, respectively. As $\ln J_1 \cdot \ln J_2 < 0$, it is easy to see that at appropriately chosen sufficiently large i and j one can make B run all finite positive values by an arbitrary small change $|\ln J_1/\ln J_2|$.

The existence of a wild Lorenz-like attractor in the 3D-Hénon map for some open region of the parameters (M_1, M_2, B) implies, by virtue of Theorem 1, that the map F_ε has a wild Lorenz-like attractor at some open set of parameters $\varepsilon \in \Delta_{ij}$ (since such attractors are robust with respect to small C^2-perturbations [16]). We also recall that in the space of parameters there exist open regions $\mathcal{N}$ (Newhouse regions) where diffeomorphisms with nontransverse heteroclinic cycles are dense[4)] (these heteroclinic cycles include the same fixed points O_1 and O_2, just the pair of heteroclinic orbits Γ_{12} and Γ_{21} is different for different values of ε). As we just explained, Theorem 1 implies that near any value of $\varepsilon \in \mathcal{N}$ that corresponds to such a heteroclinic cycle there is a small open region of parameter values that corresponds to the existence of a wild Lorenz-like attractor. As this region lies in $\mathcal{N}$, there are other values of ε in it which correspond to a new heteroclinic cycle, so we may find a smaller open subregion where one more wild Lorenz-like attractor exists. By repeating this procedure infinitely many times, we arrive at the following result.

Theorem 2. *In any neighborhood of $\epsilon = 0$ in the space of parameters there exist open domains $\mathcal{N}$ where the values of ϵ are dense for which the corresponding diffeomorphism F_ϵ has infinitely many coexisting wild-hyperbolic strange attractors of Lorenz-type.*

3. PROOF OF THEOREM 1

By [29] and [6], one can introduce C^r-coordinates $(x, y) = (x_1, x_2, y)$ in a neighborhood U_1 of the fixed point O_1 such that the so-called local map $T_{01}(\epsilon) := F_\epsilon|_{U_1}$ will, for small enough ϵ, take the form

$$T_{01}: \begin{aligned} \bar{x} &= \lambda R_\varphi x + O(x^2\ y) \\ \bar{y} &= \gamma_1 y + O(\|x\|\ y^2) \end{aligned} \tag{3.1}$$

where the (2×2)-matrixR_φ is the rotation by angle φ. Analogously, we introduce C^r-coordinates (u_1, u_2, v) in a neighborhood U_2 of O_2 such that the local map $T_{02}(\epsilon) := F_\epsilon|_{U_2}$ will, for small enough ϵ, take the form

$$T_{02}: \begin{aligned} \bar{u}_1 &= \lambda_1 u_1 + O((u_1^2 + |u_2|)v) \\ \bar{u}_2 &= \lambda_2 u_2 + O(u_1^2 + |u_2||v|) \\ \bar{v} &= \gamma_2 v + O(\|u\|\ v^2). \end{aligned} \tag{3.2}$$

In coordinates (3.1), the local stable and unstable manifolds of O_1 in U_1 are $W^s_{loc} = \{y = 0\}$ and $W^u_{loc} = \{x = 0\}$. In coordinates (3.2), the local stable and unstable manifolds of O_2 in U_2 are $W^s_{loc} = \{v = 0\}$ and $W^u_{loc} = \{u = 0\}$; the extended unstable manifold of condition D is tangent to $u_2 = 0$ at each point of $W^u_{loc}(O_2)$ (see e.g. [6, 29]). When $\epsilon = 0$, choose two

[4)] this is derived from the existence of Newhouse regions near every three-dimensional map with a homoclinic tangency [5, 31] exactly in the same way as we derived in [32] the existence of Newhouse regions for two-dimensional diffeomorphisms with a heteroclinic cycle from the analogous result [21] for two-dimensional diffeomorphisms with a homoclinic tangency pairs of heteroclinic points $M_1^- = (0,0,y^-) \in W^u_{loc}(O_1)$, $M_2^+ = (u_1^+, u_2^+, 0) \in W^s_{loc}(O_2)$, and $M_2^- = (0,0,v-) \in W^u_{loc}(O_2)$, $M_1^+ = (x_1^+, x_2^+, 0) \in W^s_{loc}(O_1)$, where M_1^- and M_2^+ belong to Γ_{12}, while M_2^- and M_1^+ belong to Γ_{21}. Let $q_1 > 0$ and $q_2 > 0$ be such integers that $M_2^+ = F_0^{q_1}(M_1^-)$ and $M_1^+ = F_0^{q_2}(M_2^-)$. Let Π_l^+ and Π_l^-, $l = 1, 2$, be some small neighborhoods of M_l^+ and M_l^-. The so-called global maps $T_{12}(\epsilon) := F_\epsilon^{q_1}|_{\Pi_1^-} : \Pi_1^- \to \Pi_2^+$ and $T_{21}(\epsilon) := F_\epsilon^{q_2}|_{\Pi_2^-} : \Pi_2^- \to \Pi_1^+$ are well-defined for small enough ϵ. In the coordinates of (3.1) and (3.2) the global maps are written as follows:

$$T_{12}: \begin{array}{ll} \bar{u} - u^+ &= A^{(1)}x + b^{(1)}(y - y^-) + O(x^2 + (y-y^-)^2), \\ \bar{v} &= (c^{(1)})^\top x + d^{(1)}(y-y^-) + O(x^2 + (y-y^-)^2), \end{array} \tag{3.3}$$

$$T_{21}: \begin{array}{l} \bar{x} - x^+ = A^{(2)}u + b^{(2)}(v - v^-) + O(u^2 + (v-v^-)^2), \\ \bar{y} = \mu + (c^{(2)})^\top u + d^{(2)}(v-v^-)^2 + O(u^2 + \|u\| \cdot |v - v^-| + |v-v^-|^3), \end{array} \tag{3.4}$$

where $d^{(1)} \neq 0$ and $d^{(2)} \neq 0$, since $W^u(O_1)$ and $W^s(O_1)$ intersect transversely at M_2^+, and since the tangency between $W^u(O_2)$ and $W^s(_1)$ at M_1^+ is quadratic. We can also, with no loss of generality, assume that $b^{(2)} = (b, 0)^\top$ (one can always achieve this by a rotation of the x-variables). Since both the global maps T_{12} and T_{21} are diffeomorphisms, we have

$$J_{12} = \det \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & b_1^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} & b_2^{(1)} \\ c_1^{(1)} & c_2^{(1)} & d^{(1)} \end{pmatrix} \neq 0, \quad J_{21} = \det \begin{pmatrix} a_{11}^{(2)} & a_{12}^{(2)} & b \\ a_{21}^{(2)} & a_{22}^{(2)} & 0 \\ c_1^{(2)} & c_2^{(2)} & 0 \end{pmatrix} \neq 0. \tag{3.5}$$

In particular, we obtain that $b \neq 0$. We also assume that

$$c_1^{(2)} \neq 0, \tag{3.6}$$

which is equivalent to condition D, as it is easy to see [6].

When the local maps T_{0l} are brought to normal forms (3.1) and (3.2), one have convenient estimates [6, 29] for the iterations T_{0l}^k. Namely, the map $T_{01}^k(\epsilon) : (x_0, y_0) \to (x_k, y_k)$ can be written in the following "cross"-form

$$\begin{array}{l} x_k = \lambda^k R_{k\varphi} x_0 + \lambda^{2k} \rho_{1k}(x_0, y_k, \epsilon)\ , \\ y_0 = \gamma_1^{-k} y_k + \lambda^k \gamma_1^{-k} \rho_{2k}(x_0, y_k, \epsilon)\ , \end{array} \tag{3.7}$$

and the map $T_{02}^k(\epsilon) : (u_0, v_0) \to (u_k, v_k)$ can be written as

$$\begin{array}{l} (u_{1k}, u_{2k}) = (\lambda_1^k u_{10}, \lambda_2^k u_{20}) + (\lambda_1^{2k} + |\lambda_2|^k) \rho_{3k}(u_0, v_k, \epsilon)\ , \\ v_0 = \gamma_2^{-k} v_k + \lambda_1^k \gamma_2^{-k} \rho_{4k}(u_0, v_k, \epsilon)\ , \end{array} \tag{3.8}$$

where the functions ρ_k and their derivatives up to the order $(r-2)$ are uniformly bounded.

Using formulas (3.3), (3.4), (3.7) and (3.8) we obtain the following expression for the first return map $T_{ij} \equiv T_{21}T_{02}^jT_{12}T_{01}^i : \Pi_1^+ \to \Pi_1^+$

$$\begin{aligned}
&\bar{u} - u^+ = A^{(1)}\lambda^i R(i\varphi)x + b^{(1)}(y - y^-) + O(\lambda^{2i}x^2 + (y - y^-)^2),\\
&\gamma_2^{-j}(\bar{v} + \lambda_1^j\rho_{4j}) = (c^{(1)})^\top \lambda^i R(i\varphi)x + d^{(1)}(y - y^-) + O(\lambda^{2i}x^2 + (y - y^-)^2),\\
&\bar{x}_1 - x_1^+ = \lambda_1^j a_{11}^{(2)}\bar{u}_1 + b(\bar{v} - v^-) + O(\lambda_1^{2j} + |\lambda_2|^j + (\bar{v} - v^-)^2),\\
&\bar{x}_2 - x_2^+ = \lambda_1^j a_{21}^{(2)}\bar{u}_1 + O(\lambda_1^{2j} + |\lambda_2|^j + (\bar{v} - v^-)^2),\\
&\gamma_1^{-i}(\bar{y} + \lambda^i\rho_{2i}) = \mu + c_1^{(2)}\lambda_1^j\bar{u}_1 + c_2^{(2)}\lambda_2^j\bar{u}_2 + d^{(2)}(\bar{v} - v^-)^2\\
&\qquad + O(\lambda_1^{2j} + |\lambda_2|^j + |\lambda_1|^j\|\bar{u}\| \cdot |\bar{v} - v^-| + |\bar{v} - v^-|^3),
\end{aligned} \tag{3.9}$$

Make a coordinate shift $u_{new} = u - u^+ + \alpha_{ij}^1$, $v_{new} = v - v^- + \alpha_{ij}^1$, $x_{new} = x - x^+ + \alpha_{ij}^3$, $y_{new} = y - y^- + \alpha_{ij}^4$, where the constants α_{ij} are of order $O(\lambda^i + |\lambda_1|^j + \gamma_1^{-i} + |\gamma_2|^{-j})$, in such a way that all constant terms in the equations for $\bar{x}_{new}$, $\bar{u}_{new}$ and $\bar{v}_{new}$ and the linear in y_{new} terms in the equation for $\bar{y}_{new}$ will vanish. After this, the coefficient of v in the equation for $\bar{x}_{2,new}$ may become non-zero (of order α) — we will return it to be zero by an additional rotation of the x-variables to a small angle. Finally, system (3.9) takes the form

$$\begin{aligned}
&\bar{u} = A^{(1)}\lambda^i R(i\varphi)x + b^{(1)}y + O(\lambda^{2i}x^2 + y^2),\\
&\gamma_2^{-j}\bar{v}(1 + O(\lambda_1^j)) = (c^{(1)})^\top \lambda^i R(i\varphi)x + d^{(1)}y + O(\lambda^{2i}x^2 + y^2),\\
&\bar{x}_1 = \lambda_1^j a_{11}^{(2)}\bar{u}_1 + b\bar{v} + O(\lambda_1^{2j}\bar{u}_1^2 + (\lambda_1^{2j} + |\lambda_2|^j)|\bar{u}_2| + |\lambda_1|^j\|\bar{u}\| \cdot |\bar{v}| + \bar{v}^2),\\
&\bar{x}_2 = \lambda_1^j a_{21}^{(2)}\bar{u}_1 + \qquad O(\lambda_1^{2j}\bar{u}_1^2 + (\lambda_1^{2j} + |\lambda_2|^j)|\bar{u}_2| + \lambda_1^j\|\bar{u}\| \cdot |\bar{v}| + \bar{v}^2),\\
&\gamma_1^{-i}\bar{y}(1 + O(\lambda^i)) = M + c_1^{(2)}\lambda_1^j\bar{u}_1 + O(\lambda_1^{2j} + |\lambda_2|^j)\bar{u}_2 + d^{(2)}\bar{v}^2\\
&\qquad + O((\lambda_1^{2j} + |\lambda_2|^j)\bar{u}^2 + \lambda_1^j\|\bar{u}\| \cdot |\bar{v}| + |\bar{v}|^3),
\end{aligned} \tag{3.10}$$

where the new coefficients may differ from the original ones to some small (of order α_{ij}) corrections. The parameter M is given by

$$M = \mu + \lambda_1^j(c_1^{(2)} + \dots) - \gamma_1^{-i}y^-(1 + \dots), \tag{3.11}$$

where dots stand for some coefficients that tend to zero as $i, j \to \infty$.

Next, we take the right-hand side of the second equation of (3.10) divided to $d^{(1)}$ and to the $(1 + O(\lambda_1^j))$-factor from the left-hand side as the new variable y — then the equation for $\bar{v}$ will become $\gamma_2^{-j}\bar{v} = d^{(1)}y$. Now, by defining $u_{new} = u - (b^{(1)}/d^{(1)})\gamma_2^{-j}v + O(\gamma_2^{-2j}v^2)$, we eliminate all terms in the equation for $\bar{u}$ which depend on y alone. Thus, we obtain

$$\begin{aligned}
&\bar{u} = \tilde{A}\lambda^i R(i\varphi)x + O(\lambda^{2i}x^2 + \lambda^i\|x\||y|), \qquad \gamma_2^{-j}\bar{v} = d^{(1)}y,\\
&\bar{x}_1 = \lambda_1^j a_{11}^{(2)}\bar{u}_1 + b\bar{v} + O(\lambda_1^{2j}\bar{u}_1^2 + (\lambda_1^{2j} + |\lambda_2|^j)|\bar{u}_2| + |\lambda_1|^j\|\bar{u}\| \cdot |\bar{v}| + \bar{v}^2),\\
&\bar{x}_2 = \lambda_1^j a_{21}^{(2)}\bar{u}_1 + O(\lambda_1^{2j}\bar{u}_1^2 + (\lambda_1^{2j} + |\lambda_2|^j)|\bar{u}_2| + \lambda_1^j\|\bar{u}\| \cdot |\bar{v}| + \bar{v}^2),\\
&\gamma_1^{-i}\bar{y}(1 + O(\lambda^i)) = M + c_1^{(2)}\lambda_1^j\bar{u}_1 + O(\lambda_1^{2j} + |\lambda_2|^j)\bar{u}_2 + d^{(2)}\bar{v}^2\\
&\qquad + O((\lambda_1^{2j} + |\lambda_2|^j)\bar{u}^2 + \lambda_1^j\|\bar{u}\| \cdot |\bar{v}| + |\bar{v}|^3),
\end{aligned} \tag{3.12}$$

where $\tilde{A} = A^{(1)} - b^{(1)}(c^{(1)})^\top/d^{(1)} + \dots$; by (3.5),

$$\det\tilde{A} \equiv J_{12}/d^{(1)} \neq 0. \tag{3.13}$$

Next, from the first two equations of (3.12), we find $\bar{u}$ and $\bar{v}$ as functions of x and y and substitute the obtained expressions into the last two equations. This gives us the following formula for the map $T_{ij} : (x, y) \mapsto (\bar{x}, \bar{y})$.

$$\begin{aligned}
&\bar{x}_1 = bd^{(1)}\gamma_2^j y + O(|\lambda_1|^j \lambda^i \|x\| + \gamma_2^{2j} y^2),\\
&\bar{x}_2 = K_1 x_1 + K_2 x_2 + O(\lambda_1^{2j}\lambda^{2i} x^2 + \gamma_2^{2j} y^2),\\
&\gamma_1^{-i}\bar{y}(1 + O(\lambda^i)) = M + c_1^{(2)}\lambda_1^j\lambda^i[(\tilde{a}_{11}\cos(i\varphi) + \tilde{a}_{12}\sin(i\varphi) + \dots)x_1\\
&\qquad +(\tilde{a}_{12}\cos(i\varphi) - \tilde{a}_{11}\sin(i\varphi) + \dots)x_2] + d^{(2)}(d^{(1)})^2\gamma_2^{2j}y^2\\
&\qquad\qquad +O((\lambda_1^{2j} + |\lambda_2|^j)\lambda^i x^2 + \lambda_1^j\lambda^i|\gamma_2|^j\|x\| \cdot |y| + |\gamma_2|^{3j}|y|^3),
\end{aligned} \tag{3.14}$$

where $K_{1,2} = O(|\lambda_1|^j\lambda^i)$ are certain coefficients whose exact values are irrelevant for us.

Further, we will choose the integers i and j such that, as they tend to infinity, the product $S_{ij} = (\lambda^2\gamma_1)^i(\lambda_1\lambda_2\gamma_2)^j \equiv (J_1)^i(J_2)^j$ will stay bounded away from zero and infinity (this can always be achieved since $\ln J_1 \cdot \ln J_2 < 0$ by assumption). Let us rescale the coordinates:

$$y = -\frac{\gamma_1^{-i}\gamma_2^{-2j}}{d^{(2)}(d^{(1)})^2}Z, \quad x_1 = -\frac{b\gamma_1^{-i}\gamma_2^{-j}}{d^{(2)}d^{(1)}}Y, \quad x_2 = -\frac{\lambda_2^j\lambda^i\gamma_1^{-i}\gamma_2^{-j}}{c_1^{(2)}d^{(2)}(d^{(1)})^2}X.$$

Then system (3.14) recasts as

$$\begin{aligned}
&\bar{Y} = Z + O(|\lambda_1|^j\lambda^i), \qquad \bar{X} = bd^{(1)}c_1^{(2)}K_1\lambda_2^{-j}\lambda^{-i}Y + O(|\lambda_1|^j\lambda^i),\\
&\bar{Z} = M_1 + M_2 Y + S_{ij}(\tilde{a}_{12}\cos i\varphi - \tilde{a}_{11}\sin i\varphi + \dots)X - Z^2 + O(|\lambda_1|^j\lambda^i),
\end{aligned} \tag{3.15}$$

where

$$M_1 = -d^{(2)}(d^{(1)})^2\gamma_1^{2i}\gamma^{2j}M, \quad M_2 = c_1^{(2)}(\lambda\gamma_1)^i(\lambda_1\gamma_2)^j(\tilde{a}_{11}\cos i\varphi + \tilde{a}_{12}\sin i\varphi + \dots). \tag{3.16}$$

The coefficient in front of M is asymptotically large as $i, j \to \infty$, hence one can make M_1 run arbitrary finite values as μ varies. The coefficient $(\lambda\gamma_1)^i(\lambda_1\gamma_2)^j \equiv S_{ij}\lambda_2^{-j}\lambda^{-i}$ in the formula for M_2 is also asymptotically large; therefore, to keep M_2 finite we will let φ vary close to a value which corresponds to $\tilde{a}_{11}\cos(i\varphi)) + \tilde{a}_{12}\sin(i\varphi) = 0$ (recall that $c_1^{(2)}(\tilde{a}_{11}^2 + \tilde{a}_{12}^2) \neq 0$ by (3.6),(3.13), so by varying φ one can achieve arbitrary finite values of M_2). In this case, the coefficient of X in the equation for $\bar{Z}$ equals to $\pm S_{ij}(\sqrt{\tilde{a}_{11}^2 + \tilde{a}_{12}^2} + \dots)$. Note that the coordinate changes we made while bringing the map T_{ij} to form (3.15) are compositions of linear transformations and transformations that are asymptotically close to identity as $i, j \to +\infty$. Therefore, the Jacobian of (3.15) coinsides, up to a factor equal to 1+asymptotically vanishing terms) with the Jacobian of T_{ij} in the original coordinates, which is easily evaluated [8] when the maps T_{01} and T_{02} are brought to form (3.1),(3.2): we thus find that the Jacobian of (3.15) equals S_{ij} times an asymptotically constant positive factor. This immediately implies that the coefficient of Y in the equation for $\bar{X}$ tends to a positive constant as $i, j \to +\infty$.

It follows that in the region of parameter values that correspond to bounded M_1 and M_2 map (3.15) can be brought to the form

$$\begin{aligned}
&\bar{X} = Y + O(|\lambda_1|^j\lambda^i), \qquad \bar{Y} = Z + O(|\lambda_1|^j\lambda^i),\\
&\bar{Z} = M_1 + M_2 Y + KS_{ij}X - Z^2 + O(|\lambda_1|^j\lambda^i),
\end{aligned} \tag{3.17}$$

where K tends to a positive constant as $i, j \to +\infty$. We complete the proof of Theorem 1 by noticing that $S_{ij} \equiv (\lambda^2\gamma_1)^i(\lambda_1\lambda_2\gamma_2)^j$ for large i and j runs arbitrary finite values as the parameter $|\ln J_1/\ln J_2|$ varies.

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