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ASYMPTOTIC NORMAL FORMS FOR EQUILIBRIA WITH A TRIPLET OF ZERO CHARACTERISTIC EXPONENTS IN SYSTEMS WITH SYMMETRY

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Asymptotic normal forms for equilibria with a triplet of zero characteristic exponents in systems with $\mathbb{Z}_q$-symmetry are listed.

1. Introduction

The purpose of this paper is to present a list of asymptotic normal forms describing the trajectory behaviour near a stability boundary of a triply degenerate equilibrium state in systems with discrete symmetry. We say *a triple instability* when a dynamical system has an equilibrium state such that the associated linearized problem has a triplet of zero eigenvalues. In such a case, as it is well known, the study is reduced to a three-dimensional system on the center manifold. Moreover, if the original system possesses a symmetry, as many systems in hydrodynamics do, then the reduced system may also inherit the symmetry.

In order to study bifurcations near a stability boundary one has to introduce some small governing parameters the number of which is at least equal to the order of degeneracy of the linear problem, or this number may even be greater provided that there is an additional degeneracy in the non-linear part. Since the unfolding parameters are small, the orbits on the center manifold may stay in a small neighborhood of the equilibrium state for a rather long time (due to the fact that there is no fast instability in the center manifold because all characteristic exponents of the truncated linearized system are nearly zero). Thus, it is reasonable to rescale the parameters and phase variables so that they become of finite values instead of asymptotically vanishing ones; the time variable must be rescaled as well.

The proposed approach is a rather general one. Its advantage is that upon the rescaling procedure has been carried out, many resonant monomials disappear. The most trivial example is a saddle-node bifurcation with a single zero eigenvalue. In this case the center manifold is one-dimensional. The Taylor expansion of the system near the equilibrium state can be written as

$$\dot{x} = \mu + x^2 + l_3 x^3 + \dots ,$$

where μ is a small governing parameter. The rescaling $x \to \sqrt{|\mu|}x$, $t \to t/\sqrt{|\mu|}$ brings the system to the form

$$\dot{x} = \pm 1 + x^2 + O\left(\sqrt{|\mu|}\right),$$

so the second degree monomial only survives in the limit $\mu \to 0$.

An analogous algorithm can be applied to the multi-dimensional case. The limit of the rescaled system as its governing parameters tend to zero, gives a description "in the main order" of the behavior of the system near a bifurcation point. We call such a limit system *an asymptotic normal form.*

The asymptotic normal forms that arise in the study of equilibria with single or double zero eigenvalues are one- or two-dimensional, respectively. The analysis of such forms is often very comprehensive so the most of efforts is applied for establishing the rigorous correspondence between the dynamics in the asymptotic normal form and that in the original system [1, 2]. The situation is different in higher dimensions.

Three- (and higher) dimensional asymptotic normal forms may exhibit a non-trivial dynamics. For example, the Shilnikov chaos was found in the asymptotic normal form corresponding to the bifurcation of triple zeros with a complete Jordan box [3]; the existence of the Lorenz attractor was shown in normal forms for the bifurcations corresponding to triply zero eigenvalues in the case of an additional symmetry [4]. Notably, the normal forms mentioned above turn out to coincide with some well-known models coming from different applications: the third-order Duffing equation, the Shimizu–Marioka system, the Lorenz model.

In this paper we will derive an infinite series of the asymptotic normal forms (ordered according to the increasing degree of degeneracy in non-linear terms) corresponding to a triple zero eigenvalue (taking also into account some non-degeneracy conditions: (2.3), (3.4) and (3.5)) in a system with $\mathbb{Z}_q$-symmetry. Namely, assuming that (x, y, z) are the coordinates in the three-dimensional center manifold and a bifurcating equilibrium state resides at the origin, we suppose that our system is equivariant with respect to a rotation over the angle $2\pi/q$ around the z-axis. We should emphasize that the cases $q = 2$ and $q \geqslant 3$ are principally different and will therefore be considered separately. The resulting asymptotic normal forms are given by systems (2.13) for $q = 2$ and (3.15) for $q \geqslant 3$. The degrees of polynomials in the right-hand side are listed in (2.13) and (3.12)–(3.14), respectively. It is worthwhile to remark that all listed systems have a natural "physical" meaning, namely, they describe the behaviour near a triple instability in the presence of a certain symmetry. Thus, this list below may be regarded as a recipe for exclusion of irrelevant terms in the non-linearity as well as for selection of those non-linear terms which are responsible for specific details of such behaviour.

2. Symmetry of order 2

Consider a system in $\mathbb{R}^3$ near an equilibrium state $O(0, 0, 0)$ with three zero characteristic exponents. We suppose that the systems possess a symmetry $(x, y, z) \longleftrightarrow (-x, -y, z)$. We will also suppose that the linear part of the system near O restricted onto the invariant plane $z = 0$ has a complete Jordan block. Then the system may locally be written as

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x(az + F(x^2, xy, y^2, z)) + yG(y^2, z), \\ \dot{z} = H(x^2, xy, y^2, z), \end{cases} \tag{2.1}$$

where neither $H(0, 0, 0, z)$ nor $F(0, 0, 0, z)$ contains linear terms.

Let us consider a three-parameter perturbation of the system in the form

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x(\mu_1 + az + F(x^2, xy, y^2, z)) + y(-\mu_2 + G(y^2, z)), \\ \dot{z} = -\mu_3 z + H(x^2, xy, y^2, z), \end{cases} \tag{2.2}$$

where $\mu = (\mu_1, \mu_2, \mu_3)$ is a small parameter; the functions F, G and H may also depend on μ.

Let us suppose that

$$a \neq 0. \tag{2.3}$$

It is then obvious that a change of the z-coordinate reduces (2.2) to the following form (with some new G and H)

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x(\mu_1 - z) + y(-\mu_2 + G(y^2, z)), \\ \dot{z} = -\mu_3 z + H(x^2, xy, y^2, z). \end{cases} \tag{2.4}$$

Let us next rescale the variables and the time:

$$x \to \delta_x x, \quad y \to \delta_y y, \quad z \to \delta_z z, \quad t \to t/\tau,$$

where δ_x, δ_y, δ_z and τ are some small quantities. Let $\mu_1 \neq 0$ and

$$\delta_y = \tau\delta_x, \quad \delta_z = \tau^2 = |\mu_1|.$$

Then, (2.4) can be recast in the form

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x(\pm 1 - z) - \lambda y + O(\tau), \\ \dot{z} = -\alpha z + H(\delta_x^2 x^2, \tau\delta_x^2 xy, \tau^2\delta_x^2 y^2, \tau^2 z)/\tau^3, \end{cases} \tag{2.5}$$

where α and λ are new rescaled parameters, no longer small:

$$\alpha = \mu_3 \,/\, \sqrt{|\mu_1|}, \quad \lambda = \mu_2 \,/\, \sqrt{|\mu_1|}.$$

The asymptotic normal form is a final limit of system (2.5) as $\mu \to 0$. Note that different choices of proportion between the scaling factors δ_x and τ yield different normal forms.

In the last equation in (2.5) those terms which contain z^2, y^3 and yz tend to zero as $\tau \to 0$. Thus, by cancelling out small terms we transform (2.5) to the form

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x(\pm 1 - z) - \lambda y, \\ \dot{z} = -\alpha z + \delta_x^2 x^2 H_1(\delta_x^2 x^2) \,/\, \tau^3 + \delta_x^2 xy H_2(\delta_x^2 x^2) \,/\, \tau^2 + \\ \qquad + \delta_x^2 y^2 H_3(\delta_x^2 x^2) \,/\, \tau + \delta_x^2 z x^2 H_4(\delta_x^2 x^2) \,/\, \tau. \end{cases} \tag{2.6}$$

The right-hand side in (2.6) is to be finite, i. e., if the Taylor expansions of the functions H_i begin with x^{2m_i} for zero values of the perturbation parameters μ_1, μ_2, μ_3, then the following inequalities must hold

$$\delta_x^{2(m_1+1)} \,/\, \tau^3 < \infty, \quad \delta_x^{2(m_2+1)} \,/\, \tau^2 < \infty, \quad \delta_x^{2(m_3+1)} \,/\, \tau < \infty, \quad \delta_x^{2(m_4+1)} \,/\, \tau < \infty.$$

Therefore, we can choose τ such that

$$\tau \sim \delta_x^{\beta}, \tag{2.7}$$

where

$$\beta = \min\left\{\frac{2}{3}(m_1+1),\ m_2+1,\ 2(m_3+1),\ 2(m_4+1)\right\}. \tag{2.8}$$

For example, in the most generic case where $H_i(0) \neq 0$ ($i = 1, \dots, 4$), the exponent β is equal to 2/3 in (2.7), (2.8). Then, system (2.6) is reduced to the form

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x(\pm 1 - z) - \lambda y, \\ \dot{z} = -\alpha z + x^2 H_1(0) + O(\tau). \end{cases} \tag{2.9}$$

In the limit $\tau \to 0$ this system becomes the Shimizu–Marioka model, where the parameters α and λ may take arbitrarily finite values.

Let us now consider an extra degeneration: $H_1(0) = 0$ and $H_1'(0) \neq 0$. In order to study bifurcations in this case one should introduce a new independent governing parameter which is the constant term of the Taylor expansion of H_1.

Let us next suppose $\beta = 1$ in relation (2.8). System (2.6) is then reduced to the following asymptotic form:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x(\pm 1 - z) - \lambda y, \\ \dot{z} = -\alpha z + x^2 \widetilde{h}_{10} + H_2(0) xy, \end{cases} \tag{2.10}$$

i. e., to the Lorenz equations. Here, $\widetilde{h}_{10} = H_1(0) \,/\, \tau$ is the third rescaled governing parameter which may take arbitrarily finite values.

The next degeneration $H_2(0) = 0$, $H_2'(0) \neq 0$ modifies the third equation in (2.10):

$$\dot{z} = -\alpha z + x^2 \widetilde{h}_{10} + \widetilde{h}_{20} xy + H_1'(0) x^4, \tag{2.11}$$

where $\widetilde{h}_{10} = H_1(0) \,/\, \tau^{3/2}$ and $\widetilde{h}_{20} = H_2(0) \,/\, \tau^{1/2}$. Here, $\beta = 4/3$.

By repeating this procedure we get a hierarchy of the asymptotic normal forms. Let us denote

$$H_i(x^2) = \sum_j^{\infty} H_{ij} x^{2j}.$$

We assume that at the moment of bifurcation the values of H_{ij} vanish for $j = 0, \dots, m_i - 1$. As before, we will consider these H_{ij}'s as additional independent small parameters.

It is obviously that in the rescaled system (2.6) there are non-zero factors in front of those terms which correspond to such m_i for which the minimum in (2.8) is achieved; all terms of higher orders vanish in the limit $\tau \to 0$. The terms of degree less then $2m_i$, which appear in H_i for non-zero parameter values, also survive after the rescaling; their normalized coefficients become the independent parameters which assume arbitrary finite coefficients.

Thus, if we get rid of asymptotically vanishing terms, system (2.6) takes the form

$$\begin{cases} \dot{x} = y, \\ \dot{y} = x(\pm 1 - z) - \lambda y, \\ \dot{z} = -\alpha z + x^2 \widetilde{H}_1(x^2) + xy \widetilde{H}_2(x^2) + y^2 \widetilde{H}_3(x^2) + z x^2 \widetilde{H}_4(x^2), \end{cases} \tag{2.12}$$

where $\widetilde{H}_i$'s are polynomials of degrees n_i such that

$$\begin{gathered} \max\left\{\frac{2}{3}(n_1+1),\, n_2+1,\, 2(n_3+1),\, 2(n_4+1)\right\} = \frac{1}{\beta} < \\ < \min\left\{\frac{2}{3}(n_1+2),\, n_2+2,\, 2(n_3+2),\, 2(n_4+2)\right\} \end{gathered} \tag{2.13}$$

(if some $\widetilde{H}_i$ vanish identically, then we let $n_i = -1$). The coefficients of $\widetilde{H}_{ij}$ are defined as follows:

$$\widetilde{h}_{ij} = H_{ij} \,/\, \tau^{s_i - 2(j+1)/\beta},$$

where $s_1 = 3$, $s_2 = 2$, $s_3 = s_4 = 1$.

It follows immediately from (2.13) that $n_3 = n_4$, i. e., the degrees of $\widetilde{H}_3$ and $\widetilde{H}_4$ are always equal. Hence, the list of the asymptotic normal forms which are given by (2.12), (2.13) can be ordered as the common degree n ($= n_3 = n_4$) increases.

The first in the list are the systems given by (2.9), (2.10) and (2.11) — they correspond to $n = -1$. For each greater value of n there are four sub-cases below. Each consecutive case corresponds to an additional degeneracy. This procedure is recurrent: the next successive fourth case corresponds to the previous one but with the value n increased by 1.

1) $n_1 = 3n + 2$, $n_2 = 2n + 1$; at the moment of bifurcation the first $(n-1)$ coefficients vanish in both H_3 and H_4, the first $2n$ and $(3n+1)$ coefficients vanish in H_2 and H_1, respectively.

2) $n_1 = 3n+3$, $n_2 = 2n+1$; at the moment of bifurcation the first n coefficients vanish in both H_3 and H_4, the first $(2n+1)$ and $(3n+2)$ coefficients vanish in H_2 and H_1, respectively.

3) $n_1 = 3n+3$, $n_2 = 2n+2$; at the moment of bifurcation the first n coefficients vanish in both H_3 and H_4, the first $(2n+1)$ and $(3n+3)$ coefficients vanish in H_2 and H_1, respectively.

4) $n_1 = 3n+4$, $n_2 = 2n+2$; at the moment of bifurcation the first n coefficients vanish in both H_3 and H_4, the first $(2n+2)$ and $(3n+3)$ coefficients vanish in H_2 and H_1, respectively.

3. Symmetry of order q $(q \geqslant 3)$

Let us consider a system in $\mathbb{R}^3$ which possesses an equilibrium state (0, 0, 0) with three zero characteristic exponents $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Let the Jacobian matrix of the system at the equilibrium state be zero, and let the system be equivariant with respect to a turn over $2\pi/q$ around the z-axis. The system near the equilibrium state can then be written as

$$\begin{cases} \dot{w} = wF_1(w\overline{w},\, w^q,\, \overline{w}^q) + \overline{w}^{q-1}F_2(w\overline{w},\, w^q,\, \overline{w}^q) + \\ \qquad + z(wG_1(w\overline{w},\, w^q,\, \overline{w}^q,\, z) + \overline{w}^{q-1}G_2(w\overline{w},\, w^q,\, \overline{w}^q,\, z)), \\ \dot{z} = H_1(w\overline{w},\, w^q,\, \overline{w}^q) + zH_2(w\overline{w},\, w^q,\, \overline{w}^q) + z^2H_3(w\overline{w},\, w^q,\, \overline{w}^q,\, z), \end{cases} \tag{3.1}$$

where $w = x + iy$ and $\overline{w} = x - iy$. We consider a three-parameter perturbation of (3.1) in the form

$$\begin{cases} \dot{w} = (\mu_1 + i\mu_2)w + wF_1(w\overline{w},\, w^q,\, \overline{w}^q) + \overline{w}^{q-1}F_2(w\overline{w},\, w^q,\, \overline{w}^q) + \\ \qquad + z(wG_1(w\overline{w},\, w^q,\, \overline{w}^q,\, z) + \overline{w}^{q-1}G_2(w\overline{w},\, w^q,\, \overline{w}^q,\, z)), \\ \dot{z} = -\mu_3 z + H_1(w\overline{w},\, w^q,\, \overline{w}^q) + zH_2(w\overline{w},\, w^q,\, \overline{w}^q) + z^2H_3(w\overline{w},\, w^q,\, \overline{w}^q,\, z). \end{cases} \tag{3.2}$$

We suppose that the main coupling term (the zw-term in the first equation in (3.2)) in non-zero at the bifurcation moment, i. e.,

$$G_1(0,\, 0,\, 0,\, 0) \neq 0. \tag{3.3}$$

We also assume that

$$H_3(0,\, 0,\, 0,\, 0) \neq 0. \tag{3.4}$$

Let $A = G_1(0, 0, 0, 0)$ and $B = H_3(0, 0, 0, 0)$. Without loss of generality we assume $B = 1$ (this can always be achieved by a linear rescaling of z). One can check that a suitable coordinate transformation

$$z \to z + \Psi(w,\, \overline{w})$$

eliminates all terms in H_2 up to any prescribed finite order, provided

$$\operatorname{Im} A \neq 0 \tag{3.5}$$

and

$$\operatorname{Re} A \neq \frac{1}{m}, \quad m = 1, 2, \ldots . \tag{3.6}$$

Condition (3.5) will be our standing assumption. If (3.6) holds, then we can assume that

$$H_2 = O\left(|w|^N\right)$$

for some sufficiently large N. If, on the contrary, $\operatorname{Re} A \cdot m = 1$ for some integer m at the bifurcation moment, then the only term that survives in H_2 is $(w\overline{w})^m$, whence

$$H_2 = h_{2m}(w\overline{w})^m + O\left(|w|^N\right)$$

in this case.

Let us rescale the phase and time variables:

$$w \to \tau^\beta w, \quad z \to \tau^\gamma z, \quad t \to t/\tau,$$

where β and γ are some quantities defined further and $\tau = \sqrt{\mu_1^2 + \mu_2^2}$. Then, the system (3.2) takes the form:

$$\begin{cases} \dot{w} = e^{i\Omega} w + wF_1(\tau^{2\beta} w\overline{w},\, \tau^{q\beta} w^q,\, \tau^{q\beta}\overline{w}^q) \,/\, \tau + \\ \qquad + \overline{w}^{q-1} F_2(\tau^{2\beta} w\overline{w},\, \tau^{q\beta} w^q,\, \tau^{q\beta}\overline{w}^q) \,/\, \tau^{1-\beta(q-2)} + \\ \qquad + z(wG_1(\tau^{2\beta} w\overline{w},\, \tau^{q\beta} w^q,\, \tau^{q\beta}\overline{w}^q,\, \tau^\gamma z) + \\ \qquad + \overline{w}^{q-1} G_2(\tau^{2\beta} w\overline{w},\, \tau^{q\beta} w^q,\, \tau^{q\beta}\overline{w}^q,\, \tau^\gamma z) \,/\, \tau^{(2-q)\beta}) \,/\, \tau^{1-\gamma}, \\ \dot{z} = -\alpha z + H_1(\tau^{2\beta} w\overline{w},\, \tau^{q\beta} w^q,\, \tau^{q\beta}\overline{w}^q) \,/\, \tau^{\gamma+1} + \\ \qquad + zH_2(\tau^{2\beta} w\overline{w},\, \tau^{q\beta} w^q,\, \tau^{q\beta}\overline{w}^q) \,/\, \tau + z^2 H_3(\tau^{2\beta} w\overline{w},\, \tau^{q\beta} w^q,\, \tau^{q\beta}\overline{w}^q,\, \tau^\gamma z) \,/\, \tau^{1-\gamma}, \end{cases} \tag{3.7}$$

where

$$\alpha = \mu_3 \,/\, \sqrt{\mu_1^2 + \mu_2^2}, \quad \Omega = \operatorname{Arg}(\mu_1 + i\mu_2)$$

are the normalized parameters.

After the normalization the monomials that have the factor τ in positive powers will disappear as $\tau \to 0$, whereas the monomials that have the factor τ^0 remain. Their coefficients play the role of some structural parameters of the system. The factor τ in negative powers is allowed only in front of the terms whose coefficients vanish at the bifurcation moment; after normalization, the corresponding terms can also remain, and their coefficients can be regarded as the normalized governing parameters (in addition to α and Ω).

We supposed (see (3.3), (3.4)) that the term zw in the first equation in (3.2) as well as z^2 in the second equation does not vanish. Hence it can be seen from (3.7) that $\gamma \geqslant 1$. In order for that those terms persist in the asymptotic normal form, we choose $\gamma = 1$. Then the normalized system is given as

$$\begin{cases} \dot{w} = e^{i\Omega} w + wF_1(\tau^{2\beta} w\overline{w},\, \tau^{q\beta} w^q,\, \tau^{q\beta}\overline{w}^q) \,/\, \tau + \\ \qquad + \overline{w}^{q-1} F_2(\tau^{2\beta} w\overline{w},\, \tau^{q\beta} w^q,\, \tau^{q\beta}\overline{w}^q) \,/\, \tau^{1-\beta(q-2)} + Azw + o(1), \\ \dot{z} = -\alpha z + z^2 + H_1(\tau^{2\beta} w\overline{w},\, \tau^{q\beta} w^q,\, \tau^{q\beta}\overline{w}^q) \,/\, \tau^2 + \\ \qquad + zH_2(\tau^{2\beta} w\overline{w},\, \tau^{q\beta} w^q,\, \tau^{q\beta}\overline{w}^q) \,/\, \tau + o(1). \end{cases} \tag{3.8}$$

By getting rid of the asymptotically vanishing terms, it can be rewritten as

$$\begin{cases} \dot w = e^{i\Omega} w + w \sum_{k=0}^{q-1} \tau^{2k\beta-1} (w\overline{w})^k R_k(\tau^{q\beta} w^q,\, \tau^{q\beta}\overline{w}^q) + \\ \quad + \overline{w}^{q-1} \sum_{k=0}^{q-1} \tau^{(2k+q-2)\beta-1} (w\overline{w})^k P_k(\tau^{q\beta} w^q,\, \tau^{q\beta}\overline{w}^q) + Azw, \\ \dot z = -\alpha z + z^2 + \sum_{k=0}^{q-1} \tau^{2k\beta-2} (w\overline{w})^k S_k(\tau^{q\beta} w^q,\, \tau^{q\beta}\overline{w}^q) + C_m z \tau^{2m\beta-1} (w\overline{w})^m, \end{cases} \tag{3.9}$$

where $R_0(0,\, 0) = 0$, $S_0(0,\, 0) = 0$; the last term arise only if the resonant relation (3.6) is violated, then the integer power m is here equal to $(\operatorname{Re} A)^{-1}$.

In the limit $\tau \to +\infty$ the terms up to the orders r_k, p_k, s_k survive, respectively, in R_k, P_k, S_k if and only if

$$\begin{aligned} &\max_{k=0,\,\dots,\,q-1} \left(2k + qr_k,\ 2(k-1) + q(p_k+1),\ \frac{2k + qs_k}{2} \right) = \frac{1}{\beta} < \\ &< \min_{k=0,\,\dots,\,q-1} \left(2k + q(r_k+1),\ 2(k-1) + q(p_k+2),\ \frac{2k + q(s_k+1)}{2} \right). \end{aligned} \tag{3.10}$$

If the monomial $z(w\overline{w})^m$ in the second equation in (3.9) is resonant, then it merges to the final asymptotic normal form in case where

$$(\operatorname{Re} A)^{-1} = m \leqslant \frac{1}{2\beta}. \tag{3.11}$$

Comparing (3.11) with (3.10) reveals that this term would appear in the resulting normal form simultaneously with the term $w(w\overline{w})^m$ in the first equation (i. e., when $r_m \geqslant 0$).

Relation (3.10) is easily resolved giving

$$\begin{aligned} r_k &= \begin{cases} \left[\frac{s_{2k}}{2}\right] & \text{if} \quad 2k \leqslant q-1, \\ \left[\frac{s_{2k-q}}{2}\right] - 1 & \text{if} \quad 2k \geqslant q, \end{cases} \\ p_0 &= \left[\frac{s_{q-2}}{2}\right], \quad p_k = r_{k-1} - 1 \quad \text{for} \quad k = 1,\, \dots,\, q-1. \end{aligned} \tag{3.12}$$

Thus, the structure of the asymptotic normal form is defined by the values of $s_0,\, \dots,\, s_{q-1}$. Moreover, it follows from (3.10) that

$$s_k \geqslant s_{k+1} \geqslant s_k - 1$$

and

$$s_0 - 1 \geqslant s_{q-1} \geqslant s_0 - 2.$$

Therefore, the string of the integers s_k has the following structure: for some integers k_0 and k_1 such that $0 \leqslant k_0 < k_1 \leqslant q-1$ and for some integer $d \geqslant 0$

$$s_0 = \ldots = s_{k_0} = d, \quad s_{k_0+1} = \ldots = s_{k_1} = d-1, \quad s_k = d-2 \quad \text{at} \quad k > k_1. \tag{3.13}$$

Furthermore, we have from (3.10)

$$qd + \max(2k_0,\, 2k_1 - q) \leqslant \frac{1}{\beta} < qd + \min(2k_0,\, 2k_1 - q) + 2$$

which gives

$$2k_0 < 2k_1 - q + 2 \quad \text{and} \quad 2k_1 - q < 2k_0 + 2,$$

i. e.,

$$\left|(k_1 - k_0 - \frac{q}{2})\right| < 1,$$

or, finally,

$$k_1 = \begin{cases} k_0 + \frac{q}{2} & \text{if} \quad q \quad \text{is even}, \\ k_1 = k_0 + \frac{q}{2} \pm \frac{1}{2} & \text{if} \quad q \quad \text{is odd}. \end{cases} \tag{3.14}$$

The obtained relations (3.11)–(3.14) describe completely the structure of the asymptotic normal form, namely, this is the following system

$$\begin{cases} \dot{w} = e^{i\Omega} w + \sum\limits_{k=0}^{q-1} (w\overline{w})^k \left(w\widetilde{R}_k(w^q, \overline{w}^q) + \overline{w}^{q-1}\widetilde{P}_k(w^q, \overline{w}^q) \right) + Azw, \\ \dot{z} = -\alpha z + z^2 + \sum\limits_{k=0}^{q-1} (w\overline{w})^k \widetilde{S}_k(w^q, \overline{w}^q) + \widetilde{C}_m z (w\overline{w})^m, \end{cases} \tag{3.15}$$

where $\widetilde{R}_k$, $\widetilde{P}_k$, $\widetilde{S}_k$ are polynomials of degrees r_k, p_k, s_k respectively, where r_k and p_k are expressed through of s_k via (3.12). The integers s_k have the structure given by (3.13), (3.14) (the negative values of some of r_k, p_k or s_k mean merely the absence of the corresponding terms in the normal form (3.15)). Here, $\widetilde{R}_0(0, 0) = 0$, $\widetilde{S}_0(0, 0) = 0$. The value of $\widetilde{C}_m$ is non-zero if and only if $m = (\operatorname{Re} A)^{-1}$ and $r_m \geqslant 0$.

Formulae (3.13), (3.14) define a natural order in the normal forms given by (3.15): the order follows the increasing of d and for each fixed d the increasing values of k_0 (in case q is odd there are also two possible values of k_1).

Thus, the list starts with the following systems

$$q = 3: \quad \begin{cases} \dot{w} = e^{i\Omega} w + \widetilde{P}_{000}\overline{w}^2 + Azw, \\ \dot{z} = -\alpha z + z^2 + \widetilde{S}_{100} w\overline{w}, \end{cases}$$

$$q > 3: \quad \begin{cases} \dot{w} = e^{i\Omega} w + Azw, \\ \dot{z} = -\alpha z + z^2 + \widetilde{S}_{100} w\overline{w}. \end{cases}$$

They correspond to $d = 0$, $k_0 = 1$ (the case $d = 0$, $k_0 = 0$ is trivial). For $q > 4$ the list of normal forms corresponding to $d = 0$ is continued by

$$q > 4,\ k_0 = 2, \ldots, \left[\frac{q-1}{2}\right] : \begin{cases} \dot{w} = e^{i\Omega} w + Azw + w \sum\limits_{1 \leqslant k \leqslant k_0/2} \widetilde{R}_{k00} (w\overline{w})^k, \\ \dot{z} = -\alpha z + z^2 + \sum\limits_{1 \leqslant k \leqslant k_0} \widetilde{S}_{k00} (w\overline{w})^k. \end{cases}$$

We recall that if in the above system $\operatorname{Re} A = 1/m$ for some positive integer m, $m \leqslant k_0$, then the term $\widetilde{C}_m z (w\overline{w})^m$ should be added to the last equation. Note that for $q \geqslant 3$ the systems above have the rotational symmetry $w \mapsto we^{i\varphi}$. Hence they can further be reduced to two-dimensional systems.

The next are the normal forms with $d = 1$. We list them only for $q = 3,\ 4$:

$$q=3: \qquad \begin{cases} \dot{w} =e^{i\Omega}w + \widetilde{P}_{000}\overline{w}^2 + Azw, \\ \dot{z} = -\alpha z + z^2 + \widetilde{S}_{100}w\overline{w} + \widetilde{S}_{010}w^3 + \widetilde{S}_{001}\overline{w}^3, \end{cases}$$

$$q=3,\ 4: \quad \begin{cases} \dot{w} =e^{i\Omega}w + \widetilde{R}_{100}w^2\overline{w} + \widetilde{P}_{000}\overline{w}^{q-1} + Azw, \\ \dot{z} = -\alpha z + z^2 + \widetilde{S}_{100}w\overline{w} + \widetilde{S}_{010}w^q + \widetilde{S}_{001}\overline{w}^q + \widetilde{S}_{200}(w\overline{w})^2 + \widetilde{C}_1 zw\overline{w}, \end{cases}$$

$$q=3,\ 4: \quad \begin{cases} \dot{w} =e^{i\Omega}w + \widetilde{R}_{100}w^2\overline{w} + \widetilde{P}_{000}\overline{w}^{q-1} + Azw, \\ \dot{z} = -\alpha z + z^2 + \sum\limits_{k=1}^{q-1} \widetilde{S}_{k00}(w\overline{w})^k + \widetilde{S}_{010}w^q + \\ \qquad + \widetilde{S}_{001}\overline{w}^q + \widetilde{S}_{110}w^{q+1}\overline{w} + \widetilde{S}_{101}w\overline{w}^{q+1} + \widetilde{C}_1 zw\overline{w}. \end{cases}$$

Here, $\widetilde{C}_1 \neq 0$ if and only if $\operatorname{Re} A = 1$.

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