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Leaky Fermi accelerators

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A Fermi accelerator is a billiard with oscillating walls. A leaky accelerator interacts with an environment of an ideal gas at equilibrium by exchange of particles through a small hole on its boundary. Such interaction may heat the gas: we estimate the net energy flow through the hole under the assumption that the particles inside the billiard do not collide with each other and remain in the accelerator for a sufficiently long time. The heat production is found to depend strongly on the type of Fermi accelerator. An ergodic accelerator, i.e., one that has a single ergodic component, produces a weaker energy flow than a multicomponent accelerator. Specifically, in the ergodic case the energy gain is independent of the hole size, whereas in the multicomponent case the energy flow may be significantly increased by shrinking the hole size.

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I. INTRODUCTION

The dynamics of a point particle moving within a closed region (billiard) with oscillating walls provides a mathematical model for studying the phenomenon of Fermi acceleration [1–7]. Such systems typically produce an increase in the particle's kinetic energy, and much effort is devoted to quantifying this phenomenon. It has been shown that collisions with periodically oscillating walls of an ergodic chaotic billiard accelerate the particle so that on average its energy grows linearly with the number of collisions and quadratically as a function of time [5,7–9]. In particular, this behavior is observed in a periodically oscillating dispersive billiard [5,9] and in a stadium with an oscillating base [8]. Examples of such billiards are shown in Figs. 1(a) and 1(d). It was discovered in [9,10] that if the ergodicity of the frozen billiard is violated, i.e., the shape of the billiard is changed in such a way that several ergodic components are created during a part of the billiard oscillation cycle, then the average energy growth is much faster, typically exponential in time. A Bunimovich mushroom deformed such that there exists particle exchange between its integrable and chaotic components corresponds to such a multicomponent, exponential accelerator [11]; see Fig. 1(c). The mushroom is a special case of a large class of billiards with mixed phase space where chaotic zones coexist with stability islands; the exponential character of acceleration at a periodic perturbation of such systems was established in [12,13]. The multicomponent accelerators can also be created by pseudointegrability [10,14] and by division of the billiard configuration space into disjoint pieces [9]; see Fig. 1(b).

Hereafter, to stress their nonergodic nature, we loosely call accelerators that attain exponential-in-time acceleration "multicomponent accelerators." Notice that multicomponent accelerators may have a finite [Fig. 1(a)] or infinite [Fig. 1(b)] number of ergodic components. In the latter case, as in Fig. 1(b), some of these components may belong to an integrable (or near-integrable) region composed of an uncountable number of ergodic components. It follows from [9–15] that under quite general conditions, such systems attain exponential acceleration, yet it is also shown that the exponential acceleration may be suppressed in these systems by choosing specific degenerate protocols of boundary oscillations.

One of the primary applications of the Fermi acceleration model is in plasma physics where it is used to study the heating of charged particles due to electromagnetic waves [4,16]. In such systems, the electrons absorb energy from the wave in the plasma sheath, deposit it in the plasma bulk, and return back to the sheath. Thus, the system is not closed and allows for entry and exit of particles. Leaky chaotic systems also emerge in numerous physical situations, such as chemical reactions, optical microcavities [17,18], and hydrodynamic flows (see the recent review and references therein [19]).

Stationary leaky billiards have been studied extensively. It was shown that the escape rate through holes in the billiard boundary depends sensitively on the hole position, size, and billiard properties [19–24]. In this paper, we consider time-dependent leaky billiards and provide estimates for energy gain for the two above-mentioned classes of the accelerators: ergodic [as in Figs. 1(a) and 1(c)] and multicomponent [Figs. 1(b) and 1(d)]. We consider the small hole size limit and observe that $\bar{N}$, the averaged number of collisions a particle spends in the leaky accelerator, is inversely proportional to the hole size for both cases. Then we demonstrate that for the ergodic case, the averaged energy gain per particle grows linearly with $\bar{N}$, whereas in the multicomponent case the averaged energy gain is much larger and is approximated by a quadratic polynomial in $\bar{N}$; see Eqs. (6) and (7) versus (14) and (16) and Fig. 3.

II. MODEL

Consider an accelerator that interacts with an ideal gas by exchange of particles through a small hole (or a few

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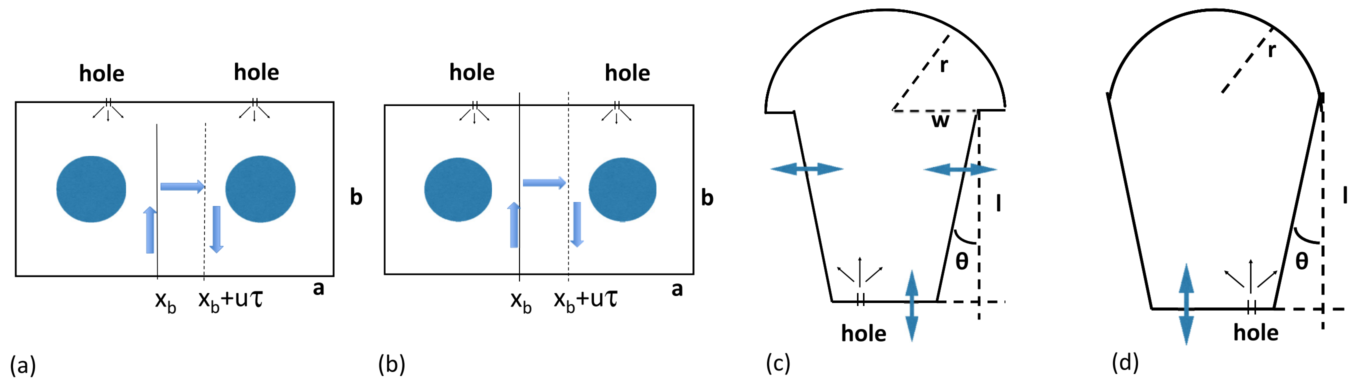

FIG. 1. (Color online) (a) Sinai accelerator, (b) divided Sinai accelerator, (c) mushroom accelerator, and (d) stadium accelerator. See the simulation section (Sec. III) for geometric specifications and dynamical properties.

small holes) on its boundary. We assume that the gas is at equilibrium, i.e., there is a stationary distribution of the particles' speed, and the distribution of the angles at which the particles move in the gas is uniform. We assume that the particles move much faster than the billiard boundary. Collisions with the moving billiard walls change the kinetic energy of the particles inside the billiard, and, on average, this may lead to an outgoing energy gain.

For a fixed kinetic energy $E_{\rm in} = \frac{v_{\rm in}^2}{2}$ (we assume the particles have a unit mass), the number density of particles entering the billiard per unit of time is proportional to $hv_{\rm in}$, where h is the size of the hole (the area of the hole for the three-dimensional case). Thus, the incoming energy flow at energy $E_{\rm in}$ is proportional to $hv_{\rm in}E_{\rm in}$. We assume that inside the billiard, the particles do not collide or interact with each other, so we can consider each of them separately. This gives us the net energy production by the accelerator per unit of time:

$$G(E_{\rm in}) = hv_{\rm in}[E_{\rm out} - E_{\rm in}], \tag{1}$$

where $E_{\rm out}$ is the averaged value of the kinetic energy at the moment of exit for a particle that enters the accelerator with the energy $E_{\rm in}$ (we average over all possible initial angles and positions in the hole, as well as over the phase of the billiard oscillations at the entry moment).

Let p_N denote the probability to exit the accelerator after N collisions with the billiards walls, and let $\bar{E}(N)$ be the corresponding averaged exit energy. Then

$$E_{\rm out} = \sum \bar{E}(N; E_{\rm in}) p_N. \tag{2}$$

We assume that the hole size h is small enough, so the effect of the hole on the statistics of the billiard is negligibly small (as in the case of the stationary Lorentz gas [23]). Specifically, we assume that $\bar{E}(N; E_{\rm in})$ can be approximated by the averaged energy of a particle in the closed (i.e., nonleaky) accelerator after N collisions. Additionally, we assume that p_N and, thus, $\bar{N}$, the averaged number of collisions before exit, do not depend on $E_{\rm in}$ nor on the wall velocity u. This is obviously true when the billiard walls are stationary, so we extrapolate this claim to the case of slowly moving boundaries. We confirm this claim numerically for the examples we consider here (see Fig. 2). In fact, the numerics show that p_N can be well approximated by the geometric distribution $p_N = \frac{1}{\bar{N}}[1 - (1/\bar{N})]^{N-1}$ [Fig. 2(c)]. The average value $\bar{N}$ in this setting is just a geometric characteristic of the billiard and the hole. The natural assumption is

$$\bar{N} \sim \frac{S}{h} \sim \frac{V}{Lh}, \quad \overline{N^2} \sim \left(\frac{V}{Lh}\right)^2, \tag{3}$$

where h is the size of the hole in the billiard boundary, S is the size of the entire billiard boundary, V is the volume occupied by the billiard, and L is the characteristic diameter of the billiard. Relations (3) are confirmed by numerical experiments [see Fig. 2(a)].

To find the net energy production (1), it remains to estimate the dependence of the averaged energy $\langle E \rangle$ of a particle in the closed accelerator on the number of collisions N. Let us recall how energy is gained in the accelerators. The reflection law for a particle hitting a moving wall is obtained by going to a coordinate frame that moves with the same velocity as the wall at the moment of collision. In the moving coordinates, we have an elastic reflection law that, after returning to the stationary frame, results in the reflection law

$$v'_\perp = 2u(t,x) - v_\perp, \quad v'_\parallel = v_\parallel, \tag{4}$$

where $u(t,x)$ is the normal velocity of the wall at the collision point x at the moment t; v and v' are the velocities before and after the collision, and the subscripts $\perp$ and $\parallel$ stand for the components of the velocity that are normal and parallel to the wall, respectively.

If the billiard is chaotic, then the correlations between the consecutive angles ϕ at which the particle hits the wall decay fast. Therefore, the process described by Eq. (4) may be approximated by a random walk with reflections: at each collision, the particle velocity undergoes a reflection and acquires an increment at a random direction.

This random walk proceeds differently for the two main classes of accelerators, namely ergodic and multicomponent [7]. In the ergodic case, the random walk becomes unbiased in the large speed limit, which means that the square of velocity (i.e., the kinetic energy) grows linearly with the number of collisions. Indeed, by taking the square of Eq. (4), the energy $E_N = \frac{1}{2}v_N^2$ after the Nth collision satisfies

$$E_{N+1} = E_N - 2u(t_N,x_N)v_N\cos\phi_N + 2u^2(t_N,x_N). \tag{5}$$

Since $|u| \ll v$, the change in the billiard shape and relative change in the energy are not significant for a large number

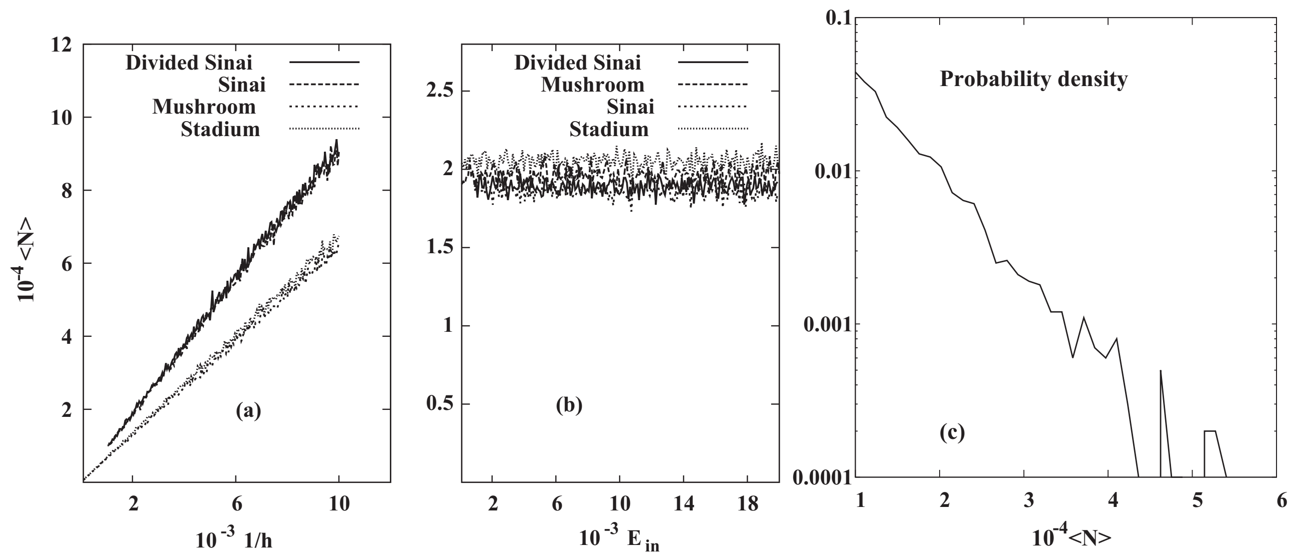

FIG. 2. The distribution and the average number of collisions; see Eq. (3). (a) Dependence on the hole size. Here, the initial energy is $E_{\rm in} = 9000$ for the Sinai and divided Sinai accelerators, and $E_{\rm in} = 1250$ for the mushroom and stadium accelerators. (b) No dependence on the initial particles' energy. Here $h = 0.0005$ for Sinai and divided Sinai accelerators, and $h = 0.000\,33$ for mushroom and stadium accelerators. (c) The exponential distribution of the exit probabilities for the stadium accelerator. Here $E_{\rm in} = 1250$ and $h = 0.000\,33$.

of consecutive collisions, so one can average Eq. (5) over the ergodic measure in the (x,ϕ) space. The second term in Eq. (5) can be much larger than the third one, but one can check (see, e.g., [7]) that after averaging over the ergodic measure and over the period of the billiard oscillation, the second term vanishes (this is a consequence of the existence of the so-called Anosov-Kasuga adiabatic invariant in the ergodic case; see [2,7,9,25–29]). Corrections to the averaging due to a slow change in the billiard shape and energy were computed in [2]. It follows from [2] that, after the averaging, the $O(uv)$ term in Eq. (5) effectively acquires a small factor of order $|u|/v$ (see also [9]). Thus, the effective change of the averaged energy per collision is of order u^2 (i.e., it is a certain portion of the kinetic energy of the wall).

It follows that in the ergodic case, the averaged energy of a particle grows as $\langle E(N)\rangle - E_{\rm in} = k\frac{\bar{u}^2}{2}N$, where $\bar{u}$ is the averaged wall speed (average of $|u|$). Thus, we conclude [see Eqs. (2) and (3)] that in the small hole limit,

$$E_{\rm out} - E_{\rm in} = k\frac{\bar{u}^2}{2}\bar{N} = k\frac{V}{Lh}\frac{\bar{u}^2}{2} \tag{6}$$

for some coefficient k that may depend on the billiard shape and on the details of the protocol of the billiard wall oscillations. By plugging this result into Eq. (1), we obtain that the energy gain rate in the ergodic case is positive, independent of the hole size h, and is given by

$$G(E_{\rm in}) = kv_{\rm in}\frac{V}{L}\frac{\bar{u}^2}{2}, \tag{7}$$

i.e., it is proportional to the kinetic energy of the billiard wall, to the volume of the billiard, and inversely proportional to the time $L/v_{\rm in}$ the gas particle with the speed $v_{\rm in}$ needs to traverse the billiard once.

Next, we investigate the case of a multicomponent accelerator. In this case, the ergodicity of the fast motion is broken, so the $O(uv)$ term in Eq. (5) does not average out. This means that the random walk (4) in the velocity space acquires a nonvanishing bias, so the particle speed is linear in N and its energy is quadratic in N. A more precise description of this process is done based on the theory developed in [7,9]. We note that the time between two consecutive collisions tends to zero as the particle speed grows, $t_{N+1} - t_N \sim L/v_N$, and it follows from Eq. (5) that $\frac{\Delta E}{\Delta t} = \frac{E_{N+1}-E_N}{t_{N+1}-t_N} \sim Lu_N \cos\phi_N E_N$, i.e., in the nonergodic case the energy changes exponentially with time, with a certain random rate. On a longer time scale, this process can be modeled by a multiplicative random walk (see [7,9,13,15]):

$$E_{n+1} = \xi_n^2 E_n, \quad v_{n+1} = \xi_n v_n, \tag{8}$$

where $E_n = \frac{v_n^2}{2}$ is the kinetic energy after n periods of the billiard oscillations, and ξ_n is the sequence of independent, identically distributed random variables, independent of the initial energy. Importantly, it is shown in [7,13,15] that this random walk cannot be decelerating and, typically,

$$\mathbb{E}\ln\xi_n > 0, \quad \mathbb{E}\xi_n > 1, \quad \mathbb{E}\xi_n^2 > 1. \tag{9}$$

If we ignore the details of particle behavior on the time scales below the period T of billiard oscillations, we can infer from Eq. (8) the following description for the behavior of the averaged energy and speed gain at time t:

$$\langle E(t)\rangle = E_{\rm in}e^{\mu t}, \quad \langle v(t)\rangle = v_{\rm in}e^{\lambda t}, \tag{10}$$

$$\begin{aligned}\langle v(s_1)v(s_2)\rangle &= \langle v^2(s_1)\rangle; \quad \langle v(s_2)/v(s_1)\rangle \\ &= \langle v^2(s_1)\rangle e^{\lambda(s_2-s_1)} \\ &= 2E_{\rm in}e^{\mu s_1}e^{\lambda(s_2-s_1)} \quad (s_2 \geqslant s_1),\end{aligned} \tag{11}$$

where $\mu = \frac{1}{T}\ln\mathbb{E}\xi^2 \ > \ 2\lambda = \frac{2}{T}\ln\mathbb{E}\xi \ > \ 0$.

Note that the number of collisions up to time t can be related to the particle speed via $LN(t) \sim \int_0^t v(s)ds$. So, the averaged number of collisions up to time t is given by

$$\langle N(t)\rangle = k_1 v_{\rm in}\frac{(e^{\lambda t}-1)}{\lambda L}$$

[see Eq. (10)] and, by Eq. (11),

$$\begin{aligned}\langle N^2(t)\rangle &= \frac{k_2}{L^2}\int_0^t\int_0^t \langle v(s_1)v(s_2)\rangle ds_1 ds_2\\ &= \frac{2k_2}{L^2}\int_0^t\int_{s_1}^t \langle v(s_1)v(s_2)\rangle ds_2 ds_1\\ &= 4k_2\frac{E_{\rm in}}{L^2}\int_0^t\int_{s_1}^t e^{\mu s_1}e^{\lambda(s_2-s_1)}ds_2 ds_1\\ &= 4k_2\frac{E_{\rm in}}{(\mu-\lambda)L^2}\left[\frac{e^{\mu t}-1}{\mu}-\frac{e^{\lambda t}-1}{\lambda}\right],\end{aligned}$$

where $k_{1,2}$ are some coefficients of order 1.

Rearranging the expressions for $\langle N\rangle$ and $\langle N^2\rangle$, we get

$$\langle E(t)\rangle - E_{\rm in} = 2k_1\mu v_{\rm in}L\langle N(t)\rangle + 4k_2\mu^2L^2\left(1-\frac{\lambda}{\mu}\right)\langle N(t)^2\rangle,$$

which, after averaging over the time t that the particle resides in the billiard, gives

$$E_{\rm out}-E_{\rm in} = \mu\frac{L^2}{T}\left[2k_1\frac{v_{\rm in}T}{L}\bar N + 4k_2\left(1-\frac{\lambda}{\mu}\right)\mu T\overline{N^2}\right]. \quad (12)$$

The rate μ in this formula is a well-defined quantity determined by a one-period run of the accelerator, and an analytic expression for μ is also available in many cases [7,9–15]. However, to compare the energy gain with that given by Eq. (6) in the ergodic case, we need to relate the rate μ with $\frac{\bar u^2}{2}$, the kinetic energy of the wall.

By Eq. (9), the exponential growth rate μ is always non-negative. The minimal value $\mu = 0$ is achieved when the distribution of particles in the billiard remains uniform during the period of billiard oscillations, as in the case of an ergodic billiard. Hence, when the deviation from the ergodic behavior is small, the rate μ is small and behaves like the square of a certain quantitative measure of this deviation. The violation of ergodicity in the exponential accelerator is caused by changes of the phase space structure of the frozen billiard as its shape changes with time. Therefore, we relate the deviation from ergodicity to the magnitude of the shape change over the period. As the billiard size is changed with the mean speed $\bar u$, the dimensionless parameter estimating the ergodicity violation is $\bar u T/L$. Thus, the rate of the energy increase over the period T is, at small $\bar u$, given by

$$\mu T \propto \left(\frac{\bar u T}{L}\right)^2. \quad (13)$$

One can also extract this relation from the formulas for μ for various cases of multicomponent accelerators (e.g., from [7,9–12]). Notice that even when $\bar u$ is not small, the ratio $\bar u T/L$ remains bounded and so does μT (we assume everywhere that the typical length scale of the billiard does not change hugely along the cycle, otherwise pathological behaviors may arise). So, Eq. (13) can be used in this case as well; it would then simply mean that the quantities on both sides of the relation are of order 1.

Plugging Eq. (13) into Eq. (12), we find

$$E_{\rm out}-E_{\rm in} = \frac{\bar u^2}{2}\left[k_1\frac{v_{\rm in}T}{L} + k_2\mu T\overline{N^2}\right], \quad (14)$$

or, in the case $\bar u T/L \ll 1$,

$$E_{\rm out}-E_{\rm in} = \frac{\bar u^2}{2}\left[k_1\frac{v_{\rm in}T}{L}\bar N + k_2\left(\frac{\bar u T}{L}\right)^2\overline{N^2}\right], \quad (15)$$

where the new coefficients $k_{1,2}$ depend on the shape of the billiard and the protocol of the wall oscillation.

The term $v_{\rm in}T/L$ is proportional to the number of collisions per period. In our setting, this number is assumed to be large. It is also important for the validity of the exponential growth model that the particle is initially fast (i.e., $v_{\rm in} \gg \bar u$) yet it remains in the billiard at least for one period. Therefore, Eqs. (12), (14), and (15) are valid under the assumption

$$1 \ll \frac{v_{\rm in}T}{L} \lesssim \bar N;$$

in particular $v_{\rm in} \lesssim \frac{V}{Th}$. When this condition is violated, our theory is not applicable.

Comparing Eqs. (14) and (15) with Eq. (6), we see that the energy gain in the multicomponent accelerator is much larger than in the ergodic case. Even if the exponential growth rate μ is very small, the coefficient of $\bar u^2\bar N$ in Eq. (14) is large whereas the corresponding coefficient in Eq. (6) is simply a constant. With the increase of μ, the quadratic in $\bar N$ term becomes dominant in Eqs. (14) and (15) and provides the main contribution to the energy gain.

Using Eq. (3) for $\bar N$ and $\overline{N^2}$, we finally find the energy production rate of Eq. (1) for the multicomponent case:

$$G(E_{\rm in}) = v_{\rm in}\frac{V}{L}\frac{\bar u^2}{2}\left[k_1\frac{v_{\rm in}T}{L} + k_2\mu T\frac{V}{Lh}\right]. \quad (16)$$

Clearly, the gain rate G can be made much larger than the gain in the ergodic case by diminishing the hole size or by increasing the incoming velocity.

III. SIMULATIONS

There are two distinct predicted dependencies of the energy gain on the hole size and on $v_{\rm in}$ for the two types of leaky accelerators. To examine these predictions, we consider two classes of leaky accelerators where each is considered with two sets of parameters—one corresponding to an ergodic case and the other to a multicomponent case.

Dispersing accelerators [Figs. 1(a) and 1(b)]: at $t = 0$, a vertical bar is inserted at a position x_b to the double Sinai billiard (a rectangle with two disks). Then the bar moves to the right with a constant velocity u until time τ and then the bar is removed. The cycle restarts at time T. We consider two cases: (1a) an ergodic case in which the bar only partially blocks the rectangle, covering 90% of the rectangle length (this case is called "Sinai" in all figures); (1b) a multicomponent case in which the bar completely divides the rectangle into two parts (hereafter "divided Sinai"). Notice that each component of the frozen billiards is ergodic and mixing [30]. These accelerators exhibit exponential-in-time energy growth in the multicomponent case and quadratic-in-time energy growth in the ergodic case [9]. To examine the leaky behavior, two holes of length h are placed on the upper rectangle boundary; the holes are shifted from the disk centers to avoid fast escaping orbits, and two holes are introduced to avoid strong dependence on the billiard oscillation phase. In all simulations, we use the following parameters: the rectangle width is $a = 4$, its height is $b = 2$, the disk radii are $1/2$, the bar velocity is $u = 0.1$, the bar is introduced at the position $x_b = 0.0915$ and removed at the time moment $\tau = 1.83$, and the period is $T = 5.49$. The initial energy and the hole size are as indicated in the figures.

Focusing accelerators [Figs. 1(c) and 1(d)]: The mushroom is a multicomponent system having an integrable component and a chaotic component [31], whereas the slanted stadium is ergodic and mixing [32]. The oscillating mushroom accelerator exhibits exponential-in-time energy growth [11] whereas the oscillating stadium exhibits quadratic-in-time energy growth [7,8].

The shape of the mushroom is determined by the following four parameters: r is the radius of the cap; w is the half-width of the hole at the bottom of the cap (it coincides with the half-width of the stem at its highest point and $w \leqslant r$); ℓ is the length of the stem; and the angle θ describes the inclination of the stem sides. When $w = r$, the mushroom becomes a slanted stadium.

For the purpose of numerical experiments, we used the following protocols: $r(t) = 1$, $w(t) = b_0 - b_1[1 - \cos(t)]$, and $\ell(t) = a_0 - a_1 \sin(t)$, $a_0 = 1$, $b_0 = 1$. For the mushroom, we set $a_1 = 0.5$ and $b_1 = 0.4$; and for the stadium, $a_1 = 0.5$, $b_1 = 0$. The hole is located at the bottom of the stem with the center displaced by 0.01 from the center of the stem. In all experiments, $\theta = 0.1111$.

In each numerical experiment, 2000 particles are injected at randomly chosen times during the period $[0,T]$ into the billiard through the holes, with random positions in the hole and entering angle. Each particle moves inside the billiard undergoing elastic collisions with the boundary until it exits by colliding with the hole. The exit time, the number of collisions until exit, and the exit speed are recorded.

Figure 2(a) demonstrates that the average number of collisions at exit, $\overline{N}$, scales linearly with $1/h$ for both the multicomponent and the ergodic cases. Figure 2(b) shows that $\overline{N}$ does not depend on the initial energy, again for both cases. Figure 2(c) shows the geometric distribution of the exit collisions for the mushroom. These results support the assumptions made in Eq. (3).

Figure 3 shows the dependence of $G/v_{\text{in}} = (E_{\text{out}} - E_{\text{in}})h$ on $1/h$ for the four billiard types. Figure 3(a) shows that the net energy flow increases linearly with $1/h$ for billiards with exponential acceleration (nonergodic case) as predicted in Eq. (16). The inset shows that for sufficiently small holes, the flow is essentially independent of the hole size for the ergodic billiards as predicted by Eq. (7).

Figure 4 shows that the average energy gain, $E_{\text{out}} - E_{\text{in}}$, grows linearly with v_{in} for the multicomponent cases as predicted by Eq. (14). The inset shows that this difference also grows with v_{in} for the ergodic case (at a slow pace). To the leading order in h, one should not see such growth according to Eq. (6). We explain the effect by the order h difference between the statistics of the closed and leaky accelerators. For example, averaging the second term of Eq. (5) over a boundary with the order h hole produces corrections of order $h\bar{u}v$, which cause an order h bias in the random walk of the velocity. Figure 5 shows the mean bias $\langle v_{\text{out}} - v_{\text{in}}\rangle/\langle N\rangle$, which is indeed present and does not vanish in the limit of large initial speed, both in the ergodic and multicomponent cases. However, the bias is much smaller in the ergodic case, confirming our conjecture that it is of order h. The dependence of the energy gain on small u for the multicomponent accelerators is more delicate to obtain numerically due to the existence of two competing terms in Eq. (16). By measuring the u dependence of the slope of the energy gain dependence on $1/h$, namely the u dependence of the slope of curves such as those shown in Fig. 3, we obtained that indeed the slope dependence on u is close to u^4 (the simulations were performed for $u \in [0.1,0.2]$ in the divided Sinai accelerator).

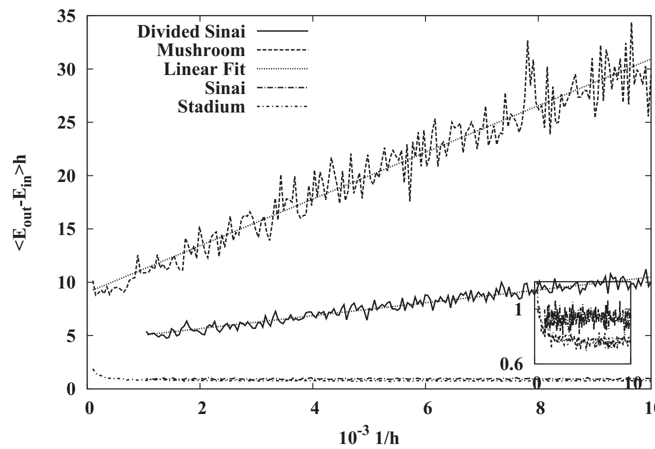

FIG. 3. Averaged energy gain dependence on the hole size. The energy gain increases linearly with $1/h$ for multicomponent billiards [see Eq. (16)], and it is independent of the hole size for the ergodic billiards (for sufficiently small holes); see Eq. (7). Here, the initial energy is $E_{\text{in}} = 9000$ for the Sinai and divided Sinai accelerators, and $E_{\text{in}} = 1250$ for the mushroom and stadium accelerators.

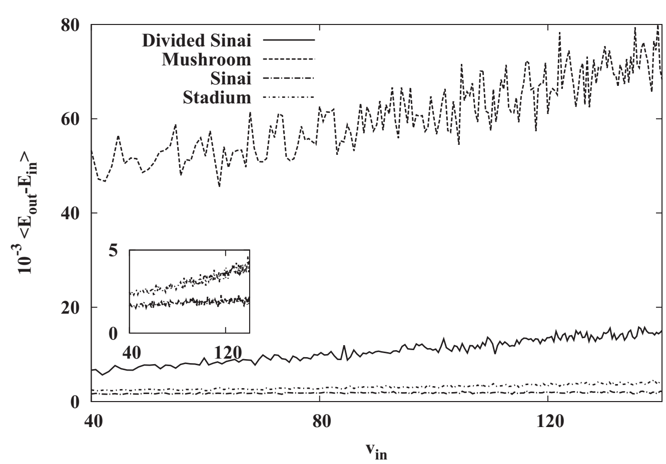

FIG. 4. Dependence of the average energy gain on the initial speed: linear growth with v_{in} for multicomponent billiards [Eq. (14)] and slow growth with v_{in} for the ergodic case, possibly due to order h corrections to Eq. (6). Here $h = 0.0005$ for Sinai and divided Sinai accelerators, and $h = 0.000\,33$ for mushroom and stadium accelerators.

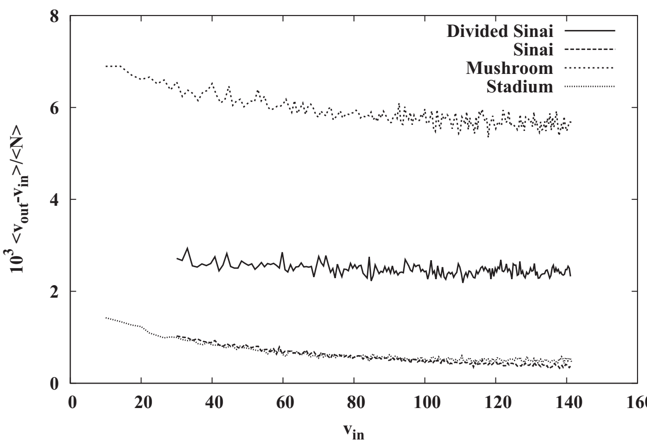

FIG. 5. The speed bias vs initial speed. Here $h = 0.0005$ for Sinai and divided Sinai, and $h = 0.000\,33$ for mushroom and stadium.

IV. DISCUSSION AND CONCLUSION

A priori, a faster accelerator does not automatically imply higher energy gain as faster particles escape earlier from the accelerator, and the net energy gain reflects the balance between the averaged escape time and the energy gained by then. In this paper, we find that in a billiard the escape is mainly determined by the average number of collisions, and from this observation we are able to find the balance between these two factors for two different types of accelerators. Our results conclusively show that multicomponent leaky accelerators produce higher net energy flow than their ergodic counterparts. Moreover, we show that the difference increases significantly when the hole size decreases.

One may worry that decreasing the hole size will lead to very long residence times. However, for the multicomponent case the residence time increases only logarithmically with the hole size. Indeed, the number of collisions is related to the residence time via $LN(t) \sim \int_0^t v(s)ds$, hence in the multicomponent case the averaged residence time is proportional to $\log \bar{N} \sim \log h$, while in the ergodic case it is proportional to $\sqrt{\bar{N}} \sim h^{-1/2}$. Hence, we expect that taking the small hole limit for gaining energy may become practical in the case of multicomponent leaky accelerators: they produce high gain at a fast pace.

Another question that naturally arises is the effect of interparticle collisions to our system. While our numerical experiments are done in a single-particle setting (the ensembles we consider correspond to different initial conditions for one particle), the general conclusion [7,9,12,13,15] on the higher acceleration rate in the nonergodic case holds true for any dimension of the system. Therefore, if we consider the gas of N colliding particles inside the three-dimensional billiard as a system in $3N$-dimensional configuration space, then we will have the same enhanced acceleration rate provided this system loses ergodicity for a substantial part of the billiard oscillation period T. We have discussed two different kinds of the ergodicity loss mechanisms. The first one is based on the separation of the billiard into two unconnected components during a part of the period. During the separation phase, the system will lose ergodicity irrespective of the number of particles, and the rate of the temperature growth will be the same as in a single-particle case; see [9]. Thus, taking collisions into account should not change the outgoing heat flow in this case.

The other mechanism corresponds to choosing the billiard shape in such a way (e.g., by combining scattering and focusing components in the boundary) that elliptic islands emerge in the phase space of the single-particle billiard. The strong heat flow we achieve in this case is due to the fact that the particle can get captured in the island, so the distribution of particles in the billiard deviates from uniform for a significant portion of the period T. Here, the interparticle collisions may restore the uniformity and, therefore, impede the heat flow. Therefore, to achieve the high heat flow in this case, the number of collisions of a typical particle with other particles per period T must be small. Then, the nonuniform particles' distribution will sustain for a substantial part of each oscillation period. This gives us the following limitation to the system parameters: $vT < S$, where S is the mean free path. In other words, the frequency of the billiard wall oscillations must be higher than the interparticle collision frequency for the gas inside the billiard. We also need the number of the particle-to-wall collisions per period to be sufficiently large, i.e., $L \ll vT$, which indicates that the size L of the billiard must be much smaller than the mean free path S, i.e., the Knudsen number of the gas inside the billiard must be sufficiently high [33].

At room temperature, the gas particles may move with the speed v of order of 500 m/s or higher; if the gas is sufficiently dilute (e.g., if $\rho \approx 10^{21}$ m^{-3}), one may take $S \sim 10^{-3}$ m or higher. This gives $T^{-1} > 1$ MHz and $L < 10$ μm, i.e., the parameters can be comparable with those of the MEMS devices. We can conclude that our results suggest that a periodic modulation of a shape of a microcavity at a radiofrequency might cause an anomalously high heat production, depending on the shape of the cavity.

Another relevant setting is that of a hot plasma, which can often be considered collisionless in tokamaks and various other settings; e.g., for plasma temperatures of order 10^6 K, the density of 10^{19} m^{-3}, and a trap whose shape is modulated at a radiofrequency, the above conditions for the validity of our model limit the size L of the trap at several meters. Moreover, the collision frequency of a plasma decreases with the increase in temperature [34], so if the plasma that enters the trap through the hole is dilute or hot enough to begin with, the collisionless approximation will only improve inside the billiard as the particles get heated up. Therefore, if the plasma traps of the periodically modulated shapes that correspond to nonergodic billiards can be created (e.g., by electromagnetic fields), then we have at least a theoretical possibility of the sustained and fast plasma heating via the mechanism suggested in this paper.

Finally, we note that performing multiparticle simulations with particles having smooth attracting or repelling potentials in oscillating accelerators of the three suggested types (ergodic, dispersing multicomponent, and mixed phase-space multicomponent) may reveal nontrivial phenomena as order parameters, such as density, are varied: phase transitions between ergodic and nonergodic states may lead to transitions between different forms of energy gain.

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