ON HAMILTONIAN SYSTEMS WITH HOMOCLINIC CURVES OF A SADDLE

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The availability of a stable homoclinic curve of a saddle equilibrium state in Hamiltonian systems is known to be their typical feature. It pertains to the fact that as far as the equilibrium state and its stable (W^s) and unstable (W^u) manifolds lie within a common energy level the manifolds W^s and W^u can then intersect along homoclinic curves transversely. Hence it may be expected that the set of all trajectories of the Hamiltonian system entirely located within the neighborhood of a homoclinic curve or of a bunch of homoclinic curves of saddle equilibrium state allows some reasonable, if not complete, description. Originally this problem for the case of homoclinic curve of the saddle-focus has been considered by R.J.Devaney /1/. He has found that the set of curves located at the energy level of a saddle-focus can be described in terms of a symbolic dynamics with countable number of symbols. Curious it is but this description turns out to be absolutely similar to that done for the structure of neighborhood of a rough homoclinic Poincare curve /2,3,/.

In this paper there are considered homoclinic bunches and a homoclinic contour with a saddle. Suppose that the system X with Hamiltonian $H \in C^3$ in some region $D \subseteq R^{2n} (n \geq 2)$ possesses the equilibrium state O. And let $\pm\lambda$, $\pm\lambda_i$ (i=2,...,n) be the roots of the characteristic equation for the point O. Suppose that O is a saddle, i.e. $0 < \lambda < Re\,\lambda_i$ (i=2,...,n).

In the vicinity of the saddle the vector field of the system is expressed as $\dot{x} = -\lambda x + \ldots$, $\dot{y} = -Ay + \ldots$,

$$\dot{u} = \lambda u + \ldots, \quad \dot{v} = A^T v + \ldots,$$

where $x \in R^1$, $y \in R^{n-1}$, $u \in R^1$, $v \in R^{n-1}$, $Sp\,A = \{\lambda_2, \ldots, \lambda_n\}$, the dots stand for the terms higher than the first order of smallness. The W_0^s in the point O touches the plane U = 0, V = 0, and the W_0^u touches the plane X = 0, Y = 0. As W^{ss} (W^{uu}) let us denote a stable (unstable) nonleading (n-1) - dimensional manifold of the saddle O. The W^{ss} touches the axis Y = 0, and the W^{uu} - the axis V = 0. The W^{ss} splits the W_0^s into two parts, the W_+^s and the W_-^s. Accordingly $W_0^u = W^{uu} \cup W_+^u \cup W_-^u$.

We shall consider that the W_+^s adjoins the W^{ss} from $X > 0$, and the W_+^u adjoins W^{uu} from $U > 0$. Let us assume that the W_0^s and the W_0^u intersect transversely along m homoclinic trajectories $\Gamma_1, \ldots, \Gamma_m$ not located in the W^{ss} and the W^{uu}. The latter means that the Γ_i enter the saddle and leave it touching the leading directions, the axes X and U respectively. We shall enumerate Γ_i so that

$$\bigcup_{i=1}^{m_1} \Gamma_i \subseteq W_+^s \cap W_+^u, \quad \bigcup_{i=m_1+1}^{m_1+m_2} \Gamma_i \subseteq W_+^s \cap W_-^u,$$

$$\bigcup_{i=m_1+m_2+1}^{m_1+m_2+m_3} \Gamma_i \subseteq W_-^s \cap W_-^u, \quad \bigcup_{i=m_1+m_2+m_3+1}^{m_1+m_2+m_3+m_4} \Gamma_i \subseteq W_-^s \cap W_+^u$$

(Here $m_1 + m_2 + m_3 + m_4 = m$). Without loss of generality, we may consider that $m_1 \neq 0$.

Let H = 0 be the level at which O is located. Let X_h stand for the system restriction to the level H = h. Let V be a small neighborhood of the bunch $\Gamma_1 \cup \ldots \cup \Gamma_m \cup O$. Let Ω_h denote the set of system X_h trajectories entirely lying in the V.

T h e o r e m 1. Suppose that $m = m_1 = 1$. Then for sufficiently small V and small $h_0 > 0$, dependent of the V, 1) at $h \in (-h_0, 0)$ the set $\Omega_h = \varnothing$, 2) at $h \in (0, h_0)$ the Ω_h consists of a single rough periodic motion L_h of the saddle type, and besides $\lim_{h \to +0} L_h = \Gamma_1 \cup O$, 3) $\Omega_0 = \Gamma_1 \cup O$.

T h e o r e m 2. For the case of m = 2 there exists such a sufficiently small neighborhood V, that 1) $\Omega_0 = \Gamma_1 \cup \Gamma_2 \cup O$, 2) at $m_1 = m_2 = 1$ or at $m_1 = m_4 = 1$ (Fig. 1a)* at small $h \neq 0$ the Ω_h comprises one periodic motion L_h of the saddle type, $\lim_{h \to +0} L_h = \Gamma_1 \cup O$, $\lim_{h \to -0} L_h = \Gamma_2 \cup O$ and 3) at $m_1 = m_3 = 1$ (Fig. 1b) the Ω_h at small h > 0 consists of two periodic mo-

*) The figures are given in the projection to the plane (X,U).

tions L_h^1 and L_h^2 of the saddle type, and $\lim_{h\to+0} L_h^i = \Gamma_i \cup O$, whereas at small h < 0 it consists of a single saddle periodic motion L_h of the "eight" type, $\lim_{h\to-0} L_h = \Gamma_1 \cup \Gamma_2 \cup O$, 4) at $m_1 = 2$ (Fig. 1c) $\Omega_h = \emptyset$ at $h < 0$, and at small $h > 0$ the $X_h|_{\Omega_h}$ is topologically equivalent to the suspension over the Bernoulli two-symbol scheme.

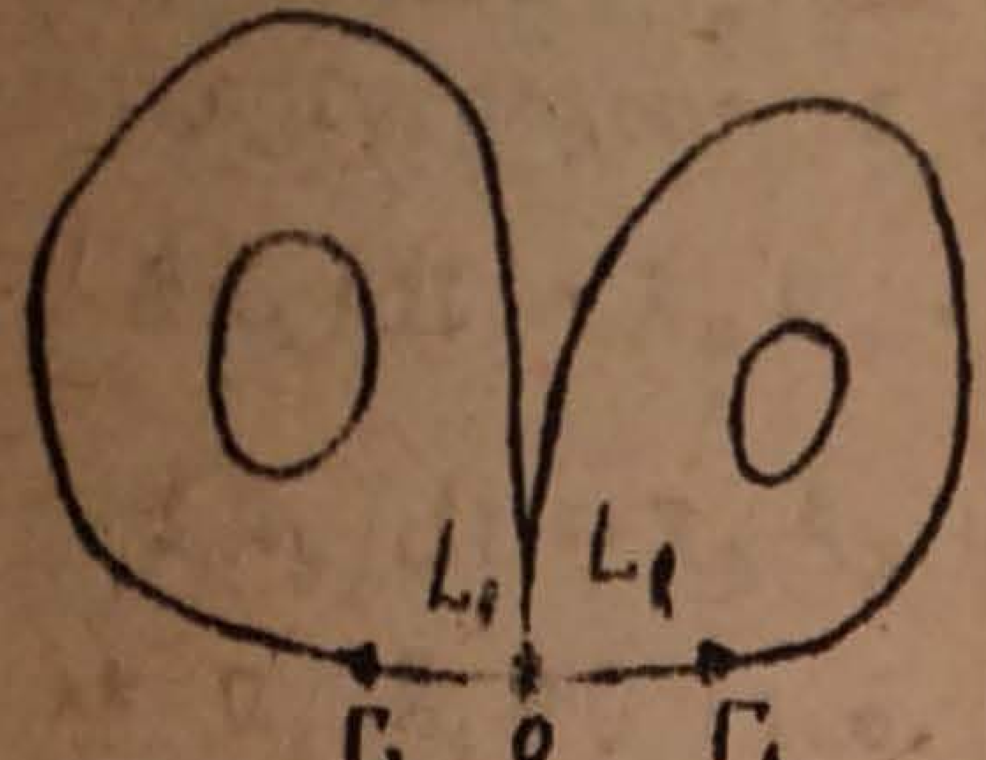

Fig. 1a

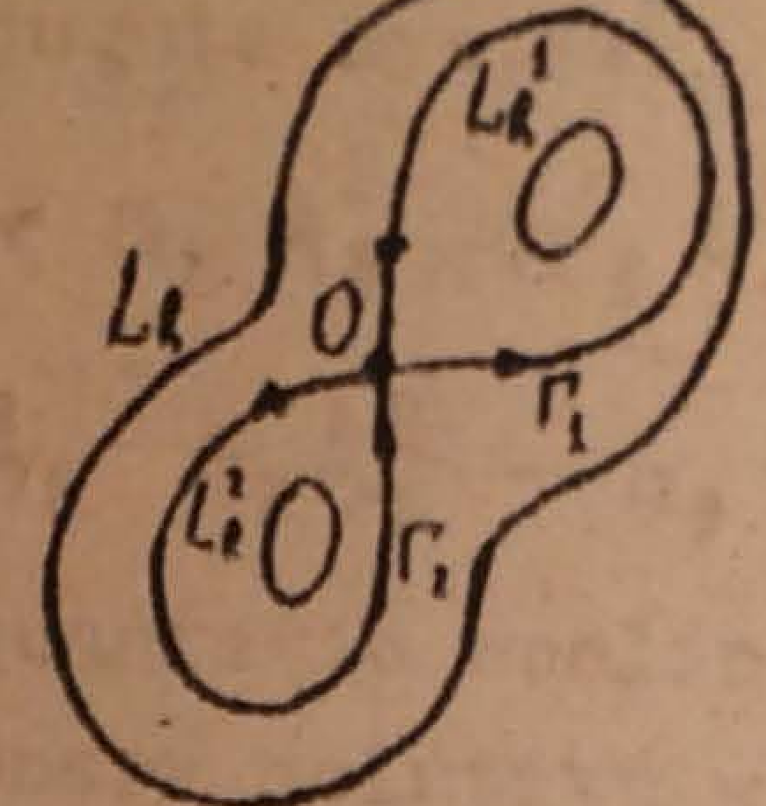

Fig. 1b

Fig. 1c

In case of $m \geq 3$ the theorems 1 and 2 are generalized in the below theorem 3.

T h e o r e m 3. For sufficiently small V and small $h_0 > 0$, dependent of the V, 1) $\Omega_0 = \Gamma_1 \cup \ldots \cup \Gamma_m \cup O$, 2) Ω_h at $h \neq 0$, $|h| < h_0$ is hyperbolic, and the $X_h|_{\Omega_h}$ is topologically equivalent to the suspension over the Markov topological chain defined in case of $h > 0$ with the help of graph G^+, and in case of $h < 0$ by the graph G^- (Fig. 2)*.

Fig. 2

*) In case of $m_3 = m_4 = 0$ the vertex 2 must be eliminated from the graphs.

From the figure it is clear that at $m \geq 3$ either G^+ or G^- has a vertex through which at least two cycles pass.

Corollary. In case of $m \geqslant 3$ the system X possesses a complicated structure.

Our further study presupposes an existence of homoclinic bunch having countable number of curves. Though here we are concerned with the most simple case only.

Let the system X at the energy level of the saddle have a saddle periodic motion L. The stable and unstable manifolds L are denoted by W_L^s and W_L^u. Suppose that there exists a homoclinic contour $\Gamma_1 \cup \Gamma_2 \cup L \cup O$ where $\Gamma_1 \subseteq W_0^u \cap W_L^s$, $\Gamma_2 \subseteq W_0^s \cap W_L^u$ and that the manifolds stated intersect along the Γ_1 and the Γ_2 transversely. Suppose also that $\Gamma_1 \not\subset W^{uu}$, $\Gamma_2 \not\subset W^{ss}$. Without loss of generality it may be considered that $\Gamma_1 \subset W_+^u$, $\Gamma_2 \subset W_+^s$. Now let us take a small neighborhood V of the contour $\Gamma_1 \cup \Gamma_2 \cup L \cup O$ and through Ω_h denote the set of trajectories of the system X_h, the trajectories lying entirely within the V.

Theorem 4. There exists such a small neighborhood V and such sufficiently small $h_0 > 0$, dependent on the V, that 1) at $h \in (-h_0, 0)$ the Ω_h consists of a single saddle cycle L_h, $\lim_{h \to 0} L_h = L$, and 2) at $h \in (0, h_0)$ the X_h / Ω_h is topologically equivalent to the suspension over the Bernoulli twosymbol scheme, and 3) the Ω_0 consists of Γ_1, Γ_2, L, O and of countable number of trajectories, homoclinic to O.

References

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