

Richness of chaos in the absolute Newhouse domain

Dmitry Turaev

Abstract. We show that universal maps (i.e. such whose iterations approximate every possible dynamics arbitrarily well) form a residual subset in an open set in the space of smooth dynamical systems. The result implies that many dynamical systems emerging in natural applications may, on a very long time scale, have quite unexpected dynamical properties, like coexistence of many non-trivial hyperbolic attractors and repellers and attractors with all zero Lyapunov exponents. Applications to reversible and symplectic maps are also considered.

Mathematics Subject Classification (2000). Primary 37C20, 37D45; Secondary 37G25, 37J40, 37E20, 37D20, 37D25, 37D30, 37C70.

Keywords. renormalization, homoclinic tangency, elliptic orbit, hyperbolic attractor, zero Lyapunov exponent, reversible system, Hamiltonian system

Decades of the study of dynamical systems with chaotic behavior revealed that with few exceptions these systems are more difficult than we would like. The diversity and variability of the types of chaotic dynamics occurring practically in any application are so great that nobody nowadays pursues the goal of a detailed mathematical description of the dynamics of a given system with chaos. The main source of difficulty is that the most of chaotic dynamical systems which emerge in natural applications appear to be *structurally unstable*. A system, i.e. a smooth map $f : M \rightarrow M$ of a smooth n -dimensional manifold M , or a smooth flow f_t on M , is called *structurally stable* if it is topologically equivalent to every close system (two systems are topologically equivalent if there exists a homeomorphism of M which maps the orbits of one system to the orbits of the other). A structurally unstable system is thus such that its orbit structure can be changed by an arbitrarily small (in some C^r -metric on M) perturbation.

Structurally unstable systems can fill open regions in the space of smooth dynamical systems. One of these regions, the so-called *Newhouse domain* \mathcal{N} , is the interior of the closure of the set of the systems which have a *homoclinic tangency* (that is an orbit of tangency between stable and unstable manifolds of a saddle periodic orbit). By [1, 2], this open set is non-empty, for the space of C^r -smooth maps on any manifold M of dimension $n \geq 2$ for any $r \geq 2$. Importantly, any generic family of maps which contains a map with a homoclinic tangency intersects \mathcal{N} for some open set of parameters [2, 4, 14, 15, 16]. As homoclinic tangencies easily

appear in a huge variety of chaotic dynamical systems for many parameter values, it follows that a great many naturally emerging models belong to the Newhouse domain for some, presumably large, regions of the parameter space¹. Studying dynamics of maps from the Newhouse domain is therefore one of the basic questions of the mathematical chaos theory.

In [3, 5, 8, 9], it was shown, however, that by an arbitrarily small (in C^r , for any $r = 2, \dots, \infty, \omega$) perturbation of any map from the Newhouse domain, one can create homoclinic tangencies of arbitrarily high orders and arbitrarily degenerate periodic points. This result shows that bifurcations of any map from \mathcal{N} are too diverse, and their complete and detailed understanding is impossible. An unfolding of tangency of order m requires m parameters, and here m can be arbitrarily large, so no finite-parameter unfolding can capture all changes in the dynamics which can occur at the perturbations of a given map $f \in \mathcal{N}$. In fact, for maps from an arbitrarily small neighborhood of $f \in \mathcal{N}$ in the space of C^r -smooth maps, the relation of topological equivalence has infinitely many independent continuous invariants (in other words, for any such neighborhood the set of the equivalence classes is infinite-dimensional); the same is true if we consider weaker equivalence relations: the topological equivalence on the set of non-wandering orbits, or on the set of periodic orbits, or even on the set of stable periodic orbits [3, 5]. The goal of this paper is to describe in precise terms the scale of this variability and dynamical richness for systems from the Newhouse domain.

In my opinion, the main characteristic feature of systems from \mathcal{N} is the absence of self-similarity: generically, the short-time behavior does not determine what will happen on longer time scales (contrary to Axiom A systems where a finite Markov partition determines the dynamics for all times). In order to describe this property, I use the following construction from [26, 27]. Let f be a C^r -map of an n -dimensional manifold M . Let B be any ball in M , i.e. let $B = \psi(U^n)$ where U^n is the closed unit ball in R^n and ψ is some C^r -diffeomorphism which takes U^n into M (we may take various maps ψ for the same ball B ; transition from one particular choice of ψ to another corresponds to a C^r -transformation of coordinates in B). We also assume that the C^r -diffeomorphism ψ is, in fact defined on some larger ball V : $U^n \subset V \subseteq R^n$. Given positive m , the map $f^m|_B$ is a return map if $f^m(B) \cap B \neq \emptyset$. By construction, the return map $f^m|_B$ is smoothly conjugate with the map $f_{m,\psi} = \psi^{-1} \circ f^m \circ \psi$ (in order the map $f_{m,\psi}$ to be properly defined, we need to also assume that $f^m(B) \subseteq \psi(V)$). The map $f_{m,\psi}$ is a C^r -map $U^n \rightarrow R^n$, and it is solely defined by the choice of the coordinate transformation ψ and the number of iterations m (the choice of the map $\psi : U^n \rightarrow M$ fixes the ball $B = \psi(U^n)$ as well). We will call the maps $f_{m,\psi}$ obtained by such procedure *renormalized iterations of f* . The set $\bigcup_{m,\psi} f_{m,\psi}$ of all possible renormalized iterations of f will be called *the dynamical conjugacy class* of f . As the balls $\psi(U^n)$ can be of arbitrarily

¹to get convinced, one may take any map of a two-dimensional disc with a chaotic behavior, find a saddle periodic point and follow, numerically, its stable and unstable invariant curves; the usual picture is that, after a number of iterations, folds in the unstable curve come sufficiently close to the stable curve, so the tangencies can be created by a slight parameter tuning

small radii, with the center situated anywhere, the dynamical conjugacy class of f captures arbitrarily fine details of the long-time behavior of f .

When we speak about dynamics of the map, we somehow describe its iterations, and the description should be insensitive to coordinate transformations. Therefore, the class of the map f , as we just have introduced it, gives some representation of the dynamics of f indeed: the larger the class, the more rich and diverse the dynamics of f is. There are some natural restrictions on this richness, as certain properties of the map f are inherited by all the maps from its class. For instance, when the topological entropy of f is zero, so is the entropy of every map from the class, any form of the hyperbolicity is inherited as well, all the maps from the class of a symplectic map are symplectic (although the symplectic form may become not a standard one), the class of a volume-contracting or a volume-preserving map contains only volume-contracting and, respectively, volume-preserving maps, an orientation-preserving map produces a class which contains orientation-preserving maps alone.

Importantly, only few of such “inheritable” properties can survive C^r -small perturbations of the map f . One of the known robust structures is the so-called *dominated splitting* (see [18]). A smooth map $f : M \rightarrow M$ of a compact n -dimensional manifold M has a dominated splitting when the tangent space at every point $x \in M$ is split into direct sum of two subspaces: $N^+(x) \oplus N^-(x) = R^n$, which depend continuously on x , which are invariant with respect to the derivative of f : $f'(x)N^+(x) = N^+(f(x))$ and $f'(x)N^-(x) = N^-(f(x))$, and which, at each $x_0 \in M$, satisfy the following requirement:

$$\begin{aligned} \lambda^-(x_0) &:= \overline{\lim}_{m \rightarrow +\infty} \sup_{\|u\|=1, u \in N^-(x_0)} \frac{1}{m} \ln \|f'(x_m) \cdots f'(x_0)u\| < \\ &< \lambda^+(x_0) := \underline{\lim}_{m \rightarrow +\infty} \inf_{\|v\|=1, v \in N^+(x_0)} \frac{1}{m} \ln \|f'(x_m) \cdots f'(x_0)v\|, \end{aligned}$$

where $x_0, x_1, \dots, x_m, \dots$ denotes the orbit of x_0 by f ; in other words, the dominance condition means that the maximal Lyapunov exponent corresponding to the subspace $N^-(x_0)$ is strictly less than minimal Lyapunov exponent corresponding to the subspace $N^+(x_0)$. There always exist trivial splittings, with $N^- = \emptyset$, $N^+ = R^n$ or $N^- = R^m$, $N^+ = \emptyset$. Non-trivial dominated splitting exists for uniformly hyperbolic systems (in this case $\lambda^-(x) < 0 < \lambda^+(x)$ for every x) and for uniformly partially-hyperbolic and pseudo-hyperbolic (volume-hyperbolic) systems. In general, there may be several dominated splittings for the same map, so we may have a hierarchy of subspaces $\emptyset = N_0^- \subset N_1^- \subset \cdots \subset N_k^- = R^n$, $R^n = N_0^+ \supset N_1^+ \supset \cdots \supset N_k^+ = \emptyset$ such that every pair N_j^-, N_j^+ corresponds to a dominated splitting. For any particular field $N_k^\pm(x)$ of the invariant subsets in this hierarchy, the linearized map restricted to the subset may exponentially contract (or expand) d -dimensional volumes for some $d \leq \dim N_k^\pm$. This volume contraction/expansion property is also inheritable by all renormalized iterations of f and it also persists at small smooth perturbations.

The general suspicion is that, perhaps, no other robust inheritable properties exist. This claim can be demonstrated for various examples of homoclinic bifur-

cations (see [24]), and can be used as a working guiding principle in the study of systems with a non-trivial dynamics:

every dynamics which is possible in U^n should be expected to occur at the bifurcations of any given n -dimensional system which has a compact invariant set without a non-trivial dominated splitting and without a volume-contraction or volume-expansion property.

This statement is not a theorem and it might be not true in some situations, still it gives a useful view on global bifurcations, as we will see in a moment.

The basic example is given by an identity map of a ball. The identity map has no kind of hyperbolic structure, neither it contracts nor expands volumes, so, according to the above stated principle, ultimately rich dynamics should be expected at the bifurcations of this map. Indeed, let us call a map f *C^r -universal* [26, 27] if its dynamical conjugacy class is C^r -dense among all orientation-preserving C^r -diffeomorphisms acting from the closed unit ball U^n into R^n . By the definition, the dynamics of any single universal map is ultimately complicated and rich, and the detailed understanding of it is not simpler than the understanding of all diffeomorphisms $U^n \rightarrow R^n$ altogether. At the first glance, the mere existence of C^r -univesal maps of a closed ball is not obvious for sufficiently large r . However, the following result is proven in [27].

Theorem 0.1. *For every $r = 1, \dots, \infty$, C^r -universal diffeomorphisms of a given closed ball D exist arbitrarily close, in the C^r -metric, to the identity map of D .*

A way to use this result is to note that, as it follows from Theorem 0.1, every time we have a periodic orbit for which the corresponding first-return map $x \mapsto \bar{x}$ is, locally, identity:

$$\bar{x} \equiv x,$$

or coincides with identity up to flat (i.e. sufficiently high order) terms:

$$\bar{x} = x + o(\|x\|^r),$$

a C^r -small perturbation of the system can make the first-return map universal, i.e. bifurcations of this orbit can produce dynamics as complicated as it only possible for the given dimension of the phase space.

In examples below, we show how powerful this observation can be. We start with the so-called *absolute Newhouse domain* \mathcal{A} in the space of C^r -smooth maps ($r \geq 2$) of any given manifold M , $\dim M \geq 2$. This domain is an open subset of the Newhouse domain such that no map from \mathcal{A} has a non-trivial dominated splitting, nor it uniformly contracts or expands volumes. The set \mathcal{A} can be constructed as the interior of the closure of the set of maps which have a particular type of *heteroclinic cycle*.

Namely, in the two-dimensional case the heteroclinic cycle is the union of 4 orbits: two saddle periodic orbits, p_1 and p_2 , such that the saddle value at p_1 is less than 1 and at p_2 it is greater than 1, and two heteroclinic orbits, Γ_{12} and Γ_{21} , such that Γ_{12} corresponds to transverse intersection of $W^u(p_1)$ and $W^s(p_2)$ (the unstable manifold of p_1 and the stable manifold of p_2), and Γ_{21} corresponds to

tangency between the other pair of invariant manifolds, $W^u(p_2)$ and $W^s(p_1)$. The saddle value is defined as the absolute value of the product of multipliers of the periodic orbit, i.e. it is the absolute value of the determinant of the derivative of the first-return map (if x_0 is a point of period l , then $f^l(x_0) = x_0$ and f^l is called the first-return map). Thus, if the saddle value is greater than 1, then the map f expands area near p_1 , and f is area-contracting near p_1 if the saddle value is less than 1. So, no map with the heteroclinic cycle of the type we just described is uniformly area-contracting, nor area-expanding. The tangency between the stable and unstable manifolds forbids the existence of a non-trivial dominated splitting. When the map f is perturbed, the tangency may disappear, however new orbits of heteroclinic tangency may appear somewhere else, and indeed, as follows from [2, 7], maps with a heteroclinic cycle of the above described type are dense (in C^r , $r \geq 2$) in a non-empty open region in the space of C^r -smooth maps; moreover, the closure of this region contains all maps with such heteroclinic cycles. This region is our domain \mathcal{A} in the two-dimensional case.

In the higher-dimensional case, where $n = \dim M > 2$, we consider heteroclinic cycles for which the saddle periodic orbits p_1 and p_2 have one-dimensional unstable manifolds, so the multipliers $\lambda_{j1}, \lambda_{j2}, \dots, \lambda_{jn}$ of the orbit p_j are such that $|\lambda_{ji}| > 1 > \max_{k \geq 2} |\lambda_{jk}|$ for each $j = 1, 2$. For each of the points p_j , we order the multipliers according to the decrease in the absolute value, i.e. $|\lambda_{jk}| \geq |\lambda_{js}|$ if $k \leq s$. We assume then that λ_{12} is real, while the rest of the multipliers λ_{1k} , $k \geq 3$, go in complex-conjugate (non-real) pairs except, maybe, for the last one, λ_{1n} , which must be real if n is odd. For the multipliers λ_{2k} , $k \geq 2$, of the orbit p_2 , we will allow only the last one, λ_{2n} , to be real if n is odd. As in the two-dimensional case, we also assume that $W^u(p_1)$ and $W^s(p_2)$ have a transverse intersection at the points of a heteroclinic orbit Γ_{12} , while $W^u(p_2)$ and $W^s(p_1)$ have a tangency at the points of the heteroclinic orbit Γ_{21} .

These conditions mean [24] that the map with such heteroclinic cycle cannot have a non-trivial dominated splitting. Indeed, if we have a dominated splitting, the spaces N^+ and N^- at a periodic point must be the invariant subspaces of the derivative of the first-return map at this point; moreover, for some $\bar{\lambda} > 0$, the space N^+ corresponds to the multipliers whose absolute value is greater than $\bar{\lambda}$, and N^- corresponds to the multipliers whose absolute value is less than $\bar{\lambda}$. As the multipliers λ_{2k} , $k \geq 2$, go in pairs of equal absolute value, for any non-trivial dominated splitting the dimension of the space N^+ at the points of the orbit p_2 must be odd. On the other hand, as the multipliers λ_{1k} with $k \geq 3$ also go in complex-conjugate pairs, the only possibility for the space N^+ at the points of the other periodic orbit, p_1 , be odd-dimensional corresponds to $\dim N^+ = 1$. Since N^+ depends on the point continuously, $\dim N^+$ should be the same at the points of p_1 as at the points of p_2 . Thus, the only possibility for a non-trivial dominated splitting occurs when at the points of the periodic orbits p_j , $j = 1, 2$, the space N^+ corresponds to the multiplier λ_{j1} (whose absolute value is greater than 1), and the space N^- corresponds to the rest of multipliers, i.e. N^+ must be tangent to $W^u(p_j)$ and N^- must be tangent to $W^s(p_j)$. By continuity, this would imply that N^+ would be tangent to $W^u(p_2)$ at every point of $W^u(p_2)$, and N^- would

be tangent to $W^s(p_1)$ at every point of $W^s(p_1)$. As the manifolds $W^u(p_2)$ and $W^s(p_1)$ are not transverse at the points of the heteroclinic orbit Γ_{21} , we find that $N^+ \oplus N^- \neq R^n$, a contradiction to the definition of the dominated splitting.

Now, assume that $\left| \prod_{k=1}^n \lambda_{1k} \right| < 1$ and $\left| \prod_{k=1}^n \lambda_{2k} \right| > 1$, i.e. the map f contracts volume at the points of p_1 and expands volume at the points of p_2 . So, the maps with the heteroclinic cycle that satisfies all these assumptions do not have a non-trivial dominated splitting and cannot be uniformly volume-contracting, nor volume-expanding. One can extract from [4, 14] that the C^r -closure of the set of the maps with such heteroclinic cycles has a non-empty interior, which is our absolute Newhouse domain \mathcal{A} in the space of n -dimensional C^r -maps.

By the definition, for any map $f \in \mathcal{A}$, by an arbitrarily small perturbation of f a heteroclinic cycle of the type we just described can be created. Typically, the tangency between $W^u(p_2)$ and $W^s(p_1)$ at the points of the heteroclinic orbit Γ_{21} is quadratic, however, by an arbitrarily small (in C^r) perturbation, this tangency can be split in such a way that a new orbit of the heteroclinic tangency between $W^u(p_2)$ and $W^s(p_1)$ can be created, and for this new orbit the order of tangency can be *infinite* [8, 9]. This contradicts the usual logic stemming from singularity theory, where small perturbations usually lead to a decrease in the degeneracy. Here, the order of degeneracy may be increased without a bound (the price is that the new heteroclinic orbit which corresponds to the flat tangency is, in some sense, much longer than the original orbit of quadratic tangency). Importantly, by an additional, arbitrarily C^r -small perturbation of the heteroclinic cycle with the flat tangency, a *periodic spot* can be created (cf. [9]). The periodic spot is a ball $D \subset M$ filled by periodic points, i.e. $f^l x \equiv x$ for every $x \in D$ and some l , the same for all $x \in D$. By applying Theorem 0.1 to the map $f^l|_D$, we thus find

Theorem 0.2. *For every $r = 2, \dots, \infty$, the C^r -universal maps form a residual subset² in the absolute Newhouse domain.*

A more dramatic formulation of this result can be as follows: dynamics of a generic map from the absolute Newhouse domain \mathcal{A} is beyond human comprehension. Indeed, just by the definition, every possible robust (i.e. common for an open set of maps) dynamical feature is present in each universal map as well. In particular, each universal map has an infinite set of attractors of all possible robust types, as well as an infinite set of repellers of all types. For example, as a corollary to Theorem 0.2, we obtain

Theorem 0.3. *For every $r = 2, \dots, \infty$, a C^r -generic map $f \in \mathcal{A}$ has infinitely many uniformly hyperbolic attractors of every possible³ topological type.*

Of course, every such map has all possible types of arbitrary uniformly-hyperbolic sets, i.e. not just attractors, also “saddles” and repellers. Similar to [7], one may show that the attractors and repellers are not separated (the closure of all the

²i.e. a countable intersection of open and dense subsets

³for a map of the n -dimensional ball, $n \geq 2$

attractors has a non-empty intersection with the closure of all the repellers) for a generic map from \mathcal{A} . Indeed, we obtain the attractors/repellers from periodic spots, which are born in an arbitrarily small neighborhood of some heteroclinic cycles; in particular, some iteration of such spot comes close to the saddle periodic orbit which is a part of the heteroclinic cycle. By taking smaller and smaller neighborhoods of the heteroclinic cycle, we find that the limit of both attractors and repellers contain the same saddles. We note that this inseparability of the set of attractors from the set of repellers means that the Conley's fundamental construction of attractor-repeller pairs [19] cannot, generically, produce completely meaningful results.

The fact that generic maps may have an infinite (countable) set of attractors is known since [1] where the genericity of the maps with infinitely many coexisting stable periodic orbits ("sinks") was proven for *area-contracting* maps from the Newhouse domain. Moreover, the closure of the set of stable periodic orbits was shown to contain a non-trivial hyperbolic set. Generic inseparability of the set of "sinks" from the set of "sources" (completely unstable periodic orbits) was proven in [7] for the absolute Newhouse domain in the space of two-dimensional maps (i.e. when no area-contraction nor area-expansion property holds). Examples with coexistence of infinitely many *non-trivial* attractors (invariant tori, Lorenz-like attractors, Benedicks-Carleson attractors) were built in [6, 10, 11, 12, 17]. Our results here show that attractors of *arbitrarily complicated nature* can coexist in unbounded number.

Maybe even more typical for the absolute Newhouse domain are strange attractors of a different nature, as described by the following

Theorem 0.4. *For every $r = 2, \dots, \infty, \omega$, a C^r -generic map $f \in \mathcal{A}$ has an uncountable (of the cardinality of continuum) set of weak attractors such that for each orbit in each of these attractors all Lyapunov exponents are zero.*

By the *weak attractor* we mean a compact, chain-transitive invariant set Y which is an intersection of a nested sequence of trapping neighborhoods, namely

$$Y = \bigcap_{i=1}^{\infty} D_i \text{ where } D_i \subseteq D_{i+1} \text{ and (the trapping property) } f(\text{cl}(D_i)) \subset \text{int}(D_i)$$

[20, 25]. This definition means that even if we add a sufficiently small bounded noise to f , the forward iterations of any point in Y will forever stay in a small neighborhood of Y (in one of the trapping regions D_i). The chain-transitivity means that for an arbitrarily small level of the bounded noise there exists a "noisy" orbit of f which connects any two points in Y , i.e. the attractor Y contains no smaller attractor. The weak attractors we construct in Theorem 0.4 are the so-called solenoids, filled by *limit-periodic* orbits. Namely, there is a monotonically increasing sequence of integers k_i such that each of the sets D_i is a union of k_i disjoint balls D_{ij} , $j = 1, \dots, k_i$, and $f(\text{cl}D_{ij}) \subset \text{int}D_{i,j+1 \bmod k_i}$ (hence k_{i+1} is always a multiple of k_i).

It is obvious that one can build such solenoids by a perturbation of periodic spots. The periodic spot itself is a chain-transitive set and it can be made a weak attractor (even asymptotically stable) by a small smooth perturbation. Every orbit

in the periodic spot has all Lyapunov exponents zero. However, the maps with the *periodic* weak attractors with zero Lyapunov exponents are not generic (the set of the maps with periodic spots is, as we explained above, dense in \mathcal{A} , but it is not residual - by Kupka-Smale theorem). Therefore, we need a solenoid construction in order to achieve the C^r -genericity. Moreover, in contrast to the previous results which have been proven so far only in the smooth category, Theorem 0.4 holds in the real-analytic case ($r = \omega$) as well. There is a hope in the contemporary dynamical systems community that some kind of non-uniform hyperbolicity or partial hyperbolicity is a typical feature for the majority of systems. Theorem 0.4 shows, however, that this cannot be fully true in the absolute Newhouse domain.

We have used a particular type of heteroclinic cycles in order to describe the richness of dynamics and bifurcations in the absolute Newhouse domain. One can, however, show that maps with other types of homoclinic and heteroclinic cycles or other bifurcating orbits whose existence prevents the map from possessing a dominated splitting and from uniform contraction/expansion of volumes (see the corresponding criteria in [24]) also belong either to the absolute Newhouse domain itself, or to its boundary. One of the easiest examples is given by the so-called *reversible* maps. Given a smooth involution R (i.e. $R \circ R = id$) of the manifold M , a map $f : M \rightarrow M$ is called reversible if $f^{-1} = R \circ f \circ R$; such maps naturally appear as Poincaré maps in time-reversible flows. Often, naturally appearing time-reversible flows are also Hamiltonian, however, *non-Hamiltonian* reversible flows are frequent too. A periodic point x of the reversible map f is called *symmetric* if $Rx = f^l x$ for some l (in other words, the set of points of the symmetric periodic orbit is invariant with respect to R). The symmetric periodic orbit is called *elliptic* if all its multipliers are simple and have absolute value 1. Obviously, the multipliers of a symmetric periodic orbit come in pairs: if λ is the multiplier, then λ^{-1} is also a multiplier. Therefore, a symmetric elliptic periodic orbit remains elliptic for an arbitrary reversible map sufficiently close (in C^1) to the original one. In other words, reversible maps with symmetric elliptic periodic orbits form an open subset in the space of all C^r -smooth reversible maps. This open subset is our absolute Newhouse domain in the reversible case, \mathcal{A}_r (note that no non-trivial dominated splitting exists at the elliptic point, nor the map can contract/expand volumes *exponentially* at such point).

It is well-known [21] that dynamics near a typical symmetric elliptic point is pretty much conservative, e.g. invariant KAM-tori may exist. However, between the tori we have resonant periodic orbits, and one can show that by a perturbation, which is arbitrarily small in C^r , $r = 1, \dots, \infty$, and which keeps the map in the reversible class, arbitrarily degenerate resonant periodic orbits (hence - periodic spots) can be born from the elliptic orbit. Even if a periodic spot sequence is symmetric, it can be split into a pair of non-symmetric spot sequences (i.e. one sequence in the pair is taken into the other spot sequence by the involution R). Behavior near a *non-symmetric* periodic orbit (e.g. near a non-symmetric periodic spot sequence) of a reversible map does no longer need to be conservative-like or in any other way to differ from the general case (cf. [13]). Thus, by applying Theorem 0.1 to the non-symmetric periodic spot sequences which emerge near the

symmetric elliptic orbit, we obtain

Theorem 0.5. *For every $r = 1, \dots, \infty$, the C^r -universal maps form a residual subset in \mathcal{A}_r . In particular, a C^r -generic map $f \in \mathcal{A}_r$ has infinitely many uniformly-hyperbolic attractors and uniformly-hyperbolic repellers of every possible topological type, and the closure of the attractors of each of such maps coincides with the closure of the repellers and contains all symmetric elliptic points.*

One may argue that the genericity notion we employ here is not necessarily adequate to the intuitive idea of “being typical”. However, if we do not insist on having an infinite set of hyperbolic attractors and are satisfied with, say, one, the corresponding maps will be open and dense in \mathcal{A}_r . Since the emergence of hyperbolic theory in the 60-s, the problem of finding a uniformly-hyperbolic attractor in a system of natural origin has been actively discussed (see also a very interesting recent discovery in [22, 23]). Ironically, Theorem 0.5 offers amazingly simple while seemingly useless solution: any reversible map with elliptic point in general position possesses a hyperbolic attractor. Of course, this is hardly what we want, as such attractor does not represent the whole of dynamics and coexists with too many other, mainly unknown, objects.

In the case of a *symplectic* map f of an even-dimensional symplectic manifold M , we restrict the definition of the dynamical conjugacy class of f by including into it only those renormalized iterations $f_{m,\psi} = \psi^{-1} \circ f^m \circ \psi$ which all preserve the same given symplectic form on M (for example, when M is a two-dimensional disc with the standard symplectic form $dx \wedge dy$, ψ can be any map with a constant Jacobian). Though this requirement restricts possible choices of the maps ψ , the balls $\psi(U^n)$ can still be of arbitrarily small sizes and situated anywhere in M , so the such defined class of f still provides a description of the behavior of f on arbitrarily fine spatial scales. With this definition of the dynamical conjugacy class we call a symplectic map C^r -universal if the C^r -closure of its class contains all orientation-preserving *symplectic* C^r -diffeomorphisms acting from the closed unit ball U^n into R^n .

Exactly like in the above discussed case of reversible maps, the maps with elliptic periodic points form an open subset, \mathcal{A}_s , in the space of C^r -smooth symplectic maps. While most of the neighborhood of the elliptic point is filled by KAM-tori, resonant periodic orbits between the tori can be arbitrarily degenerate, and periodic spots can be born out of the elliptic orbit by an arbitrarily small smooth perturbation within the class of symplectic maps. By applying a “symplectic version” of Theorem 0.1 to these spots (see [26, 9] for the two-dimensional case) we obtain

Theorem 0.6. *For every $r = 1, \dots, \infty$, the C^r -universal maps form a residual subset in the absolute Newhouse domain \mathcal{A}_s in the space of symplectic maps.*

We, of course, do not have attractors or repellers here (as symplectic maps are volume-preserving). Note also that in the two-dimensional case the set \mathcal{A}_s coincides with the usual Newhouse domain in the space of area-preserving maps, and in this case Theorem 0.6 holds true for the analytic case ($r = \omega$) as well [9, 28].

Symplectic maps appear as Poincaré maps for Hamiltonian systems restricted to a fixed energy level. Unless a special structure (uniform partial hyperbolicity) is imposed on the system, elliptic periodic orbits appear in Hamiltonian systems seemingly inevitably (e.g. they exist generically in energy levels near points of minimum of the Hamiltonian). By Theorem 0.6, dynamics near any such orbit can approximate iterations of an arbitrary symplectic map arbitrarily well. It is one of the most basic physics beliefs that the fundamental dynamical processes are described by Hamiltonian equations, the laws of nature. By Theorem 0.6, given any such process, we may record what the values of variables are at certain, arbitrarily long, discrete sequences of time values, and, for an arbitrary large set of such recordings, almost any, arbitrarily chosen Hamiltonian system (with an elliptic orbit somewhere) will reproduce all the records with an arbitrary high precision, just by an appropriate change of variables and arbitrarily fine tuning of parameters - with the only requirement that the number of degrees of freedom is determined correctly. In other words, for an arbitrary choice of the laws of nature one can still have an arbitrarily good agreement with observation by making a right choice of variables. The point of view that the laws of nature are relative, and their choice is, to a certain extent, a matter of convenience, exists for a long time (see e.g. [29]); our results here provide an additional support to it.

References

- [1] S. Newhouse, *Diffeomorphisms with infinitely many sinks*, Topology 13, 9-18 (1974).
- [2] S. Newhouse, *The abundance of wild hyperbolic sets and non-smooth stable sets for diffeomorphisms*, Publ. Math. IHES 50, 101-151 (1979).
- [3] S.Gonchenko, D.Turaev, L.Shilnikov, *On models with a structurally unstable Poincare homoclinic curve*, Sov. Math., Dokl. 44, 422-426 (1992).
- [4] S. Gonchenko, D. Turaev, L. Shilnikov, *On the existence of Newhouse domains in a neighborhood of systems with a structurally unstable Poincare homoclinic curve (the higher-dimensional case)*, Russian Acad. Sci., Dokl., Math. 47, 268-273 (1993).
- [5] S. Gonchenko, L. Shilnikov, D. Turaev, *On models with non-rough Poincare homoclinic curves*, Physica D 62, 1-14 (1993).
- [6] S. Gonchenko, D. Turaev, L. Shilnikov, *Dynamical phenomena in multi-dimensional systems with a structurally unstable Poincare homoclinic curve*, Russian Acad. Sci., Dokl., Math. 47, 410-415 (1993).
- [7] S. Gonchenko, D. Turaev, L. Shilnikov, *On Newhouse domains of two-dimensional diffeomorphisms with a structurally unstable heteroclinic cycle*, Proc. Steklov Inst. Math. 216, 76-125 (1997).
- [8] S. Gonchenko, D. Turaev, L. Shilnikov, *Homoclinic tangencies of arbitrarily high orders in the Newhouse regions*, J. Math. Sci 105, 1738-1778 (2001).
- [9] S. Gonchenko, D. Turaev, L. Shilnikov, *Homoclinic tangencies of arbitrarily high orders in conservative and dissipative two-dimensional maps*, Nonlinearity 20, 241-275 (2007).

- [10] S. Gonchenko, L. Shilnikov, D. Turaev, *On dynamical properties of multidimensional diffeomorphisms from Newhouse regions: I*, Nonlinearity 21 (2008).
- [11] S. Gonchenko, L. Shilnikov, D. Turaev, *On global bifurcations in three-dimensional diffeomorphisms leading to wild Lorenz-like attractors*, Regul. Chaotic Dyn. 14 (2009).
- [12] S.V. Gonchenko, L.P. Shilnikov, O.V. Sten'kin, *On Newhouse Regions with Infinitely Many Stable and Unstable Invariant Tori*, Proc. Int. Conf. "Progress in Nonlinear Science", Vol. I, 80-102, University of Nizhny Novgorod, 2002.
- [13] J.S.W. Lamb, O.V. Sten'kin, *Newhouse regions for reversible systems with infinitely many stable, unstable and elliptic periodic orbits*, Nonlinearity 17, 1217-1244 (2004).
- [14] J. Palis, M. Viana, *High dimension diffeomorphisms displaying infinitely many periodic attractors*, Annals of Math. 140, 207-250 (1994).
- [15] N. Romero, *Persistence of Homoclinic Tangencies in Higher Dimensions*, Erg. Th. Dyn. Syst. 15, 735-757 (1995).
- [16] P. Duarte, *Elliptic isles in families of area-preserving maps*, Erg. Th. Dyn. Syst. 28, 17811813 (2008).
- [17] E. Colli, *Infinitely many coexisting strange attractors*, Ann. Inst. Henri Poincaré. Non Linear Analysis 15, 539-579 (1998).
- [18] C. Bonatti, L.J. Diaz, M. Viana, *Dynamics Beyond Uniform Hyperbolicity: A Global Geometric and Probabilistic Perspective*, Springer, 2005.
- [19] C. Conley, *Isolated Invariant Sets and the Morse Index*, CBMS Regional Conference Series in Mathematics, Vol. 38, American Mathematical Society, Providence, 1978.
- [20] D. Ruelle, *Small random perturbations and the definition of attractors*, Lect. Notes Math. 1007, 663676 (1983).
- [21] M.B. Sevryuk, *Reversible Systems*, Lect. Notes Math. 1211, Springer, 1987.
- [22] S.P. Kuznetsov, *Example of a Physical System with a Hyperbolic Attractor of the Smale-Williams Type*, Phys. Rev. Lett. 95, 144101 (2005).
- [23] S.P. Kuznetsov, A. Pikovsky, *Autonomous coupled oscillators with hyperbolic strange attractors*, Physica D 232, 87-102 (2007).
- [24] D. Turaev, *On dimension of non-local bifurcational problems*, Bifurcation and Chaos 6, 919-948 (1996).
- [25] D. Turaev, L. Shilnikov, *An example of a wild strange attractor*, Sb. Math. 189, 137-160 (1998).
- [26] D. Turaev, *Polynomial approximations of symplectic dynamics and richness of chaos in non-hyperbolic area-preserving maps*, Nonlinearity 16, 1-13 (2003).
- [27] D. Turaev, *On dynamics of maps close to identity*, preprint MPIM2006-161 (2006).
- [28] V. Gelfreich, D. Turaev, *Universal dynamics in a neighbourhood of a generic elliptic periodic point*, Regul. Chaotic Dyn. (2010).
- [29] H. Poincaré, *La Science et l'Hypothèse*, Flammarion, 1902.

Mathematics Department, Imperial College, SW7 2AZ London, UK
E-mail: dturaev@imperial.ac.uk