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MATHEMATICS

Homoclinic Tangencies of Arbitrarily High Order in Conservative Two-Dimensional Maps

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In [1, 2], we proved that small smooth perturbations of a two-dimensional diffeomorphism with nontransversal Poincaré homoclinic orbit may lead to homoclinic tangencies of arbitrarily high order and, as a consequence, to arbitrarily degenerate periodic orbits. These results show that global bifurcations of codimension 1 may accumulate bifurcation sets of arbitrarily high codimension. On this basis, we made the conclusion that it is impossible in principle to obtain a complete description of the dynamics and bifurcations of systems with homoclinic tangencies.

Recall that systems with homoclinic tangencies are dense in open regions of the space of dynamical systems. Moreover, these regions (Newhouse regions) exist near any system with a homoclinic tangency [3, 4]; see [5] for the conservative case. The results on the density in these regions (with respect to the C^r-topology with an arbitrary finite r) of systems with infinitely degenerate periodic and homoclinic orbits both give evidence that the behavior of the trajectories of systems from Newhouse regions is extremely complicated and make it possible to establish the genericity of quite unexpected global properties of the dynamics. Thus, our results were used by the author of [6] to disprove Smale's conjecture about the genericity of the exponential growth of the number of periodic trajectories as a function of the period, and it was proved in [7] that the generic two-dimensional C^r-diffeomorphism (for finite r) from a Newhouse region is not conjugate to any C^∞-diffeomorphism.

The fact that systems with homoclinic tangencies of arbitrarily high orders are dense in Newhouse regions was proved in [1] for general smooth maps. In the two-dimensional case, this means, in particular, that $|\lambda\gamma| \neq 1$, where λ and γ are the multipliers of a saddle fixed point. Thus, our genericity condition excluded the case of area-preserving diffeomorphisms. In this paper, we show that these results can still be extended to area-preserving diffeomorphisms. Moreover we consider both the smooth and real-analytic cases in a unified way.

Let f be a two-dimensional area-preserving C^r-diffeomorphism, where $r = 2, 3, \ldots, \infty$ or $r = \omega$; the value ω corresponds to the real-analytic case. We assume that f has a saddle periodic trajectory $L(f)$ with a homoclinic orbit $\Gamma(f)$. Suppose also that $W^s(L)$ and $W^u(L)$ are quadratically tangent at the points of Γ. In general, if the stable and unstable manifolds W^s and W^u have a tangency of order m at the points of some trajectory, then we say that there is a homoclinic tangency of order m (a quadratic tangency corresponds to $m = 1$, a cubic tangency corresponds to $m = 2$, etc.). If $m > r$, then the order of tangency is said to be indefinite (or infinite). Naturally, in the analytic case, the manifolds W^s and W^u can have only a finite order of tangency (otherwise, they coincide).

Let K be a (sufficiently large) compact subset of the phase space containing the trajectories $L(f)$ and $\Gamma(f)$ together with their small neighborhoods. We define the closeness of two diffeomorphisms as follows. If r is finite, then we say that two C^r-diffeomorphisms are δ-close if the C^r-distance between them in K is at most δ. Two C^∞-diffeomorphism are said to be δ-close if they are $(r\delta)$-close in the C^r-metric on K for each $r < \dfrac{1}{\delta}$. In the real-analytic case, we take a sufficiently small complex neighborhood Q of K and say that two C^ω-diffeomorphisms are δ-close if their values at each point of Q differ by at most δ. Obviously, any diffeomorphism g close to f has a saddle periodic orbit $L(g)$ close to $L(f)$.

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Theorem 1. *Any neighborhood of f contains an area-preserving diffeomorphism* $\tilde{f}$ *that has infinitely many trajectories of homoclinic tangency of every order between the stable and unstable manifolds of the periodic orbit* $L(\tilde{f})$.

Suppose that f has also a nontrivial uniformly hyperbolic set Λ containing $L(f)$. For any diffeomorphism $\tilde{f}$ sufficiently close to f, there exists a hyperbolic set $\Lambda(\tilde{f})$ such that $f|_{\Lambda(f)}$ is topologically conjugate to $\tilde{f}\big|_{\Lambda(\tilde{f})}$.

Theorem 2. *There exists an area-preserving diffeomorphism* $\tilde{f}$ *arbitrarily close to f such that, for each pair of periodic points* P_1 *and* P_2 *from* $\Lambda(\tilde{f})$, *it has infinitely many orbits of tangency of every orders between the stable and unstable manifolds of the points* P_1 *and* P_2.

The proofs of Theorems 1 and 2 are based on the following fact: For any area-preserving diffeomorphism f with a homoclinic tangency, there exists an arbitrarily close area-preserving diffeomorphism which has a heteroclinic circle with two saddles, one transverse heteroclinic trajectory, and one trajectory of heteroclinic tangency. We speak there of circles of the third class, which have Ω-moduli, i.e., continuous invariants of local topological conjugacy on the set of nonwandering trajectories (see [8]). The existence of the Ω-moduli leads to an additional, hidden, degeneracy: the diffeomorphisms with heteroclinic contours form a bifurcation surface of codimension one, but any arbitrarily small perturbation that does not lead out from the bifurcation surface (i.e., does not destroy the heteroclinic circle) necessarily results in bifurcations of nonwandering orbits if it only changes the value of the Ω-modulus—by the very definition of the latter. Indeed, using the improved version of our perturbation technique from [2], which now includes both the conservative and real-analytic cases, we can show that any finite number of coexisting heteroclinic tangencies can be obtained by a small perturbation that does not destroy the given heteroclinic circle. Then, a small perturbation of n such tangencies gives a tangency of order n [2], again without destroying the heteroclinic cycle. This procedure can be repeated arbitrarily many times; in the limit, we obtain the required infinite set of coexisting tangencies of all orders.

It is well known (see, e.g., [9]) that bifurcations of a quadratic homoclinic tangency in the two-dimensional conservative case lead to the birth of elliptic periodic trajectories. Such a trajectory is considered nondegenerate if, first, there are no strong resonances, i.e., its multipliers $e^{\pm i\varphi}$, where $0 < \varphi < \pi$, are such that $\varphi \neq \frac{\pi}{2}, \frac{2\pi}{3}$, and secondly, the first Birkhoff coefficient B_1 in the complex normal form

$$\bar{z} = e^{i\varphi}z(1 + B_1 zz^*) + O(|z|^4)$$

is nonzero (but $\mathrm{Re}B_1 = 0$ because of conservativity). If $\frac{\varphi}{\pi}$ is irrational, then the normal form can be written as

$$\bar{z} = e^{i\varphi}z(1 + B_1 zz^* + B_2(zz^*)^2 + \ldots + B_m(zz^*)^m) + o(|z|^r), \tag{1}$$

where $m = \left[\frac{r-1}{2}\right]$ (in the case of infinite r, one can take any finite m). We say that the elliptic point is degenerate of order s if $B_i = 0$ for $i = 1, 2, \ldots, s$ and $B_{s+1} \neq 0$ (in this case, $\mathrm{Re}B_{s+1} = 0$ again). If $B_{s+1} \neq 0$, then the degenerate periodic point is stable; this follows from results of Moser. In the case of finite r, the degeneracy order is indeterminate (or infinite) if the corresponding normal form can be written as

$$\bar{z} = e^{i\varphi}z + o(|z|^r) \tag{2}$$

(i.e., all of the Birkhoff coefficients vanish).

Note that degenerate elliptic points may be born at bifurcations of higher order homoclinic tangencies. Namely, let g be an area-preserving two-dimensional diffeomorphism with saddle periodic trajectory whose stable and unstable manifolds have a tangency of order n at the points of some homoclinic orbit. Consider a generic n-parameter family g_μ, where $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ and the μ_i are the splitting parameters of the manifolds $W^s(O)$ and $W^u(O)$ at some point M^+ of the orbit of homoclinic tangency. Let Π^+ be a sufficiently small neighborhood of M^+. Suppose that $T_k \equiv f_\mu^k : \Pi^+ \to \Pi^+$ is the first-return map (such maps are well defined for sufficiently large k and small μ; see [2]).

Lemma 1. *The map* T_k *can be reduced to the form*

$$(X, Y) \mapsto (Y, -X + E_0 + E_1 Y + \ldots + E_{n-1}Y^{n-1} - \sigma Y^{n+1}) + o(1)_{k \to +\infty}, \tag{3}$$

where $|\sigma| = 1$, *by a canonical change of coordinates of class* C^{r-1} (C^r *if* $r = \infty, \omega$) *and a change of parameters of class* C^{r-n-1} (*of any finite degree of smoothness with respect to the parameters if* $r = \infty, \omega$).

Note that the new coordinates (X, Y) and the parameters $E_0, E_1, \ldots, E_{n-1}$ can take any values, i.e., the range of their covers balls (in $\mathbf{R}^2$ and $\mathbf{R}^n$, respectively) centered at zero of radius tending to infinity as $k \longrightarrow \infty$. In (5), $o(1)$ denotes the terms tending to zero uniformly on any compact set along with their derivatives up to the order $r - 2$ with respect to the coordinates and up to the order $r - n - 2$ with respect to the parameters (if $r = \infty$, then they tend to zero together with any finite number of derivatives; if $r = \omega$, then they tend to zero uniformly on a complex neighborhood of an arbitrary compact set in the plane (X, Y) together with any finite number of derivatives with respect to the parameters).

If $E_0 = 0 + \dots$ and $E_1 = 2\cos\varphi + \dots$, where the dots denote terms tending to zero as $k \longrightarrow \infty$, then map (3) has a fixed point with multipliers $e^{\pm i\varphi}$. If $\frac{\varphi}{\pi}$ is irrational, then the normal form can be written in form (1). Choosing appropriate coefficients $E_2, E_3, \dots, E^{n-1}$, we can easily make all of the Birkhoff coefficients B_j with $j < \frac{n-1}{2}$ vanish. This implies that bifurcations of single-round periodic trajectories in the family g_μ lead to elliptic periodic trajectories of any degeneracy degree up to $\left[\frac{n}{2}\right] - 1$. Using Theorem 1, we obtain the following result.

Theorem 3. *Any neighborhood of the diffeomorphism f from Theorem* 1 *contains a diffeomorphism* $\tilde{f}^*$ *having infinitely many elliptic periodic points of every degree of degeneracy.*

Recall that near any diffeomorphism with homoclinic tangency Newhouse regions exist in the space of area-preserving two-dimensional C^r diffeomorphisms for $r = 2, 3, \dots, \infty, \omega$ [5]. As in the general two-dimensional case, the dynamical properties of the conservative diffeomorphisms from these regions are extremely complicated. Thus, Theorems 1 and 3 imply the following assertion.

Theorem 4. *Diffeomorphisms having infinitely many homoclinic tangencies of arbitrary orders and elliptic periodic points of every degree of degeneracy are dense in the Newhouse regions in the space of area-preserving two-dimensional* C^r *diffeomorphisms*, *where* $r = 2, 3, \dots, \infty, \omega$.

In the smooth case ($r \le \infty$), this theorem can be strengthened. If the stable and unstable manifolds of some periodic orbit locally coincide along some curve, then we call such a pencil of homoclinic orbits a homoclinic band. If all points of some region in the phase space are periodic with the same period, then we call such a region a periodic spot.

Theorem 5. *Diffeomorphisms with homoclinic bands and periodic spots are dense in the Newhouse regions in the space of area-preserving two-dimensional diffeomorphisms of class* C^r ($r \le \infty$).

As we see, bifurcations of homoclinic tangencies and elliptic points may lead to dynamical phenomena of arbitrarily high complexity. To give a precise meaning to this assertion, we use the scheme suggested in [10]. Consider an area-preserving C^r diffeomorphism $f(r = 1, 2, \dots, \infty, \omega)$ of a 2-manifold $\mathcal{M}$. Let U denote the closed unit disk in R^2. Take an arbitrary C^r diffeomorphism ψ: $U \to \mathcal{M}$ with constant Jacobian and any positive integer n for which the map $f_{n,\psi} = \psi^{-1} \circ f^n \circ \psi$ is well defined on U (i.e., $f^n \circ \psi(U) \subseteq \psi(V)$, where V is a disk such that $V \supseteq U$ and ψ can be extended to a C^r diffeomorphism with constant Jacobian on this disk). The map $f_{n,\psi}$ is an area-preserving C^r diffeomorphism $U \to \mathrm{R}^2$. We refer to the maps $f_{n,\psi}$ obtained by such a procedure as renormalized iterations of f. The set of all renormalized iterations of f is called the dynamical class of this map.

Note that the transformations (changes of coordinates) ψ are not generally area-preserving: they preserve the standard symplectic form up to constant factors. Therefore, the image $\psi(U)$ can be a disk of an arbitrarily small radius situated anywhere. Thus, the dynamical class of f contains an information about the behavior of arbitrarily long iterations of this map on arbitrarily small scales.

Definition 1. An area-preserving C^r diffeomorphism f ($r = 1, 2, \dots, \infty, \omega$) is said to be universal (or C^r universal) if its dynamical class is dense in the space of all area-preserving orientation-preserving C^r diffeomorphisms from the unit disk U to $\mathbf{R}^2$.

According to this definition, the dynamics of any given universal map is not simpler than that of all symplectic diffeomorphisms together. Nevertheless, as the following theorem shows, universality is a generic property.

Theorem 6. *For any* $r = 2, 3, \dots, \infty, \omega$, *the* C^r *universal maps form residual subsets in the Newhouse regions in the space of area-preserving two-dimensional* C^r *diffeomorphisms.*

In the real-analytic case ($r = \omega$), the notion of a residual set is not defined uniquely. We use the following definition: A set A is residual in some subset B of the space of real-analytic diffeomorphisms if, for any f from B and any compact subset K of the phase space, K has a complex neighborhood Q such that the intersection of A with some open neighborhood X of the diffeomorphism f in the space of holomorphic maps on Q is a countable intersection of open dense subsets of X.

The proof of Theorem 6 is based on Theorem 1, Lemma 1, and the results on symplectic polynomial approximations obtained in [10]. Theorem 6 implies, in particular, that any symplectic diffeomorphism of the 2-disk can be approximated by a diffeomorphism analytically conjugate to a small perturbation of any given area-preserving map with a homoclinic tangency (this map must be restricted to a suitable region in the phase space). Moreover, the following stronger assertion is valid. Consider the space $D_{k,r}$ ($r = 2, 3, \dots, \infty, \omega$) of all k-parameter families of area-preserving orientation-preserving diffeomorphisms f_ε from the unit disk U to $\mathbf{R}^2$ of class C^r with respect to the phase variables and the parameters ε [the parameters $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k)$ range over the closed unit ball in $\mathbf{R}^k$].

Theorem 7. *The space* $D_{k,r}$ *contains a residual set* $\mathcal{D}$ *such that, for any* $g_\varepsilon \in \mathcal{D}$ *and any area-preserving two-dimensional* C^r *map f with homoclinic tangency, arbitrarily close to f in the space of area-preserving two-*

dimensional C^r maps there is a f-parameter family $\tilde{f}_\varepsilon$ of maps such that some renormalized iteration of the map $\tilde{f}_\varepsilon$ coincides with g_ε for each ε.

Thus, any dynamical phenomenon that is generic for some open set in the space of area-preserving diffeomorphisms of the 2-disk or that occurs in a generic finite-parameter family of such diffeomorphisms can be encountered arbitrarily close to any area-preserving diffeomorphism from a Newhouse region.

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