Simple Bifurcations Leading to Hyperbolic Attractors

L. P. SHIL'NIKOV AND D. V. TURAEV
Institute for Applied Mathematics & Cybernetics
Ul'janova st. 10, 603005, Nizhny Novgorod, Russia

Abstract—We prove the existence of a new stability boundary of periodic orbits in a high-dimensional case, thereby resolving the problem on a "blue sky catastrophe" in a general one-parameter family. We additionally establish the existence of a codimension-one boundary which separates the Morse-Smale systems from systems with hyperbolic Smale-Williams-like attractors. The route across this boundary is accomplished by the disappearance of a saddle-node periodic orbit. We also study the principal bifurcations of a torus breakdown which lead to Anosov attractors and to multidimensional solenoids.

Keywords—Global bifurcations, Hyperbolic attractors, Saddle-node bifurcation.

INTRODUCTION

The problems discussed in this paper are in a close relation to one of the main questions of nonlinear dynamics about the structure of boundaries of stability regions of periodic orbits, as well as to the question of finding simple global bifurcations leading to appearance of hyperbolic attractors.

The notion of a stability region is introduced in the following way. Consider a k-parameter family of n-dimensional dynamical systems given by the system of ODE

$$\dot{x} = X(x, \mu),$$

where $x = (x_1, \ldots, x_n)$ is the vector of phase variables and $\mu = (\mu_1, \ldots, \mu_k)$ is the vector of parameters. Suppose that, at $\mu = \mu^*$, the system possesses a periodic orbit $x = \varphi(t, \mu^*)$, all multipliers of which lie strictly inside the unit circle; i.e., the periodic orbit is stable (attractive). As it is well known, for all μ sufficiently close to μ^*, the periodic orbit will exist and the multipliers will remain inside the unit circle. Thus, one can consider the maximal region $\mathcal{D}$ in the parameter space where the periodic orbit exists and where all the multipliers are less than unity in absolute value (this is analogous, in a sense, to analytical continuation). The region $\mathcal{D}$ is called *the stability region* of the periodic orbit (note that the stability region can be many-sheeted in some cases).

There exist two types of the boundaries $\partial\mathcal{D}$ of stability regions:

(1) such that the periodic orbit exists for $\mu \in \partial\mathcal{D}$,
(2) such that the periodic orbit disappears at $\mu \in \partial\mathcal{D}$.

For the first case, the principal[1] boundaries of stability of periodic orbits can be easily enumerated. Let $\chi(\lambda, \mu) = 0$ be the characteristic equation of the linearized Poincaré map (the roots λ of this

---

[1]i.e., those which are typical for general one-parameter families

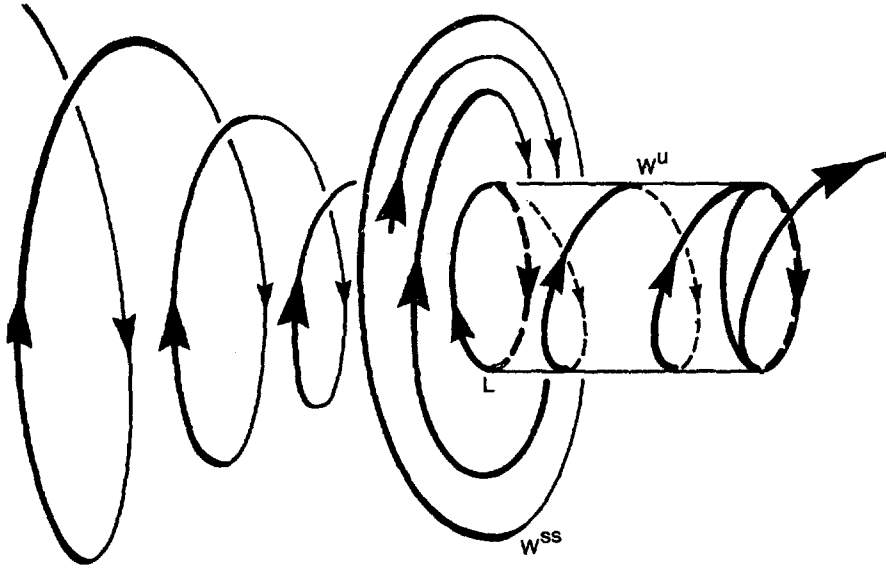Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Figure 1. The phase portrait near a saddle-node periodic orbit L. The strong-stable manifold W^{ss} of L separates a small neighborhood of it onto two regions: the node regions all orbits of which tend to L as $t \to +\infty$, and the saddle region where the two-dimensional unstable manifold W^u lies.
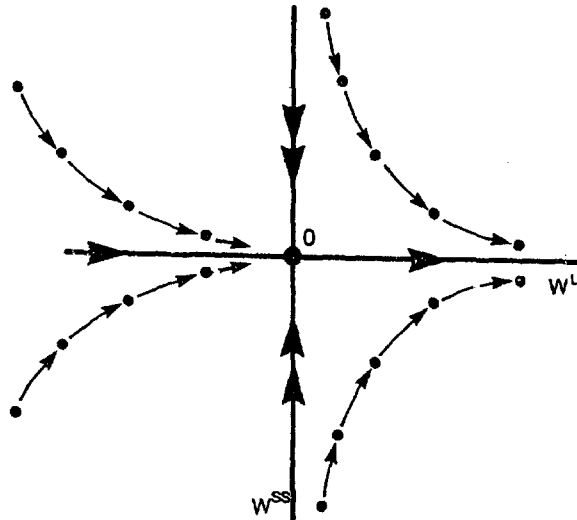

Figure 2. The phase portrait for the Poincaré map of some cross-section to a saddle-node periodic orbit.

equation are the multipliers of the periodic orbit). The loss of stability of the periodic orbit happens when some of the multipliers are lying on the unit circle. Therefore, the boundaries of stability are given by the following equations:

$$\chi(1, \mu) = 0$$
$$\chi(-1, \mu) = 0$$
$$\chi\left(e^{\pm i\omega}, \mu\right) = 0, \qquad 0 < \omega < \pi.$$

Each equation defines, in general, a codimension one surface in the parameter space. The first equation corresponds to the presence of a multiplier equal to unity. For a general one-parameter family that intersects the surface transversely, the rest of the multipliers lie strictly inside the unit circle and the first Lyapunov value does not equal to zero at the moment of bifurcation. Such periodic orbit is called *a simple saddle-node*. For this case, the phase portrait is shown in
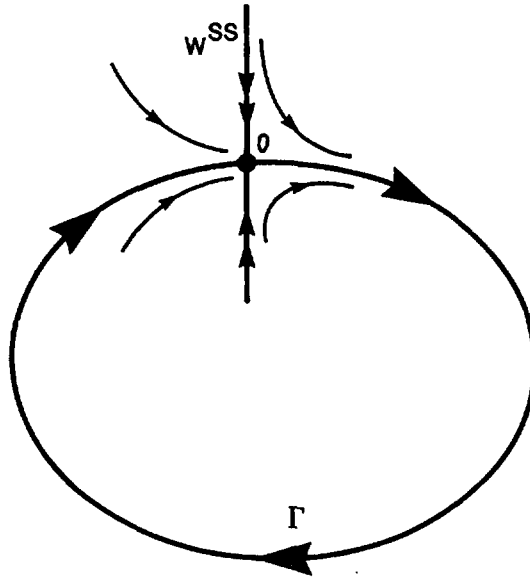
Figure 3. A homoclinic loop Γ to a saddle-node equilibrium state O on a plane.
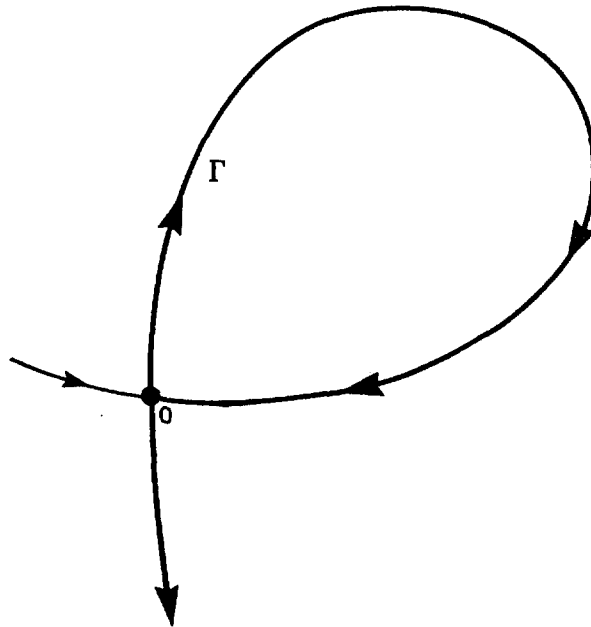

Figure 4. A homoclinic loop Γ to a saddle equilibrium state O on a plane.

Figures 1 and 2. Note that the saddle-node periodic orbit disappears when the parameter value crosses the stability boundary.

The second equation corresponds to the presence of one multiplier equal to -1. In general, the rest of the multipliers remain inside the unit circle and the first Lyapunov value L does not equal to zero. The periodic orbit does not disappear when crossing this stability boundary. If $L < 0$, then the period-doubling bifurcation takes place. If $L > 0$, then the periodic orbit loses stability at the moment of bifurcation when an unstable periodic orbit of doubled period merges into it.

The third equation corresponds to the presence of a pair of complex-conjugate multipliers $e^{\pm i\omega}$. One may assume that the rest of the multipliers lie strictly inside the unit circle, that $\omega \neq \frac{2\pi}{3}$ and $\omega \neq \frac{\pi}{2}$ and that the first Lyapunov value L does not equal to zero. The periodic orbit does not disappear when crossing such boundary, and either the birth of a stable two-dimensional torus

accompanies the loss of stability of the periodic orbit $(L < 0)$ or an unstable two-dimensional torus merges into the periodic orbit at the moment of losing stability $(L > 0)$.

These are the three principal bifurcations of the loss of stability of the periodic orbit for the case where the periodic orbit does exist on the stability boundary. As we mentioned, there are stability boundaries of another type on which the periodic orbit does not exist. The existence of such boundaries was discovered by Andronov and Leontovich. They found for two-dimensional systems of ODE that there are exactly two principal stability boundaries of the given kind: on the first boundary the stable periodic orbit disappears merging into a homoclinic loop of a simple saddle-node equilibrium state (Figure 3), and on the second boundary the stable periodic orbit merges into a homoclinic loop of a saddle equilibrium state (Figure 4) with negative saddle value.[2]

Analogous stability boundaries were found in [1] for the multidimensional case:

(1) on the first boundary, the system has a simple saddle-node equilibrium state[3] and the only orbit that leaves the saddle-node at $t = -\infty$ returns to the saddle-node as $t \to +\infty$, forming a homoclinic loop; it is assumed also that the homoclinic orbit does not lie in the strong-stable manifold (Figure 5);

(2) on the second boundary, the system has a saddle equilibrium state of the type $(n - 1, 1)$ (i.e., the characteristic exponents $\lambda_1, \ldots, \lambda_{n-1}$ have negative real parts and $\lambda_n > 0$) and one of the two orbits leaving the saddle at $t = -\infty$ returns to the saddle as $t \to +\infty$, forming a homoclinic loop (Figure 6); the saddle value

$$\sigma = \lambda_n + \max_{i=1,\ldots,n-1} \Re e\lambda_i$$

is negative.

In [1], there was shown that a stable periodic orbit merges into the homoclinic loop in both cases. When approaching the bifurcational moment, the length of the periodic orbit remains bounded whereas the period tends to infinity.

In this connection the following question arises: can there be some other types of stability boundaries of codimension one? In the present paper we show that the answer is positive. We find a new stability boundary which is an open subset of a codimension one bifurcational surface corresponding to the presence of a saddle-node periodic orbit. This open set is distinguished by some qualitative conditions determining the geometry of unstable set of the saddle-node (see Figure 7) and also by some quantitative restrictions (some value should be less than unity, see below). We will show under these conditions that when the saddle-node disappears a new stable periodic orbit arises, and both the period and the *length* of this periodic orbit tend to infinity when approaching the bifurcation moment (Theorem 1).

Actually, this is one of the possible variants of the global bifurcation of disappearance of a saddle-node periodic orbit such that all the orbits of its unstable set return to the saddle-node as $t \to +\infty$.[4] Without loss of generality, we may assume that all these orbits do not lie in the strong-stable manifold of the saddle-node.

The study of this global bifurcation has a long history. Originally, this problem appeared in the 20's in connection with the study of the phenomenon of the transition from the synchronization to the amplitude modulation regime in the van der Pol equation

$$\ddot{x} + \mu \left(1 - x^2\right) \dot{x} + \omega_0^2 x = \mu A \cos \omega t.$$

With the assumption that μ is a small parameter and that the resonance $1 : 1$ takes place (i.e., $\omega - \omega_0 \sim \mu$), Andronov and Vitt showed that the transition from the synchronization to the

---

[2]the sum of characteristic exponents

[3]i.e., an equilibrium state whose characteristic exponents lie strictly to the left of the imaginary axis except for one simple zero characteristic exponent, and the first Lyapunov value does not equal to zero

[4]Since these orbits belong to the unstable set, they tend, by definition, to the saddle-node as $t \to -\infty$ as well.
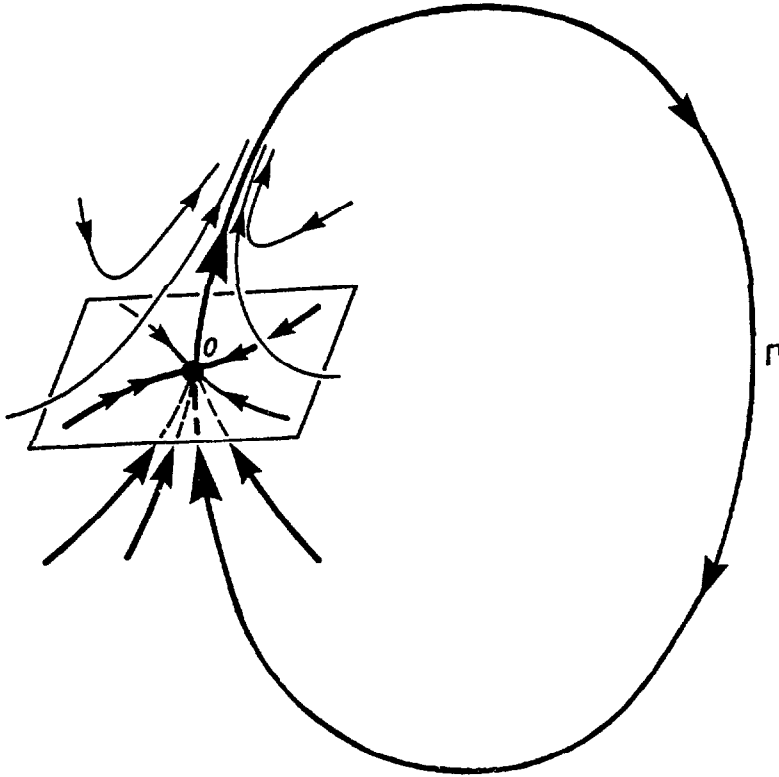
Figure 5. A homoclinic loop Γ to a saddle-node equilibrium state O for the dimension of the phase space more than two.
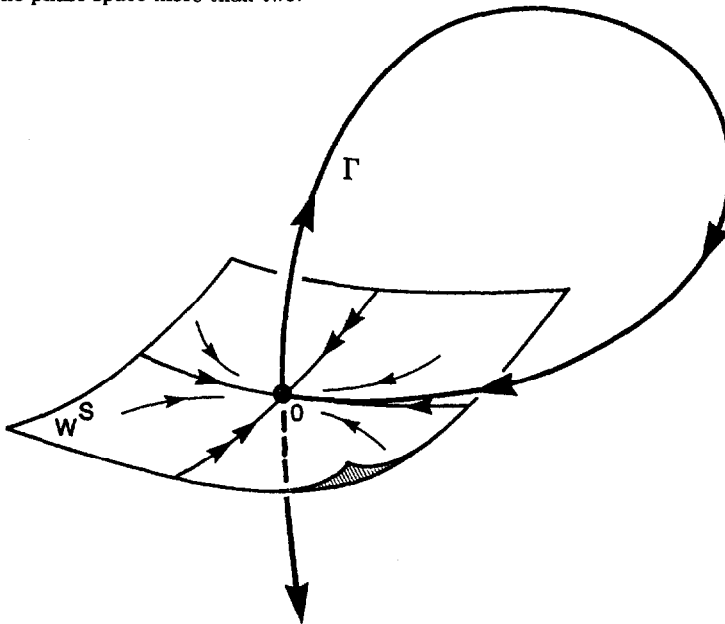

Figure 6. A homoclinic loop Γ to a saddle equilibrium state O in the multidimensional case.

amplitude modulation regime is connected with the bifurcation of the birth of a stable limit cycle from a homoclinic loop of a saddle-node equilibrium state (see Figure 3) in a time-averaged system. Returning to the initial equation, one can see that the analogous picture holds for the two-dimensional Poincaré map where the saddle-node is now the fixed point of the map and the homoclinic loop is not a single orbit, but is a continuum of orbits which compose the unstable set of the saddle-node. In that time such kind of analysis was not carried out.
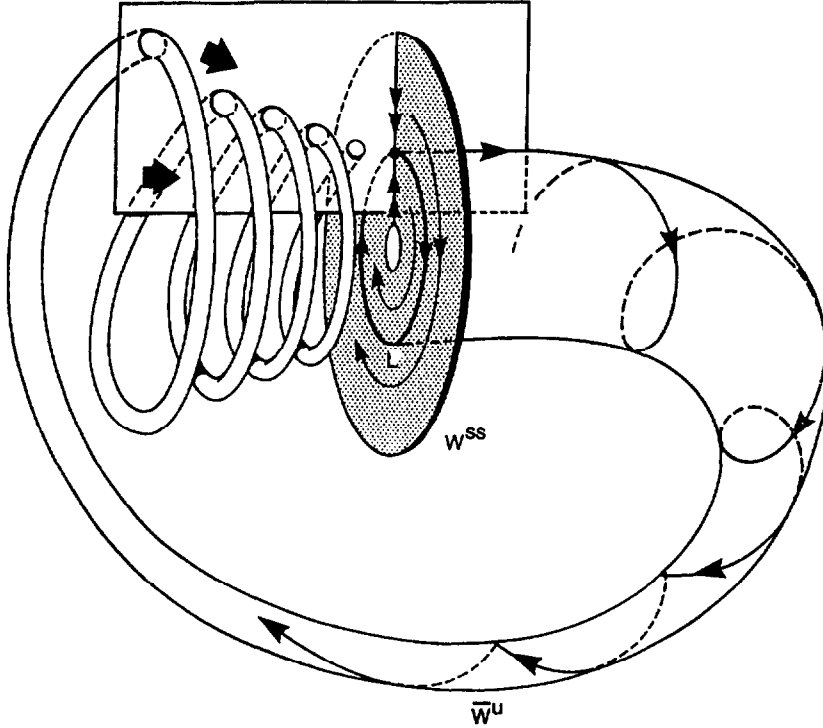
Figure 7. The configuration of the unstable manifold of a saddle-node periodic orbit L
which may lead to the blue sky catastrophe: the closure $\overline{W}^u$ is not a manifold.

The systematical study of this bifurcation was begun with the paper [2] by Afraimovich and Shil'nikov under the assumption that the dynamical system that undertakes the bifurcation is either nonautonomous and periodically depending on time, or autonomous but possessing a global cross-section (at least at that part of the phase space which is studied). Essentially, the problem was reduced to the study of a one-parameter family of C^r-diffeomorphisms $(r \geq 2)$ which has, at $\mu = 0$, a fixed point of saddle-node type such that all orbits of the unstable set of the saddle-node return to its small neighborhood and tend to it as the number of iterations of the map tends to $+\infty$.

Recall that the saddle-node point has one multiplier equal to unity and all the other multipliers are less than unity in absolute value. Near the saddle-node, the diffeomorphism has the form

$$\begin{aligned}
\bar{y} &= Ay + H(y, z) \\
\bar{z} &= z + G(y, z),
\end{aligned} \tag{1}$$

where $z \in R^1$, $y \in R^n$, A is a matrix whose eigenvalues lie strictly inside the unit circle, $H(0,0) = 0$, $H'_{(y,z)}(0,0) = 0$, $G(0,0) = 0$, $G'_{(y,z)}(0,0) = 0$. Here, the fixed point is in the origin. As it is well known, there exists a C^r-smooth invariant center manifold of the form $y = \eta(z)$, where $\eta(0) = 0$, $\eta'(0) = 0$. Restricted onto the center manifold, the map takes the form

$$\bar{z} = z + g(z), \tag{2}$$

where $g(z) \equiv G(\eta(z), z) \in C^r$, $g(0) = 0$, $g'(0) = 0$.

The saddle-node is called *simple* if $g''(0) \neq 0$. In this case equation (2) takes the form

$$\bar{z} = z + l_2 z^2 + \cdots, \tag{3}$$

where $l_2 = g''(0)/2 \neq 0$. Without loss of generality, one can assume $l_2 > 0$.
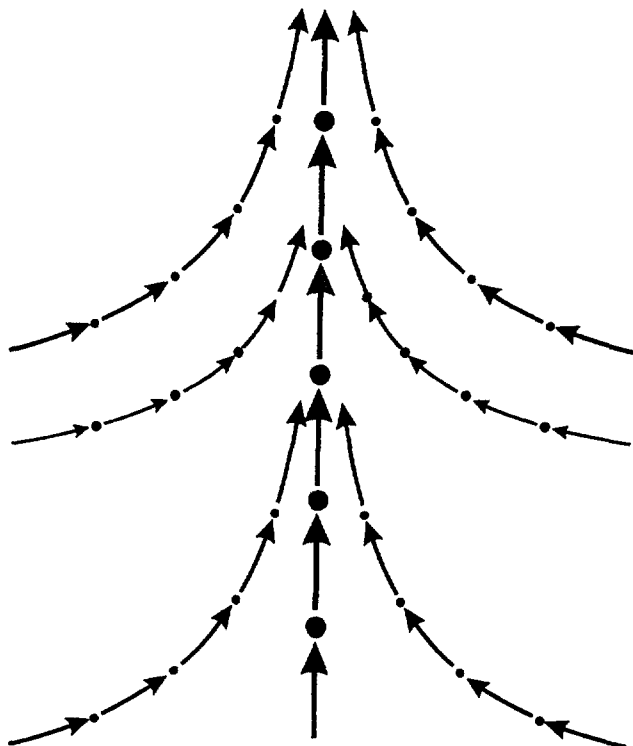
Figure 8. When the saddle-node disappears ($\mu > 0$), all orbits leave its small neighborhood.
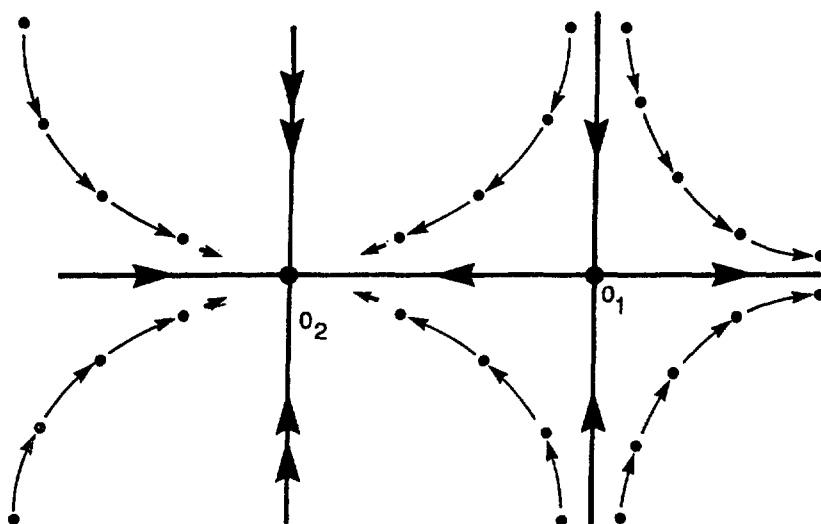
Figure 9. When $\mu < 0$, the saddle-node disintegrates onto two fixed points: saddle O_1 and stable O_2.

One can see (Figure 2) that a small neighborhood of the origin is split by the strong-stable invariant manifold $\{z = \xi(y)\}$ ($\xi(0) = 0$, $\xi'(0) = 0$) into two regions: the node region $\{x < \xi(y)\}$ and the saddle region $\{z > \xi(y)\}$. All orbits from the node region tend to the origin along the z-axis. In the saddle region, the one-dimensional unstable manifold $\{y = \eta(z),\ z > 0\}$ lies, all orbits of which tend to the origin with the iterations of the inverse map. All the other orbits from the saddle region leave the neighborhood of the origin with the iterations of both map (1) itself and its inverse.

To take into account the dependence on the parameter μ, one must consider the functions H and G from (1) to be functions on μ also. Moreover, the parameter μ is supposed to be chosen

Figure 10. The case where the set $\overline{W}^u$ is smooth.
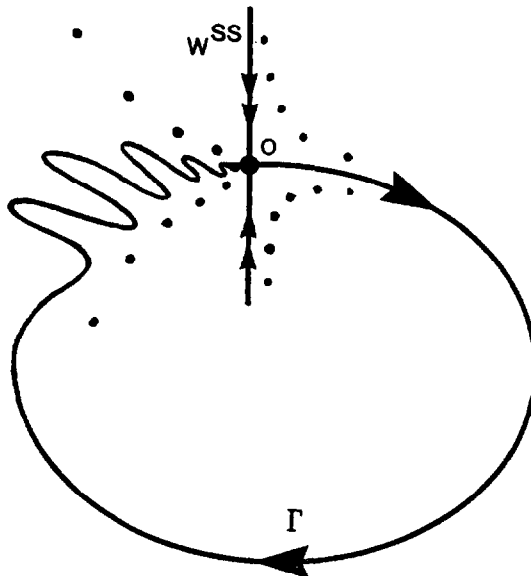
Figure 11. The case where the set $\overline{W}^u$ is a nonsmooth circle (the tangent vector does not have a limit when the point on $\overline{W}^u$ approaches O from the left).
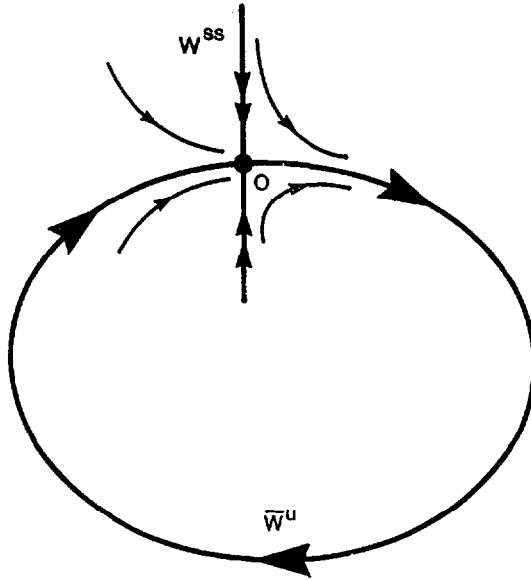
such that the saddle-node disappears when $\mu > 0$ (Figure 8), and when $\mu < 0$, it disintegrates onto two fixed points: saddle and stable (Figure 9). The restriction of the map onto the center manifold is rewritten as

$$\bar{z} = z + \mu + l_2 z^2 + \cdots . \tag{4}$$

As we mentioned, the orbits of all points of the local unstable manifold are supposed to return into the node region and tend to the origin at $\mu = 0$. The union of these orbits is denoted as $\overline{W}^u$ and this set is here homeomorphic to a circle. It occurred that the set $\overline{W}^u$ may be a smooth circle (Figure 10) or it may be nonsmooth (Figure 11).

For the smooth case there was found in [2] that when the saddle-node disappears, an attractive smooth invariant curve inherits to $\overline{W}^u$. If the map under consideration is the Poincaré map of a global cross-section for some system of ODE, then the invariant curve is the line of intersection of an invariant two-dimensional torus with the cross-section. The Poincaré rotation number on the torus tends to zero as $\mu \to +0$.

This result gave a rigorous explanation of the transition from the synchronization to the amplitude modulation in periodically forced nonlinear systems: when $\mu < 0$, the only stable regime is the stable periodic orbit which corresponds to synchronization, and the invariant torus that exists when $\mu > 0$ corresponds to modulation regime.

The case where $\overline{W}^u$ is a nonsmooth manifold was considered in a series of papers. In the aforementioned paper [2], there was established (under the so-called "big lobe" condition) the presence of the sequence of intervals (μ_i, μ_i') accumulating at $\mu = +0$ such that the system has nontrivial hyperbolic sets at $\mu \in (\mu_i, \mu_i')$. Without the big lobe condition (but for one-parameter families of a special kind), this result was proved in [3] by Newhouse, Palis and Takens. They used essentially a theorem by Block on existence of periodic orbits for endomorphisms of a circle. In [4], the results of [2,3] were extended onto the general case; there was also shown that for a "sufficiently small lobe" the intervals of parameter values corresponding to complex dynamics (hyperbolic sets) and simple dynamics (a continuous invariant curve with rational Poincaré rotation number) alternate accumulating at $\mu = +0$.

The important feature of the nonsmooth case is the existence [3,4] of parameter values in an arbitrary closeness to $\mu = +0$ which correspond to the presence of saddle periodic orbits with *structurally unstable homoclinic orbits*. According to the modern knowledge (see [5]), this leads to extremely complicated dynamics: to the Newhouse phenomenon (persistence of homoclinic tangencies, coexistence of infinitely many sinks) [6–11], to Hénon-like attractors [12,13], and to infinite degenerations [14] which make it impossible to give a complete description of all bifurcations that may occur in this case.

$\mu < 0$ $\mu > 0$

Figure 12. The local saddle-node bifurcation. For $\mu < 0$, there exist two periodic orbits: stable and saddle; the unstable manifold of the latter is homeomorphic to a two-dimensional cylinder. When $\mu > 0$, the saddle-node disappears and all orbits leave its small neighborhood.

In the present paper we point out that if an autonomous system with a saddle-node does not have a global cross-section there may be essentially more different possible cases. Let

$$\dot{x} = X(x, \mu)$$

be a one-parameter family of n-dimensional C^r-smooth ($r \geq 2$) dynamical systems which has, at $\mu = 0$, a periodic orbit of saddle-node type (i.e., such that the local Poincaré map has form (1)). When μ varies, the local bifurcations go as follows (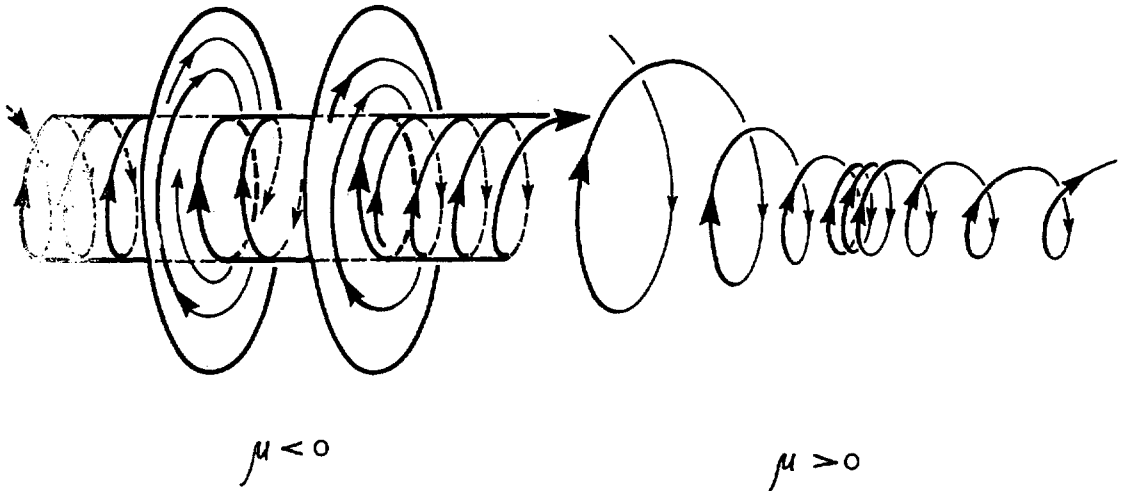Figure 12). For $\mu < 0$, there exist two periodic orbits: stable and saddle; the unstable manifold of the latter is homeomorphic to a two-dimensional cylinder. At $\mu = 0$, these periodic orbits unite in one structurally unstable periodic orbit L of the simple saddle-node type. The local unstable set W_L^u is homeomorphic to

a half-cylinder $R^+ \times S^1$. The orbit L has also the strong-stable manifold W_L^{ss} which divide a neighborhood of L into two regions: saddle and node. When $\mu > 0$, the saddle-node disappears and all orbits leave its small neighborhood. Note that if the starting point of some orbit lies in the node region, then the time which the orbit spends in a small fixed neighborhood of the saddle-node tends to infinity as $\mu \to +0$, as well as the length of the corresponding piece of the orbit.

Suppose that, at $\mu = 0$, all the orbits of W_L^u return to the node region and tend to L as $t \to +\infty$, not lying in W_L^{ss}. The union $\overline{W}^u$ of these orbits may, for instance, be a smooth two-dimensional surface: a torus, or a Klein bottle (the latter may be if the phase space is nonorientable or if the dimension n of the phase space is not less than four). Analogously to [2], the smooth invariant two-dimensional surface is preserved for $\mu > 0$. As above, if the set $\overline{W}^u$ is a nonsmooth torus, then saddle periodic orbits with homoclinic curves may appear at $\mu > 0$; the same may be in the case where $\overline{W}^u$ is a nonsmooth Klein bottle under some additional conditions [4].

Essentially different situation was earlier unknown where the set $\overline{W}^u$ is not a manifold. First, consider the following example. Let a *two*-parameter family of three-dimensional vector fields have, at some parameter values, a saddle-node periodic orbit L and a saddle-node equilibrium state O (Figure 13). Suppose that all orbits of W_L^u tend to O as $t \to +\infty$ and that the one-dimensional separatrix of O tends to L. If one of parameters is varied so that O disappears and L does not, then the set $\overline{W}^u$ will have the form shown in Figure 7. The intersection of $\overline{W}^u$ with a local cross-section S to L will be a union of a countable set of circles accumulating at the point $S \cap L$ (Figure 14). Evidently, any neighborhood of this point in the set $\overline{W}^u$ is not homeomorphic to a ball. Therefore, in this situation the set $\overline{W}^u$ is not a manifold.

Systems having a saddle-node periodic orbit with the set $\overline{W}^u$ being as shown in Figure 7 compose codimension one surfaces in the space of smooth flows in R^n ($n \geq 3$). Below, we will show (see Theorem 1) how open subsets are distinguished on these surfaces such that *for any one-parameter family X_μ that intersects such subset transversely at $\mu = 0$, the system X_μ has an attractive periodic orbit for all small $\mu > 0$,*[5] *whose period and length tend to infinity as $\mu \to +0$.*

Note the connection of this result to the problem on "the blue sky catastrophe" [15]. The original formulation was as follows. *Does there exist a continuous one-parameter family of smooth vector fields A_μ on a compact manifold such that A_μ has a closed orbit L_μ for all $\mu > 0$, and as $\mu \to +0$, the period of L_μ tends to infinity, and at $\mu = +0$, the orbit L_μ disappears on a finite distance of equilibrium states?*[6]

Virtual bifurcations of such kind were called *blue sky catastrophes* by Abraham. The first example of such catastrophe was constructed by Medvedev [16] for a one-parameter family of vector fields on a Klein bottle which has a saddle-node periodic orbit at $\mu = 0$. The Medvedev's family was of a rather special kind: the system that corresponds to $\mu = 0$ is also embedded in a one-parameter family of *conservative* vector fields, all orbits of which are closed. The Poincaré map for this conservative family has the form

$$\varphi = -\varphi + \omega(\mu) \bmod 1,$$

where $\omega \to \infty$ as $\mu \to +0$. This map has two fixed points, all other points are of period two. Actually, Medvedev used the fact that this family can be perturbed so as to have only two periodic points: one stable and one unstable fixed point; the stable fixed point corresponds to a stable periodic orbit whose period and length tend to infinity as $\mu \to +0$.

In the general one-parameter family, both the fixed points will bifurcate infinitely many times as $\mu \to +0$, changing their stability (this was noticed in [17] and studied in more detail in [18]). Formally, the blue sky catastrophe also takes place here because the structural stability of the periodic orbit under consideration was not required in the original formulation.

---

[5]We assume that the region $\mu > 0$ corresponds to the disappearance of the saddle-node L.

[6]The latter implies that the length of L_μ also tends to infinity.
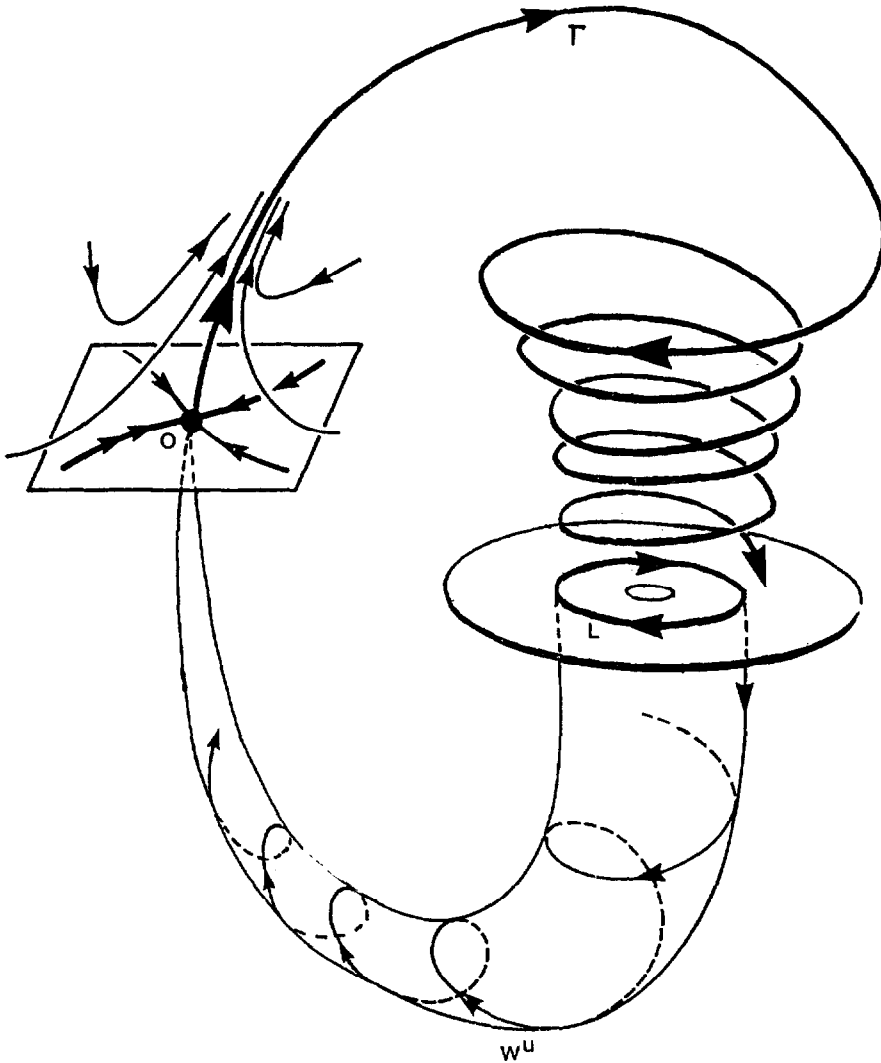
Figure 13. An example of bifurcation creating the configuration shown in Figure 7.
Here, all orbits of the unstable manifold W^u of a saddle-node periodic orbit L tend
to a saddle-node equilibrium state O as $t \to +\infty$, and the one-dimensional separatrix
of O tends to L. If one of parameters is varied so that O disappears and L does not,
then the set $\overline{W}^u$ will have the form shown in Figure 7.
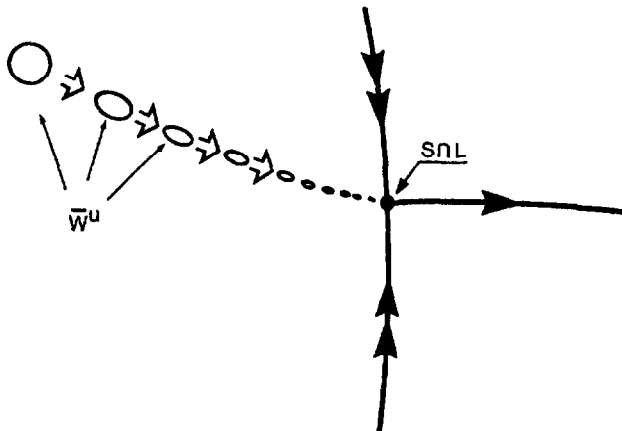
Figure 14. For the configuration shown in Figure 7, the intersection of $\overline{W}^u$ with a
local cross-section S to L is a union of a countable set of circles accumulating at the
point $S \cap L$.

Evidently, the construction proposed in the present paper gives also a solution to the blue sky problem. At the same time, it seems to be more adequate because the periodic orbit in Theorem 1 is attractive and structurally stable for all $\mu > 0$, and this property holds for an open set of one-parameter families.

The second main result of the paper is given by Theorem 2 which establishes that *in the space of smooth flows in R^n ($n \geq 4$), there exist codimension one bifurcation surfaces separating Morse-Smale systems and systems with hyperbolic attractors of the Smale-Williams solenoid type* [19,20].

We will show that the solenoid does not undergo bifurcations when approaching the boundary and period and length of any periodic orbit of the solenoid tend to infinity; i.e., we also deal with a variant of the blue sky catastrophe here.

Another attainable boundary is known [21,22] which separates Morse-Smale systems and systems with nontrivial hyperbolic sets. This boundary corresponds to the existence of a structurally unstable equilibrium state of the saddle-saddle type which has two or more homoclinic orbits. When approaching the boundary, the hyperbolic set also does not undergo bifurcations; the main difference with the cases considered in the present paper is that the hyperbolic set (the solenoid) is now attractive.

It is well known that the hyperbolic attractors have not ever been found in dynamical models arising in natural applications and that the chaotic dynamics demonstrating by the large variety of systems has a nonhyperbolic character; i.e., it is connected, for instance, with the Lorenz-like attractor [23–26], the Hénon-like attractor [12], or with quasiattractors [5,26–28]. The fact we establish here that a hyperbolic attractor can appear in a relatively simple bifurcation allows us to suppose the hyperbolic attractors can be found in dynamical systems of natural origination if the dimension of phase space is four or more. In this connection, we propose (in the Concluding Remarks) more examples of bifurcations leading to hyperbolic attractors. In the next section, we give the precise formulation of Theorems 1 and 2 and the proof for the case where the smoothness of the system with respect to the phase variables and the parameter is sufficiently high (the case of low smoothness requires more delicate calculations [29]).

MAIN RESULTS

We consider a C^r-smooth ($r \geq 2$) family of dynamical systems X_μ in R^n ($n \geq 3$) which has a periodic orbit L_0 of the simple saddle-node type at $\mu = 0$. All the orbits of the unstable manifold W^u are supposed to return to L_0 and to come into the node region as $t \to +\infty$. Herewith, any orbit of W^u is bi-asymptotic to L_0 and $\overline{W}^u$ is compact.

Let U be a small neighborhood of $\overline{W}^u$ and U_0 be a small neighborhood of L_0, $U_0 \in U$. Let us introduce coordinates in a neighborhood U_0 of L_0. We cut U_0 along some cross-section S and consider the coordinates (y, z, φ) where $\varphi \in [0,1]$ is the angular variable and (y, z) are the normal coordinates; $z \in R^1$ is a coordinate on the center manifold, $y \in R^{n-2}$ is a vector of coordinates corresponding to the multipliers less than unity in absolute value; values $\varphi = 0$ and $\varphi = 1$ correspond to the points lying on S.

We chose the coordinates such that L_0 is the curve ($y = 0$, $z = 0$); we also locally straighten the center manifold (so that it takes the form $\{y = 0\}$) and the strong-stable manifold (so that it takes the form $\{z = 0\}$).

Consider two cross-sections S_0 and S_1 to the flow X_μ which are close enough to L_0 and which have the form $z = -\varepsilon < 0$ and $z = \varepsilon > 0$, respectively.

At $\mu = 0$ (and hence, at all small μ) all orbits of W^u return to the node region $U^+ = \{z < 0\}$ in a finite time. Therefore, the flow X_μ defines a diffeomorphism T_1 by which a small neighborhood of the intersection line $l^- : \{y = 0\} = W^u \cap S_1$ is mapped into S_0. This map has the form

$$y_0 = p(\varphi_1, y_1; \mu), \qquad \varphi_0 = q(\varphi_1, y_1; \mu) \bmod 1, \tag{5}$$

where the coordinates on S_0 and S_1 are denoted as (φ_0, y_0) and (φ_1, y_1), respectively; smooth functions p and $q \bmod 1$ are 1-periodical in φ.

The curve $l^+ = T_1 l^- : \{y_0 = p(\varphi_1, 0; 0), \ \varphi_0 = q(\varphi_1, 0; 0) \bmod 1\}$ is the intersection of W^u and S_0. Note that the function q can be written in the form:

$$q(\varphi, y; \mu) = m\varphi + q_0(\varphi, y; \mu), \tag{6}$$

where q_0 is periodic in φ. The integer m defines the homotopy class of l^+ in S_0 (the sign of m defines orientation of l^+ with respect to l^-).

If the dimension n of the phase space is greater than three, then the solid torus S_0 is at least three-dimensional and the integer m may be of arbitrary value (see Figure 15 for the case $m = 2$). At $n = 3$, the cross-section S_0 is a two-dimensional annulus. Therefore, in this case, there may be only $m = 0$ (Figures 7 and 14) and $m = 1$.
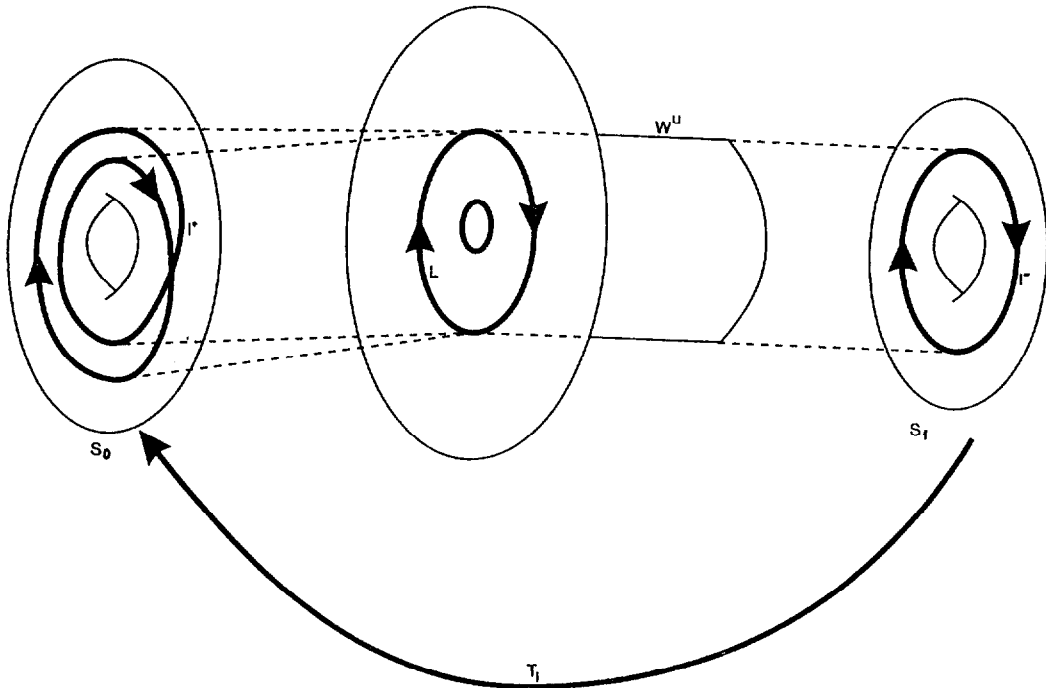

Figure 15. If the dimension of the phase space is more than 3, then the cross-sections S_0 and S_1 are, at least, three-dimensional, and the situation is possible when the curve $l^- = W^u \cap S_1$ is mapped onto the double-round curve.

Note that the structure of the set W^u is completely determined by the way W^u adjoins to L_0 from the side of the node region U^+. It is not hard to see that the intersection of $W^u \cap U^+$ with any cross-section of the kind $\{\varphi = \text{const}\}$ consists, at $m \neq 0$, of $|m|$ pieces glued at the point $\{y = 0, \ z = 0\} = L \cap \{\varphi = \text{const}\}$. It is clear that samples of W^u corresponding to different values of m are mutually nonhomeomorphic. It is also clear that W^u is a topological manifold if and only if $m = \pm 1$ (a torus or a Klein bottle, respectively).

Consider the function

$$f(\varphi) = m\varphi + q_0(\varphi, 0; 0). \tag{7}$$

Note that f is not defined uniquely because it can be changed by a transformation of coordinate φ on the cross-sections S_0 and S_1. Below we will give an algorithm how to choose the coordinates on S_0 and S_1 so that the *through map* $T_0 : S_0 \to S_1$ defined by the flow X_μ for $\mu > 0$, acts as a pure rotation in φ-coordinate

$$\varphi_1 = \varphi_0 + t(\mu), \tag{8}$$

where $t(\mu)$ is *the flight time* from S_0 to S_1 for the orbits of X_μ, $t(\mu) \to +\infty$ as $\mu \to +0$. It can be shown [29] that the function f defined modulo an arbitrary additive constant and a shift of

the origin

$$f(\varphi) \rightarrow c_0 + f(\varphi + c_1)$$

is independent of the choice of ε and of smooth coordinate transformations preserving form (8) of the through map.

Now we can formulate the main results of the paper.[7]

THEOREM 1. *Let $m = 0$ and $|f'(\varphi)| < 1$ for all φ. Then, for all small $\mu > 0$, the system X_μ has an attractive periodic orbit L_μ (nonhomotopic to L_0 in U) to which all orbits of U tend.*

THEOREM 2. *Let $|m| \geq 2$ and $|f'(\varphi)| > 1$ for all φ. Then, for all small $\mu > 0$, all orbits of U tend to a hyperbolic attractor Ω_μ topologically equivalent to the suspension over the inverse spectrum limit of the expanded map of a circle*

$$\bar\varphi = m\varphi \bmod 1. \tag{9}$$

As we mentioned above, systems close to X_0 and having a saddle-node periodic orbit close to L_0 form a codimension one bifurcational surface in the space of dynamical systems. It can also be shown that the function f depends continuously on the system on the bifurcational surface. Thus, if the conditions of either Theorem 1 or 2 are fulfilled for some system X_0, they are also fulfilled for all close systems on the bifurcational surface. This implies that Theorem 1 or 2 (depending on the value of m) is valid for any one-parameter family which intersects the surface transversely near X_0. In other words, our blue sky catastrophes may occur in general one-parameter families.

The proof of Theorems 1 and 2 is based on the calculation of the through map $T_0 : (y_0, \varphi_0) \mapsto (y_1, \varphi_1)$ from S_0 into S_1 which is defined by the orbits of X_μ for all small $\mu > 0$.

In suitable coordinates, the through map can be written in the form (see Lemma 1). Since the last two equations in (5) are independent of y, the map T_0 is written in the form

$$\begin{aligned} y_1 &= Y(\varphi_0, y_0, \mu) \\ \varphi_1 &= \varphi_0 + \tau(\varphi_0, \mu) \bmod 1, \end{aligned} \tag{10}$$

where Y is a smooth function 1-periodic in φ_0. The function τ is *the flight time* from S_0 to S_1.

Note that the flight time does not depend here on y_0. This is connected with the existence of the strong-stable invariant foliation in U_0. This foliation can be locally straightened; i.e., the coordinates in U_0 can be chosen such that the leaves of the foliations would have the form $\{z = \text{const}, \varphi = \text{const}\}$. The invariance of the foliation means that for any two points on a leaf the orbits of these points will intersect the same leaves simultaneously; i.e., at each moment of time the coordinates (z, φ) of one point coincide with the coordinates (z, φ) of the other points, independently of the value of the starting y-coordinates. Thus, in such coordinates, the flight time from the cross-section $\{z = -\varepsilon\}$ to the cross-section $\{z = \varepsilon\}$ is independent of y_0.

The flight time is smooth function periodic in φ_0. Clearly, $\tau(\varphi_0, \mu) \rightarrow \infty$ as $\mu \rightarrow +0$.

THEOREM 3. *The coordinates can be chosen such that the flight time $\tau(\varphi_0, \mu)$ is independent of φ_0; i.e.,*

$$\varphi^{\text{out}} = \varphi^{\text{in}} + \tau(\mu) \bmod 1 \tag{11}$$

in formula (10). Besides, the following estimate

$$\|Y\|_{C^{r-1}} \leq K e^{-\alpha\tau(\mu)} \tag{12}$$

holds, where K and α are same positive constants.

---

[7] We need the invariance of function f with respect to coordinate transformations to be sure that the values $\max|f'|$ and $\min|f'|$ are well-defined objects.

Inequality (12) is an obvious consequence of the fact that the flow near the saddle-node is exponentially contracting in y-variables which correspond to the multipliers lying strictly inside the unit circle. The constant α must be chosen such that the spectrum of these multipliers lies inside the circle $\{|(\cdot)| = e^{-\alpha}\}$ in the complex plane.

On the other hand, the first statement of the theorem is, technically, the most complicated part of the work. As we mentioned, the evolution of the coordinates (z, φ) is independent of y-coordinates. The corresponding differential equation is

$$\begin{aligned} \dot{z} &= \tilde{g}(z, \varphi, \mu) \\ \dot{\varphi} &= 1, \end{aligned} \tag{13}$$

where $\tilde{g}(0, \varphi, 0) \equiv 0$, $\tilde{g}(0, \varphi, \mu) > 0$ for $\mu > 0$. Note that equality (11) would be evidently fulfilled if the second equation in (13) were autonomous; i.e., if $\tilde{g}$ were independent of φ.

We prove here Theorem 3 for the case where $\tilde{g}$ is smooth enough with respect to x, φ, and μ. Actually, in this case, the nonautonomous vector field can be made "almost" autonomous in a small neighborhood of L_0 by a smooth transformation of coordinates, namely, by reduction to a normal form.

First, note that if $\tilde{g}$ is sufficiently smooth, then one can make it independent of φ at $\mu = 0$ (see [3]) by a smooth transformation of variables. Besides, using the standard normalizing procedure, any finite segment of the Taylor expansion of $\tilde{g}$ in powers of z and μ can also be made independent of φ. This implies that, for an arbitrarily large k, we can write

$$\frac{\partial \tilde{g}}{\partial \varphi} = o\left(\tilde{g}^k\right) \tag{14}$$

as $\mu \to +0$ (this estimate holds in a small neighborhood of L_0 and the larger k we choose, the smaller the size of the neighborhood may be). We can also write

$$\frac{\partial^2 \tilde{g}}{\partial \varphi^2} = o\left(\tilde{g}^k\right), \quad \frac{\partial^3 \tilde{g}}{\partial \varphi^3} = o\left(\tilde{g}^k\right), \dots \tag{15}$$

for an arbitrary fixed number of derivatives.

We consider the case where L_0 is a simple saddle-node. Then

$$\tilde{g} = l_2 z^2 + o(z)^2, \qquad l_2 \neq 0$$

at $\mu = 0$. Without loss of generality, we assume $l_2 = 1$. Suppose the family X_μ is transverse to the bifurcational surface of systems with a simple saddle-node. This allows us to assume

$$\frac{\partial \tilde{g}}{\partial \mu} \neq 0.$$

Rescaling, if necessary, the parameter μ we get

$$\tilde{g}|_{z=0} = \mu.$$

Thus, we can write

$$\tilde{g} = \mu + z^2 + O\left(|\mu||z| + |z|^3 + \mu^2\right).$$

Consider an orbit $\{z(t; \varphi_0, \mu), \varphi(t; \varphi_0, \mu)\}$ of system (13) starting at $t = 0$ with the point $(\varphi = \varphi_0,\ z = -\varepsilon)$. Since $\frac{\partial z}{\partial t}$ $(\equiv \tilde{g}(z, \varphi, \mu))$ does not vanish at $\mu > 0$, we can express t as a function of z, φ_0, and μ. By (13), we get

$$\varphi = \varphi_0 + t\left(z, \varphi_0, \mu\right)$$

and

$$t\left(z, \varphi_0, \mu\right) = \int_{-\varepsilon}^{z} \frac{ds}{\tilde{g}\left(s, \varphi_0 + t\left(s, \varphi_0, \mu\right), \mu\right)}. \tag{16}$$

Denote $u = \frac{\partial t}{\partial \varphi_0}$. By differentiating (16) with respect to φ_0, we find

$$u\left(z, \varphi_0, \mu\right) = -\int_{-\varepsilon}^{z} \left. \frac{\frac{\partial \tilde{g}}{\partial \varphi}}{\tilde{g}^2} \right|_{(s, \varphi_0 + t(s, \varphi_0, \mu), \mu)} \left[1 + u\left(s, \varphi_0, \mu\right)\right] ds. \tag{17}$$

By (14), we have $|\frac{\partial \tilde{g}}{\partial \varphi} / \tilde{g}^2| \to 0$ as $\mu \to +0$, so the function u can be found by the successive approximation method as the unique continuous solution of integral equation (17).

Note that we defined the function u at $\mu > 0$. Nevertheless, the right-hand side of the integral equation has a limit as $\mu \to +0$ (it vanishes). Therefore, the solution u has also a limit, namely, $u \to 0$ as $\mu \to +0$. Using estimates (14),(15), we can repeat these arguments for a number of derivatives of u with respect to φ_0 and μ (this number is the larger, the larger the value of k). Thus, the value of u tends to zero along with the arbitrary given number of derivatives as $\mu \to +0$.

By definition, the flight time $\tau(\varphi_0, \mu)$ equals to

$$t\left(z = \varepsilon, \varphi_0, \mu\right) \equiv t\left(\varepsilon, 0, \mu\right) + \int_0^{\varphi_0} u(\varepsilon, \phi, \mu)\, d\phi.$$

We see that the coordinate transformation

$$\varphi_0 = \varphi_0 + \int_0^{\varphi_0} u(\varepsilon, \phi, \mu)\, d\phi$$

brings the second equation of (10) to required form (11). The theorem is proved.

As it follows from Theorem 3 and from formulas (5)–(7), if μ is sufficiently small and positive, then on a small neighborhood of $l^- = W^u \cap S_1$ in S_1, there is defined the map $T \equiv T_0 T_1 :$ $(\varphi_1, y_1) \mapsto (\bar{\varphi}_1, \bar{y}_1)$ by the orbits of the flow X_μ which can be represented in the following form (we omit the indices "1"):

$$\begin{aligned} \bar{y} &= \tilde{p}(\varphi, y; \mu) \\ \bar{\varphi} &= \tau(\mu) + f(\varphi) + \tilde{q}(\varphi, y; \mu) \bmod 1, \end{aligned} \tag{18}$$

where $\tilde{p}$ and $< \tilde{q}$ are smooth functions periodic in φ; moreover, there exist C^{r-1}-limits $\lim_{\mu \to +0} \tilde{p}$ and $\lim_{\mu \to +0} \tilde{q}$

$$\tilde{p}(\varphi, y; \mu = 0) \equiv 0 \tag{19}$$

$$\tilde{q}(\varphi, y = 0; \mu = 0) \equiv 0. \tag{20}$$

These formulas mean that, as $\mu \to +0$, the map T becomes arbitrarily close (in C^{r-1}-topology) to the one-dimensional map

$$\bar{\varphi} = \tau(\mu) + m\varphi + q_0(\varphi, 0, 0) \equiv t(\mu) + f(\varphi) \bmod 1. \tag{21}$$

If $m = 0$ and $|f'(\varphi)| < 1$, then for any τ, map (21) has an attractive fixed point to which all orbits tend. The same is clearly valid for all close maps, in particular, for the map T at small $\mu > 0$. Since the map T is defined by the orbits of the flow X_μ, the fixed point corresponds to the attractive periodic orbit L_μ of X_μ; this gives us Theorem 1. It should be noted that the period of L_μ is about $\tau(\mu)$ and tends to infinity as $\mu \to +\infty$. Since the vector field of X_μ does not vanish in U, it follows that the length of L_μ tends to infinity also.

To prove Theorem 2, we note that if $|f'(\varphi)| > 1$ for all φ, then, by virtue of (19),(20), map (18) is expanding in φ and strongly contracting in y for all small y and $\mu \geq 0$. This allows us with the standard technique [25,30] to establish the existence of a continuous contracting invariant foliation with the leaves of the form

$$\{\varphi = \Phi^*(y; \varphi', \mu)\}, \tag{22}$$

where φ' is the coordinate of the intersection of the leaf with the line $\{y = 0\}$; the function Φ^* is Lipschitz with respect to y and continuous on (φ', μ) at all small $\mu \geq 0$.

Indeed, take any point P on S_0. Let $P_0 = P$, P_1, $P_2 \ldots$ be the iterations of P by the map T: $TP_i = P_{i+1}$. Note that for μ and y small enough, there exists $\lambda > 0$ such that for a smooth surface $\mathcal{L} : \{\varphi = \Phi(y)\}$ satisfying the Lipschitz condition

$$\left\| \frac{\partial \Phi}{\partial y} \right\| \leq \lambda \tag{23}$$

and containing the point P_i, the connected component $\tilde{\mathcal{L}}$ of its preimage $T^{-1}(\mathcal{L})$ that contains P_{i-1} is the surface of the similar form $\{\varphi = \tilde{\Phi}(y)\}$, where $\tilde{\Phi}$ is defined for all small y and it satisfies Lipschitz condition (23) with the same λ.[8]

Furthermore, it is also easily checked that for any two curves $\mathcal{L}_1$ and $\mathcal{L}_2$ satisfying (23) the C^0-distance between $\tilde{\mathcal{L}}_1$ and $\tilde{\mathcal{L}}_2$ is less than the C^0-distance between $\mathcal{L}_1$ and $\mathcal{L}_2$.

Thus, if $\mathcal{H}_i$ is the space of the smooth surfaces $\mathcal{L}$ containing P_i and having the form $\varphi = \Phi(y)$, where Φ satisfies (23), then the map $\tilde{T}_i : \mathcal{L} \mapsto \tilde{\mathcal{L}}$ is contracting in C^0-metric and $\tilde{T}_i(\mathcal{H}_i) \subseteq \mathcal{H}_{i-1}$. Applying a lemma from [31] ("on the fixed pont of a contracting operator in an infinite product of complete metric spaces") to the spectrum of the spaces and the maps

$$\bar{\mathcal{H}}_0 \xleftarrow{\tilde{T}_1} \bar{\mathcal{H}}_1 \xleftarrow{\tilde{T}_2} \ldots,$$

where the bars means C^0-closure, we find that there exists a unique sequence of surfaces $\mathcal{L}_i^*$: $\{\varphi = \Phi_i^*(y)\}$ such that $P_i \in \mathcal{L}_i^*$, $T\mathcal{L}_i^* \subset \mathcal{L}_{i+1}^*$, and $\mathcal{L}_i^* \in \bar{\mathcal{H}}_i$ (i.e., the value λ is the Lipschitz constant for Φ_i^*).

In other words, for any point P, there exists a unique Lipschitz surface $\mathcal{L}^*(P) \equiv \mathcal{L}_0^*$ containing P and such that all forward iterations of $\mathcal{L}^*(P)$ by the map T remain Lipschitz with the Lipschitz constant λ. By uniqueness, the surface $\mathcal{L}^*(P)$ depends continuously on the point P and on the parameter μ, and if two such surfaces have an intersection point, they coincide (see details in [25] where analogous considerations were carried out for the Lorenz-type maps).

Thus, the surfaces $\mathcal{L}^*$ compose a continuous invariant foliation of required form (22). The map T factorized with the leaves of the foliation is an exponentially expanding map of the circle for all small $\mu > 0$.[9] Therefore, the factorized map is conjugate [32] to linear expanding map (9) for all small $\mu > 0$, what gives Theorem 2 (analogously to [20,33]).

---

[8]This assertion is easily verified and we omit the calculations because this is rather standard and repeats analogous considerations in [25]. The value λ should be taken greater than $\min |f'(\varphi)| / \max |\tilde{q}_y'(\varphi, 0; 0)|$ (see (18)–(20)).

[9]To prove, consider a lift $\bar{T}$ of the map T onto the strip $-\infty < \varphi < +\infty$. By (18)–(20), one can easily check that for any two points P and P' with equal y-coordinates, the difference between y-coordinates of the ith iterations P_i and P_i' of, respectively, P and P' by the lift $\bar{T}$ is much less than the difference between their φ-coordinates for all positive i. Then, it is not hard to see that the distance between P_i' and P_i grows exponentially as $i \to +\infty$; i.e., faster than $K^i \text{dist}(P, P')$ where $K > 1$ can be chosen arbitrarily close to $\min |f'(\varphi)|$ if μ and y are small enough. Since the leaves of the foliation are uniformly Lipschitz, the distance between the iterations of any two leaves by the lift $\bar{T}$ grows asymptotically as the distance between the iterations of any two points lying one on one of the leaves and the other on the other. Thus, the distance between the iterations of any two leaves also grows exponentially.

CONCLUDING REMARKS:
THE DISAPPEARANCE OF A SADDLE-NODE TORUS

The idea of the use of the saddle-node bifurcation to produce hyperbolic attractors gives many more new examples for the case of bifurcations of invariant tori. We do not develop here the theory of the global bifurcations connected with the disappearance of a torus and restrict ourself by the study of a model situation.

Consider a one-parameter family of smooth dynamical systems

$$\dot{x} = X(x, \mu),$$

which has, at $\mu = 0$, an invariant m-dimensional torus τ^m filled by quasi-periodic orbits. We assume that the vector field takes the form

$$\dot{y} = A(\mu)y$$
$$\dot{z} = \mu + z^2$$
$$\dot{\varphi} = \Omega(\mu)$$

in a neighborhood of τ^m. Here, $z \in R^1$, $y \in R^{n-m-1}$, $\varphi \in \tau^m$. The matrix $A(\mu)$ is stable; i.e., its eigenvalues lie strictly to the left of the imaginary axis.

At $\mu = 0$, the torus is given by the equation $y = 0$. When $\mu > 0$, the torus disappears. Analogously to the case of disappearance of a saddle-node periodic orbit which we considered in the previous section, we construct two cross-sections S_0 and S_1: $\{y = \pm \varepsilon\}$ where ε is a small quantity. The map $T_0 : S_0 \to S_1$ will be defined for $\mu > 0$ by the orbits of the system. For our model case, it can be easily calculated:

$$y_1 = e^{A(\mu)t(\mu)}y_0$$
$$\varphi_1 = \varphi_0 + \Omega(\mu)t(\mu),$$

where $t(\mu) \sim 1/\sqrt{\mu} + \cdots$. Suppose that, at $\mu = 0$, all orbits of the unstable set W^u_{loc} : $\{y = 0, z > 0\}$ of the torus τ^m return in a small neighborhood of τ^m and tend to it as $t \to +\infty$. All these orbits will intersect S_0, defining thereby the map $T_1 : S_1 \to S_0$. We write this map in the form

$$y_0 = p(y_1, \varphi_1, \mu)$$
$$\varphi_0 = q(y_1, \varphi_1, \mu),$$

where p and q are smooth for all small μ. The image of the torus $W^u \cap S_0$ by the map T_1 is also a torus given by the equation

$$y_0 = p(0, \varphi_1, 0)$$
$$\varphi_0 = q(0, \varphi_1, 0).$$

The Poincaré map $T = T_0 T_1 : S_1 \to S_1$ is written as

$$\bar{y} = e^{A(\mu)t(\mu)}p(y, \varphi, \mu)$$
$$\bar{\varphi} = \Omega(\mu)t(\mu) + q(z, \varphi, \mu). \tag{24}$$

Since $t(\mu) \to +\infty$ as $\mu \to +0$ and since the spectrum of $A(\mu)$ lies strictly to the left of the imaginary axis, the norm of $e^{A(\mu)t(\mu)}$ is extremely small for small $\mu > 0$. This means that map (24) is very close to the "shortened" map of a torus

$$\bar{\varphi} = \omega(\mu) + B\varphi + q_0(\varphi), \tag{25}$$

where

$$\omega(\mu) = \Omega(\mu)t(\mu) + q(0,0,\mu)$$
$$B\varphi + q_0(\varphi) = q(0,\varphi,\mu) - q(0,0,\mu).$$

Here, B is an integer matrix and q_0 is a periodic function of φ with the mean value equal to zero. Denote

$$f(\varphi) = B\varphi + q_0(\varphi).$$

If $\|f'(\varphi)\| < 1$ for all φ (for instance, if $B = 0$ and if q_0 is small), then the shortened map is contracting for all $\mu > 0$, and the Poincaré map T is contracting also. Since a contracting map has only one fixed point, we arrive at the following statement.

PROPOSITION 1. *If $\|f'(\varphi)\| < 1$ for all φ, then the flow X_μ has a unique attractive periodic orbit for all small $\mu > 0$.*

This result is analogous to Theorem 1 and it gives us a new example of the blue sky catastrophe. An analogue of the theorems of [2,4,17] on the birth of a smooth invariant manifold at a saddle-node bifurcation is given by the following statement.

PROPOSITION 2. *If $|\det B| = 1$ and $\det f'(\varphi) \neq 0$ for all φ (i.e., if the shortened map is a diffeomorphism), then the Poincaré map has an invariant attractive m-dimensional torus for all $\mu > 0$ (correspondingly, the flow has an invariant attractive manifold homeomorphic to a skew product of the torus on a circle).*

The proof of this statement is based on the fact that if for a sequence $\mu_n \to +0$, the sequence $\omega(\mu_n) \bmod 1$ tends to some point $\omega*$, then the Poincaré map that corresponds to $\mu = \mu_n$ has, as a limit, the map

$$\bar{y} = 0$$
$$\bar{\varphi} = \omega * + f(\varphi).$$

This map has a stable invariant smooth attractive torus $\{y = 0\}$. The standard fact of the theory of normal hyperbolicity is that the invariant manifold is preserved for all close maps if the restriction of the map on the invariant manifold is a diffeomorphism and if the contraction in normal directions (y-directions in our case) is stronger than that which may take place along the directions tangential to the manifold. The latter requirement is trivially fulfilled in our case.

Note that the restriction of the Poincaré map on the invariant torus is close to the shortened map

$$\bar{\varphi} = \omega * + f(\varphi).$$

This implies, in particular, that if the shortened map is Anosov for all $\omega*$ (for instance, if the matrix B does not have eigenvalues on the unit circle and q_0 is small), then the restriction of the Poincaré map is also Anosov for all $(\mu > 0)$.[10]

We arrive, hence, at the following result.

PROPOSITION 3. *If the shortened map is Anosov for all $\omega*$, then for all $\mu > 0$ there exists a hyperbolic attractor the flow on which is topologically conjugate to the suspension over the Anosov diffeomorphism.*

The birth of hyperbolic attractors can be proved not only in the case where the shortened map is a diffeomorphism. Namely, this result holds true if the shortened map is expanding or if it is a

---

[10]Recall the definition: a diffeomorphism of a torus is called Anosov if the tangent space is decomposed into a direct sum of two subspaces (stable and unstable) such that this decomposition is continuous and invariant with respect to the differential of the map, and the differential of the map contracts (exponentially) vectors of stable subspaces and expands vectors of unstable subspaces. Among other significant properties, Anosov map are known to be structurally stable.

so-called Anosov covering. A map is called expanding if the length of any tangent vector grows exponentially under the action of the differential of the map. An example is the algebraic map

$$\bar{\varphi} = B\varphi \bmod 1,$$

where the spectrum of the integer matrix B lies strictly outside the unit circle, and any map close to it is also expanding. If $\|(F'(\varphi))^{-1}\| < 1$, then the shortened map

$$\bar{\varphi} = \omega(\mu) + f(\varphi) \equiv \omega(\mu) + B\varphi + q_0(\varphi)$$

is expanding for all $\mu > 0$.

Shub established that expanding maps are structurally stable [32]. The study of expanding maps and their connection with smooth diffeomorphisms was continued by Williams [33]. Using this work, we obtain the following result which is analogous to our Theorem 2 proved for the case of disappearance of a saddle-node periodic orbit.

PROPOSITION 4. If $\|(f'(\varphi))^{-1}\| < 1$, then for all $\mu > 0$ a hyperbolic attractor exists locally homeomorphic to the direct product of R^{m+1} on a Cantor set.

An endomorphism of a torus is called an Anosov covering if the continuous invariant decomposition of the tangent space into the direct sum of stable and unstable subspaces exists, as for an Anosov map (the difference is that the Anosov covering is not a one-to-one map, and therefore, it is not a diffeomorphism). The map

$$\bar{\varphi} = \omega^* + B\varphi + q_0(\varphi)$$

will be Anosov covering for all ω^* if, for instance, $|\det B| > 1$ and if q_0 is small enough. The following result is analogous to Proposition 4.

PROPOSITION 5. If the shortened map is an Anosov covering for all ω^*, then the system has, for all $\mu > 0$, a hyperbolic attractor locally homeomorphic to the direct product of R^{m+1} on a Cantor set.

To complete our list of examples of bifurcations connected with the disappearance of a torus, consider the following case. Suppose that the variables φ can be separated into two groups: $\varphi = (\varphi_1, \varphi_2)$, where φ_1 is k-dimensional and φ_2 is $(m - k)$-dimensional. Suppose that the shortened map takes the form

$$\bar{\varphi}_1 = \omega^1 + B\varphi_1 + q_1(\varphi_1, \varphi_2)$$
$$\bar{\varphi}_2 = \omega^2 + q_2(\varphi_1, \varphi_2),$$

where q_1 and q_2 are 1-periodic in φ. Let q_1 and q_2 be sufficiently small along with their first derivatives. Analogously to Proposition 2, we have the following.

PROPOSITION 6. If $|\det B| = 1$, then the Poincaré map has an invariant attractive k-dimensional torus.

This gives us an example of a blue sky catastrophe not for a periodic orbit but for a torus of an arbitrary (in principle) dimension. If B is a hyperbolic matrix, then the flow on the newborn torus is Anosov, and we have, thus, a hyperbolic attractor for all $\mu > 0$. If B is a hyperbolic matrix but $|\det B| > 1$, then analogously to Propositions 4 and 5 there can be established that a hyperbolic attractor homeomorphic to the direct product of R^{k+1} to a Cantor set exists for all $\mu > 0$.

REFERENCES

1. L.P. Shil'nikov, Some cases of generation of periodic motions from singular trajectories, (in Russian), *Matem. Sbornik* **61**, 443–466 (1963).

2. V.S. Afraimovich and L.P. Shil'nikov, On some global bifurcations connected with the disappearance of a fixed point of saddle-node type, *Soviet Math. Dokl.* **15**, 1761–1765 (1974).

3. S. Newhouse, J. Palis and F. Takens, Bifurcations and stability of families of diffeomorhisms, *Publ. Math. IHES* **57**, 5–71 (1983).

4. D.V. Turaev and L.P. Shil'nikov, Bifurcations of torus-chaos quasiattractors, (in Russian), In *Mathematical Mechanisms of Turbulence*, Kiev, (1986).

5. S.V. Gonchenko, L.P. Shil'nikov and D.V. Turaev, Quasiattractors and homoclinic tangencies, (this issue).

6. S.E. Newhouse, The abundance of wild hyperbolic sets and non-smooth stable sets for diffeomorphisms, *Publ. Math. IHES* **50**, 101–151 (1979).

7. S.V. Gonchenko, D.V. Turaev and L.P. Shil'nikov, On the existence of Newhouse domains in a neighborhood of systems with a structurally unstable Poincaré homoclinic curve (the higher-dimensional case), *Russian Acad. Sci. Dokl. Math.* **47**, 268–273 (1993).

8. J. Palis and M. Viana, High dimension diffeomorphisms displaying infinitely many sinks, *IMPA* (preprint) (1992).

9. S.E. Newhouse, Diffeomorphisms with infinitely many sinks, *Topology* **13**, 9–18 (1974).

10. S.V. Gonchenko and L.P. Shil'nikov, On dynamical systems with structurally unstable homoclinic curves, *Soviet Math. Dokl.* **33** (1), 234–238 (1986).

11. S.V. Gonchenko, D.V. Turaev and L.P. Shil'nikov, Dynamical phenomena in multidimensional systems with a structurally unstable homoclinic Poincaré curve, *Russian Acad. Sci. Dokl. Math.* **47**, 410–415 (1993).

12. M. Benedicks and L. Carleson, The dynamics of the Hénon map, *Ann. Math.* **133**, 73–169 (1991).

13. L. Mora and M. Viana, Abundance of strange attractors, *IMPA* (preprint) (1990).

14. S.V. Gonchenko, L.P. Shil'nikov and D.V. Turaev, On models with non-rough Poincaré homoclinic curves, *Physica D* **62**, 1–15 (1993).

15. J. Palis and C. Pugh, Fifty problems in dynamical systems, *Lecture Notes Math.*, Vol. 468, pp. 345–353, (1975).

16. V.S. Medvedev, On a new type of bifurcations on manifolds, (in Russian), *Mat. Sbornik* **113**, 487–492 (1980).

17. V.S. Afraimovich and L.P. Shil'nikov, On a bifurcation of codimension 1 leading to the appearance of a countable set of tori, *Soviet Math. Dokl.* **25**, 101–105 (1982).

18. Li Weigu and Zhang Zhi-Fen, The blue sky catastrohe on closed surfaces, Research Report, Vol. 19, Inst. of Math., Peking University, (1990).

19. S. Smale, *Bull. Am. Math. Sok.* **73**, 747–817 (1967).

20. R.F. Williams, *Topology* **6**, 473–487 (1967).

21. L.P. Shil'nikov, On a new type of bifurcations of multidimensional dynamical systems, *Soviet Math. Dokl.* **10**, 1369–1371 (1969).

22. V.S. Afraimovich and L.P. Shil'nikov, On attainable transitions from Morse-Smale systems to systems with many periodic motions, *Math. USSR Izvestija* **8**, 1235–1269 (1974).

23. V.S. Afraimovich, V.V. Bykov and L.P. Shil'nikov, The origin and structure of the Lorenz attractor, *Soviet Phys. Dokl.* **22**, 336–339 (1977).

24. R.F. Williams, The structure of Lorenz attractor, *Lect. Notes Math.*, Vol. 615, pp. 94–112, (1977).

25. V.S. Afraimovich, V.V. Bykov and L.P. Shil'nikov, On attracting structurally unstable limit sets of Lorenz attractor type, *Trans. Moscow Math. Soc.* **2**, 153–216 (1983).

26. V.S. Afraimovich and L.P. Shil'nikov, Strange attractors and quasiattractors, In *Nonlinear Dynamics and Turbulence*, pp. 336–339, Pitman, Boston, (1982).

27. V.S. Afraimovich and L.P. Shil'nikov, Invariant two-dimensional tori, their breakdown and stochasticity, *Amer. Math. Soc. Transl.* **149**, 201–211 (1991).

28. I.M. Ovsyannikov and L.P. Shil'nikov, Systems with a homoclinic curve of multidimensional saddle-focus, and spiral chaos, (in Russian), *Matem. Sbornik* **182**; English translation, *Math. USSR Sbornik* **73**, 415–443 (1991).

29. L.P. Shil'nikov and D.V. Turaev, Blue sky catastrophes and hyperbolic attractors, (to appear).

30. M. Hirsch, C. Pugh and M. Shub, Invariant manifolds, *Lecture Notes Math.*, Vol. 583, Springer-Verlag, (1977).

31. L.P. Shil'nikov, On a Poincaré-Birkhoff problem, *Math. USSR Sbornik* **3**, 415–443 (1967).

32. M. Shub, Endomorphisms of compact differentiable manifolds, *Amer. J. Math.* **91**, 175–199 (1969).

33. R.F. Williams, Expanding attractors, *Publ. Math. IHES* **43**, 169–204 (1974).