Fermi acceleration and adiabatic invariants for non-autonomous billiards

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Recent results concerned with the energy growth of particles inside a container with slowly moving walls are summarized, augmented, and discussed. For breathing bounded domains with smooth boundaries, it is proved that for all initial conditions the acceleration is at most exponential. Anosov-Kasuga averaging theory is reviewed in the application to the non-autonomous billiards, and the results are corroborated by numerical simulations. A stochastic description is proposed which implies that for periodically perturbed ergodic and mixing billiards averaged particle energy grows quadratically in time (e.g., exponential acceleration has zero probability). Then, a proof that in non-integrable breathing billiards some trajectories do accelerate exponentially is reviewed. Finally, a unified view on the recently constructed families of non-ergodic billiards that robustly admit a large set of exponentially accelerating particles is presented. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4736542]

Fermi acceleration is considered to be among the fundamental mechanisms by which particles gain non thermal energies in astrophysical shock waves. Fermi acceleration is caused by multiple reflections of a particle from moving obstacles such as magnetic mirrors. In this paper, we discuss mathematical models for a particle which moves freely inside a container with slowly oscillating walls and is reflected elastically from the boundary. As a result of multiple collisions, the particle energy may grow substantially. In this form, the problem has a clear connection with basic models in the kinetic theory of ideal gases.

I. INTRODUCTION

In 1949, Fermi proposed a mechanism for a charged particle to gain energy from collisions with structures in a moving magnetic field.[21] The particle follows lines of the magnetic field and is reflected from areas of a stronger magnetic field often called "magnetic mirrors." As the magnetic mirrors are not stationary, the particle may loose or gain energy in each collision. According to Fermi, on average, the particle collides more often in head-on collisions, hence the particle accelerates. As the magnetic field is created by other charged particles, this mechanism explains the energy transfer from a large number of slow particles to a small number of fast ones. A survey of related mathematical results can be found in Ref. 18.

The simplest mathematical model for Fermi acceleration was proposed by Ulam.[58] In the Fermi-Ulam model, the particle bounces elastically between two parallel walls, one of which oscillates and the other one is fixed. The particle gains energy in head-on collisions and looses it in overtaking ones. The higher-dimensional generalizations of the Fermi-Ulam model received much attention in the last decade. Here, the particle is considered to be inside a container with moving walls in $\mathbb{R}^d$. It is usually assumed that the particle's collisions with the walls are elastic.

The cumulative change in the particle's energy over a long period of time depends on the laws of the walls oscillations.[30,50,55] It is relatively easy to check that if the oscillations of the moving wall in the Fermi-Ulam model are random, the average amount of kinetic energy gained (or lost) by the particle in consecutive collisions is positive and the particle typically accelerates.[18,31,38] Similarly, in the higher-dimensional breathing case, the averaged energy grows when random collisions are assumed (see Sec. IV).

It is natural to ask if Fermi acceleration can be deduced from the underlying deterministic laws of motion without relying on external randomness. This problem is delicate. Fermi's argument suggests that acceleration should be observed. On the other hand, averaging arguments suggest that there should be no acceleration, at least when the billiard domain changes its shape periodically. Indeed, when the particle moves rapidly the time between consecutive collisions with the walls is small. If the motion of the walls is deterministic, their velocities and positions do not notably change between collisions. If the unit of time is chosen to make the initial velocity of the particle to be equal to one, the motion of the walls is slow. Hence, the changes in the particle energy are highly correlated, and alternating periods of acceleration and deceleration are induced by the oscillations of the moving wall. Moreover, by adiabatic theory,[1,32] the speed of the particle approximately returns to its initial value after a complete cycle of the wall oscillations (see Refs. 8, 39, and 59 and Sec. III). By this argument, most particles do not accelerate.

Which of the two points of view holds? In the Fermi-Ulam problem, the answer is mostly known and depends on the properties of the walls motion. In particular, if the motion

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of the wall is periodic in time, the Fermi-Ulam model can be interpreted as a Hamiltonian system with two degrees of freedom. Then, provided the motion is sufficiently smooth, Kolmogorov–Arnold–Moser (KAM) theory may be applied.[46–48] The resulting KAM curves present permanent barriers for the Fermi acceleration and therefore the particle energy stays bounded for all times. Thus, one proves that Fermi's intuition fails in the smooth time-periodic one-dimensional case. On the other hand, in the piecewise smooth one-dimensional case Fermi's intuition is correct.[55] Such delicate distinction demonstrates that much care is needed due to nontrivial averaging effects that emerge at high energies.

Before embarking into the discussion of the higher-dimensional case, we point out three important aspects associated with the notion of the energy growth, which should be kept in mind as they highlight the complexity and beauty of the acceleration problem. (1) One should distinguish between energy growth in a single realization and for an ensemble average. (2) There is a difference between energy growth rate with time and energy growth rate with the number of collisions. (3) Different time scales associated with this problem should be identified. These three notions are interconnected, as explained next in some more details.

(1) In a chaotic billiard, the evolution of the energy is sensitive to initial conditions and therefore it is natural to study its average or typical behaviour. We will see that in various models, the standard deviation grows quickly in time and an interpretation of a statistical description requires much care (see Sec. VI). For example, in the context of a random walk, it is easy to find examples where the typical behaviour is different from the mean, e.g., in a model of this type, almost every initial condition may lead to deceleration while the energy averaged over all initial conditions grows quickly. We also note that decelerating trajectories present an additional challenge to the theory since the decelerating particle enters a different dynamical regime where a different description of dynamics may be necessary, for example, averaging theory may be no longer justifiable.
(2) Many papers treat billiard systems as billiard maps and study statistical properties (e.g., averaged energy over an ensemble of initial conditions) as a function of collision number. This is adequate for studying a repeated experiment in which the stopping time is determined by the number of collisions and not to the more common experimental setups in which the stoping time is fixed. Moreover, billiards is often considered to be model for a diluted gas, where collisions with the walls are much more frequent than inter-particles collisions. Then, averages over an evolving ensemble of particles should refer to a common moment in physical time and not to a common number of collisions with the walls. Although these two approaches treat the same system, the results of the analysis may lead to very different conclusions. For example, the particle speed grows at most linearly as a function of the collision number, whereas it may grow exponentially or as a power law in time.[22,24,51] This distinction is especially important as the billiard breathing occurs at a fixed period. In particular, as the particle energy grows, especially if it grows exponentially in time, a large number of collisions may actually correspond to a fraction of a single oscillation period of the billiard table shape. In such cases, the problem of a proper choice of the relevant time scale becomes nontrivial. In Sec. II, we formulate the acceleration problem both in terms of time and the collision number and provide for some classes of billiards *a-priori* bounds on the energy growth for these two settings.
(3) The behavior of *slow* particles in a breathing billiard is not universal: all types of complications known for Hamiltonian dynamical systems with several degrees of freedom may emerge. In particular, the system may have sticky elliptic islands that influence the statistical behaviour of the low energy trajectories in an unpredictable way. Hence, we think that the phenomenon of Fermi acceleration can be described by some universal laws of energy growth only at the limit of sufficiently fast particles. In order to derive the universal growth laws, one applies an adiabatic theory which is applicable on time scales that increase with the particle speed.[8] As long as the speed increases, the theory becomes more accurate. However, if one allows the speed to drop, the theory falls apart. Recall that some orbits decelerate for a while, and therefore the probability to encounter slow particles increases with the ensemble size and with the duration of the simulation. The interplay between time scales associated with the initial speed, the breathing billiard period, the frozen billiard internal time scales, and the observation interval is non-trivial and must be indicated when acceleration rate is declared.

Our paper is aimed at explaining various characteristics of the energy growth in higher-dimensional generalizations of the Fermi-Ulam model. When the particle moves fast, a classical stationary billiard (the frozen billiard) provides a good approximation for the particle trajectory on short time scales. It is widely accepted that the existence (or non-existence) of Fermi acceleration strongly depends on the dynamical properties of the frozen billiard. The dynamics in the frozen billiard can vary from being regular for an integrable billiard (as in the one-dimensional Fermi-Ulam model) of mixed phase space for general tables or fully chaotic for dispersing billiards.[9,13,56]

In the case of integrable frozen billiard (e.g., the breathing ellipse), one might expect the behaviour to be similar to the smooth one-dimensional problem. However, it is important to note that, in contrast to the one-dimensional case, the KAM theory (even when applicable) does not provide any barriers that prohibit energy growth. Indeed, in a near-integrable non-autonomous Hamiltonian system with d degrees of freedom, $d+1$-dimensional KAM tori do not separate the $2d+1$-dimensional phase space and therefore cannot prevent the possible energy growth for $d \geq 2$. Accelerating trajectories in breathing elliptic billiards were detected numerically,[36,37] while the Fermi acceleration is not visible in the breathing annular billiards.[10,11]

On the other hand, numerical experiments[41,42] demonstrate that averaged particles acceleration is clearly observable in chaotic billiards. This observation led to the so-called Loskutov-Ryabov-Akinshin (LRA) conjecture which states that breathing chaotic billiards admit trajectories with unbounded energy. Similar to the one-dimensional case, a seemingly opposite conclusion is reached when averaging arguments are invoked. In Sec. III A, we explain how the Anosov-Kasuga averaging theory can be applied to the chaotic breathing billiards. This theory predicts that if the frozen Hamiltonian is ergodic on every energy level, then for the majority of initial conditions, an adiabatic invariant is approximately preserved on any fixed interval of the slow time. At the present time, the application of this theory to billiards is not rigorously justified, as ergodic billiard flows do not satisfy smoothness assumptions of the Anosov-Kasuga theory. Nevertheless, when we formally apply it to a particle in a d-dimensional breathing billiard of volume V, we see that the adiabatic invariant is given by $J = EV^{2/d}$. The conservation of this adiabatic invariant coincides with the prediction of the classical thermodynamic law for an adiabatic processes in an ideal gas (see Sec. III A). The approximate preservation of J means that for most initial conditions, the ratio of the energy to its initial value gets close to 1 each time the billiard restores its volume, for any bounded number of the boundary oscillation periods. Moreover, the accuracy of the energy conservation improves and the number of boundary oscillations can be increased when the initial value of energy is increased. We discuss this topic in more detail in Sec. III.

The adiabatic invariant is seemingly an obstacle both to the acceleration and deceleration. However, the apparent contradiction to the numerical evidence is removed if we recall that the Anosov-Kasuga invariant is preserved only on a bounded time scale, i.e., the adiabatic invariant does not stop the energy growth but just slows it down. An analysis of the energy growth in the ergodic case can be undertaken by comparing the particle chaotic motion to random collisions with the billiard boundaries, i.e., the changes in the energy can be modelled by a random process. Dolgopyat and de la Llave[18,19] pointed out (for a different though related problem) that this process should be related to a Bessel process whose parameters can be estimated by scaling invariance arguments (see also Ref. 55). In Sec. IV, we provide empirical arguments to show that in an exponentially mixing billiard with periodically oscillating boundary, the growth of the root of order 4 from the energy averaged over the boundary oscillation period can be modelled by a (non-recurrent) Bessel process. Our numerical experiments give a strong support to this statement.

One of the consequences is the quadratic growth of the ensemble averaged energy with time and linear growth with the number of collisions. These growth laws are indeed well-known from many numerical experiments. We note, however, that the exponential decay of correlations seems to be important here. In particular, the energy growth in the breathing Bunimovich stadium does not conform to our stochastic model, and numerical simulations performed in Ref. 52 for the stadium and pseudo-integrable billiards that undergo large periodic deformations suggest that these may indeed present faster than quadratic growth rates.

The slow energy growth for the majority of initial conditions does not contradict to the existence of an exceptional set which accelerates very fast. A rigorous result in this direction was obtained in Ref. 24 and is summarized here in Sec. V. We show that the existence of rapidly accelerating trajectories can be proved in the framework of a general approach to chaotic Hamiltonian systems with slowly changing parameters developed in Ref. 23 (we also note that the problem of Fermi acceleration in non-autonomous billiards is closely related to the Mather acceleration problem formulated for periodically forced geodesic flows and, more generally, for *a-priori* unstable Hamiltonian systems[6,14,15,40,45]). Notably, the main theorem of Ref. 24 essentially proves the Loskutov-Ryabov-Akinshin conjecture. Roughly, the theorem states that the existence of transverse heteroclinic connections between two unstable periodic orbits of the family of the frozen billiards implies that there exist trajectories that have exponential energy growth. Hence, the theorem implies that Fermi acceleration exists even when the frozen billiard has mixed phase space, so global hyperbolicity is not needed (positive topological entropy of the frozen billiard is enough[33]). The exponentially accelerating trajectories found in Ref. 24 form a set of zero Lebesgue measure and belong to the minority of initial conditions for which the adiabatic invariant is not preserved.

Up to 2010, numerical studies of breathing billiards produced averaged acceleration which grows at most as a power-law in time.[10,11,28,36] In Refs. 22 and 51, it was demonstrated that a change in the number of ergodic components during the oscillation of the billiard boundary leads to a dramatic increase in the energy growth. Indeed, in such a construction, the majority of initial conditions experience exponentially fast acceleration. In Sec. VI, we show that this exponential growth is the result of a general setup where an ergodic breathing component is periodically broken into two ergodic breathing components and then these components are reconnected. In this case, the adiabatic model suggests that the logarithm of energy measured once per period evolves like a positively biased random walk, and therefore the energy grows exponentially on average. The growth rates predicted from the stochastic model are in good agreement with results from numerical simulations for several non-autonomous billiards.[22,51]

The paper is structured in the following way. In Sec. II, we formulate the energy growth problem and establish *a-priori* bounds on the energy growth. In Sec. III, we explain the averaging arguments which imply that if the frozen billiards are all ergodic, the majority of orbits do not experience energy growth for a long time. In Sec. IV, we show that a statistical model provides a tame averaged energy growth that emerges in this case. In Sec. V, we sketch the proof of LRA conjecture by showing the existence of orbits that accelerate exponentially with time in chaotic breathing billiards.[24] In Sec. VI, we summarize and put into a wider context the recent constructions of Refs. 22 and 51 for which the majority of particles have exponential energy growth.

II. SETUP OF THE PROBLEM

A. Elastic reflections from a moving boundary

Consider a particle inside a time-dependent domain $D(t) \subset \mathbb{R}^d$, $t \in \mathbb{R}$. Our main examples will have $d = 2$ but the theory developed in this section is valid for any $d \in \mathbb{N}$. We assume that the particle moves along a straight line with constant velocity v until it hits the boundary $\partial D(t)$. A trajectory of the particle is fully described by a sequence $(t_m, \boldsymbol{x}_m, \boldsymbol{v}_m)$ where t_m is the time of the mth collision with the boundary, $\boldsymbol{x}_m \in \partial D(t_m)$ is the collision point and $\boldsymbol{v}_m$ is the particle velocity before the collision. We assume that the reflection law is *elastic*. This assumption establishes relation between $\boldsymbol{v}_{m+1}$ and $\boldsymbol{v}_m$, so we can define the *billiard map* $(t_m, \boldsymbol{x}_m, \boldsymbol{v}_m) \mapsto (t_{m+1}, \boldsymbol{x}_{m+1}, \boldsymbol{v}_{m+1})$, which governs the behavior of the particle trajectories.

In order to state the elastic reflection law, we describe the motion of the boundary by its normal velocity $u(\boldsymbol{x}, t)$, $\boldsymbol{x} \in \partial D(t)$. We assume that positive u correspond to the boundary moving outwards. Suppose the billiard domain is given by

$$D(t) = \{\, \boldsymbol{x} \in \mathbb{R}^d : F(\boldsymbol{x}, t) < 0 \,\},$$

where $F : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is a smooth function. Then the boundary of $D(t)$ is defined by $F(\boldsymbol{x}, t) = 0$. The boundary is smooth at the points where $\nabla_x F(\boldsymbol{x}, t) \neq 0$. Then the outer normal and the boundary normal velocity are, respectively, given by

$$\boldsymbol{n} = \frac{\nabla_x F}{\|\nabla_x F\|} \quad \text{and} \quad u = -\frac{1}{\|\nabla_x F\|}\frac{\partial F}{\partial t}. \tag{1}$$

It is important to allow for the billiard boundary to be non-smooth at certain points, as the main examples of chaotic behaviour are provided by dispersing billiards which have corners.

It is convenient to describe the collision in a coordinate system where the boundary does not move at the collision point at the moment of collision. In such coordinates, the elastic reflection law takes the standard form: the tangent component of the particle velocity is preserved and the normal component is reflected. Returning to the original coordinates, we obtain the following reflection law:

$$\bar{\boldsymbol{v}} = \boldsymbol{v} + 2(u - \boldsymbol{n} \cdot \boldsymbol{v})\boldsymbol{n}. \tag{2}$$

If the billiard boundary is smooth at the collision point, the elastic reflection law indeed determines the particle velocity after the collision. However, the particle trajectory may be not defined for an infinite number of iterations due to one of the following reasons.

- Corners and edges: The particle may hit a point where the boundary is not smooth. Then, the normal vector $\boldsymbol{n}$ is not defined, and there is no unambiguous way to determine the result of the collision.
- Degenerate tangencies: If the particle hits a point on the boundary with $u = \boldsymbol{v} \cdot \boldsymbol{n}$, then the continuation of the trajectory may leave the billiard after the collision. Indeed, let $(t_0, \boldsymbol{x}_0, \boldsymbol{v}_0)$ be a point of the billiard trajectory. If $u = \boldsymbol{n} \cdot \boldsymbol{v}_0$, the elastic reflection law (2) implies that the trajectory does not change its velocity and, consequently, is given locally by a straight line $\boldsymbol{x} - \boldsymbol{v}_0 t = \boldsymbol{x}_0 - \boldsymbol{v}_0 t_0$. Consider the auxiliary function $\varphi(t) = F(\boldsymbol{x}_0 + \boldsymbol{v}_0(t - t_0), t)$. Obviously, $\varphi(t_0) = \varphi'(t_0) = 0$. As the particle comes from the region $F < 0$, it follows that necessarily $\varphi''(t_0) \le 0$. In the non-degenerate case $\varphi''(t_0) < 0$, and the trajectory continues to the region $F < 0$ after the collision. However, in the case of a degenerate tangency, we have $\varphi''(t_0) = 0$ and φ typically changes its sign. In this case, the continuation of the orbit leaves the billiard domain $F < 0$, i.e., the orbit cannot be continued after the collision.

Even when the number of collisions is infinite, they may happen in a *finite* time. This phenomenon is called chattering.[44] It may happen when the trajectory approaches a degenerate tangency or when the orbit gets into a zero angle corner.

Since the billiard map is volume-preserving, we expect these phenomena to be relevant for a set of measure zero only. Namely, for any reasonable function F, the set of $(\boldsymbol{x}, t)$ values which correspond to the singularities of the boundary has measure zero within the boundary.[33] In this case, the subset of the phase space of the billiard map which corresponds to corners or tangencies also has measure zero, and the volume-preservation property implies that the set of initial conditions which correspond to orbits that hit the corners or get tangent to the boundary has measure zero indeed. More advanced arguments are necessary to show that the volume-preservation property implies that the chattering also occupies a set of measure zero.

The volume preservation follows from the symplecticity of the billiard map. Indeed, one can verify by a direct computation that the elastic reflection law implies that the form

$$\boldsymbol{v} d\boldsymbol{v} \wedge dt - d\boldsymbol{v} \wedge d\boldsymbol{x} = d(\boldsymbol{x} - \boldsymbol{v}t) \wedge d\boldsymbol{v} \tag{3}$$

is preserved by the billiard map. In fact, the symplecticity follows naturally, if one notices that the billiard motion is a limit of the flow defined by the Hamiltonian function

$$H = \frac{1}{2}\boldsymbol{v}^2 + W(F(\boldsymbol{x}, t)), \tag{4}$$

where W is a smooth function which grows fast from zero to infinity as F changes from negative to positive values (the limit corresponds to the infinitely steep W, i.e., $W = 0$ at $F < 0$ and $W = +\infty$ at $F > 0$). The billiard map is the limit of the Poincare map on a cross-section $F = -\delta$ (for $\delta \to +0$, see, e.g., Refs. 23, 49, and 57). The nonautonomous Hamiltonian flow preserves the standard symplectic form $dH \wedge dt - d\boldsymbol{v} \wedge d\boldsymbol{x}$. On the cross-section $F = const$, this form reduces to Eq. (3) by virtue of Eq. (4), so the Poincaré map and, hence, the limit billiard map must preserve form (3) indeed.

B. At most linear energy growth with collision number

Equation (2) immediately implies that $\bar{v}^2 = v^2 - 4(\boldsymbol{n} \cdot \boldsymbol{v})u + 4u^2$. Since $|\boldsymbol{n} \cdot \boldsymbol{v}| < v$, we see that the particle speed $\bar{v}$ after the collision is inside the interval

$$v - 2|u| \le \bar{v} \le v + 2|u| \,. \tag{5}$$

Therefore, *the particle speed grows (or decays) at most linearly as a function of the number of collisions.* More precisely, if $(\boldsymbol{x}_m, \boldsymbol{v}_m, t_m)$ is a billiard trajectory, then

$$v_m \le v_0 + 2m\|u\| \tag{6}$$

where $\|u\| = \sup_{\boldsymbol{x}\in\partial D(t), t\in\mathbb{R}}|u|$ is the sup-norm of the boundary velocity. Accordingly, the particle kinetic energy $E = \frac{1}{2}v^2$ grows at most quadratically with the number of collisions.

C. At most exponential energy growth with time

When the particle speed grows, the time between consecutive collisions decreases, so the question of the energy growth rate with time is not trivial. As we mentioned, the particle may go through infinitely many collisions in finite time, and one can easily construct examples where the energy becomes unbounded in finite time (e.g., consider a particle bouncing in the normal direction between two parallel walls which tend towards each other with a finite speed). The next theorem provides a simple condition under which such behavior is impossible, and the energy does not grow faster than exponentially in time.

Theorem 1. *If $F \in C^2(\mathbb{R}^{d+1})$ has bounded C^2-norm, and there is $\alpha \in \mathbb{R}$ such that $\|\nabla_x F\| > \alpha > 0$ for all $x \in \partial D(t)$ and all $t \in \mathbb{R}$, then there are constants $C_1, C_2 > 0$ such that for any trajectory of the billiard flow*

$$E(t) \le (E_0 + C_1)\exp C_2 t \tag{7}$$

provided the trajectory is well-defined on [0, t].

Proof. Suppose the particle hits the boundary at $\boldsymbol{x}_m \in \partial D(t_m)$ with velocity $\boldsymbol{v}_m$. Let $\delta_m = t_m - t_{m-1}$ denote the time between the collisions. Then $\boldsymbol{x}_{m-1} = \boldsymbol{x}_m - \boldsymbol{v}_m\delta_m$. We note that $F(\boldsymbol{x}_m - \boldsymbol{v}_m\delta_m, t_m - \delta_m) = F(\boldsymbol{x}_m, t_m) = 0$, so the Taylor series expansion gives

$$-\delta_m \sum_{j=0}^{d} \frac{\partial F}{\partial x_j}(\boldsymbol{x}_m, t_m) v_m^j + \delta_m^2 R_2 = 0 \,, \tag{8}$$

where we denote $x_0 \equiv t$ and $v_m^0 = 1$ to spare notation (v^j denotes the jth component of the vector $\boldsymbol{v}$). The remainder R_2 is written in the form

$$R_2 = \int_0^1 (1-s) \sum_{i,j=0}^{d} \frac{\partial F}{\partial x_i \partial x_j}(\boldsymbol{x}_m - s\boldsymbol{v}_m\delta_m, t_m - s\delta_m) v_m^i v_m^j \, ds \,. \tag{9}$$

Since the energy $E_m = \frac{v_m^2}{2}$, Eq. (2) implies that the change of energy after one collision is given by

$$E_{m+1} - E_m = 2u_m(u_m - \boldsymbol{v}_m \cdot \boldsymbol{n}_m) \,, \tag{10}$$

where u_m is the boundary normal velocity at the mth collision and $\boldsymbol{n}_m$ is the outer normal to the boundary at the point $\mathbf{x_m}$. Taking into account Eq. (1), we get

$$u_m - \boldsymbol{v}_m \cdot \boldsymbol{n}_m = -\frac{1}{\|\nabla_x F\|} \sum_{j=0}^{d} \frac{\partial F}{\partial x_j} v_m^j \,.$$

Substituting this into Eq. (10) and using Eq. (8), we get

$$\frac{E_{m+1} - E_m}{\delta_m} = -\frac{2u_m R_2}{\|\nabla_x F\|} \,.$$

Equation (9) implies that $|R_2| \le C(1 + E_m)$ where C depends on the C^2-norm of F. Since $\|\nabla_x F\|$ is bounded away from zero and u is uniformly bounded, we get

$$\frac{E_{m+1} - E_m}{t_m - t_{m-1}} \le C_0(1 + E_m). \tag{11}$$

This implies Eq. (7) by induction in m. □

Remark. As we mentioned, it is possible that some orbits are not defined for all times. In the setting of Theorem 1, the boundary does not have any singularity; however, trajectories of a degenerate tangency are possible. Therefore, we can guarantee the fulfillment of estimate (7) for all times only for a set of initial conditions which has a full measure but not necessarily coincides with the whole of the phase space. It should be stressed that the constants C_1 and C_2 stay the same for all orbits.

In Theorem 1, the assumption of the non-vanishing of the gradient of the function F implies that the boundary does not have corners. This is too restrictive, as many billiards discussed in the paper do have corners. However, one can generalize the result to cover piecewise smooth boundaries. In particular, for planar billiards with piecewise smooth boundaries, the theorem holds true provided the angles between the smooth boundary components that join at the corner points stay bounded away from zero for all times.

III. ADIABATIC THEORY FOR A FAST PARTICLE

In the study of Fermi acceleration, we unavoidably meet the situation when the particle moves fast compared to the billiard boundary and the theory of *adiabatic invariants* may become applicable. Adiabatic invariants are quantities which stay approximately constant for periods of time that are large enough to allow a noticeable variation in the particle energy and the billiard shape. They appear naturally in slow-fast Hamiltonian dynamics. Adiabatic theory provides a tool for describing the evolution of the particle energy on time scales which involve a large number of collisions with the boundary, and we will persistently use adiabatic invariants in Secs. IV–VI.

It should be noted that the classical adiabatic theory is developed for smooth systems while billiards do have various singularities, so more work is still needed to complete a mathematically rigorous adiabatic theory for multi-dimensional billiards. On the other hand, numerical evidence[8] strongly supports validity of the adiabatic theory in time-dependent billiards.

A. Ergodic adiabatic invariants

According to the Anosov-Kasuga averaging theory,[1,32] a system with a slowly changing in time Hamiltonian approximately preserves the Anosov-Kasuga invariant. Namely, given a one-parameter family of Hamilton functions $H(x,p;\tau)$, assume that for each fixed value of τ the corresponding Hamiltonian system, called the *frozen* system,

$$\dot{x} = \partial_p H(x,p;\tau), \quad \dot{p} = -\partial_x H(x,p;\tau)$$

is ergodic with respect to the Liouville measure on every energy level. If the parameter τ changes slowly with time ($\tau = \varepsilon t$), the energy must no longer be preserved

$$\dot{E} = \frac{dH}{dt}(x(t),p(t);\varepsilon t) = \varepsilon \partial_\tau H.$$

Since ergodicity allows replacing time averages by a space average, this equation suggests[8] that for the majority of initial conditions, the evolution of the energy is determined by $\langle \partial_\tau H \rangle_E$, the average value of $\partial_\tau H$ over the energy level:

$$E(t_1) - E(t_0) = \int_{t_0}^{t_1} \dot{E}\, dt \approx \varepsilon \langle \partial_\tau H \rangle_{E(t_0)} (t_1 - t_0)\,.$$

Actual theoretical justification of this conclusion is not trivial and is given by Anosov averaging theorem.[1] As the evolution of the energy is essentially determined by the initial energy only, one can roughly model the evolution of phase variables in our Hamiltonian system with a slowly changing parameter as follows: the flow shifts a level of constant energy for the frozen system to a level of constant energy which correspond to the new value of the parameter, while on the level itself, the fast motion is ergodic. Since the Hamiltonian flow preserves volumes in the phase space, it follows that the volume bounded by the energy level is to be preserved.

To be more precise, let $J(E,\tau)$ be the volume of the part of the phase space defined by the inequality $H(x,p;\tau) < E$. Kasuga proved the following theorem:[32] for any $\delta > 0$ and any τ^*, for all sufficiently small ε and all initial conditions outside a set of a small measure, the inequality

$$|J(H(x(t),p(t);\varepsilon t),t) - J(H(x(0),p(0);0),0)| < \delta \qquad (12)$$

holds for all $t \in [0, \varepsilon^{-1}\tau^*]$, i.e., on the intervals of the slow time τ of order one. The measure of the exceptional set vanishes with ε.

The quantity J is called an Anosov-Kasuga ergodic adiabatic invariant or, sometimes, an ergodic adiabatic invariant.

Now consider a particle inside a d-dimensional billiard with an oscillating boundary. When the particle energy is large, the particle moves much faster than the boundary does, so we may formally consider this system as a slow-fast Hamiltonian one and expect that the Anosov-Kasuga theory is applicable.

In order to apply this theory, we introduce the small parameter $\varepsilon = 1/E^{1/2}(0)$ and the rescaled time $\tau = \varepsilon t$. In the new units, the particle energy is given by $\tilde{E} = \varepsilon^2 E$. In particular, the initial energy equals to one. For the billiard flow, the volume of the phase space bounded by the energy $\tilde{E}$ equals to the product of the billiard domain, whose volume we denote by $V(\tau)$, and the ball $v_1^2 + \ldots + v_d^2 \le 2\tilde{E}$ in the particle momenta space (for simplicity we have chosen the particle to have the unit mass). Hence, the Anosov-Kasuga adiabatic invariant equals to $c_d \tilde{E}^{d/2} V(\tau)$ where the constant c_d depends on the dimension d only. If the Anosov-Kasuga theory is indeed applicable, according to Eq. (12) we get $|\tilde{E}^{d/2}(\tau)V(\tau) - \tilde{E}^{d/2}(0)V(0)| < \delta$. Coming back to the original variables, we may expect that for ergodic billiards with slowly moving boundaries

$$\left| \frac{E^{d/2}(\tau)V(\tau) - E^{d/2}(0)V(0)}{E^{d/2}(0)} \right| < \delta \qquad (13)$$

for all $\tau \in [0, \tau^*]$ outside a set of small measure.

Denote

$$J = E^{d/2} V. \qquad (14)$$

Equation (13) implies that $|J(\tau) - J(0)|/J(0)$ is small for the majority of initial conditions provided $J(0)$ is sufficiently large. It is interesting to observe that it does not follow that the changes in J are small but only that the changes of $\log J$ are small. We arrive at the following statement.

Proposition 1 (conjecture). *Let $D(\tau)$ be a family of ergodic billiard tables, δ, τ^*, K_1, and K_2 be positive numbers, and S_ε be the set of all initial conditions such that $K_1\varepsilon^{-2} \le J(0) \le K_2\varepsilon^{-2}$. Then the measure of the subset of initial conditions for which*

$$|\log J(t) - \log J(0)| < \delta \;\; \text{for all} \;\; t \in [0, \tau^*] \qquad (15)$$

converges to the full measure of the set S_ε as $\varepsilon \to 0$.

A rigorous proof of this claim requires an extension of Anosov-Kasuga adiabatic theory to billiard-like systems.

Note that in a mixing case, the conservation of the adiabatic invariant is expected to be much more accurate than the prediction of Eq. (15) (see Ref. 8 for more discussion and numerical evidence). Our numerical experiments confirm that $\log \frac{J(t)}{J(0)}$ decays as $J(0)^{-1/2}$ for an exponentially mixing billiard, and as $J(0)^{-1/2} \log J(0)$ for a breathing stadium (see Sect. IV). This puts a restriction on the rate of the Fermi acceleration in the ergodic mixing case: for the majority of initial conditions we expect $J^{1/2}$ to grow at most linearly in time (cf. Ref. 22).

It is interesting to compare Proposition 1 and Theorem 1. In order to simplify the comparison, we suppose that the billiard boundary oscillates periodically with period T and analyse the change in the energy over the period. We have $V(0) = V(T)$ and $\log \frac{J(T)}{J(0)} = \frac{d}{2} \log \frac{E(T)}{E(0)}$. If $E(0)$ is sufficiently large, Eq.(11) implies that $\left|\log \frac{E(T)}{E(0)}\right| < 2C_0$ for all billiard trajectories which are defined till $t = T$, while Eq. (14) implies that $\left|\log \frac{E(T)}{E(0)}\right|$ stays close to zero for the majority of the initial conditions.

In the kinetic theory of ideal gases, the energy per molecule is proportional to the temperature T. The adiabatic compression law of the ideal gas reads $T^{d/2}V = \text{const}$, where d is the number of the degrees of freedom of the molecule. Let us consider the ideal monoatomic gas of N particles in a

d-dimensional container D of volume V as an ergodic billiard in the Nd-dimensional configuration space

$$D^N \cap \{\|\boldsymbol{x}_j - \boldsymbol{x}_i\| \geq 2\rho, \quad i,j = 1,\ldots N\},$$

where $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N$ are the positions of the atoms in D, and ρ is the radius of an atom. Then, in the limit $\rho \to 0$, the adiabatic invariant J for such billiard is proportional to $(NT)^{Nd/2}V^N$, and formula (15) gives us

$$\frac{T(\tau)^{d/2}V(\tau)}{T(0)^{d/2}V(0)} = 1 + O(N^{-1}),$$

i.e., it coincides with the adiabatic compression law as $N \to +\infty$.

The standard definition of an adiabatic process asserts that there is no heat exchange with the surroundings. In our setting, this just means the reflection law is elastic. We also require that the pace of the billiard volume change is slow enough in comparison with the particle motion (this is always satisfied in the kinetic theory remit). Under these conditions, the Anosov-Kasuga theory provides an analogue of classical thermodynamics for the case of a finite (i.e., not necessarily large) number of particles.

B. One-dimensional billiard

One-dimensional static billiards are simple integrable systems, so the adiabatic invariant has a different nature and plays a role different from that in a higher-dimensional case. Namely, let us consider the particle bouncing in a straight-line segment of length $L(t)$. In this case, Anosov-Kasuga theory leads to the same adiabatic invariant as the classical Ehrenfest theory of adiabatic invariants (see Refs. 2, 34, and 35). In any case, one finds the adiabatic invariant $J = |v|L(\tau)$ which coincides with (14) at $d = 1$. However, it is also possible in the integrable case to find small corrections to J, after which it is preserved with a better accuracy at large v. As the general theory is typically developed for smooth systems, let us derive the corresponding formulas directly from the laws of the particles motion (computations for various non-smooth settings can also be found in Refs. 25 and 55).

For simplicity, we assume that only one end of the segment oscillates, while the second end is fixed. So, the configuration space is given by $x \in [0, L(t)]$. The elastic reflection law at the moving end $x = L(t)$ is

$$\bar{v} = 2L'(t) - v, \tag{16}$$

where t is the moment of collision, v is the particle velocity before the collision, and $\bar{v}$ is the velocity right after it (cf. Eq. (2)).

Let t_n be the moment of the nth collision, and let v_n and $\bar{v}_n$ be the velocities before and after the collision at $t = t_n$. We assume that the wall moves much slower than the particle, so $v_n > 0$ and $\bar{v}_n < 0$. The particle speed is preserved during the flight from the right end to the left end and back, so $v_{n+1} = -\bar{v}_n$, and we have the following equation for the billiard map:

$$\begin{aligned} v_{n+1} &= v_n - 2L'(t_n), \\ t_{n+1} &= t_n + \frac{L(t_{n+1}) + L(t_n)}{v_{n+1}}. \end{aligned} \tag{17}$$

The equation for the change in the speed is given by Eq. (16), and the second line in Eq. (17) just says that the velocity times the time between the collisions equals the length of the particle path. At large values of v_n, the difference between t_{n+1} and t_n is small, and the implicit function theorem implies that Eq. (17) indeed defines the map $(v_n, t_n) \mapsto (v_{n+1}, t_{n+1})$.

It is easy to check that this map preserves an area form

$$(v_{n+1} - L'(t_{n+1}))dv_{n+1} \wedge dt_{n+1} = (v_n - L'(t_n))dv_n \wedge dt_n.$$

It follows that if we define

$$\mathcal{E} = \frac{1}{2}(v - L'(t))^2, \tag{18}$$

which is the particle's pre-collision kinetic energy in the coordinate system moving with the right wall, then the area form $(v - L'(t))dv \wedge dt$ (the one preserved by the map) takes the standard form $d\mathcal{E} \wedge dt$ (since $d\mathcal{E} = (v - L'(t))(dv - L''(t)dt)$). One can also see that the area form coincides with Eq. (3), as $dx = L'(t)dt$ at the boundary.

Since the change in t is small at large v, we deduce from Eq. (17) that

$$\frac{\Delta v}{\Delta t} \approx -\frac{L'(t)}{L(t)}v.$$

So, $v_n(\tau_n)$ can be approximated by a solution of the differential equation

$$\frac{dv}{dt} = -\frac{L'(t)}{L(t)}v.$$

This equation gives $vL(t) = \text{const}$, so we conclude that the product $vL(t)$ is approximately preserved by Eq. (17). This means that as the length of the segment increases, the speed decreases, and vice versa. If $L(t)$ oscillates periodically, then the speed v will also oscillate around a constant value for a long time.

In fact, the corrected adiabatic invariant

$$J = \sqrt{2\mathcal{E}}L(t) = (v - L'(t))L(t) \tag{19}$$

is preserved with a better accuracy.

Namely, as a direct computation shows, map (17) takes the following form in the coordinates (J, t):

$$\begin{aligned} J_{n+1} &= J_n + O\left(\frac{1}{J_n^2}\right), \\ t_{n+1} &= t_n + \frac{2L^2(t_n)}{J_n} + O\left(\frac{1}{J_n^2}\right). \end{aligned} \tag{20}$$

One can immediately see that the value of J does not deviate essentially from the initial one for $O(J^2)$ collisions. As the time $(t_{n+1} - t_n)$ between two consecutive collisions is of the order $\frac{1}{J}$, it follows that J is approximately preserved on time intervals of the order J. For uniformly bounded L and L', this means that the particle speed $v = \frac{J}{L(t)} + L'(t)$ stays at a bounded distance from its (sufficiently large) initial value for time intervals of the order v.

When $L(t)$ is periodic in time and sufficiently smooth, KAM-theory guarantees that v stays close to its initial value forever[47] (for all sufficiently large initial values). Indeed, by Eq. (20), the values of J and t after the kth collision are $O(\frac{k}{J^2})$-close (uniformly for all $k \ll J^2$) to the time-k shift by the differential equation

$$\dot{J} = 0, \quad \dot{\tau} = \frac{2L^2(\tau)}{J}. \tag{21}$$

The curves $J = \text{const}$ are invariant with respect to this equation. The motion on the invariant curve is a rotation with the average frequency $\omega(J) = \frac{2}{T}\int_0^T L^2(\tau)d\tau$, where T is the period of the wall oscillations. Since $\omega'(J) \neq 0$, and since both the map (20) and its approximation (21) preserve the same area form $d\mathcal{E} \wedge dt$ (where $\mathcal{E} = \frac{J^2}{2L^2}$, see Eq. (19)), the KAM theorem implies that the majority of these invariant curves persist for the map (20) at large J (just the shape of the curves may change slightly). An initial point between any two such curves will never leave the region between them, so the value of $J_n - J_0$ stays bounded for all n in this case.

Summarizing, in the one-dimensional billiard with periodically and smoothly oscillating boundary, the deviation of the energy of a sufficiently fast moving particle from its initial value stays bounded for all times $\tau \in (-\infty, +\infty)$. The reason is the existence of the adiabatic invariant J given by Eq. (19) in combination with KAM theory. The oscillations of the particle speed are governed by the law

$$J(t) = vL(t) \approx \text{const}, \tag{22}$$

or, more accurately,

$$v = \frac{J_0}{L(t)} + u(t) + O\left(\frac{1}{J_0}\right), \quad J_0 = \text{const}, \tag{23}$$

where $u(t) = L'(t)$ is the velocity of the oscillating wall. Thus, for the corrected adiabatic invariant, we have proved the following refinement of Proposition 1: for all initial conditions with sufficiently high initial energy

$$\log \frac{J(t)}{J(0)} = \frac{u(t)}{J(0)} + O(J^{-2}(0)).$$

C. Adiabatic theory for billiards on the plane

For the two-dimensional case, the Anosov-Kasuga adiabatic invariant is $J = EV(t)$, where V is the billiard area. Let us give the derivation of this fact which does not rely on the general Hamiltonian formalism, and which is similar to the derivation of the ideal gas laws in kinetic theory. At a time t, the particle hits the boundary of the billiard at a point x at an angle φ to the inward pointing normal to the boundary. The particle energy at the moment of collision is E, so the speed is $v = \sqrt{2E}$, and the velocity of the boundary is $u(x;t)$ (the boundary velocity is normal to the boundary; we assume that positive u correspond to the boundary moving outwards). We assume $v \gg |u|$. Note that the normal component of the particle velocity can still be smaller than $|u|$ for trajectories nearly tangent to the boundary. Then, an inwards moving particle may collide with the boundary. This event corresponds to $|\varphi| > \frac{\pi}{2}$, which is impossible in a static billiard where the angle of incidence is always smaller than $\frac{\pi}{2}$. As the nearly tangent initial conditions occupy a small portion of the phase space and, with the speed growing, this portion becomes smaller and smaller, we simply exclude such trajectories from the consideration. Anyway, near the tangent trajectories, we do not have a reason to believe in the validity of the Anosov-Kasuga averaging. So we further assume $|\varphi| < \frac{\pi}{2}$.

Denote by $(\bar{v}_\perp, \bar{v}_\parallel)$ the particle post-collision velocity, with $\bar{v}_\perp$ and $\bar{v}_\parallel$ being its normal and tangent to the boundary components, respectively. The reflection law is

$$\bar{v}_\parallel = v\sin\varphi, \quad \bar{v}_\perp = 2u - v\cos\varphi. \tag{24}$$

This gives

$$\bar{E} = E - 2u\sqrt{2E}\cos\varphi + 2u^2, \tag{25}$$

where $\bar{E} = \frac{1}{2}(\bar{v}_\parallel^2 + \bar{v}_\perp^2)$ is the particle energy after the collision. If the next collision to the boundary happens at the time moment $\bar{t}$ at a point $\bar{x}$ with the incidence angle $\bar{\varphi}$, then

$$\bar{t} = \tau + \frac{\|\bar{x} - x\|}{\bar{v}}. \tag{26}$$

If $v \gg u$ and the incidence angle φ is bounded away from $\pm\frac{\pi}{2}$, Eq. (24) implies that the reflection angle ψ is $O(v^{-1})$-close to $-\varphi$, which corresponds to a static billiard. Thus, we may write

$$(\bar{x}, \bar{\varphi}) = B_t(x, \varphi) + O\left(\frac{1}{v}\right), \tag{27}$$

where B_t is the static billiard map frozen at time t. Namely, we take the billiard table at the moment t and issue a ray from the point x in the direction which makes the angle $-\varphi$ with the inward pointing normal to the boundary. Then the x-component of $B_t(x, \varphi)$ is the point where this ray intersects the boundary again (we do not move the boundary now). The φ-component of $B_t(x, \varphi)$ is the incidence angle at this point, and we also denote by $L(x, \psi; t)$ the length of the segment between x and this point.

Note that formula (27) is true only provided the orbit is not nearly tangent at $\bar{x}$, i.e., we also assume $\bar{\varphi}$ is bounded away from $\pm\frac{\pi}{2}$ (otherwise a small change in the direction of the outgoing ray could lead to the trajectory missing the collision near $\bar{x}$). By Eq. (27)

$$\|\bar{x} - x\| = L(x, -\varphi; t) + O\left(\frac{1}{v}\right).$$

Now, from Eqs. (25) and (26), we find

$$\log \bar{E} = \log E - \frac{4u(x;t)\cos\varphi}{v} + O\left(\frac{1}{v^2}\right),$$
$$\bar{t} = t + \frac{L(x,-\varphi;t)}{v} + O\left(\frac{1}{v^2}\right). \tag{28}$$

This formula and Eq. (27) together provide an approximation for the non-autonomous billiard map at high velocities. Indeed, these equations provide information about positions, times, and velocities at consecutive collisions of the particle with the billiard boundary.

Now we assume that at every t, the frozen billiard is ergodic. Namely, at every t the billiard map B_t is known to preserve the measure $\cos\varphi dx d\varphi$, where dx is the infinitesimal length element of the billiard's boundary, and we assume that B_t is ergodic with respect to this measure for all t. Since in the limit $v \to +\infty$ the map (28) converges to the identity and the map (27) is ergodic, we may apply the Anosov averaging theory[1] to the systems (27) and (28) at large v. Namely, by this theory, we can average the right-hand side of Eq. (28) over (x, φ) with respect to the measure $\cos\varphi dx d\varphi$, and then expect that the iterations of Eq. (28) will stay close to the iterations of the averaged map (except for a small measure set of initial conditions in the (x, φ)-space) for any finite interval of the slow time τ (i.e., for $O(v)$ iterations), provided v is large enough.

Let us write down the averaged system. We must compute

$$\int u(x;t)\cos^2\varphi dx d\varphi \quad \text{and} \quad \int L(x,-\varphi;t)\cos\varphi dx d\varphi,$$

where integrals are taken for $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and for x in the boundary of the billiard table frozen at the time t. As u is the normal velocity of the boundary of the billiard, it is obvious that

$$\int u(x;\tau)dx = \frac{dV}{dt},$$

where $V(t)$ is the billiard's area. Note also that $L(x,\varphi)\cos\varphi dx$ is the area element of the parallelogram with the sides L and dx. For each fixed φ the union of these parallelograms covers the billiard table, their interiors do not intersect, and when x runs through the entire billiard's boundary, each parallelogram is counted exactly twice (each segment L has two ends). Thus,

$$\int L(x,-\varphi;t)\cos\varphi dx = 2V(t).$$

So, the averaged map (28) is given by

$$\log \bar{E} = \log E - \frac{\pi V'(t)}{\ell v} + O\left(\frac{1}{v^2}\right),$$
$$\bar{t} = t + \frac{\pi V(t)}{\ell v} + O\left(\frac{1}{v^2}\right), \tag{29}$$

where ℓ is the length of the billiard's boundary (when performing the averaging, we divide the integrals to the total volume of the (x, φ) phase space, i.e., to $\iint \cos\varphi dx d\varphi = 2\ell$). Note that

$$\log \bar{E} + \log V(\bar{t}) = \log E + \log V(t) + O\left(\frac{1}{v^2}\right)$$

by virtue of Eq. (29). Since $J = EV(t)$, then

$$\log \bar{J} = \log J + O\left(\frac{1}{v^2}\right), \tag{30}$$

which is equivalent to $\bar{J} = J + O(1)$.

For $O(v)$ iterations of this map (i.e., for a finite interval of time t), the total change in $\log J$ is small. So, by the reasoning given above, in an ergodic billiard with slowly oscillating boundary the particle energy changes in such a way that on finite intervals of the time the value of $\log(EV(t))$ stays nearly constant for the majority of initial conditions. This means that the relative change in the value of the adiabatic invariant $J = E(t)V(t)$ is small, i.e., $E(t)V(t)/(E(0)V(0))$ stays nearly constant on any finite intervals of t provided $E(0)$ is large enough.

Formula (30) can be obtained in a similar way for any dimension d. Recall however that the derivation here is only formal: the Anosov theory is proven for smooth dynamical systems, while the billiard maps we consider here are, typically, non-smooth due to singularities which appear in the map near tangent trajectories or near trajectories hitting the billiard corners and due to chattering. Therefore, the question of the rigorous derivation of the boundedness of E/E_0 on finite time intervals remains open.

IV. STATISTICAL MODEL FOR THE ACCELERATION IN THE ERGODIC CASE

In the case when the frozen billiard remains ergodic for all frozen time, adiabatic theory described in Sec. III predicts that for the majority of initial conditions the Fermi acceleration is relatively slow (if present at all). If additionally the frozen billiard is chaotic, e.g., mixing, it is natural to ask if it is possible to derive a probabilistic model for the Fermi acceleration (see, e.g., Refs. 5, 12, 18, 29, and 55).

In a derivation and analysis of such models, it is necessary to take into account that, when the particle energy is low enough and its velocity is of the same order as the velocity of the billiard wall, the system loses its slow-fast structure, and all types of complications known for Hamiltonian dynamical systems with several degrees of freedom emerge. In particular, the system may have sticky elliptic islands which would change the statistical behavior of the low energy trajectories in an unpredictable way. Therefore, the stochastic model we propose here can be valid only under the condition that the energy stays larger than a certain fixed value $\hat{E}$. In other words, one should take the initial energy value E_0 so large that the probability of the trajectory to get (on the time interval under consideration) to the energy values under the threshold $\hat{E}$ is sufficiently small.

When the particle's speed is large, $v \gg u$, the change in the energy E at a collision is $\sim \sqrt{E}$ and the time interval between two consecutive collisions is of the order of $1/\sqrt{E}$.

Letting t_n and E_n be, respectively, the time of the nth collision and the post-collision kinetic energy, we can write

$$\frac{E_{n+1}-E_n}{t_{n+1}-t_n}=E_n\xi_n.$$

We can consider this equation as a definition of a bounded sequence ξ_n, which can be interpreted as a realization of a random variable whose distribution may depend on E_n and t_n. Since the intervals $t_{n+1}-t_n$ are small (of the order of $\frac{1}{\sqrt{E}}$), we may compare this equation with a stochastic ordinary differential equation of the form

$$dE=E\xi(t)dt,$$

where $\xi(t)$ is a bounded random function (a realization of a random process).

It is clear that in a breathing billiard, the changes in the energy are correlated over a large number of collisions ($\sim 1/\sqrt{E}$) as the adiabatic theory predicts that the particle will mostly accelerate when the billiard volume decreases and decelerate when the volume increases. On the other hand, in a chaotic frozen billiard, correlations between consecutive collisions decay fast. Thus, while energy changes in consecutive collisions with the moving wall exhibit the common trend, the fluctuations from this trend can be assumed non-correlated. We try to eliminate the trend by considering the evolution of the adiabatic invariant $I=J^{2/d}=EV(t)^{2/d}$ instead of the energy. We have

$$d\log I=\eta(t)dt, \tag{31}$$

where $\eta(t)=\xi(t)+\frac{2}{d}\frac{V'(t)}{V(t)}$. The approximate preservation of $\log I$ means that $\mathbb{E}(\eta)$ tends to zero as I grows. Since the natural small parameter here is $(v/u)^{-1}\sim I^{-1/2}$, we estimate

$$\mathbb{E}(\eta)\sim I^{-1/2} \tag{32}$$

in agreement with Eqs. (29) and (30).

Let us now scale I and t as follows: $I=E_0w^4$ and $t=\tau\sqrt{E_0}$, where $E_0\sim v_0^2$ is the initial value of energy (note that $w\sim\sqrt{v/v_0}$). Equation (31) transforms into

$$\frac{dw}{d\tau}=w\frac{\sqrt{E_0}}{4}\eta(\tau\sqrt{E_0}). \tag{33}$$

Note that in the new variables, the expected value of the right-hand side is $\sim\frac{1}{w}$, and the interval of the decay of correlations is $\sim\frac{M}{E_0w^2}$, where M is the number of collisions sufficient to neglect the correlations in the frozen billiard. We assume that E_0 is sufficiently large. Since we have a fast oscillating random function in the right-hand side, we may replace it by its average plus white noise. The power of the white noise must be taken equal to the integral of the correlation function. The correlation function has amplitude $\sim w^2E_0$ and the support of size $\sim\frac{1}{E_0w^2}$, so the power of the white-noise should be a constant (independent of w), which we denote as σ^2. We put the average of the right-hand side to be equal to aw^{-1} with a constant a. This gives us the following stochastic differential equation:

$$dw=\frac{a}{w}d\tau+\sigma dB_\tau, \tag{34}$$

where B is the standard Wiener process.

Note that Eq. (34) should be valid for the increments of time τ larger than the interval of the decay of correlations, i.e., for $\Delta\tau\gtrsim\frac{1}{E_0}$. This corresponds to small increments of the non-rescaled time t of order of $\frac{1}{\sqrt{E_0}}$.

By assuming that the change of billiard shape is periodic in t, we obtain that the coefficients a and σ are periodic in t, which means they are fast-oscillating functions of τ. We therefore replace a and σ by their averages over t, i.e., we assume them to be τ-independent constants. In this way, we effectively perform here an averaging over the billiard oscillations, so $w(\tau)$ in Eq. (34) becomes, up to a constant factor, the value of the square root of the particle speed, averaged over the oscillation period.

Now, in order to evaluate the coefficient a, we use the idea proposed to us by Dolgopyat (see also Refs. 12, 16, and 17). The evolution of the probability density ρ for the random variable $w(\tau)$ defined by Eq. (34) is given by the Kolmogorov-Fokker-Planck (KFP) equation

$$\frac{\partial\rho}{\partial\tau}=\frac{\partial}{\partial w}\left(-\frac{a}{w}\rho+\frac{\sigma^2}{2}\frac{\partial\rho}{\partial w}\right).$$

Since the total phase volume is preserved, it follows that the probability density $\rho(w)=w^{2d-1}$ is stationary (indeed, the phase space volume below the energy $E\sim w^4$ is of the order $E^{d/2}\sim w^{2d}$, so the portion of the phase space volume corresponding to the interval $[w,w+dw]$ is $\sim w^{2d-1}dw$). In other words, the volume preservation property implies that the right-hand side of the KFP-equation vanishes for $\rho=w^{2d-1}$, which gives

$$a=\left(d-\frac{1}{2}\right)\sigma^2. \tag{35}$$

So, the KFP-equation is

$$\frac{\partial\rho}{\partial\tau}=\frac{\sigma^2}{2}\frac{\partial}{\partial w}\left(-(2d-1)\frac{\rho}{w}+\frac{\partial\rho}{\partial w}\right). \tag{36}$$

The stochastic differential equation (34) becomes

$$dw=\sigma^2\frac{2d-1}{2}\frac{d\tau}{w}+\sigma dB_\tau, \tag{37}$$

Note that we can always make an additional scaling of time such that σ would become equal to any given constant. By making the coordinate transformation $E=w^4E_0$, we obtain the stochastic differential equation (in Stratonovich form) for the evolution of the energy

$$dE=\frac{2d-1}{8}\sqrt{E}dt+E^{3/4}\circ dB_t, \tag{38}$$

where we scale time to make the coefficient in front of $E^{3/4}$ to be equal to 1. In Ito form, this equation is recast as

$$dE = \frac{d+1}{4}\sqrt{E}dt + E^{3/4}dB_t. \tag{39}$$

We stress again that the proposed stochastic differential equations are meant to describe only the evolution of energy averaged over the oscillation period, so the coefficients are time-independent. Note also that they give a more accurate description of the energy change then, e.g., the model of Ref. 29 (which corresponds to our coefficient a taken to be zero).

As we mentioned, the above computations can be valid only until the particle energy stays above a certain threshold $\hat{E}$. In the rescaled variables, this means that $w(\tau)$ must stay above $\sqrt[4]{\hat{E}}/E_0$. The stochastic process described by Eq. (37) (Bessel process in dimension $2d$) is known to be transient at $d>1$, which means here that the probability of the particle starting at $w=1$ to ever get below $\sqrt[4]{\hat{E}}/E_0$ tends to zero as $E_0 \to +\infty$ (the probability is $\sim E_0^{-(d-1)/2}$). Therefore, we conclude that the stochastic description provided by Eq. (37) can be valid for most of trajectories if E_0 is large enough.

The expected value of the (rescaled) energy is given by

$$\bar{E}(\tau) = \int w^4 \rho(w,\tau)dw.$$

Integrating Eq. (36) by parts, we find

$$\frac{d\bar{E}}{d\tau} = 4\sigma^2(d+1)\bar{v}(\tau),$$

where $\bar{v}(\tau) = \int \rho(w,\tau)w^2 dw$ is the expectation of the particle speed (rescaled to its initial value $v_0 = \sqrt{2E_0}$). From Eq. (36) again, we have

$$\frac{d\bar{v}}{d\tau} = 2\sigma^2 d\int \rho(w,\tau)dw = 2\sigma^2 d$$

(since ρ is a probability density, its integral equals to 1). Thus,

$$\bar{v} = 1 + (2\sigma^2 d)\tau, \quad \bar{E} = 1 + 4\sigma^2(d+1)\tau + 4\sigma^4 d(d+1)\tau^2.$$

Returning to the non-scaled energy and time, we find

$$\bar{v} = v_0 + ct, \quad \bar{E} = (1+1/d)\frac{\bar{v}^2}{2} - \frac{v_0^2}{2d}, \tag{40}$$

where

$$c = 2\sqrt{2}\sigma^2 d. \tag{41}$$

Note that this formula gives us also the standard deviation for the particle speed distribution

$$s_v = \sqrt{2\bar{E} - \bar{v}^2} = \sqrt{\frac{\bar{v}^2 - v_0^2}{d}} < \frac{1}{\sqrt{d}}\bar{v}\,.$$

As $s_v(t) \sim \bar{v}$, it follows that for an N-particle ensemble the statistical average speed $\langle v\rangle_N$ should follow the law:

$$\log(\langle v\rangle_N - v_0) - \log t \approx \log c, \tag{42}$$

with the same accuracy $\sim N^{-1/2}$ for all times. Similar conclusion

$$\log(\langle E\rangle_N - E_0) \approx \begin{cases} 2\log ct + \log((1+1/d)/2), & \text{for } \gtrsim t^*, \\ \log ct + \log(2(1+1/d)v_0), & \text{for } t \lesssim t^*, \end{cases} \tag{43}$$

where $t^* = \frac{2v_0}{c(1+1/d)}$, can be derived for the ensemble average of the energy. We recall that the particle energy undergoes large oscillations during one period of the boundary motion, according to the law $I = EV(t)^{2/d} \approx const$. The above relations are valid only for speed or energy averaged over the period. A practical way to remove the oscillations is to record the value of particle energy once per period, exactly at the same value of the boundary oscillation phase.

In order to verify the stochastic differential equation (37), we recall that one period T of the boundary oscillation corresponds to the small interval $\Delta\tau = T/\sqrt{E_0}$ of the rescaled time. This interval is still much larger than the decay of correlation time ($\sim E_0^{-1}$) for the right-hand side of Eq. (33), so the stochastic description (37) should be valid on this interval. Integrating Eq. (37) on the interval $\Delta\tau$ with the initial condition $w=1$ gives us

$$\Delta w = \sigma^2\frac{2d-1}{2}\Delta\tau + \sigma\mathcal{N}(0,1)\sqrt{\Delta\tau},$$

where $\mathcal{N}(0,1)$ is the standard Gaussian distribution. Returning to the non-rescaled energy and time, we obtain that at sufficiently large initial energies the change in $E^{1/4}$ for one period has the Gaussian distribution

$$\sqrt[4]{E(T)} - \sqrt[4]{\mathrm{E}(0)} = \mathcal{N}(\mu, \mathrm{D}), \tag{44}$$

with the variance, and mean are given, respectively, by

$$D = \sigma^2 T, \quad \mu = \left(d - \frac{1}{2}\right)\frac{\sigma^2 T}{\sqrt[4]{E_0}}.$$

One can rewrite these relations as

$$\mu = \left(d - \frac{1}{2}\right)\frac{D}{\sqrt[4]{E_0}}, \quad D = \frac{cT}{2\sqrt{2d}}, \tag{45}$$

where c is the growth rate of the mean particle speed (see Eqs. (40) and (41)).

In order to test the theory, we conducted a numerical experiment with the "Bunimovich ice-cream" (slanted stadium) billiard. The billiard shape is sketched on Figure 1. In our experiments, the bottom line oscillates periodically with period $T=1$. We have generated 2.5×10^5 random initial conditions for a fixed value of E_0 and evaluated the energy after one period of oscillation of the billiard. The histograms on Figure 1 illustrate the distribution of $E^{1/4} - E_0^{1/4}$ with $E_0 = 10^5$ and 10^9, respectively. It is clearly seen that the histograms shapes are apparently close to the normal distribution and, in spite of the 10^4 fold increase in the initial

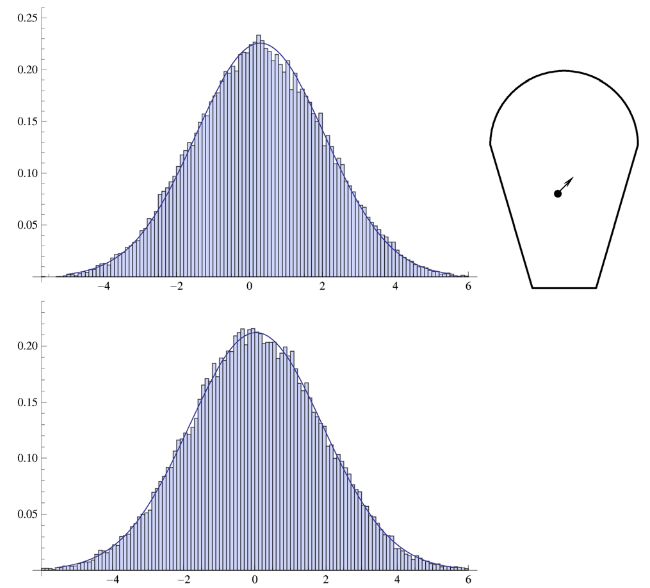

FIG. 1. The histograms illustrate the distribution of the energy for 2.5×10^5 random initial conditions after one period of oscillation of the boundary in the "Bunimovich ice-cream" billiard (right) with the oscillating bottom boundary. The trajectories have fixed initial energy $E_0 = 10^5$ (top) and 10^9 (bottom).

energy, the parameters of the distributions do not vary notably. A more quantitative check can be done in the following way: the first of the equations (45) implies that $d = \frac{1}{2} + \mu\sqrt[4]{E_0}/\mathrm{D}$. In our experiments, this identity is satisfied with relative error of less than 10%.

We also note that numerical experiments with the stadium do not agree with Eq. (44): the standard deviation $\sqrt{D}$ for the numerically obtained histograms does not stabilize as E_0 grows and scales as $\log E_0$ instead. We explain this by the presence of a family of non-hyperbolic periodic orbits in the stadium, which might impede the decay of correlations. In static billiards logarithmic corrections were observed earlier in Refs. 3 and 4. This shows that our main underlying assumption on the independence of the energy increments at consecutive collisions is not always automatically justified for any chaotic billiard and requires a proper amount of hyperbolicity in the billiard phase space.

V. EXPONENTIALLY ACCELERATING PARTICLES: RIGOROUS RESULTS

In this section, we show that a combination of normal hyperbolicity arguments[20,27] with one-dimensional adiabatic theory can be used to prove existence of trajectories with unbounded energy in a d-dimensional breathing billiard. The arguments follow Ref. 24 and rely on the general theory of Ref. 23.

Before coming to the acceleration mechanism, we need to explain why one-dimensional adiabatic invariants appear in a higher-dimensional billiard. Suppose that for every $\tau \in \mathbb{R}$, the billiard domain $D(\tau) \subset \mathbb{R}^d$ contains a non-degenerate periodic orbit $L(\tau)$. Note that $L(\tau)$ is a closed path composed of a finite sequence of straight-line segments. Let $\ell(\tau)$ denote the length of $L(\tau)$. Let $(\boldsymbol{p}, \boldsymbol{q})$ be, respectively, the momentum and position of the billiard particle. We assume that the particle has a unit mass, which in particular implies that $|\boldsymbol{p}| = v = \sqrt{2E}$.

We note that periodic orbits of the flow of the frozen billiard form two-parameter families: one parameter is the frozen time τ and the second one is the energy E, since the particle can go around the polygon L at different speeds. We denote this collection of periodic orbits by $L(E, \tau)$. If we had a smooth flow instead of the billiard flow, we would notice that at each fixed τ this collection of orbits is a smooth symplectic invariant manifold, the restriction of the frozen system to this manifold would be an integrable Hamiltonian flow, so we would conclude that the action is an adiabatic invariant for the restriction of the full system onto the invariant manifold (due to the normal hyperbolicity, the invariant manifold persists when the parameter τ is allowed to change slowly[23]). Although the billiard flow is not smooth, this fact still holds true, as we show below (it is a first step in construction of exponentially accelerating orbits).

The standard definition of the action of a periodic orbit gives

$$J = \oint_{L(E,\tau)} \boldsymbol{p}\, d\boldsymbol{q} = \sqrt{2E}\ell(\tau)\,. \tag{46}$$

The action depends only on the energy E of the particle and on the parameter τ. This definition coincides with the adiabatic invariant for a one-dimensional billiard of volume $V(\tau) = \ell(\tau)$ defined in Sec. III.

The period of the frozen orbit $L(E, \tau)$ can be expressed in terms of the action

$$T(E,\tau) = \frac{\ell(\tau)}{v} = \frac{\ell(\tau)}{\sqrt{2E}} = \frac{\partial J}{\partial E}\,. \tag{47}$$

Suppose the periodic orbit $L(E, \tau)$ hits the boundary of the frozen billiard at a sequence of points $z_j(\tau)$, $j = 1, \ldots, m$. We let $z_0 = z_m$ and define $\boldsymbol{r}_j = z_j - z_{j-1}$, $j = 1, \ldots, m$. The vectors $\boldsymbol{r}_j$ represent the straight-line segments which form the periodic orbit. Let $\boldsymbol{n}_j$ denote the internal unit normal to the boundary at z_j. Then the normal velocity of the boundary at the jth collision point is given by $u_j := \dot{z}_j \cdot \boldsymbol{n}_j$, where the dot stands for the derivative with respect to τ.

Equation (46) implies $J = v\sum_{j=1}^{m} |\boldsymbol{r}_j|$, so

$$\frac{\partial J}{\partial \tau} = v\sum_{j=1}^{m} \frac{\boldsymbol{r}_j \cdot \dot{\boldsymbol{r}}_j}{|\boldsymbol{r}_j|} = v\sum_{j=1}^{m} \frac{\boldsymbol{r}_j \cdot (\dot{z}_j - \dot{z}_{j-1})}{|\boldsymbol{r}_j|}.$$

In this sum each of $\dot{z}_j$ appears twice, and taking into account the periodicity of the orbit, we can reorder the sum

$$\frac{\partial J}{\partial \tau} = v\sum_{j=1}^{m} \left(\frac{\boldsymbol{r}_j}{|\boldsymbol{r}_j|} - \frac{\boldsymbol{r}_{j+1}}{|\boldsymbol{r}_{j+1}|}\right) \cdot \dot{z}_j\,.$$

In the frozen billiard, the angle of incidence is equal to the angle of reflection; therefore, the tangent component of the difference cancels while the normal component doubles

$$\frac{\partial J}{\partial \tau} = 2v\sum_{j=1}^{m} \frac{\boldsymbol{r}_j \cdot \boldsymbol{n}_j}{|\boldsymbol{r}_j|}\boldsymbol{n}_j \cdot \dot{z}_j = 2v\sum_{j=1}^{m} u_j \frac{\boldsymbol{r}_j \cdot \boldsymbol{n}_j}{|\boldsymbol{r}_j|}\,. \tag{48}$$

Up to this moment we have considered the frozen billiard only.

Now suppose that a trajectory of the non-autonomous billiard has initial energy E and its initial conditions are sufficiently close to $L(E, \tau)$. This trajectory stays near a periodic orbit of the frozen billiard for some time. In particular, it hits the boundary near the points z_j, $j = 1,\dots,m$. After making one roundtrip near $L(E, \tau)$, the energy of the particle changes

$$\Delta E := E(\tau + T) - E(\tau) = \sum_{j=1}^{m} \Delta E_j,$$

where ΔE_j is the change of the energy at the collision near z_j. According to the elastic collision law,

$$\Delta E_j = 2(-u_j \boldsymbol{v}_j \cdot \boldsymbol{n}_j + u_j^2)\,, \tag{49}$$

where $\boldsymbol{v}_j$ is the velocity of the particle before the collision. If the boundary velocity is much smaller than v, and the particle moves very close to the periodic orbit, we can neglect u_j and replace v_j by $v\boldsymbol{r}_j$. We get

$$\Delta E \approx -2v\sum_{j=1}^{m} u_j \frac{\boldsymbol{r}_j \cdot \boldsymbol{n}_j}{|\boldsymbol{r}_j|} = -\frac{\partial J}{\partial \tau}, \tag{50}$$

where we used Eq. (48) to get the last equality.

The time of this roundtrip is close to the period of the frozen periodic orbit (the length of the trajectories and the energies are approximately the same). Then Eqs. (47) and (50) imply that in the breathing billiard after one roundtrip near the frozen periodic orbit

$$\Delta \tau \approx \frac{\partial J}{\partial E} \text{ and } \Delta E \approx -\frac{\partial J}{\partial \tau}\,.$$

Consequently, the energy of the particle closely follows a solution of the differential equation

$$\dot{h} = -\frac{\partial J}{\partial t} \Big/ \frac{\partial J}{\partial E} = -\frac{1}{T}\frac{\partial J}{\partial t} = -2h\frac{\dot{\ell}}{\ell}\,. \tag{51}$$

We see that the time derivative of the energy is not small. On the other hand, the full time derivative

$$\dot{J}(\tau, h(\tau)) = \frac{\partial J}{\partial \tau} + \frac{\partial J}{\partial h}\dot{h}$$

vanishes along solutions of the differential equations (51) due to Eq. (47). Therefore, if the particle stays close to $L(\tau)$, the action J is an adiabatic invariant, i.e., the energy of the particle changes in such a way that $J(h, \tau)$ remains approximately constant during a long period of time.

Equation (51) can be easily integrated

$$h(\tau) = h(0)\,\frac{\ell^2(0)}{\ell^2(\tau)}\,. \tag{52}$$

We see that the energy stays bounded unless the length of the periodic orbit vanishes. It is convenient to let $w = -2\dot{\ell}/\ell$, then Eq. (51) takes the form

$$h^{-1}\dot{h} = w(\tau)\,. \tag{53}$$

We note that if the trajectories on $L(E, \tau)$ are hyperbolic then the theory of normal hyperbolicity[20] implies that there are trajectories of the non-autonomous billiard which stay near $L(E, \tau)$ as long as their energy does not drop below certain lower bound.

It is seen from Eq. (52) that a trajectory, which stays near L, does not accelerate. The possibility of accelerating trajectories depends on the dynamics of the frozen billiard. Suppose the frozen billiard has chaotic dynamics at least in a part of the phase space. In other words, the frozen billiard has a Smale horseshoe. This can be described by the existence of a pair of hyperbolic periodic orbits, L_a and L_b, connected by a pair of transversal heteroclinics (there are infinitely many such pairs within the chaotic set, we just choose any two of them). The dynamics near the horseshoe can be described using symbolic dynamics: for any sequence composed of symbols a and b, there is a trajectory which switches between L_a and L_b following the order prescribed by the sequence.

In Ref. 23, we used normal hyperbolicity arguments in the spirit of Refs. 53 and 54 in order to show that this property is inherited by slowly changing systems, which includes as a special case the billiard map under quite general assumptions on its boundary. We proved that in the presence of the Smale horseshoe, the slow non-autonomous billiard with a fast particle inside has trajectories which switch between two small neighborhoods of L_a and L_b in an arbitrary, prescribed in advance order. This freedom can be used to achieve an optimal strategy for the acceleration. Let us study the behavior of a trajectory which stays near L_a if $w_a(t) > w_b(t)$ and stays near L_b otherwise (we assume that the changes of the billiard shape are not very large, so the periodic orbits L_a and L_b persist in the frozen billiard for all t). The energy of the particle changes in time approximately as a solution of the differential equation (53) with $w = w_a$ or $w = w_b$ depending on time t

$$h^{-1}\dot{h} = W(t) := \max\{\, w_a(t), w_b(t)\,\}\,.$$

This equation has an obvious solution

$$h(t) = h(0)\exp\int_0^t W(t)\,dt\,. \tag{54}$$

We note that under very general assumptions, these solutions are not bounded. For example, if the oscillation of the boundary is periodic, then ℓ_a and ℓ_b are periodic functions of time. Then, $w_{a,b}$ are derivatives of periodic functions and therefore have zero mean $\bar{w}_{a,b} = 0$. Since W is the maximum of two periodic functions with zero mean, its mean $\bar{W}$ is positive provided w_a and w_b do not coincide over the entire period or, equivalently, provided $\ell_a(t)/\ell_b(t)$ is not constant. Then, we conclude from Eq. (54) that the energy of the particle oscillates around the exponentially growing function

$$h(t) \approx h(0) \exp \bar{W} t \,. \tag{55}$$

In the arguments above, the periodicity assumption substantially simplified the analysis. In fact, the periodicity is not an essential part of the argument: the only important things are that the billiard shape is changing (i.e., ℓ_a/ℓ_b varies with time), that the boundary motion is slow, and that the phase space of the frozen billiard retains the Smale horseshoe for all times. Then, both periodic and non-periodic cases can be treated by the proposed method, see the corresponding theory in Refs. 7 and 23.

A simple example illustrating the above described acceleration mechanism by switching between two saddle periodic orbits of the frozen billiard is given by the Sinai billiard (a rectangle minus a disc[56]). We can also consider a slightly more general billiard obtained after replacing the disc by an ellipse (see Figure 2). The billiard domain is given by

$$\left\{ |x| \le W, |y| \le 1, \left(\frac{x}{R}\right)^2 + \left(\frac{y - \psi(t)}{Q}\right)^2 \ge 1 \right\},$$

where $0 < R < W$, $0 < Q < 1$, and $\max_t |\psi(t)| + Q < 1$. The intersection of the symmetry axis $x = 0$ with the billiard table consists, at every given t, of two disjoined segments: $[-1, \psi(t) - Q]$ and $[\psi(t) + Q, 1]$, which correspond to two periodic orbits of the frozen billiard. Since the elliptic boundary is dispersive, these periodic orbits are hyperbolic, and the existence of transverse heteroclinics that connect them follows from Ref. 56. While the results above imply the existence of exponentially accelerating orbits, numerical experiments show only a slow energy growth (close to be quadratic in time) for randomly chosen initial conditions. However, in the limit $Q = 0$ where the ellipse is flattened to a straight-line segment and the periodic orbits become non-hyperbolic, this behavior changes drastically. We will provide more details in Sec. VI C.

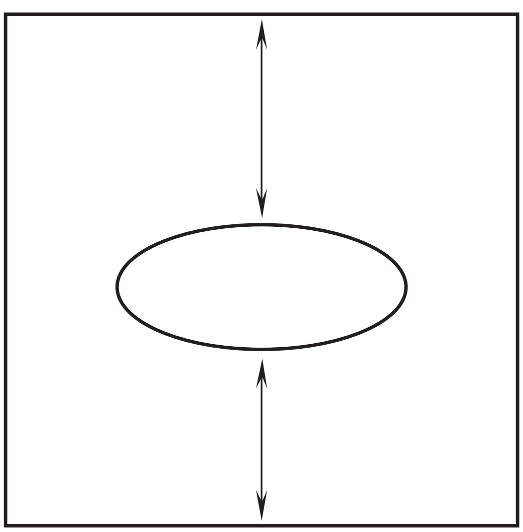

FIG. 2. Two periodic orbits in the elliptic Sinai billiard.

VI. EXPONENTIAL ACCELERATORS: MODELS AND MODELS OF MODELS

The construction described in Sec. V leads to a rigorous proof for the existence of trajectories along which the energy growth is exponentially fast in time. If the billiard is ergodic, this behavior must be exceptional as in this case the majority of trajectories should follow the predictions of the Anosov-Kasuga averaging theory and therefore accelerate much slower (see Sec. IV). The situation may change drastically if the billiard is not ergodic as *many* trajectories are forced to switch randomly between different adiabatic regimes while the billiard shape changes. It was recently noticed in Ref. 22 that this phenomenon can lead to exponential acceleration for the majority of particles.

Here, we describe this mechanism first for one-dimensional billiards, and then we build a general theory which provides explicit estimates for the energy growth rates. The acceleration is easily observable in numerical experiments and the rates are in good agreement with theoretical predictions. Nevertheless, a mathematically rigorous proof for the validity of the mechanism is still missing.

A. One-dimensional billiard with a separating wall

In the Fermi-Ulam model, the particle bounces between two moving walls. Suppose that the walls move periodically with period $T = 1$. Let us slightly modify the model by inserting at $t = t_{\text{in}} < T$ a separating wall. The separating wall also moves until it is deleted at $t = t_{\text{rm}} < T$. Then the side walls return to their initial position at $t = T$ and the process repeats periodically. The billiard cycle is sketched in Figure 3.

Let us fix the notation. Let $V(t)$ denote the length of the interval and $V_1(t)$ denote the length of the interval on the left of the temporary middle wall. Of course, $V_1(t)$ is meaningful for $t (\mathrm{mod}\, T) \in [t_{\text{in}}, t_{\text{rm}}]$ only. We define

$$p = \frac{V_1(t_{\text{in}})}{V(t_{\text{in}})} \quad \text{and} \quad \alpha = \frac{V_1(t_{\text{rm}})}{V(t_{\text{rm}})}$$

to describe the position of the middle wall at the moment of insertion and removal, respectively.

In this model, the particle is trapped either on the left or on the right from the separating wall. We assume this to be a

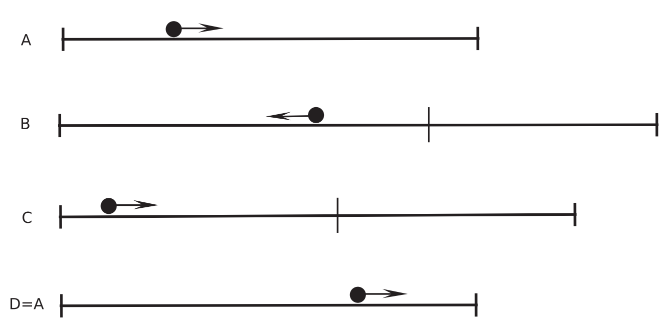

FIG. 3. A sketch of the billiard cycle for the one-dimensional billiard with a separation wall.

random event. The corresponding probabilities are proportional to the length of the intervals, i.e., are equal to p and $1-p$, respectively. We also suppose that the particle moves according to the adiabatic law. So if it starts with velocity $v(0)$ at $t=0$, then $v(t_{\rm in}) = v(0)V(0)/V(t_{\rm in})$. If the particle is trapped on the left of the moving wall, then $v(t_{\rm rm}) = v(t_{\rm in})V_1(t_{\rm in})/V_1(t_{\rm rm})$ and $v(T) = v(t_{\rm rm})V(t_{\rm rm})/V(T)$. So we get

$$v(T) = \frac{V(t_{\rm rm})V_1(t_{\rm in})}{V(t_{\rm in})V_1(t_{\rm rm})}\, v(0) = \frac{p}{\alpha}\, v(0)\,.$$

In a similar way, we use the adiabatic theory to derive the particle velocity for the particle trapped on the right from the moving wall. We conclude that

$$\frac{v(T)}{v(0)} = \begin{cases} \dfrac{p}{\alpha} & \text{with probability } p \\ \dfrac{1-p}{1-\alpha} & \text{with probability } 1-p. \end{cases}$$

Then, we can easily find the expectation

$$\mathbb{E}\left(\frac{v(T)}{v(0)}\right) = \frac{p^2}{\alpha} + \frac{(1-p)^2}{1-\alpha} \geq 1\,.$$

Since a quadratic function has only one minimum, the equality is strict provided $p \neq \alpha$. This model suggests that after one cycle, the particle velocity is multiplied by a factor larger than one on average. Assuming consecutive cycle to be uncorrelated, we conclude that on average the energy is to grow exponentially with the number of billiard cycles. We analyze the energy growth for a generalization of this model in Sec. VI B.

B. A model for the energy growth rates

The example from Sec. VI A suggests a probabilistic model for a particle in a billiard which periodically separates into two ergodic components. We postpone discussion of a physical realization for this process and consider the following discrete random process.

We start with some $E_0 > 0$. On each step of the process, this variable is multiplied either by a constant with probability p or by another constant with probability $1-p$. The values of the constants are taken to model the adiabatic change of the energy over a full cycle in a d-dimensional billiard. More precisely, we assume that at each step, we have the same probability distribution

$$\frac{E_1}{E_0} = \begin{cases} \left(\dfrac{p}{\alpha}\right)^{2/d} & \text{with probability } p \\ \left(\dfrac{1-p}{1-\alpha}\right)^{2/d} & \text{with probability } 1-p. \end{cases} \tag{56}$$

If $d=1$, it coincides with the model from Sec. VI A, except that we are now monitoring the energy instead of the speed.

Let E_n denote the energy achieved after n steps and assume that the steps are independent. It is natural to look at this problem in the logarithmic scale. Indeed, we see that $\log E_n$ follows an asymmetric one-dimensional random walk (if $p \neq \alpha$) with $\log E_n - \log E_{n-1}$ having probability distributions

$$\log\frac{E_1}{E_0} = \begin{cases} \dfrac{2}{d}\log\dfrac{p}{\alpha} & \text{with probability } p \\ \dfrac{2}{d}\log\dfrac{1-p}{1-\alpha} & \text{with probability } 1-p. \end{cases} \tag{57}$$

It is easy to see that the left and right displacements are not equal. The random walk is biased to the right for all $p \neq \alpha$. Indeed, let m_1 be the expectation of $\log\frac{E_1}{E_0}$, then

$$m_1 := \mathbb{E}\left[\log\frac{E_1}{E_0}\right] = \frac{2}{d}\left(p\log\frac{p}{\alpha} + (1-p)\log\frac{1-p}{1-\alpha}\right). \tag{58}$$

Differentiating with respect to p, we easily find that m_1 has a unique minimum at $p=\alpha$. Moreover, m_1 vanishes if $p=\alpha$. Consequently, if $p \neq \alpha$, the expectation $m_1 > 0$ and on average $\log E_n$ grows linearly in n.

Thus, the energy itself is an exponent of a biased random walk and therefore grows exponentially. Indeed, since

$$\frac{1}{n}\log\frac{E_n}{E_0} = \frac{1}{n}\sum_{k=1}^{n}\log\frac{E_k}{E_{k-1}}\,,$$

the law of large numbers implies that almost surely the limit $n \to \infty$ exists and does not depend on a realization of the random process. So, we can define *a single-orbit growth rate* by

$$r_{\rm so} = \lim_{n\to\infty}\frac{1}{n}\log\frac{E_n}{E_0} = m_1\,. \tag{59}$$

Then almost surely

$$\lim_{n\to\infty} e^{-m_1 n}E_n = E_0\,.$$

When a billiard is described by this energy growth model, most of its trajectories should follow this law. In principle, the validity of this model can be checked by evaluating $\frac{1}{n}\log\frac{E_n}{E_0}$ for a small selection of billiard trajectories. From the computational point of view this idea is typically not practical. Indeed, the standard deviation for $\log\frac{E_1}{E_0}$ is given by

$$\sigma_1 = \frac{2}{d}\sqrt{p(1-p)}\left|\log\frac{p(1-\alpha)}{(1-p)\alpha}\right|. \tag{60}$$

A typical dependence of m_1 and σ_1 on p for a fixed α is shown on Figure 4. It is clearly seen that for the majority of parameters $\sigma_1 > m_1$, and the ratio m_1/σ_1 diverges when $p \to \alpha$.

The standard deviation of $\frac{1}{n}\log\frac{E_n}{E_0}$ is $\sigma_1/\sqrt{n}$, and a relatively large number of steps is required to estimate the mean reliably. At the same time, the particle energy grows exponentially and therefore the particle moves faster and faster. Then the number of particle collisions with the boundary per one step grows exponentially (each step corresponds to a full cycle of the billiard shape change). This makes even moderate

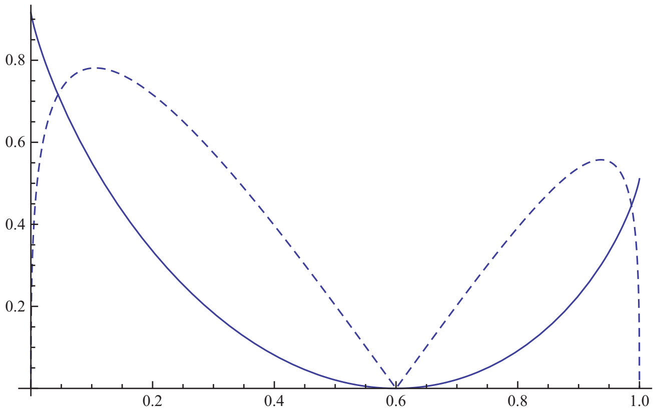

FIG. 4. The expected value (58) (solid line) and the standard deviation (60) (dashed line) for $\ln\frac{E_1}{E_0}$, as a function of p for $\alpha = 0.6$.

values of n difficult to reach, as doubling n roughly squares the number of collisions with boundary and the time required for numerical simulations grows quickly.

It is interesting to note that in our model, the average energy growth over a large ensemble of initial conditions placed on a common energy level substantially differs from the energy growth of a typical particle. Let us consider an ensemble of trajectories for a relatively small number of steps n. We note that the way the average is evaluated is of great importance. Indeed, the independence assumptions regarding the process steps implies that

$$\mathbb{E}\left[\frac{E_n}{E_0}\right] = \mathbb{E}\left[\prod_{i=1}^{n}\frac{E_i}{E_{i-1}}\right] = \prod_{i=1}^{n}\mathbb{E}\left[\frac{E_i}{E_{i-1}}\right] = \mathbb{E}^n\left[\frac{E_1}{E_0}\right] = e^{n\log q_1},$$

where q_1 is the expectation of $\frac{E_1}{E_0}$. An elementary computation gives expectation and standard deviation for $\frac{E_1}{E_0}$

$$q_1 := \mathbb{E}\left(\frac{E_1}{E_0}\right) = \frac{p^{1+2/d}}{\alpha^{2/d}} + \frac{(1-p)^{1+2/d}}{(1-\alpha)^{2/d}}, \tag{61}$$

$$s_1 := \sqrt{\mathbb{E}\left(\frac{E_1^2}{E_0^2}\right) - \mathbb{E}^2\left(\frac{E_1}{E_0}\right)} = \sqrt{(1-p)p}\left|\frac{p^{2/d}}{\alpha^{2/d}} - \frac{(1-p)^{2/d}}{(1-\alpha)^{2/d}}\right|. \tag{62}$$

We define the *ensemble growth rate* by

$$r_{\text{ens}} = \frac{1}{n}\log\mathbb{E}\left[\frac{E_n}{E_0}\right] = \log q_1.$$

By construction, the right-hand side is independent of n. This rate is observed when an average energy is evaluated over a large ensemble of initial conditions after a fixed number of cycles. Taking into account the concavity of the log function, we conclude that

$$\log q_1 \geq m_1 \geq 0.$$

It can be checked straightforwardly that both equalities are achieved only when $p = \alpha$. Therefore the ensemble growth rate is larger than the individual growth rate. It is remarkable as it implies that on average the energy grows much faster than the energy of a typical trajectory. Apparently this phenomenon is due to relatively rare realisations with higher than typical energy growth.

It is not easy to numerically observe the ensemble rate even for moderate values of n because[51] the variation of E_n grows at a faster exponential rate than r_{ens} with the increase of n, and therefore large ensembles are required. Indeed, the standard deviation for the energy gain after n cycles is

$$s_n = \sqrt{\mathbb{E}\left[\frac{E_n^2}{E_0^2}\right] - \mathbb{E}^2\left[\frac{E_n}{E_0}\right]} = \sqrt{\mathbb{E}^n\left[\frac{E_1^2}{E_0^2}\right] - \mathbb{E}^{2n}\left[\frac{E_1}{E_0}\right]}$$
$$= e^{n\log q_1}\sqrt{\left(1+\frac{s_1^2}{q_1^2}\right)^n - 1},$$

where q_1, s_1 are the mean and standard deviation for E_1/E_0 given by Eqs. (61) and (62). For an ensemble of K particles, the standard deviation of the statistical average from the theoretical mean value is $s_n/\sqrt{K}$. Hence, for a fixed K, the ensemble rate r_{ens} is observed only for a finite number of the boundary oscillation cycles, namely, for

$$n \ll \frac{\log K}{\log\left[1 + (s_1/q_1)^2\right]}.$$

If we take the average of E_n/E_0 over an ensemble and change the number n of cycles, then for a fixed ensemble, we expect to observe first the ensemble growth rate when n is small, then some oscillations associated with the growth of the standard deviation, and later, when n is sufficiently large, the smaller single-orbit rate.

Again, the last transition is easy to see in a numerical simulation of the stochastic process but it is very difficult to observe directly in numerical simulations of a billiard. Indeed, in an exponential accelerator, the particle velocity grows exponentially with n and therefore the number of collision per boundary cycle also grows exponentially; therefore, the computation time also grows exponentially in n.

The central limit theorem implies that $\frac{1}{n}\log E_n/E_0$ is distributed approximately normally according to $\mathcal{N}(m_1, \frac{\sigma_1}{\sqrt{n}})$, where the parameters are defined by Eqs. (58) and (60). Then, the average over an ensemble of K random initial conditions could be determined with the standard error of $\frac{\sigma_1}{\sqrt{nK}}$. We checked that this behaviour is reproduced in numerical experiments with billiards of various shapes and estimated the single-orbit growth rate (see, e.g., Fig. 9). We note that taking an average over an ensemble of initial conditions improves accuracy and allows us to estimate r_{so} without tracing particles with extremely high velocities.

Special care should be taken in interpretation of the energy growth rates. For example, if we suppose that $\log E_n$ undergoes an asymmetric random walk with probabilities different from the ones used in this section, we can easily produce examples where $r_{\text{so}} < 0 < r_{\text{ens}}$. Then while most particles lose energy exponentially fast, the average energy of an ensemble grows exponentially fast. Thus, the result that $m_1 > 0$ for all $p \neq \alpha$ is especially significant as it means

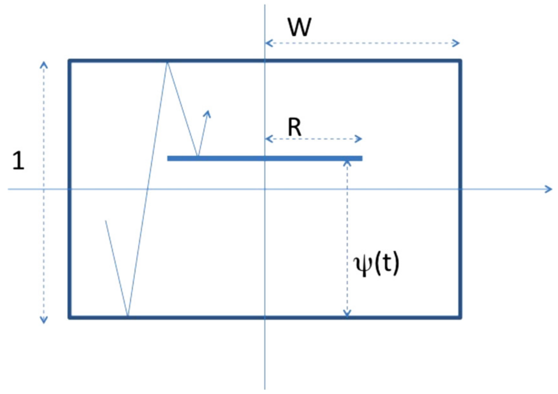

FIG. 5. A rectangular billiard table with an oscillating rod inside.

not only that the ensemble energy grows but also that for most individual particles energy grows.

Applying the probabilistic model described in this section to deterministic systems like time-dependent billiards does not have a rigorous mathematical justification yet. Nevertheless, it provides quantitative predictions for the energy growth rates, which are in good agreement with results of our numerical experiments.

C. Rectangular billiard with an oscillating rod

The first example of a periodically oscillating billiard that produces an exponential growth of energy for majority of orbits was proposed in Ref. 51. The billiard can be viewed as a two-dimensional analog of the one-dimensional billiard from Sec. VI A. The corresponding billiard table is sketched on Figure 5.

Historically, it was obtained from the oscillating elliptic Sinai billiard (compare with Figure 2) in the limit of vanishing minor semi-axis.

More precisely, we consider an oscillating horizontal rod suspended inside a rectangular billiard of width $2W$ and height 1. The rod is symmetric with respect to the y-axis, it has length $2R$, $0 < R < W$, and $y = \psi(t)$ $(0 < |\psi(t)| < 1)$ describes the height at which the rod is suspended.

For any frozen t, this billiard is "pseudo-integrable:" it has an integral of motion independent from the energy since both horizontal speed v_x and the vertical speed v_y are preserved separately. However, the dynamics of the billiard is chaotic for almost all fixed v_x, v_y (see Refs. 26, 43, and 60 for a more detailed discussion). We note this billiard is closely related to a geodesic flow on a genus-2 closed surface endowed by a flat metric with two singular points at the end points of the rod (as well as to interval exchange mappings).

Let us assume that the particle starts at time $t = 0$ near the left wall. The particle travels up to the right wall, reflects keeping the horizontal speed the same, and returns to the left wall again. Let $T = 2W/v_x$ be the horizontal travel time. The vertical component of the velocity after one round trip from the left wall can be modelled by the distribution from the previous section with $d = 1$, $p = \psi(t_{\rm in})$, and $\alpha = \psi(t_{\rm rm})$, where $t_{\rm in} = (W - R)/v_x$ and $t_{\rm rm} = (W + R)/v_x$. Therefore, the particle will accelerate on average provided $p \neq \alpha$.

We can model the acceleration due to multiple cycles by a sequence of independent random steps with different probability distributions (56) by setting the distribution parameters to be $p_k = \psi(t_{\rm in} + Tk)$ and $\alpha_k = \psi(t_{\rm rm} + Tk)$. Then the central limit theorem implies that $\frac{1}{n}\log\frac{E_n}{E_0}$ is distributed approximately normally $\mathcal{N}(m, \sigma_n)$, where $\sigma_n = O(n^{-1/2})$ and

$$m = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} m_k,$$

where m_k are given by Eq. (58) with p replaced by p_k and α by α_k.

The value of the single-orbit energy growth rate m depends on the properties of the function ψ and the period of the particle's horizontal oscillation. If ψ is periodic in time: $\psi(t + T_0) = \psi(t)$ for some $T_0 > 0$ (the minimal period of ψ), then m_k can be considered as a function evaluated at a point of a trajectory of the map $t \mapsto t + T \pmod{T_0}$. If T/T_0 is irrational, the map is ergodic and we can replace the average over the trajectory by the average over the circle. Applying this argument to Eq. (58) with $p = \psi(t + t_{\rm in})$ and $\alpha = \psi(t + t_{\rm rm})$, we get

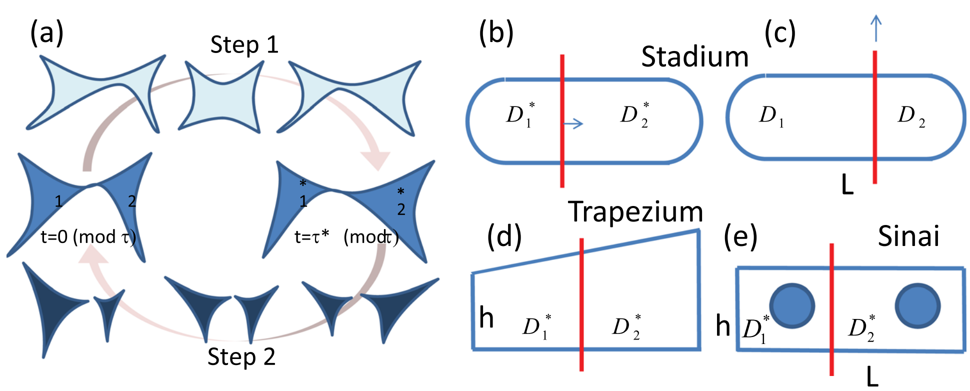

FIG. 6. The robust accelerator. General scheme (a); Divided stadium at the beginning of the first (b) and second (c) stages of the bar motion; (d) Trapezium; (e) "Double" Sinai billiard. Here $L = 2h = 2$; the radius of the discs in (e) is $\frac{1}{6}$, the distance between the centers is $\frac{4}{3}$; the inclination angle of the trapezium is $10°$. At the compression stage, the bar displacement is 0.2 for trapezium, 0.28 for stadium, 0.18 for Sinai billiard, so the growth rate stays the same: $\mathcal{R} \approx 0.08$. Reprinted with permission from V. Gelfreich, V. Rom-Kedar, K. Shah, and D. Turaev, Phys. Rev. Lett. **106**, 074101 (2011). Copyright © 2011 The American Physical Society.

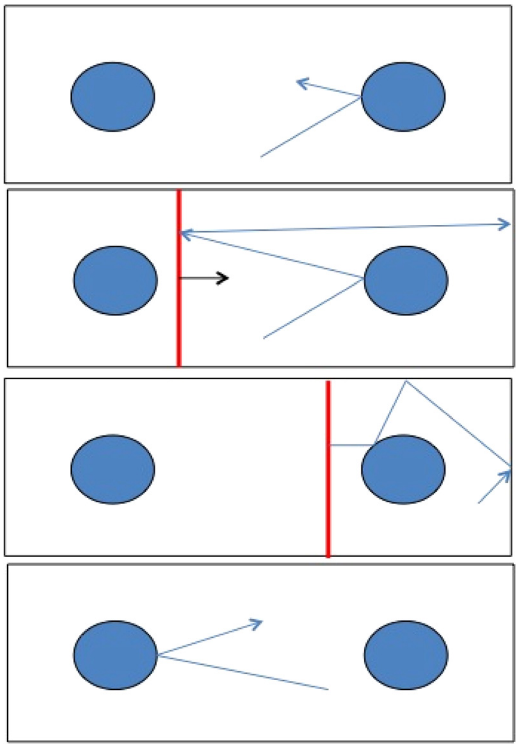

FIG. 7. A sketch of the billiard cycle.

$$m = \frac{1}{T_0}\int_0^{T_0} (\psi(t)\log\frac{\psi(t)}{\psi(t_0+t)} + (1-\psi(t))\log\frac{1-\psi(t)}{1-\psi(t_0+t)})dt, \qquad (63)$$

where $t_0 = t_{rm} - t_{in} = 2R/v_x$. The function under the integral is non-negative and it vanishes only if $\psi(t) = \psi(t+t_0)$. Therefore, m is positive unless ψ is both t_0 and T_0 periodic. The exceptional situation may arise only if ψ is constant or if t_0/T_0 is an integer, which happens when $v_x = 2R/kT_0$ for some $k \geq 1$. Thus, *if ψ is a non-constant T_0-periodic function and v_x is such that T/T_0 is irrational and t_0/T_0 is not an integer, then $m > 0$ and the particle energy increases almost surely exponentially in time.*

The particle may also accelerate in the resonant cases, for example, when T is an integer multiple of T_0. Indeed, in this case, all cycles are described by the same probability distributions: $p_k = p_0$, $\alpha_k = \alpha_0$ for all k, as the rod returns to its initial position each time when the particle returns to the right wall. Therefore, the energy of the particle increases like in the model of Sec. VI A.

Since both T and t_0 depend on the particle's horizontal velocity, the particle acceleration is sensitive to its initial conditions. In particular, for a fixed initial energy, the single-orbit energy growth rate depends on the initial direction of the particle motion. It is easy to see[51] that the ensemble rate may also vary depending on the initial distribution in v_x.

The model described in this section is non-robust in several aspects. First, we have seen that the single orbit rate of the exponential acceleration depends sensitively on v_x. We note that this instability is not too important for evaluation of ensemble rates because the full measure of initial conditions leads to the non-resonant acceleration described by the rate (63) which depends on v_x regularly.

More importantly, the model is not stable with respect to small changes in the billiard shape—if the boundaries are not perfectly horizontal or vertical, the horizontal component of the particle velocity v_x is no longer constant and the model described in this section becomes obviously inapplicable.

D. Robust exponential accelerators

In this section, we describe a class of accelerators that provide an exponential in time energy growth as proposed in Ref. 22. These billiards are constructed in such a way that the effect is robust with respect to changes in the form of the billiard. Thus, they provide an effective mechanism of the energy transfer from the slow (the moving wall) to the fast (the particle inside the billiard) components of the system. Note also that since the energy growth is exponential, it is sustained even in the presence of a small linear dissipation.

These billiards correspond to a physical realisation of the stochastic model described in Sec. VI B. The machine works as follows. We take a d-dimensional domain $D(t)$ and deform it slowly. The shape-deformation cycle of a

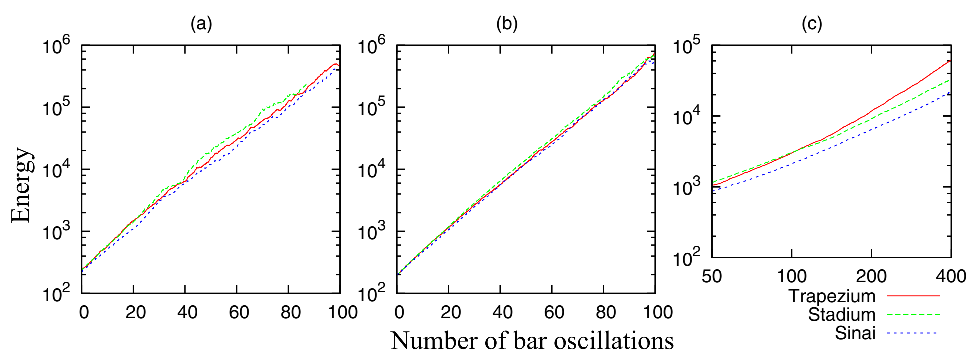

FIG. 8. Ensemble energy growth for the robust accelerators. The energy is shown in the logarithmic scale. (a) Exponential growth for the three geometries of Figs. 6(b)–6(e), where the bar is introduced and removed abruptly. (b) The same as (a) with slow and smooth bar motion (the bar velocity is continuous, piecewise linear in time). (c) When the bar is 90% of the billiard's height and does not divide the billiard, the energy growth is not exponential: $E \sim t^{1.65\div1.85}$. Reprinted with permission from V. Gelfreich, V. Rom-Kedar, K. Shah, and D. Turaev, Phys. Rev. Lett. **106**, 074101 (2011). Copyright © 2011 The American Physical Society.

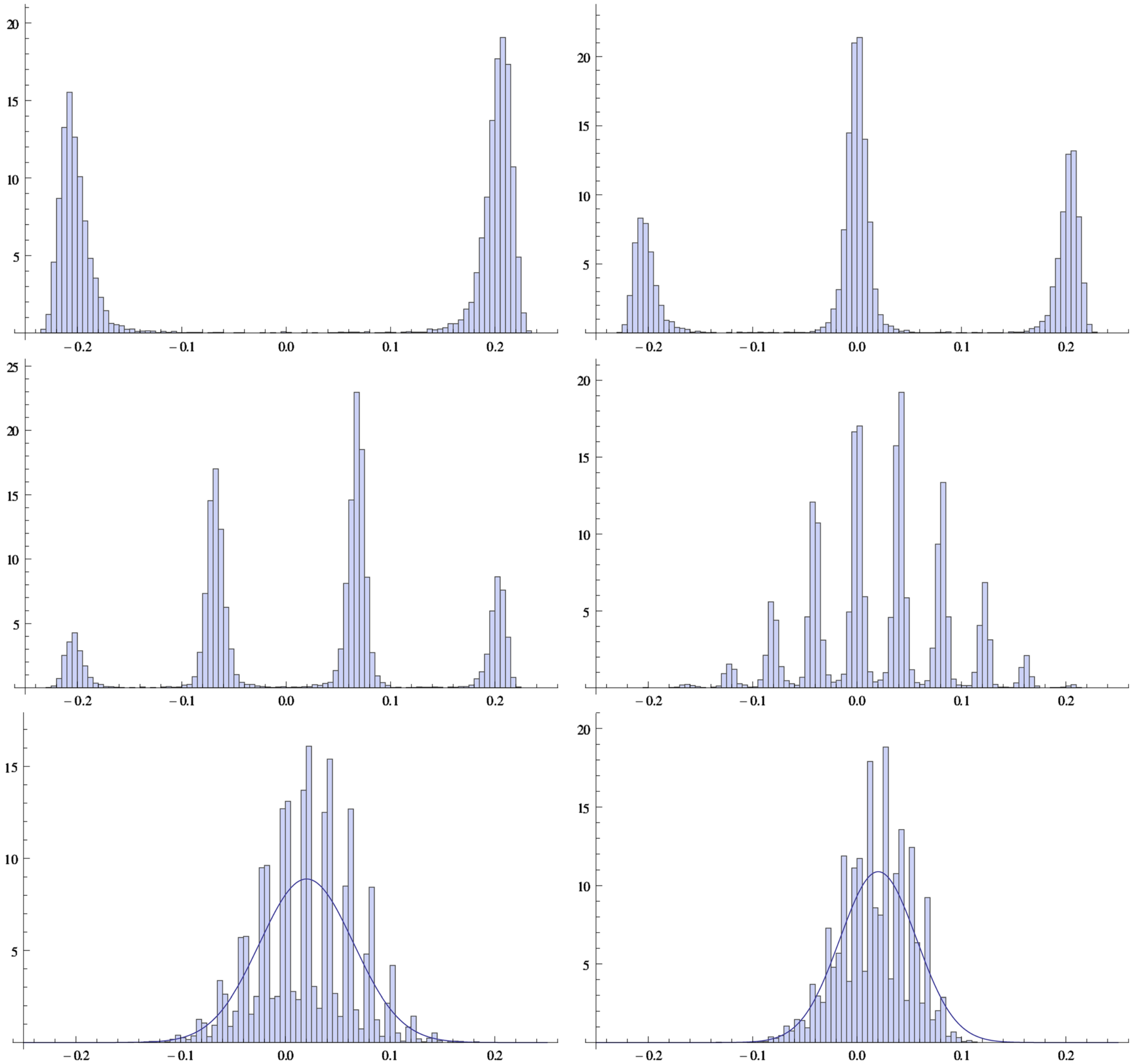

FIG. 9. Histogram of the change in log E per cycle in the stadium with a periodically inserted moving separator: (a) $n=1$, (b) $n=2$, (c) $n=3$, (d) $n=10$, (e) $n=20$. The Gaussian corresponds to the theoretical prediction $\mathcal{N}(m_1, s_1/\sqrt{n})$. We used $K=10^4$ randomly chosen initial conditions for the histograms, $E_0 = 1.28\times 10^6$.

period T consists of 2 steps (see Fig. 6(a); specific realizations we used in the numerical simulations are shown in Figs. 6(b)–6(e)). At step 1, for $t\in(0, t_{\rm in})\,(mod\, T)$, the domain is connected, and at each fixed moment of t, the corresponding billiard dynamical system is ergodic and mixing with respect to the standard Liouville measure. At $t=t_{\rm in}$, the billiard domain separates into two connected components, $\mathcal{D}_1^*$ and $\mathcal{D}_2^*$. At step 2, for $t\in(t_{\rm in}, t_{\rm rm})\,(mod\,\tau)$, the two components change their shape while remaining disjoint, and at each fixed t, each component defines an ergodic and mixing billiard dynamical system. At $t=t_{\rm rm}$, the two components reconnect again and at $t=T$ return to the same shape as at $t=0$. Then, the process repeats. An example of this cycle is shown on Fig. 7.

Let $V(t)$ be the volume of $\mathcal{D}(t)$ at time t, and $V_{1,2}(t)$ be the volumes of the two components at step 2. During step 2, we have $V(t)=V_1(t)+V_2(t)$. Define

$$p=\frac{V_1(t_{\rm in})}{V(t_{\rm in})}\quad\text{and}\quad \alpha=\frac{V_1(t_{\rm rm})}{V(t_{\rm rm})}\,.$$

If we assume that the particle moves according to the adiabatic laws and that the probability to be captured in $\mathcal{D}_1^*$ is proportional to its volume, then the evolution of the particle energy is described by the distribution (57). Indeed, if the particle moves according to the adiabatic theory and is trapped in $\mathcal{D}_1^*$, then

$$\log\frac{E(T)}{E(0)} = \log\frac{E(T)}{E(t_{\rm rm})}\frac{E(t_{\rm rm})}{E(t_{\rm in})}\frac{E(t_{\rm in})}{E(0)}$$
$$= -\frac{2}{d}\log\frac{V(T)}{V(t_{\rm rm})}\frac{V_1(t_{\rm rm})}{V_1(t_{\rm in})}\frac{V(t_{\rm in})}{V(0)} = \frac{2}{d}\log\frac{p}{\alpha}\,.$$

In a similar way, if the particle is trapped in D_2^*, then

$$\log\frac{E(T)}{E(0)} = \frac{2}{d}\log\frac{1-p}{1-\alpha}\,.$$

Consequently, the adiabatic theory implies that the energy follows the random process described in Sec. VI B. In particular, if $p \neq \alpha$, the process leads to the exponential acceleration of particles.

The numerical experiments of Ref. 22 are in good agreement with the growth rates predicted by the probabilistic models. Those experiments include, in fact, double cycle machines, when the separator is introduced two times per period in such a way that the billiard cycle becomes time-reversible, i.e., it is symmetric when t is replaced by $-t$. The examples of the ensemble energy growth are shown in Figures 8(a) and 8(b). We see good quantitative agreement with the theoretical predictions.

Figure 8(c) shows results of a modified experiment when the separating rod is stopped before it reaches its upper position and therefore the ergodicity of the billiard is not broken. In the experiment, we left the gap of around 10% of the rod's length. The graphs clearly shows that the acceleration is not exponential.

Finally Figure 9 illustrates the difference, as well as the similarity, between the stochastic model and the billiard dynamics. In these experiments, we have started with a large ensemble of initial conditions with a fixed energy E_0. The histograms show distributions of $\log\frac{E(nT)}{E_0}$ for $n = 1, 2, 3, 10, 20$. In the stochastic model, the trajectories follow adiabatic law precisely, while in the billiard they form a bell-shaped blocks centered near values predicted by the stochastic model. As n grows, the histograms show a clear tendency to approach the normal distribution, in agreement with the stochastic model.

ACKNOWLEDGMENTS

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