Billiards: A singular perturbation limit of smooth Hamiltonian flows

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Nonlinear multi-dimensional Hamiltonian systems that are not near integrable typically have mixed phase space and a plethora of instabilities. Hence, it is difficult to analyze them, to visualize them, or even to interpret their numerical simulations. We survey an emerging methodology for analyzing a class of such systems: Hamiltonians with steep potentials that limit to billiards. [http://dx.doi.org/10.1063/1.4722010]

Very little is known regarding the dynamics in high-dimensional, far-from-integrable systems. Until recently, in such systems, local analysis near fixed points and periodic orbits or geometrical analysis near specific homoclinic or heteroclinic structures have been the only available analytical tools. Numerical studies of such systems are possible, yet, due to the mixed phase space property, these are difficult to interpret. Here, we survey a methodology which we (the authors and, in part, in collaboration with Anna Rapoport) developed in the last decade–the near-billiard paradigm. In this paradigm, we can study the local and global properties of classes of multi-dimensional smooth systems by analyzing the singular billiard limit for various types of multi-dimensional systems. Notably, billiards provide a rich playground for dynamicists. Billiards can be integrable, near-integrable, of mixed phase space or uniformly hyperbolic (yet singular), and in many cases, their complex and rich dynamics have been understood in great detail. Billiards and simple impact systems are commonly used to model the classical and semi-classical motion in systems with steep potentials (e.g., in kinetic theory, chemical reactions, cold atom's motion, microwave dynamics). However, the correspondence between the smooth motion and the singular billiard model occurs to be not immediate. This correspondence is the main topic of the present article which summarizes the works of Refs. 1–8. On one hand, we show that a proper limit may be formulated, so that some basic dynamical properties of the billiard are inherited by the smooth flow (Sec. II and parts of Sec. V). On the other hand, more surprisingly, we show that some of the crucial features of the billiard flow are not shared by the smooth systems (Secs. III–V). Nonetheless, even in this latter case, we are able to learn about the properties of the smooth flow by devising singular analysis tools.

I. INTRODUCTION

The original motivation of our work is related to the Boltzmann-Sinai ergodic hypothesis. From the mathematical point of view, this hypothesis states that the gas of elastically colliding hard spheres is an ergodic system. While this prominent problem is still unresolved, the work on it led to fundamental developments in the theory of dynamical systems.[9–14] The starting point of this analysis is the observation that the dynamics of a gas of n hard spheres in a d-dimensional spatial domain is governed by a semi-dispersive billiard in an Nd-dimensional space.[9,10,14]

The "smooth Boltzmann gas" corresponds to the next order approximation where the motion is modeled by a (Hamiltonian) system of classical particles which pair-wisely interact with each other via a smooth steep repelling potential. At large kinetic energies, the interaction between two particles becomes essential only when they come very close to each other, i.e., at very short intervals of time that correspond to a near collision. As Einstein wrote: "Boltzmann very correctly emphasizes that the hypothetical forces between the molecules are not an essential component of the theory as the whole energy is of kinetic kind."[15] In other words, the hard-spheres system appears as a universal model for the interaction of classical particles at high kinetic energies. The huge number of degrees of freedom in a typical molecular system implies that statistical means should be employed for the analysis. This is the main motivation behind the quest to prove the Boltzmann-Sinai ergodic hypothesis.

We propose that one has to actually address the question of how the statistical properties of the hard-sphere model are translated back to the case of a smooth steep potential. Following the Fermi-Pasta-Ulam numerical experiments and the subsequent discovery of high-dimensional integrable systems, it was realized that the large number of degrees of freedom is insufficient for justifying the statistical approach.[16] One needs *instability* to allow the system to "forget" its initial state, so a universal probability distribution could establish itself in the space of system states.

We notice that instability in a dynamical system is a differential property (having to do with the rate at which close-by initial conditions diverge in time). Hence, to transfer the statistical description of the hard-sphere system to the smooth Boltzmann gas, one needs to control the *derivatives* of the approximation error. Since the hard-sphere system has singularities, this becomes a delicate issue. In the series of

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works that are reviewed here, we develop methodologies which could be used to address this problem.

In particular, we show that the specific instability of dispersive billiards (i.e., a uniformly hyperbolic structure), cannot universally survive a smooth regularization of the billiard. Since the uniform hyperbolicity of the dispersive billiards appears to be the underlying mechanism of the ergodicity of hard-sphere systems, it follows that the hypothesis that the statistical properties of the smooth Boltzmann gas are potential-independent and similar to those of the hard-sphere gas could be correct on a finite time scale, yet it cannot be true in the infinite time limit.

This time scale must, for all practical purposes, be large enough in systems with huge numbers of particles. However, for small number of degrees of freedom, the changes in statistics can become observable (see, e.g., Ref. 33). Thus, our results stress the importance of analyzing the finite-time behavior of the system and of analyzing how this behavior scales with the number of degrees of freedom. These issues become increasingly more relevant as experimental and numerical capabilities develop.

More generally, billiards and impact systems arise in a wide variety of science and engineering applications.[17–19] The singularities in models with impacts often lead to ambiguous results: it is not always clear how to continue solutions through singularities, especially in systems with friction. A natural method of resolving such difficulties is to recall that the impact system is, in many cases, a simplified model for forces that grow very fast across certain boundaries, the surfaces of impact. So, regularizing impact systems by smooth models with sharp growing forces near the boundary is a natural approach. One can then study the smooth model and see what conclusions survive in the limit.[17,20,21] However, one must also be sure that the result is independent of the particular choice of the smooth approximation.

This last question might seem to be easy for the frictionless case where the energy is preserved and the forces are normal to the impact boundary. Under these conditions, for any smooth regularization, one recovers the universal elastic collision law: the angle of reflection is approximately equal to the angle of incidence. This observation can indeed be enough when we are interested in some topological properties of the system. However, if we seriously want to study the dynamics, we must analyze differential properties. Such an analysis leads to the non-trivial question: "Under which conditions on the smooth regularization of the billiard, the derivative of the difference between the reflection angle and the incidence angle with respect to initial conditions is close to zero?"

We formulate the above ideas by the following singular perturbation problem. Consider Hamiltonian flows induced by a one-parameter family of steep potentials depending on a steepness parameter ϵ,

$$H = \sum_{i=1}^{n} \frac{p_i^2}{2} + V(q;\epsilon), \;\; V(q;\epsilon) \underset{\epsilon\to 0}{\to} \begin{cases} 0, & q \in D\backslash\partial D, \\ \mathcal{E}, & q \in \partial D. \end{cases} \qquad (1)$$

Here $(q,p) \in R^n \times R^n$, $D \subset R^n$, and ∂D is a piecewise smooth. The potential $V(q;\epsilon)$ is non-negative C^{r+1}-smooth function. The derivatives of $V(q;\epsilon)$ near ∂D grow without no bound as $\epsilon \to 0$. The vector $\varepsilon = (\mathcal{E}_1, \mathcal{E}_2, \ldots)$ provides the limit values of the steep potential on each of the smooth connected components of the boundary. On each such component, the constant $\mathcal{E}_i$ may be finite or infinite. Our goal is to compare the behavior of the orbits of system (1) at sufficiently small ε with the billiard flow in D.

The persistence results of Refs. 1, 2, and 5 are concerned with comparing the behavior near regular billiard orbits—orbits that hit the boundary of D at non-zero angles (see Sec. II). In Ref. 5, we show that for regular reflections the time-shift map by the billiard is C^r-close to the smooth flow for *arbitrary dimension and geometry*. Moreover, we prove that a certain billiard limit may be used for developing an asymptotic expansion for approximating regular reflections of the smooth flow. We find bounds on the error terms of the approximation (and its derivatives, up to order r) and next order corrections for a large class of potentials. In this way, a perturbational tool for analyzing far-from-integrable Hamiltonian systems is developed. This may be used to establish quantitative persistence results, for example, periodic orbits and separatrix splitting (see Refs. 5 and 8 and Table II). These persistence results were utilized to prove the existence of a large collection of chaotic hyperbolic orbits in any infinite set of sufficiently small scatterers and in convex domains with small scatterers.[22,23] We think that these tools, which may be thought of as the analog of the near-integrable Melnikov technique in the near-billiard limit, will be further used to examine finite ϵ effects in specific applications.

Singular orbits are those billiard orbits which are *tangent* to the boundary or those which hit the *corners* (i.e., the points where the billiard boundary is not smooth). Section III summarizes the two-dimensional behavior near singular orbits, and Sec. IV summarizes the higher-dimensional results.

In Refs. 1 and 3, we studied the behavior of smooth orbits that are close to the billiard orbits of *non-degenerate* (i.e., quadratic) tangency in two-dimensional dispersing billiards. While the orbit of the smooth system is still close to the billiard orbit in this case, there can be no closeness with derivatives (since the billiard map is not smooth at tangent orbits). We derive the normal form for the return map generated by the smooth flow near a periodic tangent billiard orbit (where all reflections but the tangent one are regular and occur at dispersing components). Notably, this formula describes the smooth system behavior in a region where there is no correspondence with the billiard motion. Analysis of this return map leads to a proof that *stability islands* emerge from such tangent periodic orbits of two-dimensional dispersive billiards. This is the main result of Refs. 1 and 3. It shows that even though dispersive billiards are ergodic,[10,24] the ergodicity is not typically inherited by the smooth-potential approximations (yet in special cases the "soft billiard" potentials can produce ergodic behavior[9,25–32]). Experiments with an atom-optic system[33] confirm the drastic change of statistical properties at the transition from a dispersive billiard to its smooth-potential approximation due to the emergence of stability islands out of singular orbits.

In Ref. 4, we addressed the behavior of two-dimensional smooth systems near billiard orbits that hit a corner. The billiard map is typically discontinuous at the corner orbit. We show that in the Poincaré map generated by the smooth system the discontinuities are "sewn" by means of a *corner scattering function* which can be determined via the analysis of the scaled limit of the potential at the corner. This limit is not integrable, so no explicit formulas for the scattering function exist; however, one can study its properties using qualitative methods. A surprising finding is that the scattering function is often *non-monotone*, i.e., the billiard discontinuities are not smoothed in the "most economic" way. In particular, the range of the reflection angles generated by the smooth system near the billiard corner may be larger than that achieved by smoothing the discontinuous billiard limit, namely, it is not determined by the billiard geometry alone. In the two-dimensional case, the non-monotone scattering function appears near corners of angles $\frac{\pi}{n}$, where $n \geq 2$ is an integer. We show that billiard corner orbits with outgoing angles corresponding to the extremal values of the scattering function produce elliptic islands in the smooth system. Thus, one should expect the emergence of stable periodic orbits in the smooth system when the corner angle varies across $\frac{\pi}{n}$, e.g., when the corner angle tends to zero.

Notably, the underlying mechanism of ergodicity loss is purely geometrical; it is based on the fact that orientation in the momenta space is flipped at every collision.[1–3]

In Ref. 6, we employ these observations (for a corner with an additional symmetry) to show that elliptic orbits appear in systems with steep smooth potentials that limit to Sinai billiards for *arbitrarily large dimension*. While the examples considered in Ref. 6 cannot be directly linked to the smooth many particles case, this construction of a stability island (here—a positive measure set filled by quasiperiodic orbits) in multi-dimensional highly unstable systems supports our conjecture that the systems of many particles interacting via a steep repelling potential are, typically, not ergodic.

Finally, we summarize the implications of the above results on chaotic scattering.[8,34–37] There, billiard rays come from infinity, hit some scatterers that lie in a bounded domain, and then escape again. With the steep potential methodology, we are able to analyze the correspondence between scattering by hard core obstacles (billiards) and scattering by steep smooth hills. In particular, with this correspondence, we are able to establish the existence of a hyperbolic repeller with fractal structure in a smooth Hamiltonian flow.

The paper is ordered as follows. In Sec. II, we analyze the case of regular reflections. In Sec. III, we study the behavior of smooth systems near singular billiard orbits for the two-dimensional case. A multidimensional example is considered in Sec. IV. In Sec. V, we apply the results to the scattering problem, and in Sec. VI, we list some open problems and perspective directions.

II. PERSISTENCE RESULTS FOR BILLIARD-LIKE POTENTIALS

We begin the review by formulating precisely what we mean by "approximating the smooth motion by billiards". To this aim, we first define the billiard flow, what are regular billiard reflections, and non-degenerate tangential billiard reflections. We then introduce the notion of *billiard-like potentials* on a domain D. Briefly, these are one-parameter families of C^{r+1}-smooth potentials, V_ϵ, that are essentially constant inside D and grow fast at the boundary of D. The growth rate approaches infinity as the steepness parameter ϵ approaches zero. All the works that are reviewed here are concerned with studying the flows induced by families of mechanical Hamiltonian systems with billiard-like potentials at sufficiently small ϵ values.

We establish first that for such potentials the behavior near the boundary usually limits (in the C^r topology) to the billiard reflections.[1,2,5] Then, we show that next order corrections to the billiard approximation may be found, with prescribed error estimates.[5] We end this section by recalling that these results imply that non-singular non-parabolic periodic orbits and hyperbolic sets of the limit billiard flow persist for sufficiently small ϵ values.[1,2,5] Utilizing the perturbation analysis, these persistence results become quantitative.[5]

A. Smooth reflections limit to billiard reflections

The first main step in the theory appears technical: it consists of proving that under specific natural conditions on $V(q;\epsilon)$, the *regular* billiard reflections are indeed close (and so are their derivatives) to the smooth flow reflections (see below for precise definitions of these concepts). Smooth trajectories that limit to non-degenerate tangent reflections are only C^0-close to the limiting map. Thus, this initial step formulates under what conditions the limiting process makes sense. Moreover, this step enables to subsequently use standard dynamical systems tools that relate two nearby maps. Here, the closeness of derivatives is essential as it allows to use persistence and structural stability arguments (see Sec. II B).

More precisely, consider a domain D inside $\mathbb{R}^d$ or inside a flat torus $\mathbb{T}^d$. Assume that the boundary ∂D consists of a finite number of C^{r+1} ($r \geq 1$) smooth (d–1)-dimensional submanifolds Γ_i,

$$\partial D = \Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_n.$$

The boundaries of these submanifolds, when exist, form *the corner set* of ∂D:

$$\Gamma^* = \partial\Gamma_1 \cup \partial\Gamma_2 \cup \ldots \cup \partial\Gamma_n.$$

The billiard flow is defined to be the inertial motion of a point mass inside D accompanied by elastic reflections at the boundary ∂D. Let $q \in D$ and $p \in \mathbb{R}^d$ denote the particles coordinates and momenta. Denote the billiard flow by $\rho_t = b_t\rho_0$, where $\rho_t = (q_t, p_t)$ and $\rho_0 = (q_0, p_0)$ are two inner phase points (i.e., $q_{0,t}$ are both in the interior of D). If the piece of trajectory which connects q_0 with q_t does not have tangencies with the boundary, then ρ_t depends C^r-smoothly on ρ_0. On the other hand, ρ_t loses smoothness at any point q_0 whose trajectory is tangent to the boundary at least once on the interval (0,t). Notice that ρ_t is not defined if at some $t_s < t$ the trajectory ρ_{t_s} hits the corner set.

A tangency may occur only if the boundary is not strictly convex in the direction of motion at the point of tangency. A tangency is called *non-degenerate* if the curvature in the direction of motion does not vanish. If the billiard boundary is strictly concave (strictly dispersing), then all the tangencies are non-degenerate. On the other hand, if the billiard's boundary has saddle points (or if the billiard is semi-dispersing), then there always exist directions for which the tangency is degenerate.

The billiard flow may be expressed as a formal Hamiltonian system,

$$H_b = \frac{p^2}{2} + V_b(q), \quad V_b(q) = \begin{cases} 0, & q \in int(D), \\ +\infty, & q \notin D. \end{cases}$$

Theorem 1 states that the smooth Hamiltonian flow defined by $H = \frac{p^2}{2} + V(q;\epsilon)$ limits in a natural sense to this billiard flow when the family of smooth potentials $V(q;\epsilon)$ satisfies the four conditions below. Condition **I** guarantees that inside D the motion is close to inertial motion. Condition **III** insures that the particle cannot penetrate the boundary. Condition **II** implies that the boundary is repelling and that the reaction force is normal to the boundary, so the reflection law limits to standard billiard reflection law (angle of reflection equals to the angle of incidence). Condition **IV** is less intuitive—it is needed for the smooth closeness results and for preventing the particle from sliding along the boundary.

Condition I. *For any fixed (independent of ϵ) compact region $K \subset int(D)$, the potential $V(q;\epsilon)$ diminishes along with all its derivatives as $\epsilon \to 0$,*

$$\lim_{\epsilon\to 0} ||V(q;\epsilon)|_{q\in K}||_{C^{r+1}} = 0. \tag{2}$$

Let $N(\Gamma^*)$ denote a fixed (independent of ϵ) neighborhood of the corner set and $N(\Gamma_i)$ denote a fixed neighborhood of the boundary component Γ_i. Define $\tilde{N}_i = N(\Gamma_i)\backslash N(\Gamma^*)$ (we assume that $\tilde{N}_i \cap \tilde{N}_j = \emptyset$ when $i \neq j$). Assume that for all small $\epsilon \geq 0$, there exists *a pattern function*

$$Q(q;\epsilon) : \bigcup_i \tilde{N}_i \to \mathbb{R}^1$$

which is C^{r+1} with respect to q in each of the neighborhoods $\tilde{N}_i$ and it depends continuously on ϵ (in the C^{r+1}-topology, so it has, along with all derivatives, a proper limit as $\epsilon \to 0$). Further assume that in each of the neighborhoods $\tilde{N}_i$ the following is fulfilled.

Condition IIa. *The billiard boundary is composed of level surfaces of $Q(q;0)$,*

$$Q(q;\epsilon = 0)|_{q\in\Gamma_i\cap\tilde{N}_i} \equiv Q_i = \text{constant}. \tag{3}$$

In the neighborhood $\tilde{N}_i$ of the boundary component Γ_i (where $Q(q;\epsilon)$ is close to Q_i), define *a barrier function* $W_i(Q;\epsilon)$, which is C^{r+1} in Q, continuous in ϵ and does not depend explicitly on q, and assume that there exists ϵ_0 such that

Condition IIb. *For all $\epsilon \in (0,\epsilon_0]$, the potential level sets in $\tilde{N}_i$ are identical to the pattern function level sets, and thus*

$$V(q;\epsilon)|_{q\in\tilde{N}_i} \equiv W_i(Q(q;\epsilon) - Q_i;\epsilon), \tag{4}$$

and

Condition IIc. *For all $\epsilon \in (0,\epsilon_0]$, ∇V does not vanish in the finite neighborhoods of the boundary surfaces, $\tilde{N}_i$, thus*

$$\nabla Q|_{q\in\tilde{N}_i} \neq 0, \tag{5}$$

and for all $Q(q;\epsilon)|_{q\in\tilde{N}_i}$,

$$\frac{d}{dQ}W_i(Q - Q_i;\epsilon) \neq 0. \tag{6}$$

In this way, the rapid growth of the potential across the boundary is described in terms of the barrier functions alone. Note that by Eq. (5), the pattern function Q is monotone across $\Gamma_i \cap \tilde{N}_i$, so either $Q > Q_i$ corresponds to the points near Γ_i inside D and $Q < Q_i$ corresponds to the outside or vice versa. To fix the notation, we adopt the first convention.

Condition III. *There exists a constant (may be infinite) $\mathcal{E}_i > 0$, such that as $\epsilon \to +0$ the barrier function increases from zero to $\mathcal{E}_i$ across the boundary Γ_i:*

$$\lim_{\epsilon\to+0} W_i(Q;\epsilon) = \begin{cases} 0, & Q > Q_i, \\ \mathcal{E}_i, & Q < Q_i. \end{cases} \tag{7}$$

By Eq. (6), for small ϵ, Q could be considered as a function of W and ϵ near the boundary: $Q = Q_i + \mathcal{Q}_i(W;\epsilon)$. Condition IV states that for small ϵ a finite change in W corresponds to a small change in Q:

Condition IV. *As $\epsilon \to +0$, for any fixed W_1 and W_2 such that $0 < W_1 < W_2 < \mathcal{E}_i$, for each boundary component Γ_i, the inverse barrier function $\mathcal{Q}_i(W;\epsilon)$ tends to zero uniformly on the interval $[W_1, W_2]$ along with all its $(r+1)$ derivatives.*

The use of the pattern and barrier functions reduces the d-dimensional Hamiltonian dynamics in arbitrary geometry to a 1-dimensional dynamics, thus allowing direct asymptotic integration of the smooth problem. This is the main tool, introduced first in Ref. 1 for the two-dimensional case and in Ref. 5 for the general d-dimensional case that enables the analysis of these high-dimensional nonlinear problems. Barrier functions satisfying the above conditions include ($W = \frac{\epsilon}{Q^\alpha}, e^{-\frac{Q^\alpha}{\epsilon}}, \epsilon \log Q$).

Notably, the theory applies also to the following common setting. Consider a potential $V(q)$ which does not depend on any small parameter. Assume V is bounded inside a certain region D and grows to infinity at the boundary of D. Then, at sufficiently high energy value h, the kinetic energy prevails inside D so the motion there is essentially inertial until the particle arrives at a thin boundary layer near ∂D. By rescaling the Hamiltonian and momenta: $\hat{H} = H/h, \hat{p} = p/\sqrt{h}$, we obtain the Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2} + \epsilon V(q)$ where $\epsilon = 1/h$. Then, conditions **I–IV** are satisfied for reasonable choices of $V(q)$ that approach infinity at ∂D (including classical models like Coulomb and Lennard-Jones potentials).

Given a domain D, the one-parameter family of potentials $V(q,\epsilon)$ is called a family of **billiard-like potentials on D** if for any $\epsilon > 0$, $V(q,\epsilon)$ is a C^{r+1}-smooth function which satisfies the four conditions I-IV.

Theorem 1 (Refs. 1, 5, and 38). *Given a family of billiard-like potentials $V(q;\epsilon)$ on D, let h_t^ϵ denote the Hamiltonian flow defined by*

$$H = \frac{p^2}{2} + V(q;\epsilon), \tag{8}$$

on an energy surface $H = H^* < \hat{\mathcal{E}} = \inf(V(q;\epsilon)|_{\partial D})$, and let b_t denote the billiard flow in D. Let ρ_0 and $\rho_T = b_T\rho_0$ be two inner phase points, so that on the finite time interval [0,T] *the billiard trajectory of ρ_0 has a finite number of collisions. Assume all these collisions are either regular reflections or non-degenerate tangencies. Then $h_t^\epsilon\rho \underset{\epsilon\to 0}{\to} b_t\rho$, uniformly for all ρ close to ρ_0 and all t close to T. If, additionally, the billiard trajectory of ρ_0 has no tangencies to the boundary on the time interval* [0,T], *then $h_t^\epsilon \underset{\epsilon\to 0}{\to} b_t$ in the C^r-topology in a small neighborhood of ρ_0, and for all t close to T.*

The proof of the theorem includes integration of the equations of motion at different components of the boundary layer, according to the rate at which the steep potential changes, see Refs. 5 and 38 for complete details.

We conclude that the map defined by the billiard flow from a local section at ρ_0 to a local section at ρ_T is C^r-close (respectively, C^0-close) to the corresponding family of maps that are defined by the smooth potential $V(q,\epsilon)$, as long as this segment contains only regular collisions (respectively, regular collisions and some non-degenerate tangencies). Using structural stability arguments, we can immediately conclude that for sufficiently small ϵ regular non-parabolic periodic orbits persist and that hyperbolic sets persist as well. Such persistence results are in-line with the common intuition that the motion under steep potential is well approximated by billiard (in Sec. III, we show that this intuition is incorrect near non-regular reflections).

Next, we provide error estimates for this approximation.

B. Corrections and error estimates of the billiard approximation

Theorem 1 implies that return maps of the billiard flow and of the smooth flows are close. We derive error estimates and next order corrections for such return maps by considering a family of auxiliary billiard flows in a modified domain D^ϵ. The analysis also provides a good global section for the smooth flow that may be utilized in numerical simulations. Indeed, it is shown that the boundary of the auxiliary domain, ∂D^ϵ, provides a transverse section to regular orbits of the smooth flow. More precisely, the smooth flow defines a map Φ^ϵ on the set of regular (non-tangent) phase-points,

$$S_\epsilon = \{\rho = (q,p) : q \in \partial D^\epsilon, \langle p, n(q)\rangle > 0\}. \tag{9}$$

We show that to leading order Φ^ϵ is well approximated by the corresponding billiard map B^ϵ in D^ϵ and provide the explicit expression for the next order correction and bounds on the error terms.

To construct the domain D^ϵ, we define, for each boundary component Γ_i, three boundary layer parameters $(\nu_i, \eta_i(\nu_i), \delta_i)$ all tending to zero with ϵ. The parameter ν_i equals to the value of the potential on the ith boundary of D^ϵ. It is chosen so that the inverse barrier function, $\mathcal{Q}_i(W;\epsilon)$, tends to zero along with all its derivatives uniformly for $H^* \ge W \ge \nu_i$ (see Fig. 1). The small parameter η_i equals to the corresponding level of the inverse barrier function on ∂D^ϵ: $\eta_i(\epsilon) = \mathcal{Q}_i(\nu_i;\epsilon)$. The parameter δ_i controls the closeness to inertial motion in the region D^ϵ_{int}. More precisely, D^ϵ_{int} is the region bounded by the surfaces $Q(q;\epsilon)|_{q\in\tilde{N}_i} = Q_i + \delta_i(\epsilon)$ together with $\partial N(\Gamma^*)$ (i.e., excluding the corner neighborhoods). The values of $\delta_i(\epsilon)$ are chosen so that the potential V tends to zero uniformly along with all its derivatives in D^ϵ_{int}. By conditions **I** and **IV**, we may choose the parameters ν_i, η_i, δ_i such that $\eta_i \ll \delta_i$, namely, $D^\epsilon_{int} \subset D^\epsilon$ (see below).

To each set of the boundary layer parameters ν_i, η_i, δ_i, we associate C^{r+1} bounds on $\mathcal{Q}_i$ in D/D^ϵ (denoted by $M_i^{(r)}$) and on V in D^ϵ_{int} (denoted by $m^{(r)}$),

$$M_i^{(r)}(\nu_i;\epsilon) = \sup_{\substack{\nu_i \le W \le H^* \\ 0 \le l \le r+1}} |\mathcal{Q}_i^{(l)}(W;\epsilon)|, \tag{10}$$

$$m^{(r)}(\delta;\epsilon) = \sup_{\substack{q \in D^\epsilon_{int} \\ 1 \le l \le r+1}} ||\partial^l V(q;\epsilon)||. \tag{11}$$

Condition **IV** implies that the $M_i^{(r)}$'s approach zero as $\epsilon \to 0$ for any fixed $\nu > 0$; hence, the same holds true for any sufficiently slowly tending to zero $\nu(\epsilon)$, i.e., the required $\nu_i(\epsilon)$ exist. Similarly, condition **I** implies that $m^{(r)}$ approaches zero as $\epsilon \to 0$ for any fixed δ; therefore, the same holds true for any choice of sufficiently slowly tending to zero $\delta_i(\epsilon)$. As $m^{(r)} \to 0$, it follows that within D^ϵ_{int} the flow of the smooth Hamiltonian trajectories is C^r-close to the free flight, i.e., to the billiard flow. It is established in Ref. 5 that by using an appropriate change of coordinates in each of the three regions (inside D^ϵ_{int}, in $D^\epsilon \backslash D^\epsilon_{int}$, and outside of D^ϵ), the equations of motion may be written as differential equations integrated over a finite interval with a right hand side which tends to zero in the C^r-topology as $\epsilon \to 0$. Thus, Picard iterations supply, in addition to the error estimates for the zeroth

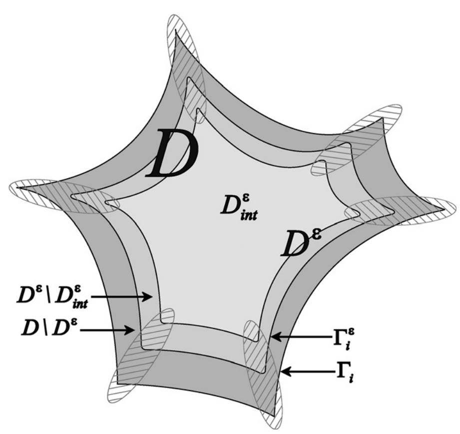

FIG. 1. The partition of the domain D into regions. D^ϵ_{int} is an interior region in which $V(q,\epsilon)$ is smaller than $m^{(r)}(\delta;\epsilon)$ in the C^r topology. D^ϵ is the auxiliary billiard, so on its ith boundary components $V(q,\epsilon) = \nu_i$. Clearly, $D^\epsilon_{int} \subset D^\epsilon$. The boundary of D^ϵ provides a global section S_ϵ for regular reflections of the smooth flow (see Ref. 5).

order approximation, higher order corrections to the return map. We summarize below the explicit formulae for the first order corrections. These formulas may be useful in future applications—they may play in the near-billiard context the same role as the Melnikov analysis does for near-integrable systems.

The map Φ^ϵ on S^ϵ (see Eq. (9)) is composed of an interior flight part and a reflection part: ($\Phi^\epsilon = R^\epsilon \circ F^\epsilon$).

The interior map F^ϵ (see Fig. 2): Let $q \in \Gamma_j^\epsilon$ for some j, and assume that the segment $q + p\tau$ with $\tau \in [0, \tau_\circ^\epsilon(p,q)]$ that connects Γ_j^ϵ with Γ_i^ϵ lies inside D^ϵ so that $q + p\tau_\circ^\epsilon(q,p) \in \Gamma_i^\epsilon$. Further assume that the reflections at $\Gamma_{j,i}^\epsilon$ are non-tangent, so there is some $c > 0$ such that $\langle p, n(q)\rangle > c$ and $\langle p, n(q + p\tau_\circ^\epsilon(q,p))\rangle < -c$. Then, the free flight map $F^\epsilon : (q,p) \mapsto (q_{\tau^\epsilon}, p_{\tau^\epsilon})$ for the smooth Hamiltonian flow is $O_{C^r}(m^{(r)} + \nu_i + \nu_j)$-close to the free flight map $F_\circ^\epsilon$ of the billiard in D^ϵ and is given by

$$q_{\tau^\epsilon} = q + p\tau^\epsilon + \int_0^{\tau^\epsilon} \nabla V(q + ps; \epsilon)(s - \tau^\epsilon)ds + O_{C^{r-1}}((m^{(r)} + \nu_i + \nu_j)^2),$$
$$p_{\tau^\epsilon} = p - \int_0^{\tau^\epsilon} \nabla V(q + ps; \epsilon)ds + O_{C^{r-1}}((m^{(r)} + \nu_i + \nu_j)^2). \tag{12}$$

The flight time $\tau^\epsilon(q,p)$ is $O_{C^r}(m^{(r)} + \nu_i + \nu_j)$-close to $\tau_\circ^\epsilon(p,q)$ and is uniquely defined by the condition $Q(q_{\tau^\epsilon}; \epsilon) = Q_i + \eta_i(\epsilon)$,

$$\tau^\epsilon(q,p) = \tau_\circ^\epsilon(q,p) + \frac{\left\langle \nabla Q, \int_0^{\tau_\circ^\epsilon} \nabla V(q + ps; \epsilon)(\tau_\circ^\epsilon - s)ds \right\rangle}{\langle \nabla Q, p\rangle} + O_{C^{r-1}}((m^{(r)} + \nu_i + \nu_j)^2), \tag{13}$$

where ∇Q is taken at the billiard collision point $q + p\tau_\circ^\epsilon(p,q)$ where $Q_i(q + p\tau_\circ^\epsilon(p,q); \epsilon) = Q_i + \eta_i(\epsilon)$.

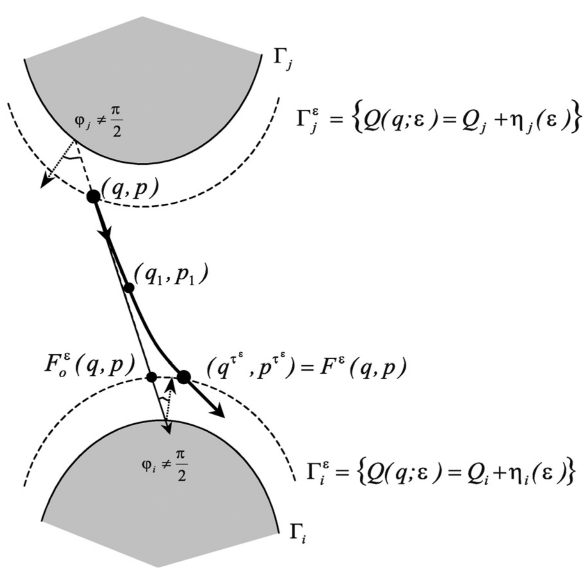

FIG. 2. Free flight between boundaries Γ_i^ϵ and Γ_j^ϵ. A smooth trajectory is marked by a bold line, and an auxiliary billiard trajectory is marked by a solid line (see Ref. 5).

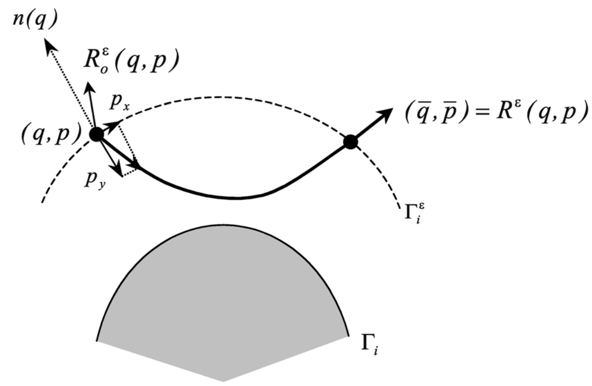

FIG. 3. Reflection from the boundary Γ_i. A smooth trajectory is marked by a bold line. The auxiliary billiard trajectory changes its direction according to the usual reflection law from the boundary of (D^ϵ), namely, Γ_i^ϵ (see Ref. 5).

The reflection map R^ϵ (see Fig. 3): To formulate the reflection law R^ϵ for the smooth orbit, we need to define several geometrical entities. Consider a point $q \in \Gamma_i^\epsilon$ and let the momentum p be directed outside D^ϵ (i.e., towards the boundary) at a bounded from zero angle with Γ_i^ϵ. The smooth trajectory of (q,p) spends a small time $\tau_c^\epsilon(q,p)$ outside D^ϵ and then returns to Γ_i^ϵ with the momentum directed strictly inside D^ϵ. Let p_y and p_x denote the components of momentum, respectively, normal and tangential to the boundary Γ_i^ϵ at the point q,

$$p_y = \langle n(q), p\rangle, \quad p_x = p - p_y n(q). \tag{14}$$

We assume that the unit normal to Γ_i^ϵ at the point q, $n(q)$, is oriented inside D^ϵ, so $p_y < 0$ at the initial point. Denote by $Q_y(q;\epsilon)$ the derivative of Q in the direction of $n(q)$,

$$Q_y(q;\epsilon) := \langle \nabla Q(q;\epsilon), n(q)\rangle.$$

Recall that the surface Γ_i^ϵ is a level set of the pattern function $Q(q;\epsilon)$, and thus, we may study how the normal $n(q)$ changes as one moves along the level set Γ_i^ϵ (in the tangential plane) and as one moves to nearby level sets (in the normal direction). Let $K(q;\epsilon)$ denote the derivative of $n(q)$ in the directions tangent to Γ_i^ϵ, and let $l(q;\epsilon)$ denote the derivative of $n(q)$ in the direction of $n(q)$. Obviously, Q_y is a scalar, K is a matrix, and l is a vector tangent to Γ_i^ϵ at the point q. Note that $Q_y \neq 0$ by virtue of condition IIc. Define the integrals

$$I_1 = I_1(q,p) = 2\int_0^{-p_y} Q_i'\left(\frac{1 - p_x^2 - s^2}{2}; \epsilon\right)ds$$
$$I_2 = I_2(q,p) = 2\int_0^{-p_y} Q_i'\left(\frac{1 - p_x^2 - s^2}{2}; \epsilon\right)s^2 ds, \tag{15}$$

and the vector J,

$$J(q,p) = \left[-\frac{I_2(q,p)}{p_y} l(q;\epsilon) + I_1(q,p)K(q;\epsilon)p_x\right] / Q_y(q;\epsilon). \tag{16}$$

Notice that J is a vector tangent to Γ_i^ϵ at the point q and that by Eq. (10),

$$I_{1,2} = O_{C^r}(M_i^{(r)}), \quad J = O_{C^{r-1}}(M_i^{(r)}). \tag{17}$$

Lemma 3 of Ref. 5 asserts that for sufficiently small $\epsilon \le \epsilon_0$ the reflection map $R^\epsilon : (q,p) \mapsto (\bar{q}, \bar{p})$ is given by

$$\begin{aligned}\bar{q} &= q + O_{C^r}(M_i^{(r)}) = q + p_x \tau_c^\epsilon(q,p) + O_{C^{r-1}}((M_i^{(r)})^2),\\ \bar{p} &= p - 2n(q)p_y + O_{C^r}(M_i^{(r)})\\ &= p - 2n(q)p_y - p_y J(q,p) - n(q)\langle p_x, J(q,p)\rangle\\ &\quad + O_{C^{r-1}}((M_i^{(r)})^2),\end{aligned} \tag{18}$$

where the collision time of the smooth Hamiltonian flow is estimated by

$$\tau_c^\epsilon(q,p) = O_{C^r}(M_i^{(r)}) = -\frac{1}{Q_y(q;\epsilon)} I_1(q,p) + O_{C^{r-1}}((M_i^{(r)})^2). \tag{19}$$

Combining the two maps, we established in Ref. 5:

Theorem 2. *Assume $V(q;\epsilon)$ is a billiard potential family on D and choose δ_i's and ν_i's such that $\delta_i(\epsilon), \nu_i(\epsilon), m^{(r)}(\epsilon), M_i^{(r)}(\epsilon) \to 0$ as $\epsilon \to 0$. Then, on the cross-section S_ϵ (see Eq. (9)) near orbits of regular reflections (that is, given any constant $C > 0$, near the points $(q,p) \in S_\epsilon$ such that $\langle n(q), p\rangle \ge C$ and $|\langle n(\bar{q}), \bar{p}\rangle| \ge C$ where $(\bar{q},\bar{p}) = B^\epsilon(q,p)$), for all sufficiently small $\epsilon \le \epsilon_0$, the Poincaré map Φ^ϵ of the smooth Hamiltonian flow is defined, and it is $O(m^{(r)} + \nu + M^{(r)})$-close in the C^r-topology to the billiard map $B^\epsilon = R_\circ^\epsilon \circ F_\circ^\epsilon$ in the auxiliary billiard table D^ϵ. Furthermore,*

$$\begin{aligned}\Phi^\epsilon &= R^\epsilon \circ F^\epsilon = B^\epsilon + O_{C^r}(m^{(r)} + \nu + M^{(r)})\\ &= (R_\circ^\epsilon + R_1^\epsilon) \circ (F_\circ^\epsilon + F_1^\epsilon) + O_{C^{r-1}}((m^{(r)} + \nu + M^{(r)})^2)\\ &=: B^\epsilon + \Phi_1^\epsilon + O_{C^{r-1}}((m^{(r)} + \nu + M^{(r)})^2),\end{aligned}$$

where $\nu = \max_i \nu_i$, $M^{(r)} = \max_i M_i^{(r)}$, $\Phi_1^\epsilon = O_{C^{r-1}}(m^{(r)} + \nu + M^{(r)})$, and the leading and first order corrections $F_{o,1}^\epsilon$ and $R_{o,1}^\epsilon$ are explicitly given by Eqs. (12)–(19) and $\Phi_1^\epsilon = R_\circ^\epsilon \circ F_1^\epsilon + R_1^\epsilon \circ F_\circ^\epsilon$.

Furthermore, we notice that this methodology also tells us how close the smooth and the billiard trajectories are along their entire path:

Theorem 3. *Under the same conditions as in Theorem 2, given a finite T and a regular billiard trajectory in* [0,T], *the time t map of the smooth Hamiltonian flow and of the corresponding auxiliary billiard are $O(\nu + m^{(r)} + M^{(r)})$-close in the C^r-topology for all $t \in T \backslash T_R$, where T_R is the finite collection of impact intervals each of them of length $O(|\delta| + M^{(r)})$.*

C. Persistence of periodic orbits and hyperbolic sets

The (C^1)-closeness of the billiard and smooth flows after one regular reflection leads, using structural stability arguments, to persistence of regular periodic and homoclinic orbits. The above error estimates allow us to establish quantitative version of these persistence results:

Theorem 4 (Ref. 5). *Consider a family of Hamiltonian systems with billiard-like potential $V(q,\epsilon)$ on D. Let $P^b(t)$ denote a T-periodic, non-parabolic, non-singular orbit of the billiard flow. Then, for any choice of $\nu(\epsilon), \delta(\epsilon)$ such that $\nu(\epsilon), \delta(\epsilon), m^{(1)}(\epsilon), M^{(1)}(\epsilon) \to 0$ as $\epsilon \to 0$, for sufficiently small ϵ, the smooth Hamiltonian flow has a uniquely defined periodic orbit $P^\epsilon(t)$ of period $T^\epsilon = T + O(\nu + m^{(1)} + M^{(1)})$, which stays $O(\nu + m^{(1)} + M^{(1)})$-close to P^b for all t outside of collision intervals (finitely many of them in a period) of length $O(|\delta| + M^{(1)})$. Away from the collision intervals, the local Poincaré map near P^ϵ is $O_{C^r}(\nu + m^{(r)} + M^{(r)})$-close to the local Poincaré map near P^b. In particular, if P^b is hyperbolic, then P^ϵ is also hyperbolic and, inside D^ϵ, the stable and unstable manifolds of P^ϵ approximate $O_{C^r}(\nu + m^{(r)} + M^{(r)})$-closely the stable and unstable manifolds of P^b on any compact, forward-invariant or, respectively, backward-invariant piece bounded away from the singularity set in the billiard's phase space; furthermore, any transverse regular homoclinic orbit to P^b is, for sufficiently small ϵ, inherited by P^ϵ as well.*

Such results may be utilized to establish the existence of specific orbits when two small parameters are involved. Consider a family of billiard tables D_γ, where γ corresponds to some geometrical parameter. For example, in Ref. 39, D_0 is an ellipsoid and D_γ is a family of perturbed shapes, where γ measures their closeness to the ellipsoid. For $\gamma \neq 0$, the billiard map in D_γ has transverse homoclinic orbits with splitting angle of order γ (see Ref. 39). Then, provided $(\nu + m^{(1)} + M^{(1)}) \ll \gamma$, the smooth flow associated with the two-parameter billiard potential families $V(q;\gamma,\epsilon)$ on D_γ also has transverse homoclinic orbits. This inequality provides a bound on $\epsilon(\gamma)$. More generally, when, for sufficiently small γ, a certain $\gamma-$robust property in the C^1 topology may be proved, the smooth flows attain the same property provided $(\nu + m^{(1)} + M^{(1)}) \ll \gamma$.

To provide concrete bounds on ϵ, assume hereafter that the behavior of the potential near the boundary dominates the estimate; we say that $V(q;\epsilon)$ is *boundary dominated*, if $V(q;\epsilon)$ and its derivatives are smaller in the interior of D_{int}^ϵ (i.e., in the region bounded by the surfaces $Q(q;\epsilon) = Q_i + \delta_i(\epsilon)$) than on the boundary of this domain. This means that for boundary dominated potentials $m^{(r)}(\delta;\epsilon) = \sup_{q \in D_{int}^\epsilon} ||\partial^l V(q;\epsilon)|| = \sup_{q \in \partial D_{int}^\epsilon} ||\partial^l V(q;\epsilon)||$ (here, $l = 1, \ldots, r+1$). In this case, one may choose (ν, δ) so as to minimize the error bounds. Table I summarizes the resulting errors of the billiard approximation for several commonly used potentials. The last column in the table is achieved by insisting that the error (the third column) is smaller than γ for $r = 1$. Namely, if γ represents a measure of the C^1 robustness of some dynamical property (e.g., of the transversality of homoclinic points), the last column shows how small should ϵ be to ensure that this property persists for the smooth Hamiltonian flow.

III. EFFECTS OF SINGULARITIES: THE EMERGENCE OF ISLANDS OF STABILITY IN TWO-DIMENSIONAL FLOWS

Section II shows that regular hyperbolic billiard orbits persist in the smooth and sufficiently steep flows, namely, that the common intuition that smooth flows may be replaced by billiards is justified in such cases. Here, we show that this

TABLE I. Error estimates for several potentials, assuming the boundary-domination property. We denote $\beta_r = \frac{1}{r+2+\frac{1}{\alpha}}$ at $\alpha \geq 1$ and $\beta_r = \frac{\alpha(r+2)}{(r+1+\alpha)(r+2+\frac{1}{\alpha})}$ at $\alpha \leq 1$ (see Ref. 5).

Potential	Boundary width	Error	Impact intervals	C^1 robustness
$W(Q;\epsilon)$	$\eta(\epsilon)$	$m^{(r)} + \nu + M^{(r)}$	$\|\delta\| + M^{(r)}$	$\epsilon_g(\gamma)$
$e^{-\frac{Q}{\epsilon}}$	$\epsilon\|\ln\epsilon\|$	$\sqrt[r+2]{\epsilon}$	$\sqrt[r+2]{\epsilon}$	$o(\gamma^3)$
$e^{-\frac{Q^2}{\epsilon}}$	$\sqrt{\epsilon\|\ln\epsilon\|}$	$\left(\frac{\epsilon}{\|\ln\epsilon\|}\right)^{\frac{1}{2(r+2)}}$	$\left(\frac{\epsilon}{\|\ln\epsilon\|}\right)^{\frac{1}{2(r+2)}}$	$o(\gamma^6)$
$\left(\frac{\epsilon}{Q}\right)^{\alpha}$	$\epsilon^{\frac{r+2}{r+2+1/\alpha}}$	$\epsilon^{\frac{1}{r+2+\frac{1}{\alpha}}}$	ϵ^{β_r}	$o(\gamma^{3+\frac{1}{\alpha}})$

approximation *fails* near singularities of the billiard flow in the two-dimensional case. Indeed, we prove that tangent homoclinic orbits, tangent periodic orbits, and some of the orbits that have end points in corners give rise to stable periodic and quasiperiodic motion (hereafter—stability islands) in the smooth case. These results may be applied to families of Sinai billiards that admit such singular trajectories. They imply that even though the smooth reflections are as close as possible to those of the billiard (as shown in Sec. II), global properties such as ergodicity are destroyed by the islands. Thus, even when the decay of correlations for the billiard map is exponential, the correlations for the smooth flow, for any finite ϵ, have recurrences and do not decay at all in the islands. The prevailing conjecture, supported by simulations, is that the mere existence of such islands leads to a power-law decay of the correlations in the chaotic component due to "stickiness" to the islands boundaries. We thus propose that even though the singularity-induced islands are small for small ϵ, their influence on the decay of correlations in the chaotic component may be important.

To establish these results, we consider two-parameter families of billiard-like potentials $V(q;\gamma,\epsilon)$ of the billiard family D_γ. The geometrical parameter γ is introduced to unfold the billiard trajectory singularity.

In Sec. III A, we consider the unfolding of tangent periodic and homoclinic orbits, see Fig. 4.[1,3] We assume that at $\gamma = 0$ the billiard table D_0 has a tangent periodic/homoclinic orbit and prove that the smooth flow has a stable periodic orbit near this singular orbit. For the tangent periodic orbit case, we find the normal form of the local return map. We then prove that this map has stable (elliptic) periodic orbits for certain parameter values. In the (γ,ϵ) parameter plane, these values form a stability wedge which emanates from the origin. The dependence of the island phase-space area and of the width of the stability wedge on ϵ and on the energy level is found (see Theorem 5 below). Notably, we see that independent of the regularization of the billiard (the particular choice of the billiard-like potential), the existence of a tangent periodic orbit always implies the existence of a stability region in the (γ,ϵ) plane. On the other hand, the normal form of the return map depends on the potential in a non-trivial fashion (see Table II). Selecting a path inside, this wedge of stability down to the ϵ-axis defines a one-parameter family of Hamiltonian flows $h_t(\epsilon,\gamma(\epsilon))$ that converge to the billiard flow and for which elliptic islands exist for all $\epsilon < \epsilon_0$, namely, for arbitrarily small ϵ. Hence, even though the dispersive billiard is mixing, such smooth regularizations of it are non-ergodic for arbitrarily small ϵ. The size of these islands decreases with ϵ, typically as a power law (see Table II).

In Sec. III B, we consider the unfolding near corners.[4] To this aim, we assume that at $\gamma = 0$ the billiard table D_0 has a sequence of regular reflections which begins and ends at a corner (termed a *corner polygon*). We prove that under some additional prescribed conditions, such a polygon may produce stable periodic orbits in a γ,ϵ wedge which emanates from the origin. The normal form for the return map near the stable orbit turns out to be the area preserving Henon map. Here, in contrast to the tangent case, the existence and the stability of a periodic orbit which limits to the corner polygon depend on both the form of the smooth potential and the billiard geometry. Namely, taking two different regularizations of a given billiard family with a corner polygon, one regularization may produce a stable periodic orbit, whereas the other may have no periodic orbits limiting to this corner polygon.

Now, consider an arbitrary one-parameter family of dispersing billiards D_γ. One would like to characterize the appearance of islands for sufficiently small ε as a function of γ. Since dispersive billiards are hyperbolic, it is clear that for sufficiently small ε the only mechanism for creating islands is the behavior of the smooth system near singular orbits of the billiard, namely, near tangent orbits and near orbits which enter a corner. Generically, if no special symmetries are imposed, D_0 has many near-tangent periodic orbits, but no tangent ones. We conjecture that for generic families, a small deformation of D_0 to D_γ can transform a near-tangent periodic orbit of period n to a tangent one for some γ of order λ^{-n}, where $n \gg 1$. This implies that for sufficiently small ε, very small (size $\delta_{\tan}(\varepsilon)\lambda^{-n}$) islands will appear in the smooth Hamiltonian approximation to the billiard flow in D_γ. On the other hand, we expect D_0 to have many corner polygons and, in particular, corner polygons with only one edge—a minimizing cord (a segment emanating from one of the corners which has a straight angle reflection from the boundary). Typically, these corner polygons will have the angles ϕ_{in}

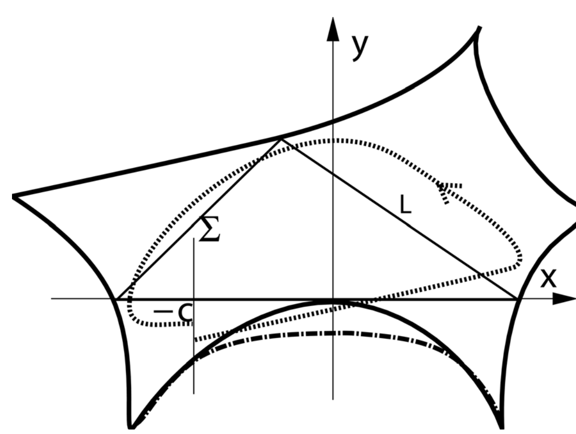

FIG. 4. Tangent periodic orbits. The solid thick boundary corresponds to the billiard table D_0, and the dotted dashed boundary corresponds to its deformation for some $\gamma > 0$. L is a simple tangent periodic orbit of D_0, whereas for $\gamma > 0$, it is the regular hyperbolic orbit L_γ^-. The return map to Σ is provided by Eq. (22) and Table II (see Ref. 3).

and ϕ_{out} in general position, i.e., ϕ_{out} will not be an extremum of the scattering function for the given ϕ_{in}. So, according to our results, only a saddle periodic orbit can be born from any such polygon at sufficiently small ε. However, due to the transitivity, we can expect sufficiently long corner orbits for which ϕ_{out} will be close to the extremum of the scattering function. Hence, some small islands can be obtained from these orbits after γ is tuned appropriately.

Note that in applications where one needs to tailor a billiard table with some given properties the idea of small perturbation of the billiard boundary is, in fact, irrelevant, so one can consider large changes in γ as well. Then, producing low period tangent orbits or minimizing cords with any given values of (ϕ_{in}, ϕ_{out}) is very easy. In this way, one can produce elliptic islands of a visible size in families of billiard-like potentials with mixing limiting billiard. For example, the experimental works of Kaplan *et al.*[33] shows that elliptic islands that arise due to corners significantly influence the statistics of escape from cold atom optical traps.

A. Islands produced by tangencies

Consider the family of dispersing billiards D_γ and assume that at $\gamma = 0$, the billiard table D_0 has a simple tangent periodic orbit L (i.e., L has a single tangent collision at a point where the boundary has non-vanishing curvature). We assume that the dependence of D_γ on γ is in general position, so that the tangent periodic orbit disappears, say, at $\gamma < 0$, whereas at the opposite sign of γ, two periodic orbits are born, see Fig. 4. One of these periodic orbits (L_γ^-) passes near the former point of tangency without hitting the boundary, and the other (L_γ^+) has a regular reflection close to that point. Away from the bifurcation point the persistence results imply that the smooth system has similar structure at sufficiently small ϵ; hence, one concludes that the smooth system must also have a bifurcation value γ_ϵ at which the two periodic orbits $L^{\pm}_{\gamma_\epsilon,\epsilon}$ collide and disappear. Namely, the tangent periodic orbit of the singular system becomes a parabolic periodic orbit of the smooth system. Moreover, just before the coalescence of the orbits, one of them must become linearly stable due to index arguments. In Ref. 1, we prove that the above scenario actually occurs (see also Refs. 40 and 41). More precisely, we prove that for each fixed sufficiently small ϵ, there is an interval of γ values for which the smooth flow has a linearly stable periodic orbit. The underlying geometrical mechanism for the creation of this non-hyperbolic behavior is a horseshoe bifurcation near the tangent periodic orbit. Determining under which conditions such a bifurcation occurs in non-dispersive billiards is an interesting problem.

To establish the existence of elliptic islands and to find their size, we explicitly construct the return map near the linearly stable periodic orbit. To this aim, we further assume that near the tangent collision point, for any given energy level H, we can define a boundary layer region so that near it the energy H may be scaled out. More precisely, we assume that the billiard-like potential family $(V(q;\gamma,\epsilon))$ satisfies the following *scaling assumption:*

[S] *There exist some* $\delta = \delta(\epsilon, H) > 0$, $\beta = \beta(\epsilon, H)$, *and* $\nu(\epsilon, H)$ *such that* $\delta, \beta, \nu/H \to 0$, *as* $\epsilon \to 0$, *and the function*

$$\tilde{W}_\epsilon(\tilde{Q}) = \frac{W(\delta\tilde{Q} + \beta, \epsilon) - \nu}{H\delta^{3/2}} \tag{20}$$

converges as $\epsilon \to 0$ *to a* C^{r+1} *function* $\tilde{W}_0(\tilde{Q})$, *either for* $\tilde{Q} > 0$ *or for all real* $\tilde{Q}$. *The convergence is* C^{r+1}*-uniform on any closed finite interval of values of* $\tilde{Q}$ *from the domain of definition. Furthermore, the integral*

$$\int_1^{+\infty} \tilde{W}'_\epsilon(q)\frac{dq}{\sqrt{q}} \tag{21}$$

converges uniformly for all sufficiently small ϵ.

This scaling assumption is satisfied by all the potentials that we examined so far and serves to determine the dependence of the scaling parameters on ϵ and the energy (see Table II). The following theorem is the main result of Ref. 3:

Theorem 5. *Consider a family of dispersing billiards* D_γ *having a simple non-degenerate tangent periodic orbit at* $\gamma = 0$. *Consider a two-parameter family of* C^r, $r \geq 5$, *smooth Hamiltonian flows* $h_t(\epsilon, \gamma)$ *with billiard-like potentials approximating the billiard flows as* $(\epsilon, \gamma) \to 0$. *Assume that the barrier function near the point of tangency satisfies the scaling assumption [S] for some* $\delta(\epsilon, H)$ *and that the associated function* F *is such that the range of values of* $F'(v)$ *includes* R^- *(all negative values).*

Then, for small ϵ, *in the* (ϵ, γ) *plane, there exists a wedge* $C^-\delta(\epsilon, H) < \gamma < C^+\delta(\epsilon, H)$ *(with some constant* $C^\pm$*) such that for all parameter values in this wedge, on the energy level* H, *there exist elliptic islands of width proportional to* $\delta(\epsilon, H)$.

The proof of this theorem is constructive. The asymptotic normal form, as $\epsilon \to 0$, of the return map of the

TABLE II. Islands scalings near tangent periodic orbits.

Potential $W(Q;\epsilon)$	Islands' scaling $\delta(\epsilon,H)$	E-shift $\nu(\epsilon,H)$	Q-shift $\beta(\epsilon,H)$	$\tilde{W}_0(\tilde{Q})$	Return map $F(v)$
$e^{-\frac{Q}{\epsilon}}$	ϵ	0	$-\epsilon\ln(\delta^{\frac{3}{2}}H)$	$e^{-\tilde{Q}}$	$\frac{\sqrt{\pi}}{2}e^{-v}$
$(1-Q^\alpha)^{\frac{1}{\epsilon}}$	$\frac{\epsilon^{\frac{1}{\alpha}}}{\alpha}(-\ln(\epsilon^{\frac{3}{2\alpha}}H))^{\frac{1}{\alpha}-1}$	0	$\left(-\epsilon\ln(\delta^{\frac{3}{2}}H)\right)^{\frac{1}{\alpha}}$	$e^{-\tilde{Q}}$	$\frac{\sqrt{\pi}}{2}e^{-v}$
$\left(\frac{\epsilon}{Q}\right)^\alpha$	$\left(\frac{\epsilon^\alpha}{H}\right)^{\frac{1}{\alpha+3/2}}$	0	0	$\frac{1}{\tilde{Q}^\alpha}$	$\frac{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}{2\Gamma(\alpha)}\frac{1}{v^{\alpha+\frac{1}{2}}}$
$\epsilon\lvert\ln Q\rvert^\alpha$	$\left(\frac{\alpha\epsilon}{H}\left(\frac{2}{3}\lvert\log\frac{\alpha\epsilon}{H}\rvert\right)^{\alpha-1}\right)^{\frac{2}{3}}$	$\epsilon\lvert\ln\delta\rvert^\alpha$	0	$\lvert\ln\tilde{Q}\rvert$	$\frac{\pi}{2\sqrt{v}}$

Hamiltonian flow in an $O(\delta)$-neighborhood of the tangent periodic trajectory is the two-dimensional map,

$$\bar{u} = v, \quad \bar{v} = \xi\left(v + \frac{a}{\sqrt{\kappa}}F(v)\right) - (\xi - 2)\Gamma - u, \qquad (22)$$

where

$$F(v) = -\int_0^{+\infty} \tilde{W}_0'(v + x^2)dx = -\frac{1}{2}\int_v^{+\infty} \tilde{W}_0'(Q)\frac{dQ}{\sqrt{Q - v}}. \qquad (23)$$

$F(v)$ is well defined, by Eq. (21), either on R^+ or on R (i.e., on the domain of definition of $\tilde{W}_0$) and it is C^r-smooth. Table II provides its form for various potentials. The parameter κ is the billiard's curvature at the tangency. ξ is the sum of the singular multipliers of L, which are the multipliers of L_γ^- for $\gamma \to 0^+$ (i.e., the multipliers of L if one disregards the influence of the tangent point). Since we consider here dispersive geometry, it follows that $|\xi| > 2$ and the sign of ξ equals to $(-1)^n$, where n is the number of reflections of L_γ^- at sufficiently small γ. The parameter $\Gamma = \gamma/\delta$ is the rescaled unfolding parameter, and a is a geometrical parameter defined by the billiard return flow near L_0 at $\epsilon = \gamma = 0$ ($a > 0$ for dispersive billiards). To complete the proof, the return map (22) is analyzed. One shows that it has a fixed point, that its eigenvalues have to sweep the unit circle as Γ sweeps a finite interval and that the Birkhoff coefficient cannot be identically zero along this interval. Hence, one concludes that the return map (22) has an elliptic island of finite size (in the rescaled variables u and v).

In the original, non-scaled variables and parameters, the island exists in a δ-size wedge of γ values and its area is of order $\sqrt{H}\delta^2/\xi a$ in the (y, p_y) cross-section. Table II presents the calculation of the scaling and shifting factors and the return map function $F(v)$ for several typical potentials. Notably, islands appear at any fixed H value and scale as a power law in (ϵ). Their scaling for large energies $H = H(\epsilon)$ may be calculated as long as $\delta(\epsilon, H) \to 0$ as $\epsilon \to 0$, see Table II. Finally, we notice that ξ, the sum of the singular multipliers of L is expected to increase exponentially with the number of reflections of L. Hence, the tangent orbits with the smallest period are expected to have the largest islands.

The above results are derived for two-dimensional dispersing billiards, or, more generally, for tangent periodic orbits with exactly one tangent collision and with all the regular collisions occurring with dispersing parts of the boundary components. It may be interesting to study the behavior near tangencies when some of the components are focusing and the behavior when more than one tangency occurs (with or without symmetry). A generalization of this construction to higher-dimensional settings is possible and should lead to a proof of the existence of center-saddle periodic orbits, namely, to proving non-hyperbolic behavior for arbitrary small ϵ even when the limiting billiard is hyperbolic.

B. Islands produced by corners

The other mechanism for creating elliptic motion in smooth Hamiltonian families that limit to dispersing billiards is corners. Here, we consider a sequence of regular billiard reflections that begins and ends at the same corner point of the billiard. Such a sequence is denoted by P_0 and is called a **corner polygon** (see Fig. 5). Notice that such a sequence is not an orbit of the billiard. Denote by θ the angle created by the billiard boundary arcs joining at this corner, and define ϕ_{in}, ϕ_{out} to be the angles created by the corner polygon with the corner bisector (notice the different directions of ϕ_{in} and ϕ_{out}). As opposed to the tangent singularity, such a corner polygon may produce a number (possibly zero) of periodic orbits of the smooth flow. The number and the stability of the emerging orbits depend on *both* the billiard geometry (in particular, on $\phi_{in}, \phi_{out}, \theta$) *and on the form of the potential*. Here, we assume that the potentials are billiard-like (satisfying conditions I–IV of Sec. II) and that there is sufficient repulsion from the corner regions so that trajectories cannot remain in the corner for unbounded times (the scattering assumption). We show that usually the produced periodic orbits are hyperbolic. Yet, by introducing an additional geometrical parameter, γ, it is often possible to create wedges in the plane of parameters (γ, ϵ) which corresponds to the existence of stable periodic orbits.

To provide precise statements, we recall first the behavior of the billiard near corners and then the behavior of smooth flows near corners.

Billiard reflections near a corner may be characterized by the outgoing angles $\Phi_\pm(\varphi; \theta)$. Provided $\theta > 0$, an incoming parallel ray enters the corner with an angle φ and exits a neighborhood of the corner after a finite number of reflections ($N_\theta := \lfloor \frac{\pi}{\theta} \rfloor$ or $N_\theta + 1$). The angles that the outgoing trajectories of the parallel ray make with the corner bisector are close (up to corrections associated with the curvature of the billiard boundary near the corner) to one of two possible angles $\Phi_\pm(\varphi; \theta)$. The angle $\Phi_+(\varphi; \theta)$ is realized if the upper boundary is hit first, and $\Phi_-(\varphi; \theta)$ is realized otherwise. The billiard scattering angles $\Phi_\pm(\varphi; \theta)$ and the number of reflections at

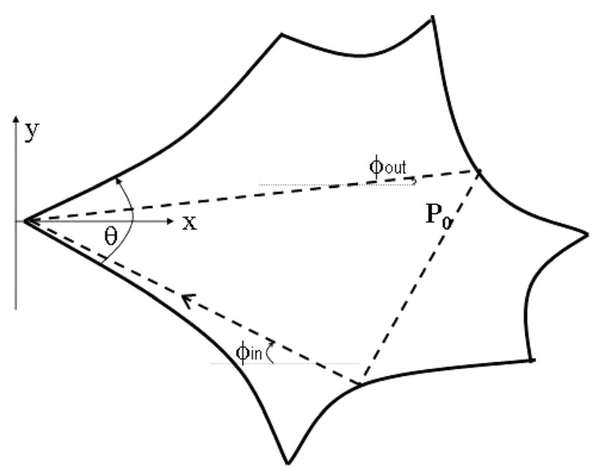

FIG. 5. Corner polygon geometry. The return map to the cross section $x = R$ is constructed by following the regular billiard reflections at $x > R$ and the properties of the scattering function near the corner (see Ref. 4).

the corner may be explicitly computed by geometrical means, see, e.g., Ref. 4. When $\theta = \frac{\pi}{N}$ the two scattering angles are equal: $\Phi_+(\varphi;\theta) = \Phi_-(\varphi;\theta) = (-1)^{N+1}\varphi$.

To describe the behavior of smooth billiard-like systems near the corners, we introduce an additional ingredient, the *scattering function*. This function captures the main features of the scattering by the potential at the corner point. To define the scattering function, we make some natural scaling assumptions on the potential V near the corner. Let (x,y) denote Cartesian coordinates with the x-axis being the bisector of the billiard corner, and the origin at the corner point (see Fig. 5). We assume there exists a scaling

$$(\bar{x},\bar{y}) = \frac{1}{\delta(\varepsilon)}(x - x_\varepsilon, y - y_\varepsilon)$$

such that in the scaled coordinates the potential has a finite limit as $\varepsilon \to 0$,

$$V(x_\varepsilon + \delta\bar{x}, y_\varepsilon + \delta\bar{y};\varepsilon) \to V_0(\bar{x},\bar{y}).$$

Let the level set $V_0(\bar{x},\bar{y}) = h$ be a hyperbola-like curve, which asymptotes the lines $\bar{y} = \pm\bar{x}\tan\frac{\theta}{2} + c_\pm$ as $\bar{x}\to\infty$. This level curve bounds an open wedge $V_0 \le h$ which extends towards $\bar{x} = +\infty$. For the scaled system given by the Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V_0(\bar{x},\bar{y}), \tag{24}$$

every trajectory with energy $H = h$ lies in this wedge.

Under some natural assumptions on V, we show that the solutions to the scaled equations go towards $\bar{x} = +\infty$ as $t \to +\infty$ and as $t \to -\infty$, and that they always have asymptotic incoming ($\phi_{in} = -\lim_{t\to-\infty}\arctan\frac{p_y(t)}{p_x(t)}$, $|\phi_{in}| \le \frac{\theta}{2}$) and outgoing angles ($\phi_{out} = \lim_{t\to+\infty}\arctan\frac{p_y(t)}{p_x(t)}$, $|\phi_{out}| \le \frac{\theta}{2}$). Moreover, there is a well defined limiting scattering function $\phi_{out} = \Phi^0(\phi_{in},\eta)$, where η is a scattering parameter of a parallel beam entering the wedge at $x = +\infty$ with incoming angle ϕ_{in}. This scattering function Φ^0 carries the needed information on the dynamics near the corner. In particular, the range of $\Phi^0(\phi_{in},\cdot)$ provides the interval I of allowed outgoing angles. While $I_{bill}(\phi_{in}) = [\Phi_{-(-1)^{N_\theta}}(\phi_{in}), \Phi_{+(-1)^{N_\theta}}(\phi_{in})] \subseteq I(\phi_{in})$, there are natural examples in which the interval $I(\phi_{in})$ is strictly larger than the billiard scattering range $I_{bill}(\phi_{in})$. For example, when $\theta = \frac{\pi}{N}$ the interval $I_{bill}(\phi_{in})$ degenerates to a point, so generically, we expect $I(\phi_{in})$ to be strictly larger than $I_{bill}(\phi_{in})$ when θ is close to $\frac{\pi}{N}$. Moreover, for such θ values, the scattering function must be non-monotone (see Ref. 4 for an explicit example).

A corner polygon of the billiard is said to be non-degenerate if $\phi_{out} \in I(\phi_{in})$, and infinitesimally small changes in ϕ_{out} change the return position of the trajectory so that the corner is missed. A billiard corner polygon with incoming angle ϕ_{in} and outgoing angle ϕ_{out} may produce a periodic orbit of the smooth Hamiltonian flow (8) at small nonzero ε if and only if $\phi_{out} = \Phi^0(\phi_{in},\eta)$ for some η. Given a $\phi_{out} \in I$, there exists a set (discrete, in general) of η's such that $\phi_{out} = \Phi^0(\phi_{in},\eta)$. To produce a periodic orbit, the scattering function needs to be monotone and the global behavior near the corner polygon needs to be non-degenerate:

Theorem 6 (Ref. 4). *Consider a family $V(x,y;\varepsilon)$ of billiard-like potentials limiting to a billiard in D and satisfying the scattering assumption and the corner scaling assumption. Assume D has a non-degenerate corner polygon with incoming and outgoing angles (ϕ_{in},ϕ_{out}). Then, for sufficiently small ε, for every η such that $\phi_{out} = \Phi^0(\phi_{in},\eta)$ and $\frac{\partial}{\partial\eta}\Phi^0(\phi_{in},\eta) \neq 0$, the Hamiltonian family has a hyperbolic periodic orbit which, as $\varepsilon \to 0$, limits to the billiard corner polygon.*

To prove this theorem, a return map of the smooth flow to an interior cross section of the corner polygon is constructed. The outer part of the map is well approximated by the regular billiard reflections, whereas the behavior near the corner is controlled by the scattering function. The monotonicity of the scattering function together with the hyperbolicity of the outer map allows to prove that saddle periodic orbits are created. If, on the contrary, ϕ_{out} corresponds to a maximum or minimum of $\Phi^0(\phi_{in},\eta)$ as a function of η then elliptic motion may emerge. As the appearance of such an extremum is a codimension-1 phenomenon, to obtain a robust picture, it is necessary to consider here an additional parameter and to ensure that the appropriate non-degeneracy conditions are set. We thus introduce, as in the tangent case, a geometrical parameter γ which is responsible for regular changes in the geometry of the billiard. The corresponding two-parameter family of billiard-like potentials $V(x,y;\varepsilon,\gamma)$ is called a tame perturbation of the billiard-like potential $V(x,y;\varepsilon,0)$ if the barrier functions W do not depend on γ, the pattern functions Q, defined in some neighborhood of the open boundary arcs without the corners, are C^r-smooth with respect to γ and the scaled potentials V_ε depend C^r-smoothly on γ as well. Finally, this tame family is called non-degenerate if some explicit expression does not vanish (the return map to the corner along the regular reflections must change with γ in a generic fashion).

Theorem 7 (Ref. 4). *Consider a family of billiard-like potentials $V(x,y;\varepsilon)$ limiting to a billiard in a domain D and satisfying the scattering assumption and the corner scaling assumption with a scaling parameter $\delta(\varepsilon)$. Assume D attains a non-degenerate corner polygon with incoming and outgoing angles (ϕ_{in},ϕ_{out}). Let $V(x,y;\varepsilon,\gamma)$ be a one-parameter tame perturbation of $V(x,y;\varepsilon)$, satisfying the non-degeneracy assumption. Then, for every η^* such that $\phi_{out} = \Phi^0(\phi_{in},\eta^*)$ is a strict extremum (i.e., $\frac{\partial}{\partial\eta}\Phi^0(\phi_{in},\eta^*) = 0$ and $\frac{\partial^2}{\partial\eta^2}\Phi^0(\phi_{in},\eta^*) \neq 0$), there exists a wedge of width $\delta^2(\varepsilon)$ in the (ε,γ) parameter plane in which the Hamiltonian flow defined by the potential $V(x,y;\varepsilon,\gamma)$ has elliptic islands of size $O(\delta^2)$, where the islands limit to the billiard corner polygon as $\varepsilon \to 0$.*

To prove this theorem, the return map is again constructed. Here, using the extremal behavior of the scattering function, it is shown that in some rescaled coordinates the return map becomes, to leading order, the area-preserving Hénon map. Moreover, it is established that if the corner polygon has $n+1$ edges, then the bifurcation coefficient a in the Hénon map is proportional to λ^{2n}/δ^2 and the rescaling of the phase space area includes factors proportional to λ^{3n}/δ^4. Hence, the size of the islands, in both parameter space and phase space, decreases exponentially with the number of reflections and as a power law with ε. The non-monotonicity of the scattering function naturally arises when its range is larger than the billiard scattering range $I_{bill}(\phi_{in})$ as occurs when θ is close to $\frac{\pi}{N}$.

The stability of the corner-passing periodic orbits is solved here in terms of the scattering function Φ which is defined only by the potential at the corner, and is almost independent of the geometrical properties of the underlying billiard (the genericity condition is the only place where the geometry enters: this condition is always fulfilled if the billiard is dispersive and the corner polygon is never tangent to the boundary, while in the non-dispersive billiard where the boundary contains convex components, this condition may be violated, but it may always be achieved by a small smooth perturbation of the boundary). This fact is somewhat surprising in view of the behavior near tangencies. In particular, it shows that contrary to the previously studied cases (of non-singular periodic orbits and of tangent periodic orbits), the existence of the periodic orbit which limits to a corner polygon is not determined by the billiard geometry alone.

In Theorem 7, which is concerned with the general case, we cannot know if a stable periodic orbit is produced by a corner polygon without computing the potential-dependent scattering function. Unfortunately, there seems to be no explicit formulas which would relate the scattering function to the potential V. We prove that $\Phi^0(\phi,\eta)$ is a smooth function and that as $\eta\to\pm\infty$ it approaches the billiard scattering angles $\Phi_\pm(\phi;\theta)$. Finding an analytical form for Φ and for its critical values is probably an unsolvable question in the general case. Indeed, it is known[42] that in the case $V_0(\bar{x},\bar{y}) = e^{\bar{y}-k\bar{x}} + e^{-\bar{y}-k\bar{x}}$ (here, $k=\tan\frac{\theta}{2}$, so $k\in(0,1)$), the system given by Eq. (24) has no other analytic integrals which are polynomial in momenta for $k\neq 1$ and $k\neq 1/\sqrt{3}$ (i.e., when the corner angle θ differs from $\pi/2$ and $2\pi/3$). The non-existence of meromorphic integrals for this system is proven in Ref. 43 (based on the method of Ref. 44) for $k\neq 4/(m(m-1))^2, m\in Z$. While we conclude that Φ cannot be expected to be explicitly written, it is straightforward to recover it numerically.

Nonetheless, there is one case in which we can prove the creation of elliptic islands by using only asymptotic information about the scattering function. This occurs when a billiard corner polygon bifurcates into a regular periodic orbit of the billiard: a billiard periodic orbit may detach from the corner point under a small perturbation of the boundary if and only if $\phi_{out} = \Phi_\pm(\phi_{in},\theta)$. In terms of the scattering function Φ, this case corresponds to $\eta=\pm\infty$ and it is not covered by the above mentioned Theorems 6 and 7. The behavior of the corner-passing periodic orbits of the Hamiltonian flow at non-zero ε has in this case a more profound relation with the billiard geometry:

Theorem 8 (Ref. 4). *Consider a dispersing billiard-like family, with a non degenerate corner polygon satisfying* $\phi_{out} = \Phi_+(\phi_{in})$. *If* $\Phi^0(\phi_{in},\eta)$ *is monotone, then, for sufficiently small* ε, *an elliptic periodic orbit is produced by the billiard corner polygon if* $(\frac{\pi}{\theta} - \left[\frac{\pi}{\theta}\right] - \frac{1}{2})\theta < \phi_{in} < \frac{\pi}{2}$.

Note that the nature of the billiard flow at the corner is highly sensitive to the numerical properties of θ, with bifurcation points at $\theta_N = \frac{\pi}{N}$ and $\theta_N^* = \frac{\pi}{N+\frac{1}{2}}$. The influence of these bifurcations on the limiting Hamiltonian flow has yet to be studied—it may produce nontrivial dynamics (e.g., the analysis of Sec. IV). The effect is especially relevant for small angles.

IV. FULLY ELLIPTIC ORBITS IN MULTI-DIMENSIONAL BILLIARD-LIKE POTENTIALS

The possibility of extending the two-dimensional results regarding the destruction of ergodicity by the smooth potentials to higher dimensions is not obvious. Intuitively, one could argue that in the higher-dimensional setting there will be always enough unstable directions to destroy any stability region and might conclude that the above results are inherently two-dimensional. From a mathematical point of view, the appearance of islands of stability is natural in Hamiltonian systems which are not hyperbolic or partially hyperbolic, see Ref. 45 and references therein for the (C^1)-version of this conjecture. However, a specific family of systems like (1), limiting to the hyperbolic Sinai billiards, may turn out to be non-generic (see the introduction in Ref. 6), and it is unknown if generic (C^1)-perturbations are relevant in the framework of mechanical systems. Crucially, it is not immediately obvious whether the Hamiltonian systems under consideration are partially hyperbolic or not.

In fact, the analysis we did for the two-dimensional case (Sec. III) can be carried out onto higher dimensions in order to show that the smooth approximation of any dispersive billiard cannot be uniformly hyperbolic. The arguments are based on the same geometrical structure which ensures the uniform hyperbolicity of the dispersive billiards themselves. It was noted by Krylov[46,47] that the key to the Boltzmann conjecture is a characteristic instability of the hard-sphere gas in the space of dimension two and higher. It is related to the convex shape of the colliding bodies (so it does not take place in one-dimensional systems). Sinai showed that this instability is an inherent property of dispersive billiards and built with co-workers a deep mathematical theory which indeed relates this instability to the statistical properties of the hard-spheres gas.[9–11,24] For a general dispersive billiard (a domain with a piece-wise concave boundary), the Krylov-Sinai instability is expressed as follows: a parallel beam diverges after reflecting from the boundary. In the phase space, this translates to a cone-preservation property: the positive cone $dq\cdot dp\geq 0$ is mapped inside itself by the derivative of the billiard flow (here, q is the vector of coordinates and p is the vector of momenta). This is equivalent to the hyperbolicity of the billiard flow: at each point of every regular orbit, there are stable and unstable subspaces invariant with respect to the derivative of the billiard map, the unstable directions belong to the positive cone and the stable directions belong to its complement.[9,10,48] This cone structure is independent of the particular shape of the dispersive billiard, so it creates a universal reason for the ergodicity of any such billiard.

However, the dispersing property has other universal consequences. As the unstable subspace belongs to the positive cone, it is uniquely parameterized by momenta, and since every collision changes the orientation in the momenta space, it follows that the orientation in the unstable subspace flips at every collision.[1,6] We can, therefore, always have a continuous family of initial conditions such that the flow map (for some fixed time) will keep the orientation of the unstable subspace for some initial conditions (those having an even number of regular collisions at the given time interval) and will change the orientation for the other initial conditions (those having an odd number of regular collisions); the transition happens at initial conditions that have singular orbits. There can be no uniformly hyperbolic *smooth* flow with the same behavior (as there will be no singular orbits at which such transition can happen). Therefore, the uniformly hyperbolic structure of dispersive billiards cannot survive any smooth approximation of the billiard potential.

These arguments do not preclude the existence of some hypothetical partially hyperbolic structure in a dispersive billiard of dimension higher than 2. However, in Ref. 6, we (together with Rapoport) showed that no such universal structure could exist which cannot be destroyed by the smooth approximation. Indeed, we showed that in any dimension fully elliptic orbits appear in a predictable way in smooth systems that are arbitrarily close to Sinai billiards,[6] thus providing the first explicit mechanism for the creation of stable periodic orbits in high-dimensional smooth near dispersing-billiard systems.

In our constructions, the stability zones in the n-dimensional settings are created by trajectories that enter a corner point. At the corner, n codimension-1 surfaces meet in a symmetric fashion, so that the corresponding solid angle is controlled by a single geometrical parameter μ (see Fig. 6 and Eq. (25)). We find, first numerically for the 3-dimensional case,[7] and then analytically for the general case,[6] that a corresponding smooth steep potential family has a stable orbit in wedges of parameters (μ, ε) that extend towards the μ axis. Thus, our main result is:

Theorem 9 (Ref. 6). *There exist families of analytic billiard potentials that limit (in the sense of Sec. II A, see Eqs. (2)–(8)) to Sinai billiards in n-dimensional compact domains (in particular, for any finite n such billiards are hyperbolic, ergodic, and mixing), yet, for arbitrarily small ε, the corresponding smooth Hamiltonian flows have fully elliptic periodic orbits.*

To establish this result, we generalize the geometrical construction depicted in Fig. 6 to n-dimensional billiards depending on a geometrical parameter μ (see more details below). These billiards are Sinai billiards for any $\mu \in (0, 1)$ and depend smoothly on μ. We then consider families of symmetric potentials $W(q; \mu, \varepsilon)$ that limit as $\varepsilon \to 0$, for any fixed μ, to these billiards. These potentials preserve the symmetry $(q_i \leftrightarrow q_j)$; hence, the motion along the diagonal is invariant for all ε values. We thus establish that for

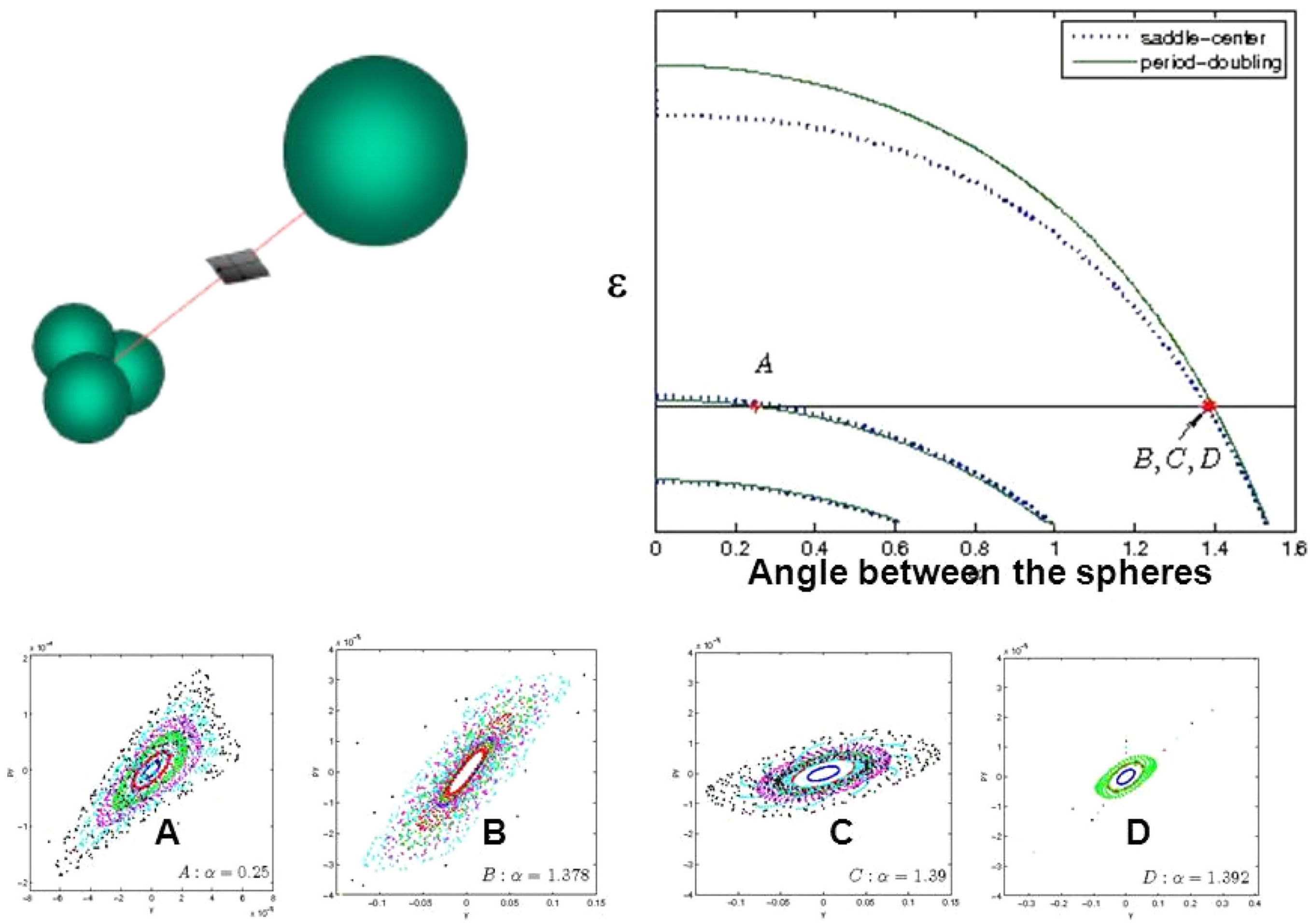

FIG. 6. The billiard geometry and the stability islands in the three-dimensional case (the geometrical parameter in the bifurcation diagram is related to μ, yet is defined a bit differently, see Ref. 7 for details).

sufficiently small ε, these Hamiltonian flows have a periodic orbit $\gamma(t,\mu,\varepsilon)$ along this diagonal and prove that the $2n$ Floquet multipliers of this orbit may be found by solving a *single* second order linear equation with a time-periodic coefficient. This coefficient depends on μ, ε, and n as parameters, and it approaches a sum of delta-like functions as $\varepsilon \to 0$. For certain classes of $W(x;\mu,\varepsilon)$ (e.g., when $W(x;\mu,\varepsilon)$ decays as a power-law in the distance to the scatterers), we are able to analyze the asymptotic behavior of the emerging linear second order equation: we prove that for these potentials there are countable infinity values of μ, one of them given by $\frac{1}{\sqrt{n}}$ (i.e., bounded away from $\mu = 0, 1$), from which a wedge of stability region in the (μ,ε) plane emerges. Namely, we prove that for any n, for arbitrarily small ε, there exists an interval of μ values at which $\gamma(t,\mu,\varepsilon)$ is linearly stable.

Next, we provide a few more details regarding the construction and the definition of μ and $W(q;\mu,\varepsilon)$. These are then utilized to state precisely the stability results regarding the smooth systems, including some estimates on the ε-dependence of the parameter range of μ at which elliptic behavior appears.

Define the n-dimensional billiard's domain D as the region exterior to $(n+1)$ spheres S^{n-1}: one sphere Γ_{n+1} of radius R which is centered on the diagonal at a distance L from the origin, i.e., at the point $\frac{1}{\sqrt{n}}(L,\ldots,L)$, and n spheres $\Gamma_1,\ldots,\Gamma_n$ of radius r, each centered along a different principle axis at a distance $0 \le l \le r\sqrt{\frac{n}{n-1}}$ from the origin, i.e., the sphere Γ_k is centered at $(0,\ldots,l_k,\ldots,0)$ (Fig. 6). To obtain a bounded domain, we enclose this construction by a large n-dimensional hyper-cube centered at the origin (we will look only at the local behavior near the diagonal connecting the radius-r spheres $\Gamma_1,\ldots,\Gamma_n$ to the radius-R sphere Γ_{n+1} and thus we will not be concerned with the form of the outer boundary). The diagonal line $(\xi,\ldots,\xi)$ intersects the radius-R sphere in the normal direction and the spheres $\Gamma_1,\ldots,\Gamma_n$ at their common intersection point $P_c = (\xi_c,\ldots,\xi_c)$, where $\xi_c = \frac{l}{n} + \frac{1}{\sqrt{n}}\sqrt{r^2 - l^2(1-\frac{1}{n})}$. Thus, for $L > R + \sqrt{n}\,\xi_c$, the diagonal line defines a *corner ray*

$$\gamma = \left\{\left\{(\xi,\ldots,\xi)\,|\,\xi \in \left(\xi_c, \frac{L-R}{\sqrt{n}}\right)\right\}\right.$$

that starts at the corner P_c, gets reflected from the radius-R sphere, and returns to P_c (and then gets stuck as there is no reflection rule at the corner).

Notice that the dynamics in the billiard is unchanged when all the geometrical parameters are proportionally increased; hence, with no loss of generality, we may set $r=1$ and regard all the other parameters as scaled by r. It is convenient for us to express the scaled l and L through

$$\mu = \sqrt{1 - \left(1 - \frac{1}{n}\right)\frac{l^2}{r^2}}, \quad d = \frac{L - R - \sqrt{n}\,\xi_c}{r}. \tag{25}$$

Consider the smooth motion in the scaled billiard region, governed by the Hamiltonian equation (1), i.e.,

$$H = \sum_{i=1}^{n} \frac{p_i^2}{2} + W(q_1,\ldots,q_n;\ \varepsilon), \tag{26}$$

with

$$W(q;\varepsilon) = \frac{1}{n}\sum_{k=1}^{n} V\left(\frac{Q_k}{\varepsilon}\right) + V\left(\frac{Q_{n+1}}{\varepsilon}\right), \tag{27}$$

where $Q_k(q)$ (the pattern function of Sec. II) is the distance from x to Γ_k,

$$Q_k(q) = \sqrt{\sum_{i=1}^{n} q_i^2 - 2lq_k + l^2} - 1 (k = 1,\ldots,n), \tag{28}$$

$$Q_{n+1}(q) = \sqrt{\sum_{i=1}^{n}\left(q_i - \frac{L}{\sqrt{n}}\right)^2} - R$$

(recall that we scale $r=1$). The potentials associated with the r-spheres (i.e., $V(\frac{Q_k}{\varepsilon})$) are multiplied by the $1/n$ factor so that for all n values the potential height near the corner is of the same magnitude as the potential near the R-sphere.

The C^{k+1} ($k \ge 1$) smooth function V satisfies at $z > 0$,

$$V(z) > 0, \quad V'(z) < 0, \tag{29}$$

so the potentials are repelling. We further assume that $V''(z)$ decays sufficiently rapidly for large z (similar to the assumptions in Sec. III B), so there exists some $\alpha > 0$ such that

$$V''(z) = O\left(\frac{1}{z^{2+\alpha}}\right) \text{ as } z \to +\infty. \tag{30}$$

One can take, for example, power-law, Gaussian, or exponential potentials: $V(z) = \left(\frac{1}{z}\right)^{\alpha}$, $(\alpha > 0)$, $V(z) = \exp(-z^2)$, $V(z) = \exp(-z)$. These potentials naturally appear in applications (e.g., the Gaussian form arises in the problem of cold atomic motion in optical traps,[33] whereas the power-law and exponential potentials are abundant in various classical models of atomic interactions).

The potential $W(q;\varepsilon)$ given by Eqs. (27) and (28) is symmetric with respect to any permutation of the q_i's $(i = 1,\ldots,n)$. This strong symmetry enables us to derive a one-degree of freedom equation for the motion along the diagonal. To study the stability of the periodic orbit, one needs to linearize the Hamiltonian equations of motion, solve the corresponding $2n$-dimensional linear system with the time-periodic coefficients for a set of $2n$ orthonormal initial conditions, and find the stability of the associated $(2n \times 2n)$-dimensional monodromy matrix. Such a computation finally leads to a set of $2n$ Floquet multipliers (2 of which are trivially one). The symmetric form of the potential, together with some change of coordinates, allows to reduce this formidable task to a much simpler one—to solving a single second order homogeneous equation with a time periodic coefficient: $\ddot{y} + a(t)y = 0$. Indeed, we establish that the $2n$ Floquet multipliers are simply $(1, 1, \lambda, \frac{1}{\lambda}, \ldots, \lambda, \frac{1}{\lambda})$, where λ is the eigenvalue of the monodromy matrix of this second order linear equation. The periodic coefficient $a(t)$ is explicitly found in terms of the diagonal periodic orbit (which also depends on the energy level h) and the geometric parameters of the problem. In particular, the dependence of $a(t)$ on n turns out to be particularly simple, allowing to study in a transparent manner the role of the dimension n. A careful (non-trivial) analysis of the scattering properties of the second order equation leads to the following general result:

Theorem 10. *Suppose the potential function V satisfies Eqs. (29) and (30). Then, given any $h \in (0, 2V(0))$, any natural $n \geq 2$, and any positive d and R, there exists a tending to zero countable infinite sequence $1 \geq \mu_0 > \mu_1 = 1/\sqrt{n} > \ldots > \mu_k > \ldots > 0$ such that arbitrarily close to every point $(\mu = \mu_k, \varepsilon = 0)$, there are wedges of (μ, ε) at which the orbit γ is linearly stable.*

More detailed information regarding the wedges character may be obtained in two specific cases as explained next. First, we may estimate the wedges height in ε provided the scaling parameter $\beta = \frac{1-\mu^2}{(n-1)\mu^2}$ is sufficiently small and the following integral (which is well defined for potentials satisfying Eqs. (29) and (30)) is positive,

$$I(h) = \frac{2}{\sqrt{h}} \int_{V^{-1}(h/2)}^{+\infty} V''(z) \frac{dz}{\sqrt{h - 2V(z)}}.$$

Theorem 11. *Provided $I(h) > 0$, the diagonal periodic orbit γ is stable for (μ, ε) values in the wedge enclosed by the two curves,*

$$\varepsilon_0^+ = I(h) \frac{1-\mu^2}{(n-1)\mu^2} \left(1 + \frac{1}{d+R}\right)^{-1} + o\left(\frac{1-\mu^2}{(n-1)\mu^2}\right),$$

and

$$\varepsilon_0^- = I(h) \frac{1-\mu^2}{(n-1)\mu^2} \left(1 + \frac{1}{d}\right)^{-1} + o\left(\frac{1-\mu^2}{(n-1)\mu^2}\right).$$

In particular, notice that the wedge height is polynomial in n-namely, the region of stability does not shrink exponentially with the dimension as one may have expected.

Second, the wedges structure for the power-law potential case may be described in detail (see Ref. 6 for numerical verifications of these formulas),

Theorem 12. *Consider the power-law potential $V(Q, \varepsilon) = \left(\frac{\varepsilon}{Q}\right)^{\alpha}$. Then, for sufficiently small ε and μ, there exists an infinite number of disjoint stability tongues in the (μ, ε) plane at which $\gamma(t; \mu, \varepsilon, n)$ is linearly stable. For sufficiently large k, the kth stability zone emanates from the μ axis near the bifurcation value,*

$$\mu_k \approx \frac{1}{k} \sqrt{\frac{2(\alpha+1)}{\alpha(n-1)}},$$

and extends up to the ε-axis, intersecting it near

$$\varepsilon_k \approx (h/2)^{1/\alpha} \frac{(\alpha+1)}{\alpha(n-1)} \frac{4}{\pi^2 k^2} \left(\int_0^{\pi/2} (\sin\theta)^{1/\alpha} d\theta\right)^2,$$

at a stability interval of length

$$(\Delta\varepsilon)_k \approx \frac{4\varepsilon_k}{\pi k G(0,\alpha) d(1 + \frac{d}{R})} \left(\frac{4\alpha(\alpha+1)}{n-1} \frac{(2\varepsilon_k)^{\alpha}}{h}\right)^{1/2(\alpha+1)},$$

where $G(0, \alpha) > 0$ is (non-trivially) computable function which depends only on α.

Rewriting the above formulas (with $(c_{1,2})$ as shorthand notation), we obtain

$$\varepsilon_k \approx \frac{h^{1/\alpha}}{n-1} c_1(k, \alpha), \quad \frac{(\Delta\varepsilon)_k}{\varepsilon_k} \approx \frac{1}{\sqrt{n-1}} \frac{c_2(k, \alpha)}{d(1 + \frac{d}{R})}.$$

In particular, to leading order, the wedges relative heights are independent of the energy level and decay as $(1/\sqrt{n})$ with the dimension.

The analysis is performed only for the symmetric case. It is quite possible that one may extend it to the nearly symmetric case. Indeed, it is easy to break the symmetry, by, for example, multiplying the terms $V(Q_k(x)/\varepsilon)$ in Eq. (27) by slightly different coefficients,

$$V_k^{pert}(q; \varepsilon) = (1 + \delta a_k) V_k(q; \varepsilon), \tag{31}$$

where a_k are uniformly distributed on the unit interval (i.e., we consider the case in which each sphere has a slightly different potential). The phase portraits of the perturbed motion with $\delta = 0.001$ are shown in the right column of Fig. 7 (we do verify that the projection plots of $X = ||x - \gamma(0)||, P = ||\dot{x} - \dot{\gamma}||$ remain bounded, namely, that there is no instability

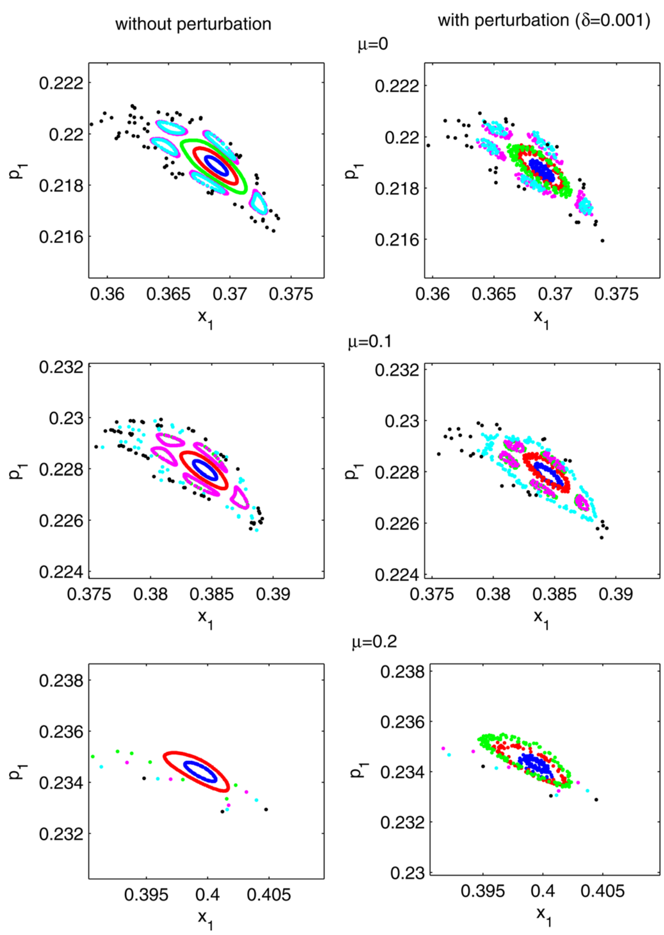

FIG. 7. Islands in a 20-dimensional symmetric (left) and asymmetric (right) systems. Return map projection to the (x_1, p_1) plane is shown (see Ref. 6).

in any direction of the 20-dimensional phase space). We see that such a modification appears to preserve the elliptic character of the periodic orbit (this is not an obvious statement as the Flouqet multipliers of $\gamma(t)$ are in the strong 1:1:...:1 resonance).

The considered example is clearly highly symmetric and is not directly linked to the smooth many particle case (the "smooth Boltzmann gas"). Nonetheless, the possibility of explicitly constructing stable motion in smooth n degree of freedom systems that limit to strictly dispersing billiards is now established. Thus, the grand program of proving that arbitrarily large systems of particles interacting via a steep repelling potential may, at arbitrarily high energies, have fully elliptic orbits appears to be reachable in a few years work.

V. SCATTERING PROBLEMS

Scattering problems with smooth steep potentials appear in diverse field of physics (notably chemical reactions) and are thus of practical significance. In these problems, one considers a potential which rapidly decays to zero outside a compact domain (called the scattering region) and examines the evolution of a ray of initial conditions that hit the scattering region. Mathematically, the key ingredient for analyzing such problems is the detection and characterization of the compact invariant set associated with the potential. The methods described in Secs. I–IV may be applied to study such questions when the potential is steep, or at high energy levels (if the potential is unbounded). We describe here some of the results achieved, together with Rapoport, in the two-dimensional setting.[8]

We start with the formal definition of the scattering map for the limit billiard case. Namely, consider a scattering billiard in $\mathbb{R}^2$; that is, a collection D of disjoint hard-wall obstacles (D_i) which reside strictly inside the centered at zero disc of a sufficiently large radius $\bar{R}$. We call D a *Sinai scatterer* if each D_i is bounded by a finite number of C^{r+1}-smooth *strictly convex* (when looked from inside of D_i) arcs that meet each other at non-zero angles.

Let a particle come from infinity with the momentum $(p_x, p_y) = \sqrt{2h}(\cos\varphi_{in}, \sin\varphi_{in})$, where h is the energy (recall that the energy is conserved). The parallel rays corresponding to the same value of φ_{in} are distinguished by their impact parameter b_{in}; the absolute value of b_{in} equals to the distance between a particular ray and the origin. Given the energy value, the incoming orbit is uniquely defined by the data (φ_{in}, b_{in}). For typical initial conditions, orbits that come from infinity must go to infinity (by Poincare recurrence theorem). The outgoing angle and impact parameter are determined by elastic reflections from the obstacles D_i. Thus, the scattering map $\mathcal{S} : (b_{in}, \varphi_{in}) \to (b_{out}, \varphi_{out}, t_{out})$ is defined. The interaction time t_{out} is defined as $t_{out} = \lim \bar{R} \to +\infty \frac{\mathcal{L}\bar{R} - 2\bar{R}}{\sqrt{2h}}$, where $\mathcal{L}\bar{R}$ denotes the length of the orbit inside the centered at zero disc of radius $\bar{R}$. For a fixed φ_{in}, the scattering functions Φ and T are defined as

$$(\Phi(b), T(b)) = (\varphi_{out}(b, \varphi_{in}), t_{out}(b, \varphi_{in})).$$

The scattering functions and their fractal properties had been extensively studied by numerical simulations, see, e.g., Refs. 34–37.

We call (b_{in}, φ_{in}) regular, if the corresponding orbit makes a finite number of reflections from the obstacles before leaving the scattering region, and all these reflections are regular (i.e., the orbit does not visit the corner points and all the reflections are non-tangent). Then all close-by initial conditions are also regular and the scattering map is C^r-smooth.

The complement to the set of regular initial conditions is a compact set of measure zero. There are exactly two sources for non-smooth behavior of $\mathcal{S}$: interior singularities that are associated with singular reflections from the scatterers, and trapping singularities which correspond to the number of reflections tending to infinity.

The interior singularities have a simple signature in terms of the scattering map. Namely, if the trajectory is tangent to the scatterer boundary at one of the reflections, then $\Phi(b)$ has a square root singularity fold. Notice that both $\Phi(b)$ and $T(b)$ are continuous across the tangent singularity line (see Fig. 8). If the trajectory ends up in a corner, then the scattering map $\mathcal{S}$ is not defined at that point, having a discontinuous behavior of both $\Phi(b)$ and $T(b)$ along this corner singularity line.

The trapping singularities have a more complex signature. Denote by Σ_Λ the measure zero set of all (b_{in}, φ_{in}) for which the orbit is trapped in the scattering region (i.e., it makes an infinite number of reflections and does not go to infinity). Note that the scatterer D may have a nontrivial compact invariant set Λ, i.e., the set of all orbits that stay bounded for all time (from $-\infty$ to $+\infty$). The above defined set Σ_Λ consists of all the initial conditions (b_{in}, φ_{in}) belonging to the *stable manifold* of Λ.

Recall that we consider here Sinai scatterers, so all regular orbits in Λ are hyperbolic. If the hyperbolic set Λ is bounded away from the singulariy set (corners and tangent collisions) in the phase space, we call the Sinai scatterer *regular*. In this case, the scattering function near the trapping singularities has a characteristic self-similar structure, e.g., for every φ_{in} the values of b_{in} which correspond to the trapping form a Cantor set which is diffeomorphic to a transverse section of the stable manifold of Λ.

The set of regular Sinai scatterers is robust under smooth perturbations. Indeed, the uniformly hyperbolic invariant set Λ is structurally stable, i.e., a sufficiently small smooth deformation of D does not change the symbolic dynamic description of the dynamics on Λ. In order to see that the set of regular Sinai scatterers is non-empty, consider three identical circular disks of radius a that are centered at the vertices of an equilateral triangle with edges of length R. When $R > 3a$, the invariant set Λ is bounded away from any tangent trajectory, and Λ is fully described by symbolic dynamics on 3 symbols with a simple transition matrix.[35]

We call the Sinai scatterer *singular* if the set of trapping singularities is no longer separated from the interior singularities, and the simple Cantor-set structure is lost. This is due to the fact that the invariant set Λ now includes tangent and/or corner orbits. For example, as the three discs of the above mentioned example get closer, tangent orbits are created; At each such bifurcation point the phase space partition changes and, as the scatterers get closer, the transition matrix

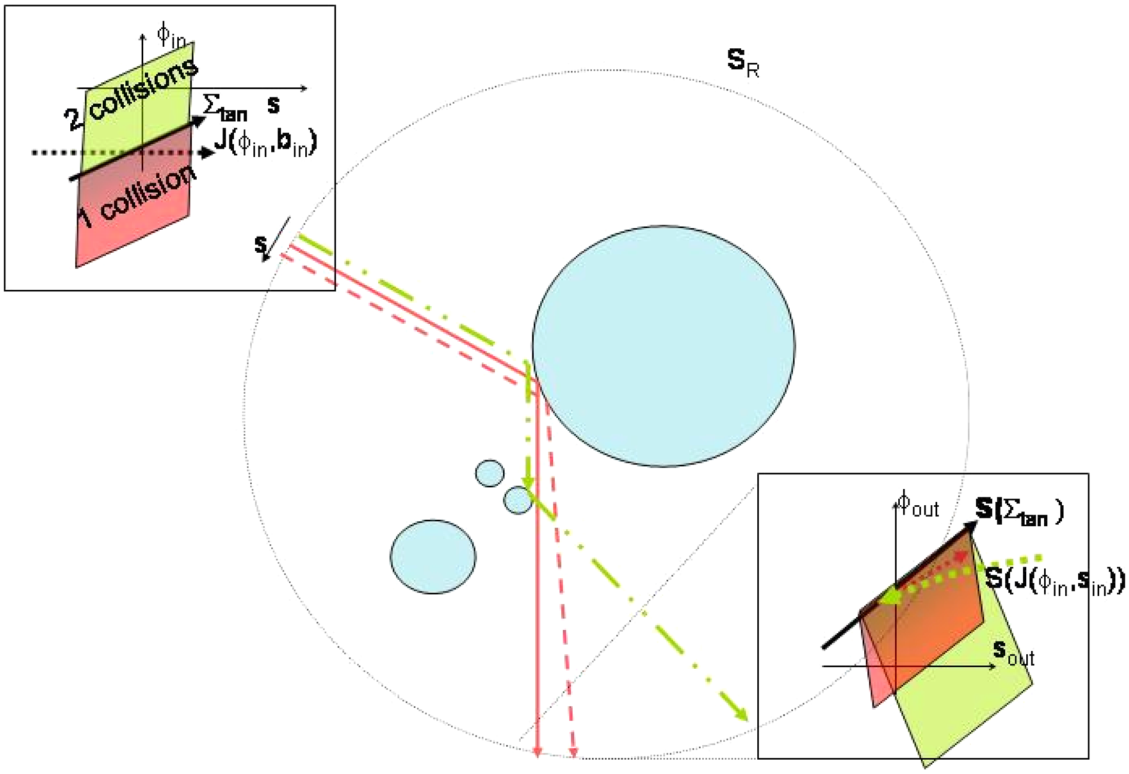

FIG. 8. The scattering map (see Ref. 8).

becomes more complex—in the limit at which the discs touch each other, infinite Markov partition is needed and the invariant set has a full measure.[35] For a general, say, one-parameter family of Sinai scatterers, it is plausible that parameter values that produce singular scatterers fill non-empty intervals. More precisely, we suspect that tangent bifurcations of (Λ) occur on a dense set of parameter values belonging to these intervals (similar to the Newhouse phenomenon[49,50]). It may be quite challenging to study the structure of the scattering function for these singular Sinai scatterers.

Now, consider the system (1) with a steep potential that limits to a regular Sinai scatterer. Namely, we consider smooth systems that satisfy conditions **I-IV** of Sec. II in the domain $\mathcal{D} = R^2 \backslash D$, with an additional assumption that the potential decays sufficiently fast as $|q| \to \infty$. For example, we consider potentials of the form $V(q;\varepsilon) = \sum_{i=1}^{n} E_i V_i\left(\frac{Q_i(q)}{\varepsilon}\right)$, where $Q_i|_{q\in\partial D_i} = 0$, $Q_i|_{q\in\mathcal{D}} > 0, E_i \geq \mathcal{E} > 0$, and, for some $\alpha > 0$,

$$V_i(0) \geq 1,\ V_i'(z) < 0, V_i(z) = O\left(\frac{1}{z^\alpha}\right) \quad \text{for} \quad z \gg 1. \qquad (32)$$

By construction, for all energies $h \in (0,\mathcal{E})$, for sufficiently small ε, the Hill's region of the smooth flow approaches $\mathcal{D}$.

Since the potential decays fast, every trajectory which tends to infinity is asymptotic to a straight line, so there are well-defined asymptotic direction φ and impact parameter b. Thus, one may define the scattering map of the smooth flow, $\mathcal{S}^\varepsilon : (b_{in},\varphi_{in}) \to (b^\varepsilon_{out},\varphi^\varepsilon_{out},t^\varepsilon_{out})$. Applying the regular smooth limit results of Ref. 5 (see Sec. II C) to this setup ensures that near regular initial data (b_{in},φ_{in}) the smooth scattering map converges to the billiard scattering map as $\varepsilon \to 0$, along with all derivatives. Thus, for smooth-potential approximations of regular Sinai scatterers, one expects the scattering map to have the same structure as for the billiard limit. Namely, we have the following result.

Theorem 13. *Consider the Hamiltonian system (1) with a rapidly decaying at infinity billiard-like potential* $V(q,\varepsilon)$ *in the complement* $\mathcal{D}$ *to a regular Sinai scatterer* D*. Let* Λ_h *be the maximal compact invariant set of the billiard in* $\mathcal{D}$*. Then, for all small* ε*, the maximal compact invariant set* Λ^ε_h *in the energy level* $h \in (0,\mathcal{E})$ *is topologically conjugate to* Λ_h*. Moreover, the local stable and unstable manifolds of* Λ^ε_h *are* C^r *close to the local stable and unstable manifolds of* Λ_h*.*

The theorem implies that the scattering function has the same qualitative structure for all small ε. In particular, it has the perfect self-similar behavior associated with regular hyperbolic scattering, see Refs. 34–37. In the case of *singular* Sinai scatterers, the behavior of the scattering function for the smooth flow is quite different. In particular, one cannot expect structural stability as ε varies. As our results suggest, the main effect of the smooth potential is to destroy the hyperbolicity of the invariant set of the singular Sinai scatterer. Indeed, billiard singularities give rise to elliptic periodic orbits of the smooth scatterer.

Numerous numerical studies of scattering by smooth potentials demonstrate that elliptic islands play a significant role in the structures of scattering maps. In particular, it has been proposed that the existence of such elliptic component leads to "fat" fractal behavior of the scattering function.[51,52]

However, in general smooth systems, it is difficult to describe invariant sets and to isolate the scattering signature of each ingredient. Examining the behavior of smooth systems that limit to singular scatterers provides a method for studying such effects. Indeed, by utilizing the singular mechanisms for creating stability islands (Sec. III), we can examine the scattering by small stability islands with control over the size and structure of the islands and of the rest of the invariant set. In Ref. 8, two symmetric geometrical settings of singular Sinai scatters (with corners—Figs. 9(a) and 9(b) and with tangencies—Figs. 9(c) and 9(d)) were thus examined.

From these studies, the following scenario emerges (see Fig. 10). Let μ^* denote a bifurcation value for which the billiard invariant set has a singularity (e.g., a tangent periodic orbit or a corner polygon). Then, as discussed in Sec. III, under some conditions on the potential and the geometry,[3,4] a stability wedge in the (μ, ε) plane emanates from $(\mu^*, 0)$, i.e., the smooth flow has stable periodic orbit for all parameters in this wedge (as in Fig. 6). For a fixed μ value intersecting this wedge, there exists an interval of ε values, $[\varepsilon^-(\mu), \varepsilon^+(\mu)]$, at which the periodic orbit is stable. Fixing such a "generic" μ value close to μ^*, where at μ the billiard invariant set is hyperbolic and non-singular and $\varepsilon^{\pm}(\mu)$ are small; the following sequence of bifurcations occurs as ε is increased from zero:

1. For a sufficiently small ε, the non-hyperbolicity effects are small so the scattering function looks self-similar, and its fractal dimension approaches that of the billiard scattering function at the given value of μ. Discontinuities in the billiard scattering function may lead to additional singular components in the scattering function of the smooth flow (fourth column of Fig. 10).
2. Increasing ε towards and through the interval $[\varepsilon^-(\mu), \varepsilon^+(\mu)]$ leads to a sequence of Hamiltonian bifurcations of the hyperbolic periodic orbits that produces elliptic orbits. These bifurcations appear in the scattering function as the merge between several unresolved regions. For ε inside the wedges of stability, the signature of non-hyperbolic chaotic scattering shows up—the density of singularities is large and does not appear to converge to a discrete set as further magnifications are employed. We notice that the stability interval $[\varepsilon^-(\mu), \varepsilon^+(\mu)]$ indicates the stability property of a single periodic orbit. At least near the period-doubling end of this interval, there exist a cascade of other periodic orbits that are stable; hence, the visible non-hyperbolicity interval is certainly larger than $[\varepsilon^-(\mu), \varepsilon^+(\mu)]$ (second and third columns of Fig. 10).
3. Beyond the stability interval, intervals of seemingly hyperbolic scattering or other interval of stability (that stem from other stability wedges) can be encountered (first column of Fig. 10).

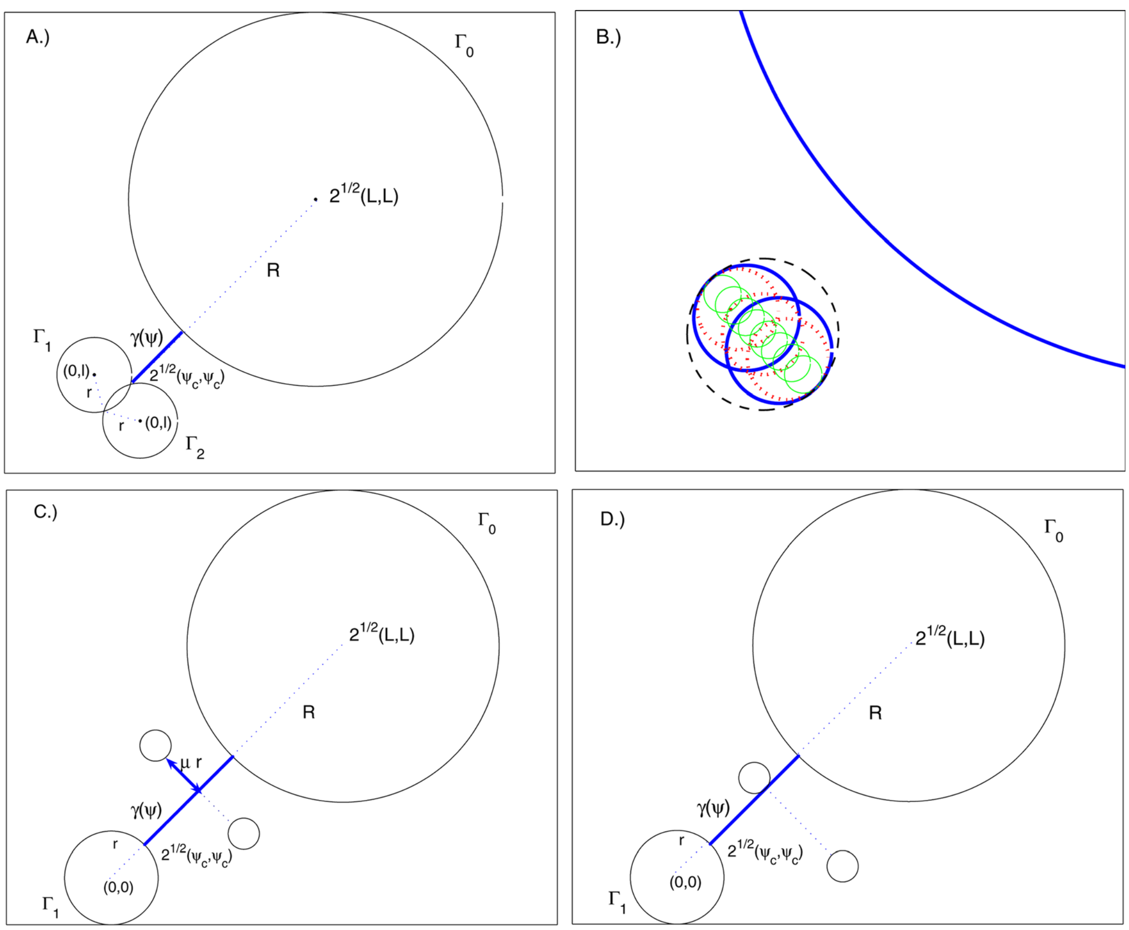

FIG. 9. Singular Sinai scatters. (a) and (b) have corner singularities, whereas (c) and (d) provide a simple realization for a tangent bifurcation of the invariant set (see Ref. 8).

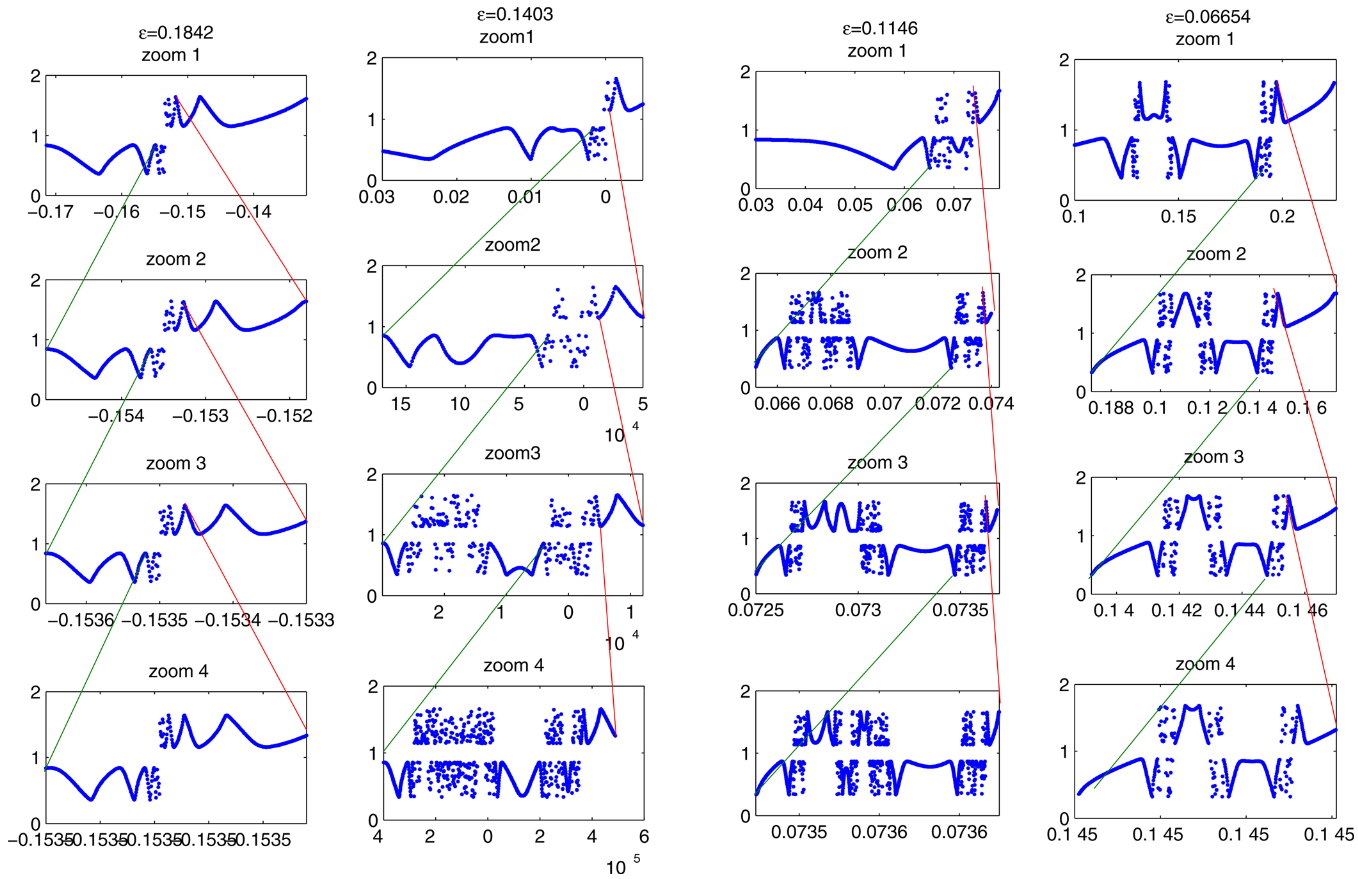

FIG. 10. Self-similar (outer columns) and singular (middle columns) scattering functions: close-ups of unresolved regions of the scattering function $\Phi_2(s;\epsilon)$ for the corner case with $n=2, \mu=0.9$, and $\epsilon = 0.1842, 0.1403, 0.1146, 0.06654$ are shown. The second and third values correspond to values that produce elliptic islands (see Ref. 8).

4. A larger increase in ε is problem-specific and may involve topological changes of the corresponding Hill's region (i.e., homoclinic bifurcations). In our examples, it finally leads to the reduction of the invariant set to one unstable periodic orbit and then to the destruction of the invariant set.

The above description suggests that by choosing a one-parameter family of steep potentials $(\mu, \varepsilon(\mu)) \to (\mu^*, 0)$, such that $\varepsilon(\mu) \in (\varepsilon^-(\mu), \varepsilon^+(\mu))$ for all μ values the fractal dimension of the corresponding scattering function is close to two for arbitrarily small ε. On the other hand, we have seen that for a fixed $\mu \neq \mu^*$, for sufficiently small ε, hyperbolic-like chaotic scattering is observed. Thus, near μ^*, the fractal dimension of the scattering function can be controlled by varying the appropriate combination of $\varepsilon, (\mu - \mu^*)/\varepsilon$ and h as derived from the form of the return map near the singular orbit (see Sec. III).

VI. CONCLUSIONS AND PERSPECTIVES

The near-billiard paradigm allows to analyze a variety of dynamical properties of multi-dimensional non-integrable Hamiltonian systems. We conclude by listing some of the problems that require further research.

One future direction concerns establishing the correspondence between the smooth system and its singular limit near regular collisions for a wider class of impact systems. This may include time-dependent billiards, billiards on non-flat manifolds, billiard motion with added magnetic field, impact systems where the potential does not vanish inside the billiard domain, systems with dissipation, and non-elastic reflection law. For all these system classes, after the regular collisions are studied, one should analyze the effect of smoothing the system on the behaviour near singular orbits. This includes studying the grazing bifurcation[18,54] and its smooth approximations.

For the billiards themselves, the analysis of singular orbits and their transformations at a smooth approximation is far from being complete. For example, we have not studied the general behavior near corner orbits in dimension higher than 2 (in Ref. 6, we considered only a particular case of a very symmetric multi-dimensional corner). Nor have we studied the behavior near corner angles large than π. We have proved the birth of elliptic islands out of tangent periodic orbits of dispersive billiards—what happens in the non-dispersive case, when the tangent periodic orbit hits both convex and concave boundary components? What happens near degenerate tangencies, e.g., when the billiard orbit has a cubic tangency to the boundary? Such tangencies can appear near an inflection point of the billiard boundary (in the two-dimensional case) or near any boundary point where the curvature form is not sign-definite (in the multi-dimensional case). In the billiard, an orbit with degenerate tangency cannot, typically, be continued past the tangency point, so the question of how the smooth approximation flow behaves near the inflection and saddle boundary points is quite nontrivial. This problem also includes the analysis of how smoothing the system affects chattering (this phenomenon corresponds to a billiard orbit that approaches an inflection/saddle point by making infinitely many collisions in a finite interval of time[53]). More generally, billiard trajectories can make a long journey along a convex boundary component, hitting it at a small angle. What happens with such orbits when the billiard potential is replaced by its smooth steep approximation?

Further, one should study periodic orbits that have several tangencies to the billiard boundary (as well as corner orbits that also have tangencies). What is the normal form of the Poincaré map for the smooth system near such an orbit? Is it true that given a (semi)dispersive k-dimensional billiard with a periodic orbit that has (k–1) tangencies to the boundary, a fully elliptic periodic orbit can be born for an arbitrarily steep smooth approximation? We proved this in the case $k=2$ in Ref. 3, the higher-dimensional case is open. A positive answer will give a working tool for establishing the *non-ergodicity* of the motion in steep repelling potentials in any dimension of the configuration space. A natural conjecture is that periodic or corner orbits that make any given number of tangencies to the billiard boundary can be obtained by an arbitrarily small (in C^∞) deformation of the dispersive billiard domain. Once this conjecture is proven, one should be able to show that for any ergodic dispersive billiard, in any dimension, the ergodicity can be ruined by arbitrarily steep smooth approximation of the potential. Applying the technique to a system of high energy particles interacting via a repelling hard-core potential would show that the non-ergodicity is quite typical for the smooth Boltzmann gas. In particular, one would be able to create stable configurations out of many particles interacting by a repelling potential at arbitrarily large kinetic energies.

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