



Simple Scenarios of Onset of Chaos in Three-Dimensional Maps

Alexander Gonchenko*, Sergey Gonchenko[†] and Alexey Kazakov[‡]

*Research Institute of Applied Mathematics and Cybernetics,
Lobachevsky State University of Nizhny Novgorod,
ul. Ulyanova 10, Nizhny Novgorod 603005, Russia*

**agonchenko@mail.ru*

†gonchenko@pochta.ru

‡kazakovdz@yandex.ru

Dmitry Turaev

*Department of Mathematics, Imperial College London,
London SW7 2AZ, United Kingdom*

dturaev@imperial.ac.uk

Received February 2, 2014

We give a qualitative description of two main routes to chaos in three-dimensional maps. We discuss Shilnikov scenario of transition to spiral chaos and a scenario of transition to discrete Lorenz-like and figure-eight strange attractors. The theory is illustrated by numerical analysis of three-dimensional Henon-like maps and Poincaré maps in models of nonholonomic mechanics.

Keywords: Strange attractor; chaotic dynamics; spiral attractor; torus-chaos; homoclinic orbit; three-dimensional Henon map; Celtic stone; unbalanced ball; nonholonomic constraint.

1. Shilnikov Chaos in Flows

In 1965, Shilnikov discovered that a homoclinic loop to a saddle-focus can imply chaos. The notion itself did not exist then; the “chaos theory” emerged and became popular only 10–20 years later. Chaos was found in many nonlinear models of hydrodynamics, optics, chemical kinetics, biology, etc. It also occurred that strange attractors in models of various origins often have a spiral structure, i.e. the chaotic orbits seem to move near a saddle-focus homoclinic loop. That the homoclinic loop to a saddle-focus with a positive saddle value implies chaos — this is Shilnikov theorem [Shilnikov, 1965, 1970], but why is the converse also so often true, how can chaos imply a homoclinic loop to a saddle-focus? This question quite preoccupied Shilnikov in the mid-80s. He found [Shilnikov, 1986] that if a system depends on a parameter and evolves, as

the parameter changes, from a stable (“laminar”) regime to a chaotic (“turbulent”) motion, then this process is naturally accompanied by a creation of a saddle-focus equilibrium in the phase space and, no matter what particular way to chaos the system chooses, it is also natural for the stable and unstable manifolds of this saddle-focus to get sufficiently close to each other, so a creation of a homoclinic loop becomes easy.

This idea is not mathematically formalizable, it is an empirical statement, which makes it even more important: as it is not mathematics, it cannot be derived from any abstract notion. It relates the beginning of the route to chaos (Andronov–Hopf bifurcation) with the end (formation of a spiral attractor) in a simple and model-independent way. In this paper, we further develop this idea (see also [Gonchenko *et al.*, 2012b]) and discuss new basic

scenarios of chaos formation which should be typical for three-dimensional maps and four-dimensional flows (higher dimensions will, surely, bring more diversity). The first of these scenarios (Sec. 2) has already been mentioned in [Shilnikov, 1986]. The other scenario (see Sec. 3.2) is not related to saddle-foci and is more “Lorenz-like”, however it does not require the symmetry the classical Lorenz system possesses.

First, we recall in more detail the scenario from [Shilnikov, 1986] of the creation of spiral chaos. Shilnikov considered a smooth three-dimensional system

$$\dot{x} = X(x, R) \tag{1}$$

that depends on a certain parameter R (the choice of notation for the parameter had a hydrodynamic motivation; one may think of R as being somehow related to Reynolds number). Let the increase of R lead the system from a stable regime to a chaotic one. That is, at some $R < R_1$ the system has a stable equilibrium state O , at $R = R_1$ it loses stability, the new stable regime also loses stability with the increase of R , and so on. Without additional symmetries or degeneracies, or other equilibria coming into play, it is natural to assume that the loss of stability at $R = R_1$ corresponds to the Andronov–Hopf bifurcation, so that a single periodic orbit L is born from O at $R > R_1$, and this periodic orbit inherits the stability of O . The point O is a saddle-focus at $R > R_1$, and at small positive values of $R - R_1$ the two-dimensional unstable manifold W^u_O is a disc with boundary L . As the system evolves towards chaos with R increasing, the stable periodic orbit L may also lose the stability, via a period-doubling or a secondary Andronov–Hopf bifurcation that corresponds to a birth of an invariant torus from L . In any case, before the periodic orbit loses stability, its multipliers must become complex at $R > R_2$ for some $R_2 > R_1$ (the multipliers of L are real positive at R close to R_1 , so they must become complex before one of them becomes equal to -1). At $R > R_2$ the manifold W^u_O will wind onto L and form a funnel-type configuration (Fig. 1). This funnel will attract all orbits from some open region D . After the funnel is formed, the creation of a homoclinic loop to O as R grows further becomes very natural: the throat of the funnel may become smaller or it may change its position, so that W^u_O and W^s_O may start getting closer to each other until a homoclinic loop is formed at some $R = R_3$. If the complex

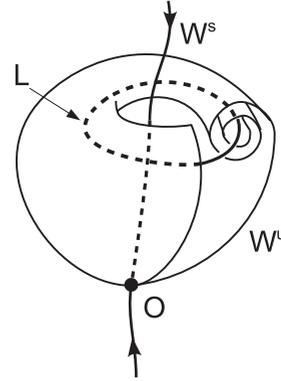


Fig. 1. A funnel-type configuration of $W^u(O)$.

characteristic exponents of the saddle-focus O are nearer to the imaginary axis than the real negative one (this condition is automatically fulfilled at $R = R_1$, so we may assume it continues to hold at $R = R_3$ too), then the existence of the homoclinic loop to O implies complex orbit behavior (infinitely many suspended Smale horseshoes) in a neighborhood of the loop [Shilnikov, 1965, 1970]. In case the throat of the funnel can be cut by a cross-section such that all the orbits that intersect the cross-section come inside, the unstable manifold W^u_O (more precisely, its part from O till the cross-section, plus the cross-section itself) will bound a forward-invariant region; at $R = R_3$ the attractor which lies in this region will contain the homoclinic loop and the chaotic set around the loop. The orbits in this set spiral around the saddle-focus, so the characteristic shape of the “spiral attractor” can be observed. When R changes the loop splits, but a large portion of the chaotic set will survive; also new, multi-round homoclinic loops may appear, etc. One can have this scenario of the transition to chaos for n -dimensional systems with any $n \geq 3$, e.g. just by adding $(n - 3)$ contracting directions.

The main point of this observation is that the Andronov–Hopf bifurcation of the stable equilibrium O not only creates a stable periodic orbit L , it also transforms O to a saddle-focus, and instead of following details of the further evolution of the stable regimes (the periodic orbit L , the periodic orbit born from L after, for example, the first period-doubling, etc.) it may be more useful for the understanding of the transition to chaos to continue to watch what happens to the primary equilibrium O and how the shape of its unstable manifold evolves. Studying typical dynamical features of attractors that can exist in a Shilnikov funnel

could be an interesting research direction. A model for the Poincaré map in the funnel was proposed in [Shilnikov, 1986]. Based on the analysis of this map, a birth of an invariant torus in the funnel was studied in [Afraimovich & Vozovoi, 1988, 1989]. In [Belykh *et al.*, 2005] there was shown that a certain type of funnel is consistent with the existence of a hyperbolic Plykin attractor. The wild attractor built in [Bamon *et al.*, 2005] can also be inscribed in a Shilnikov funnel (in dimension $n \geq 5$).

The above described scenario appears to give the simplest (hence, the most general) route to chaos. It involves a very small number of objects responsible for chaos formation: the equilibrium O , its unstable manifold, and the periodic orbit L . However, there can be more complicated schemes. For example, Shilnikov also noticed that the Andronov–Hopf bifurcation at $R = R_1$ can be different from what is described above. Namely, we assumed that this bifurcation is soft, i.e. the equilibrium O is stable at the bifurcation moment and its stability is transferred to the stable periodic orbit L at $R > R_1$. However, there can also be a subcritical Andronov–Hopf bifurcation at $R = R_1$: the periodic orbit L can be saddle, it exists then at $R < R_1$, and at the moment L merges with O (i.e. at $R = R_1$ already) the equilibrium O becomes a (weak) saddle-focus. The unstable manifold W_O^u at $R = R_1$ is the limit of the unstable manifold of L . Thus, already at $R = R_1$, the manifold W_O^u may have a nontrivial shape, e.g. it may form a funnel, so a large forward-invariant region associated to this funnel is created at $R = R_1$. If a chaotic set Λ (not necessarily an attractor) had already been formed at $R < R_1$ inside this region, then we can observe a sudden transition from the stable regime O to a well-developed spiral chaos at $R = R_1$. A similar way of a sudden transition from a stable equilibrium to a large invariant torus was considered in [Afraimovich & Vozovoi, 1988, 1989]. The chaotic set Λ can be created in several ways. For example, at some R smaller than R_1 a saddle-node periodic orbit can emerge and, as R grows, decompose into a saddle periodic orbit L and a stable periodic orbit L_+ . In the three-dimensional case the stable manifold of L is two-dimensional, and L divides it into two halves. Let one of the halves tend to O and the other half, W_L^{u+} to L_+ . As R grows, the orbit L_+ may lose stability in some way and, eventually, homoclinic intersections of W_L^{u+} with W_L^s may form. The homoclinic to a saddle periodic

orbit is accompanied by a nontrivial hyperbolic set Λ' [Shilnikov, 1967]. If L keeps the homoclinics as it merges with O , then the weak saddle-focus O will have a homoclinic loop at $R = R_1$. Chaotic dynamics associated with this so-called Shilnikov–Hopf bifurcation was studied in [Belyakov, 1980; Bosh & Simo, 1993]. If L loses its homoclinics near $R = R_1$, a portion Λ of the hyperbolic set Λ' may still survive until $R = R_1$.

In systems with symmetry, instead of the Andronov–Hopf bifurcation a pitchfork bifurcation may happen to a symmetric stable equilibrium O . Then, instead of a stable periodic orbit L , a pair of stable, symmetric to each other equilibria O_1 and O_2 will be born; the equilibrium O will become a saddle with one-dimensional unstable manifold that tends to $O_{1,2}$. After the equilibria $O_{1,2}$ acquire complex characteristic exponents, the unstable separatrices of O will start winding around $O_{1,2}$; the further increase of a parameter may lead then to formation of a symmetric pair of homoclinic loops and chaos like in the Lorenz model or in systems with “double-scroll” attractors [Arneodo *et al.*, 1981, 1985; Khibnik *et al.*, 1993]. In dimension $n \geq 4$ a symmetric wild Lorenz-like attractor may emerge in this way [Turaev & Shilnikov, 1998]. Without a symmetry, similar scenarios are also possible (see e.g. [Shilnikov & Shilnikov, 1991]): in a system with a stable equilibrium O_1 a saddle-node equilibrium may emerge which decomposes into a saddle equilibrium O and a stable equilibrium O_2 , so that one separatrix of O tends to O_2 and the other tends to O_1 . After that, as parameters change, chaos may form around these three equilibria and their unstable manifolds.

Returning to the simplest scenario, note that the spiral attractor formed in the funnel does not need to be the “true” strange attractor. Bifurcations of a homoclinic loop to a saddle-focus can lead to the birth of stable periodic orbits along with the hyperbolic sets [Ovsyannikov & Shilnikov, 1987]. Therefore, stable periodic orbits can coexist with hyperbolic sets in the funnel. If the period of these orbits is large, or their domains of attraction are narrow, then they will be practically invisible and the attractor will appear chaotic. Such attractors were called quasiattractors in [Afraimovich & Shilnikov, 1983a]. We discuss this notion in more detail in Sec. 3.1. We also give conditions (following [Ovsyannikov & Shilnikov, 1987; Turaev & Shilnikov, 1998]) for the absence

of stable periodic orbits and the true chaoticity of the attractor.

2. Shilnikov Scenario for Maps

The second basic scenario which was described in [Shilnikov, 1986] requires the dimension n of the system to be at least 4. We assume that system (1) has a stable periodic orbit L , which undergoes a soft Andronov–Hopf bifurcation at $R = R_1$ (i.e. its multipliers cross the unit circle and a stable two-dimensional invariant torus is born from L). One may consider a cross-section S to L , then the point $O = S \cap L$ is a saddle-focus fixed point of the Poincaré map on S . The intersection of the invariant torus with the cross-section is an invariant curve C ; it bounds the unstable manifold W_O^u . At small $R - R_1$ a neighborhood D of $W_O^u \cup C$ is an absorbing domain (the image by the Poincaré map of the closure of D lies strictly inside D), and $W_O^u \cup C$ is the attractor in D . We assume that for the entire range of R values under consideration there exists a continuously dependent on R absorbing domain D which contains O along with W_O^u . As R grows, the manifold W_O^u may start winding onto C , and a funnel will form. Then W_O^u may come closer to the stable manifold W_O^s , so at a certain interval of values of $R > R_1$ the saddle-focus fixed point O will have homoclinic orbits in D . The corresponding attractor in [Shilnikov, 1986] was called *Poincaré attractor*. The idea was that when we do not consider this attractor on a cross-section and look at it in the phase space of the continuous-time dynamical system (1), it will appear different from the spiral attractor described in the previous section. The main element of the spiral attractor is an equilibrium state and its unstable manifold, the main element of the Poincaré attractor is the saddle-focus periodic orbit L and homoclinics to it (transverse homoclinics to periodic orbits were discovered by Poincaré, so the name).

We, however, will focus more on the attractors of discrete-time dynamical systems, i.e. we will not assume that the map under consideration is the Poincaré map for some smooth flow. Then the chaotic attractor in the funnel formed by the unstable manifold of a saddle-focus fixed point O of our map can have a shape very similar to that of the spiral attractor for systems with continuous time. Therefore, we will also call it spiral or Shilnikov attractor, or discrete Shilnikov attractor. One of the

differences of this attractor from the spiral attractor for flows is that, in the case of maps, the homoclinics to O exist for intervals of parameter values (not for a discrete set of parameter values as it is typical for flows). The boundaries of such interval correspond to homoclinic tangencies.

In the figures below we show the discrete Shilnikov attractor in the three-dimensional Henon-like map

$$\bar{x} = y, \quad \bar{y} = z, \quad \bar{z} = M_1 + Bx + M_2z - y^2 \quad (2)$$

(this map emerges in the study of homoclinic tangencies in multidimensional systems [Gonchenko et al., 1993]). In Fig. 2(i), one notices that the attractor (here — the numerically obtained limit of iterations of a randomly chosen initial point) is strikingly similar to the spiral attractor for flows. The beginning of the route to the spiral chaos is quite flow-like here: the closed invariant curve C [Fig. 2(b)] bifurcates as a single entity to a double-round invariant curve [Fig. 2(c)]; the double-round curve loses stability and bifurcates to the 4-round curve [Fig. 2(e)]. Next, the bifurcation scenario changes: the invariant curve does not double anymore, it loses smoothness, gets destroyed, and chaos is created [Fig. 2(f)].

In general, it is difficult for a closed invariant curve to bifurcate as a single object. When we change parameters, a pair of resonant periodic orbits, saddle L and stable L_+ , emerge and become visible on the invariant curve. The transition to, say, double-round closed invariant curve would require a simultaneous period-doubling of the two resonant periodic orbits, which is a codimension-2 phenomenon, i.e. it would require a special parameter tuning. Therefore, when a resonance materializes on the invariant curve, it is more natural to expect a breakdown of the invariant curve following one of Afraimovich–Shilnikov scenarios [Afraimovich & Shilnikov, 1983b]. In particular, the stable resonant periodic orbit L_+ can itself undergo the Andronov–Hopf bifurcation and become a saddle-focus L_{sf} ; the unstable manifold of the L_{sf} will be bounded by a multicomponent closed invariant curve (the number of components equals to the period of L_{sf}). As parameters change, the unstable manifold of this periodic saddle-focus can form a periodic funnel and a periodic spiral attractor can form inside it.

An example of such behavior is shown in Fig. 3. The periodic spiral attractor consists of several disjoint components; their number equals to the period

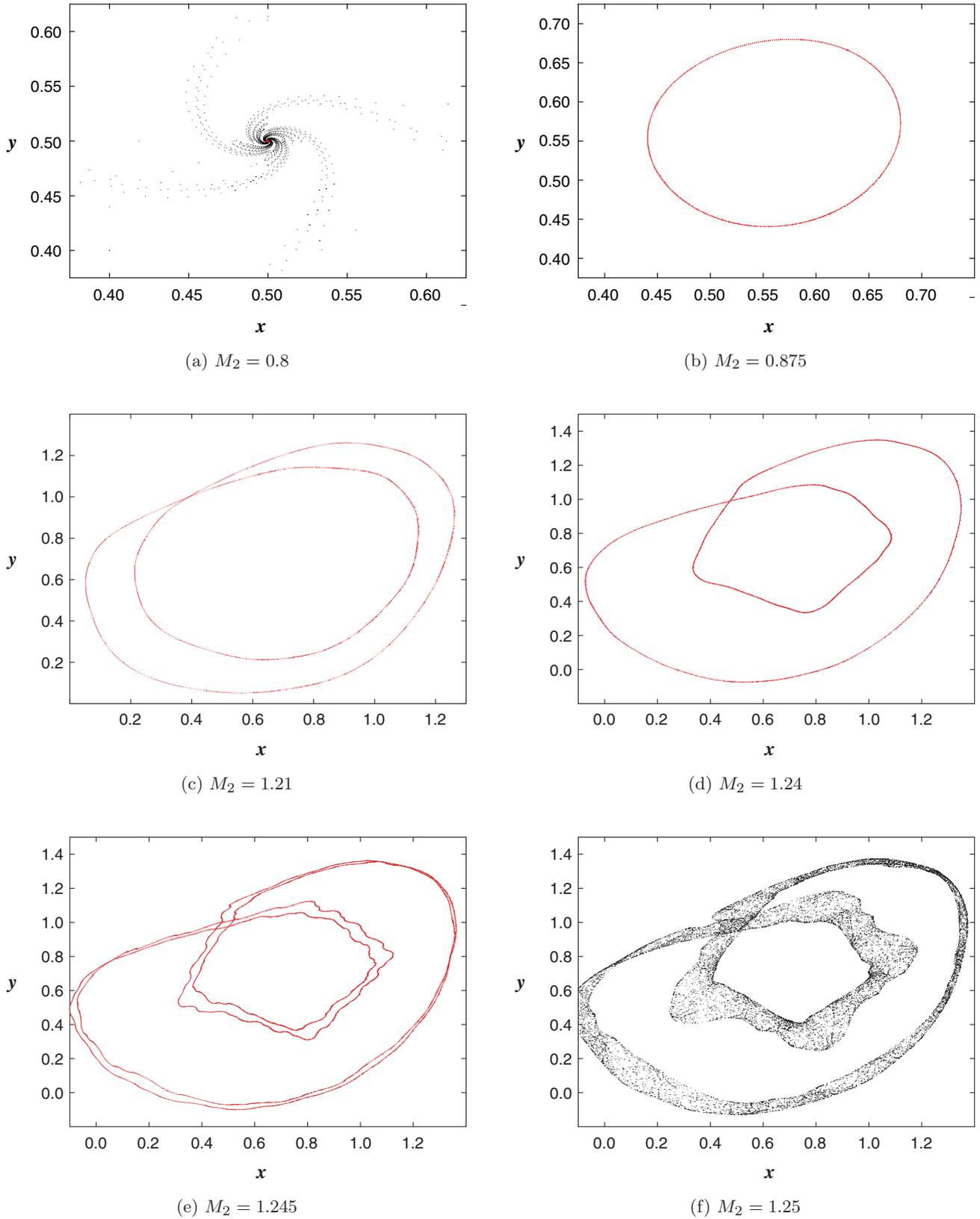
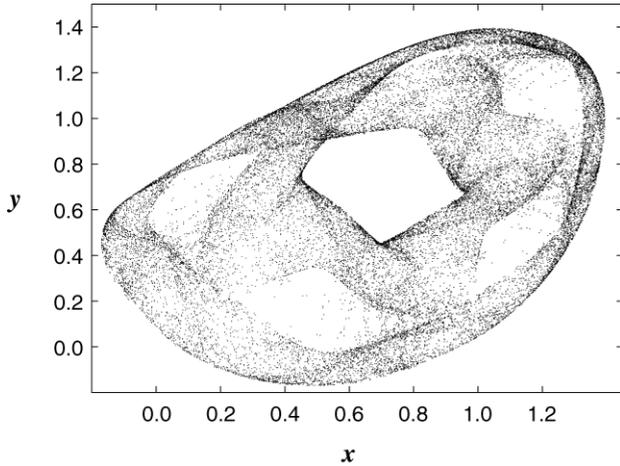
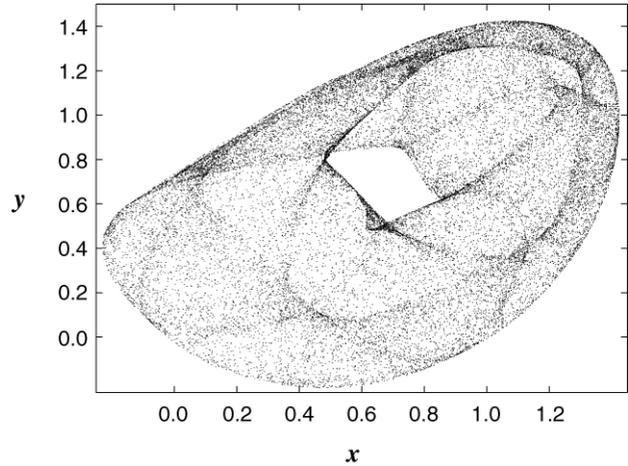


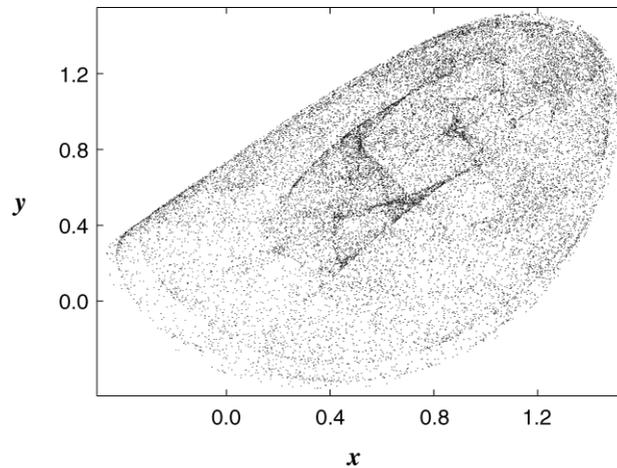
Fig. 2. Evolution of attractors of Henon-like map (2) at fixed $B = 0.7$ and $M_1 = 0$: (a) a stable fixed point, (b) a stable closed invariant curve, (c) the invariant curve has doubled, (d) and (e) the second doubling and loss of smoothness, (f)–(h) breakdown of the invariant curve and onset of chaos and (i) Shilnikov attractor.



(g) $M_2 = 1.258$



(h) $M_2 = 1.27$



(i) $M_2 = 1.29$

Fig. 2. (Continued)

of the resonant saddle-focus [see Fig. 3(b)]. Note that the components may collide to each other [Figs. 3(c) and 3(d)] as parameters change. This means that the two-dimensional unstable manifold of the resonant saddle-focus L_{sf} that is used to bound the components of the periodic spiral attractor starts to intersect the codimension-1 stable manifold of the other resonant periodic orbit L . The attractor now includes both of the resonant periodic orbits; we call such attractor “super-spiral”. Another example of such type of attractor is given in Fig. 4.

Recall that these resonant spiral/superspiral attractors exist within the funnel formed by the unstable manifold of the original saddle-focus fixed point. As parameters change, the stable and

unstable manifolds of the resonant periodic orbits may intersect the unstable and, respectively, stable manifold of this fixed point, so the periodic structure of the attractor may be lost and it may start to look more flow-like. The interplay between the original fixed point and the resonant periodic orbits can proceed in many different ways. An example is shown in Fig. 5 where the shape of attractor is determined by the two-dimensional unstable manifold of the fixed point and the one-dimensional unstable manifold of an orbit of period 3.

Resonance 1:3 is strong in the sense that if a periodic point undergoes the Andronov–Hopf bifurcation with a pair of multipliers $e^{\pm i\varphi}$ where φ is close to $\frac{2\pi}{3}$, then the closed invariant curve that is born at this bifurcation may fast get destroyed by

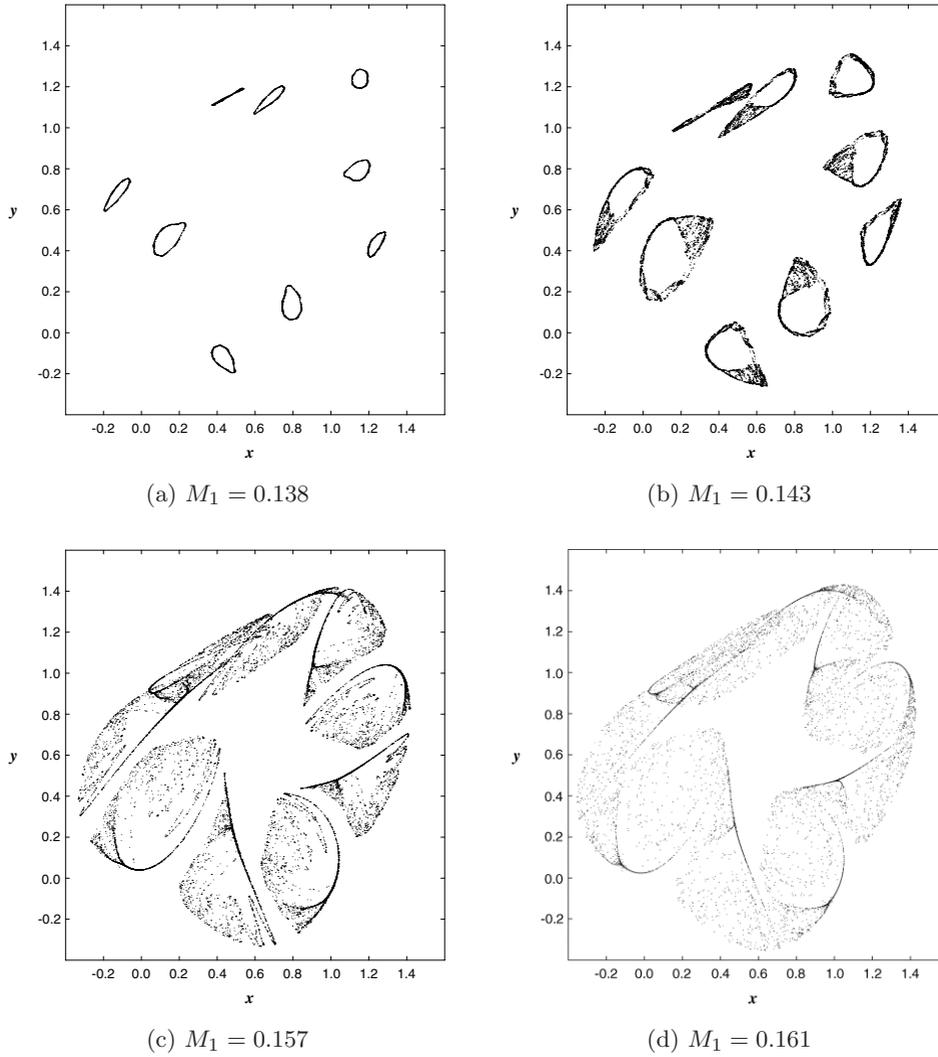


Fig. 3. Attractors of map (2) for $B = 0.7$, $M_2 = 1.055$. (a) 9-component closed periodic curve, (b) spiral attractor “sitting” on the resonant orbit of period 9, (c) the components of period-9 spiral attractor start to collide and (d) “super-spiral” attractor.

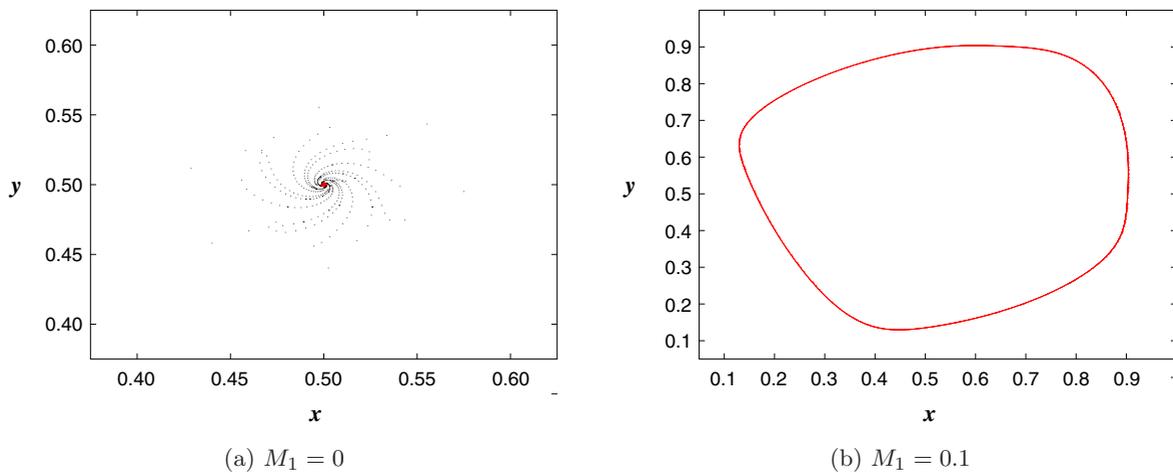


Fig. 4. Attractors of map (2) at $B = 0.7$, $M_2 = 0.8$: (a) stable fixed point, (b) and (c) stable closed invariant curve, (d) and (e) loss of smoothness and breakdown of the invariant curve and (f) super-spiral attractor “sitting” on resonant points of period 4.

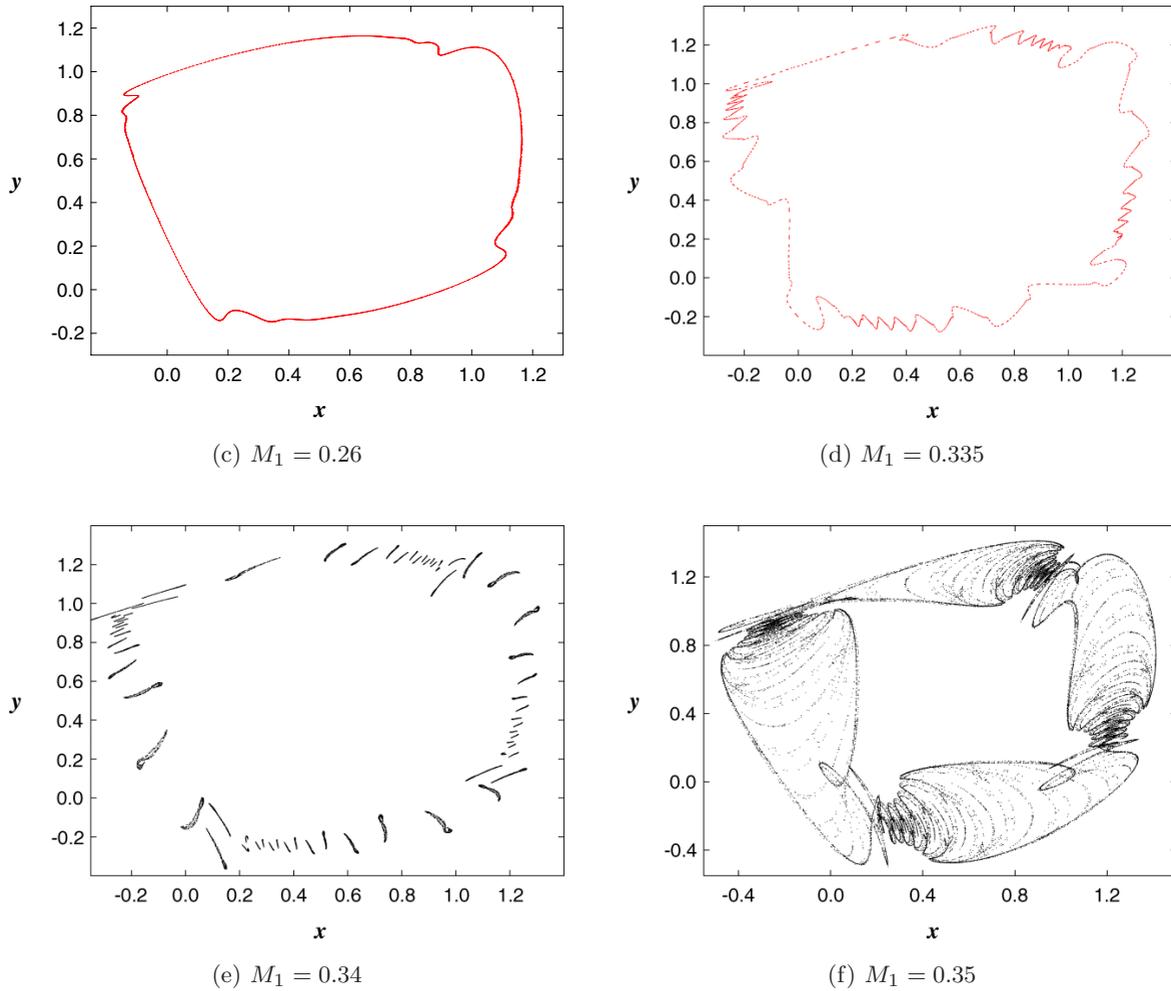


Fig. 4. (Continued)

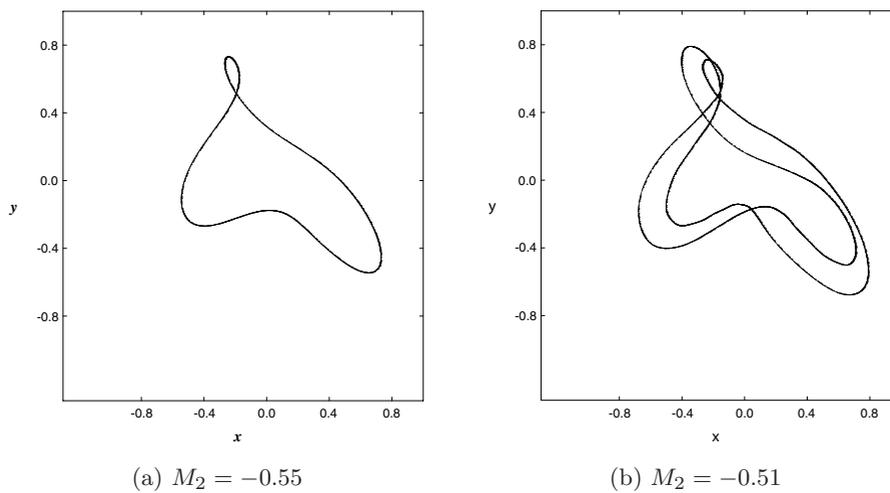


Fig. 5. Attractors of map (2) at $B = 0.7$, $M_1 = -0.195$: (a) closed invariant curve, (b) the curve has doubled, (c) and (d) period-3 orbit gets involved, and chaotic attractor is created, (e) and (f) the saddle-focus fixed point gets included into the attractor.

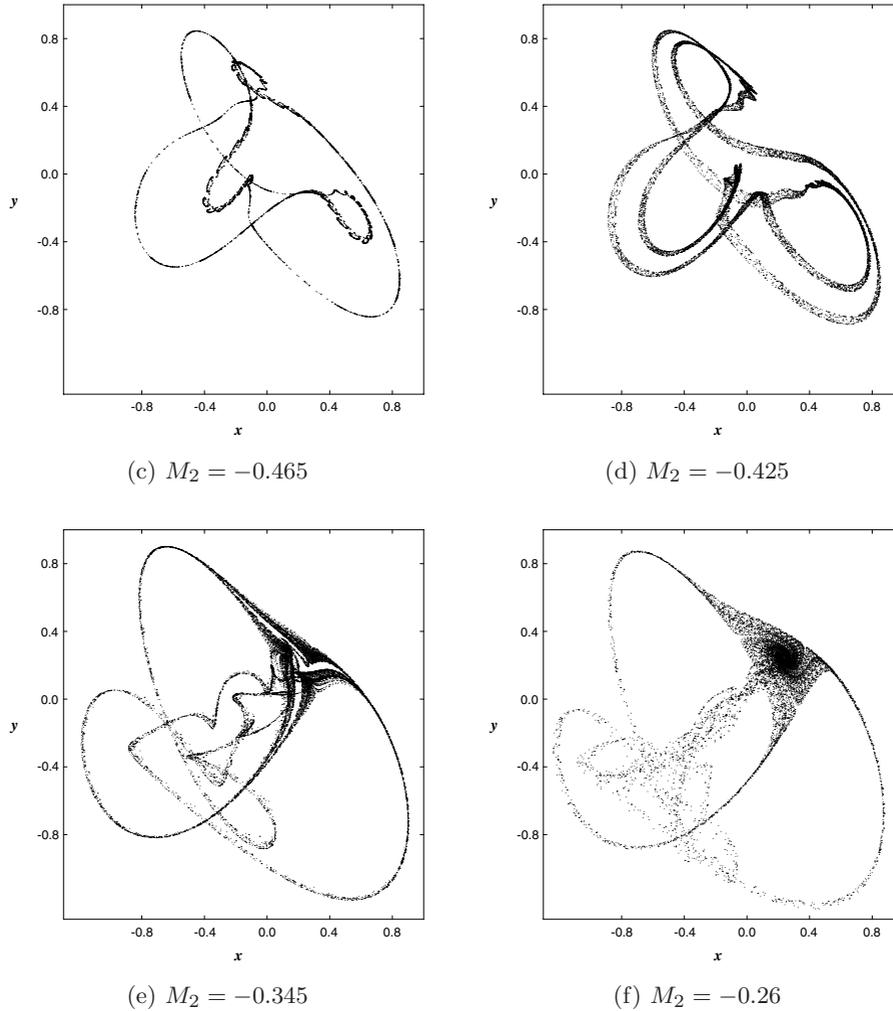


Fig. 5. (Continued)

colliding with a homoclinic structure of a nearby orbit of triple period; the corresponding bifurcation diagram is in Fig. 6 (see [Gavrilov, 1977; Arnold, 1977; Kuznetsov, 1998]). This determines the triangle shape of the funnel. Since the zone in the parameter space that is associated with the resonance 1:3 is quite wide, this characteristic shape should be observed quite often.

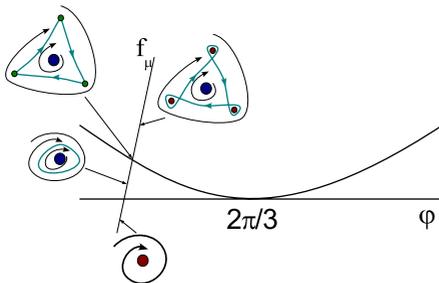


Fig. 6. Resonance 1:3.

3. Lorenz-Like Scenario for Maps

3.1. Quasiattractors and true strange attractors. Pseudohyperbolicity

It is well-known that hyperbolic attractors and Lorenz attractors are two types of “true” chaotic attractors. Namely, every orbit in such attractor has positive maximal Lyapunov exponent and this property is robust (it persists at small changes of the system). Hyperbolic attractors are structurally stable; Lorenz attractors are not, but their chaoticity is persistent [Afraimovich *et al.*, 1977, 1982; Guckenheimer, 1976; Williams, 1977; Guckenheimer & Williams, 1979].

We note that this property (of keeping “strangeness” at small smooth perturbations) does not seem to hold for many “physical” attractors observed in numerical experiments, where the

apparent chaotic behavior can easily correspond to a stable periodic orbit with a very large period (plus inevitable noise); see more discussion in [Newhouse, 1974; Afraimovich & Shilnikov, 1983a]. In particular, Hénon-like strange attractors [Benedicks & Carleson, 1991; Mora & Viana, 1993] that are often in two-dimensional maps may transform into stable long-period orbits by arbitrarily small changes of parameters [Ures, 1995]. The same is true for spiral attractors of various types and, in particular, for Poincaré–Shilnikov attractors presented in the previous section. The point is that homoclinic tangencies to the saddle-focus periodic orbit can emerge within the spiral attractor. When three-dimensional volumes are contracted, bifurcations of such tangency lead to the birth of periodic sinks [Gonchenko et al., 1993, 1996; Gonchenko et al., 2012c].

In general, nontransverse homoclinics and heteroclinics are ubiquitous in nonhyperbolic attractors. Without special restrictions [Turaev, 1996] such bifurcations lead to the birth of stable long-period orbits, so “windows of stability” emerge in chaos, and the better the accuracy of observations the more of these stability windows can be seen. This makes the whole concept of strange attractor questionable (in respect to its applicability to reality). In order to resolve this problem Afraimovich and Shilnikov introduced the term *quasiattractor*, or ε -quasiattractor [Afraimovich & Shilnikov, 1983a], that means an attractive closed invariant set which contains a saddle periodic orbit with a transverse homoclinic (i.e. a chaotic component) and may contain stable periodic orbits too, but the period of every stable orbit must be larger than ε^{-1} . So, for ε small enough, even if there are stable periodic orbits within the attractor, they will not be recognized.

The spiral attractor discussed in the previous section is, thus, a quasiattractor which contains a transverse homoclinic to a saddle-focus fixed point or a periodic orbit. The discrete Lorenz-like and figure-eight attractors which we discuss below are examples of a strange attractor (maybe, a quasi-

attractor) which contains a transverse homoclinic to a fixed or periodic point which is a saddle, i.e. its leading (nearest to the unit circle) multipliers are real. An important feature of these attractors is that they can, under certain conditions, be true strange attractors, i.e. one can guarantee the robust absence of stable periodic orbits.

A universal structure which prevents the birth of stable periodic orbits was proposed in [Turaev & Shilnikov, 1998]. Namely, if an attractor has the so-called *pseudohyperbolicity* property, then neither the system itself nor any close system can have stable periodic orbits in a certain neighborhood of the attractor. This property (the term volume-hyperbolicity can also be used in the same context [Bonatti et al., 2005]) is formulated as follows. Let a map F (the case of a flow is treated analogously) have an absorbing domain \mathcal{D} (a strictly forward-invariant neighborhood of an attractor A) and let the tangent space at each point $x \in \mathcal{D}$ admit a decomposition into the direct sum of two subspaces E_x^{ss} and E_x^{uc} which are invariant with respect to the differential DF and which depend continuously on x . Moreover, let DF be strongly contracting along E^{ss} and let it expand in volume in E^{uc} . Then the map is pseudohyperbolic in \mathcal{D} , and every orbit in the attractor $A \subset \mathcal{D}$ has positive maximal Lyapunov exponent; moreover, this property persists at small smooth perturbations of the system (see [Turaev & Shilnikov, 1998, 2008] for more detail). Note that one can derive easily verifiable sufficient conditions for the pseudohyperbolicity, as given by the following result (a reformulation of Lemma 1 of [Turaev & Shilnikov, 1998]).

Lemma 1. *Let a map T be defined on a closure of an open region \mathcal{D} , and $T(\text{cl}(\mathcal{D})) \subset \mathcal{D}$. Suppose that in some coordinates (x, z) on \mathcal{D} the map $T : (x, z) \mapsto (\bar{x}, \bar{z})$ can be written as $\bar{z} = f(x, z)$, $\bar{x} = g(x, z)$ where f, g are at least C^2 -smooth, and $\det(\frac{\partial g}{\partial x}) \neq 0$. Denote $A = \frac{\partial f}{\partial z} - \frac{\partial f}{\partial x}(\frac{\partial g}{\partial x})^{-1}\frac{\partial g}{\partial z}$, $B = \frac{\partial f}{\partial x}(\frac{\partial g}{\partial x})^{-1}$, $C = (\frac{\partial g}{\partial x})^{-1}\frac{\partial g}{\partial z}$, $D = (\frac{\partial g}{\partial x})^{-1}$. If*

$$\max \left\{ \sup_{(x,z) \in \mathcal{D}} \sqrt{\|A\| \|D\|}, \sup_{(x,z) \in \mathcal{D}} \|A\|, \sup_{(x,z) \in \mathcal{D}} \sqrt{|\det D|} \right\} + \sqrt{\sup_{(x,z) \in \mathcal{D}} \|B\| \sup_{(x,z) \in \mathcal{D}} \|C\|} < 1,$$

then the attractor of the map T in the absorbing domain \mathcal{D} is pseudohyperbolic.

This lemma is based on the Afraimovich–Shilnikov “annulus principle” [Afraimovich & Shilnikov,

1974a, 1974b, 1977; Shilnikov et al. 1998, 2001] which gives sufficient conditions for the existence of what is now called a dominated splitting. It also generalizes the hyperbolicity conditions proposed in

[Afraimovich *et al.*, 1977, 1982] for the Poincaré map of the Lorenz attractor.

Hyperbolic and Lorenz attractors satisfy the pseudohyperbolicity property, however there are other pseudohyperbolic attractors. For example, in [Turaev & Shilnikov, 1998] an example of a *wild-hyperbolic* strange attractor was constructed for a four-dimensional flow. Unlike hyperbolic and Lorenz attractors, wild-hyperbolic ones may contain homoclinic tangencies. However, these tangencies are such that their bifurcations do not lead to stable periodic orbits (as the conditions from [Gonchenko *et al.*, 1993, 1996, 2008] for the birth of periodic sinks from homoclinic tangencies are automatically violated by the pseudohyperbolicity).

Another example of a wild-hyperbolic attractor with the pseudohyperbolicity property can be obtained by a small time-periodic perturbation of a flow that possesses a Lorenz attractor [Turaev & Shilnikov, 2008]. By taking a discrete forward orbit of the corresponding Poincaré map (the map for the period of the perturbation), we obtain a strange attractor which looks quite similar to the canonical (continuous time) Lorenz attractor. We call such attractors *discrete Lorenz attractors* (see exact definitions in [Gonchenko *et al.*, 2013b]). Importantly, a normal form for the bifurcations of periodic points with the triplet of multipliers $(-1, -1, +1)$ is an (exponentially small) periodic perturbation of the Shimizu–Morioka system [Shilnikov *et al.*, 1993]; this system is known to have a Lorenz attractor [Shilnikov, 1986, 1993]. Therefore, discrete Lorenz-like attractors can appear at the bifurcations of an arbitrary map which has a periodic orbit that undergoes the $(-1, -1, +1)$ -bifurcation [Shilnikov *et al.*, 1993; Gonchenko *et al.*, 2005; Gonchenko *et al.*, 2013b].

In particular, a class of Henon-like maps was considered in [Gonchenko *et al.*, 2013b]:

$$\bar{x} = y, \quad \bar{y} = z, \quad \bar{z} = Bx + f(y, z), \quad (3)$$

where f is a smooth function. The Jacobian of such map is constant and equals B . The fixed points are given by $x = y = z = x_0$, $x_0(1 - B) = f(x_0, x_0)$. The characteristic equation at the fixed point is $\lambda^3 - A\lambda^2 - C\lambda - B = 0$, where $A = f'_z(x_0, x_0)$, $C = f'_y(x_0, x_0)$. At $(A = -1, C = 1, B = 1)$, the fixed point has multipliers $(-1, -1, +1)$. Take a smooth three-parameter family of maps (3) which at zero parameter values has a fixed point with multipliers $(-1, -1, +1)$, and let the fixed point exist

for a region of parameter values adjoining to zero. Move the origin to the fixed point. The map takes the form

$$\begin{aligned} \bar{x} &= y, \\ \bar{y} &= z, \\ \bar{z} &= (1 - \varepsilon_1)x + (1 - \varepsilon_2)y - (1 + \varepsilon_3)z \\ &\quad + \alpha y^2 + \beta yz + \gamma z^2 + \dots, \end{aligned} \quad (4)$$

where $\varepsilon_{1,2,3}$ are small, $\alpha = \frac{1}{2}f''_{yy}(x_0, x_0)$, $\beta = f''_{yz}(x_0, x_0)$, $\gamma = \frac{1}{2}f''_{zz}(x_0, x_0)$, and the dots stand for cubic and higher order terms.

Lemma 2 [Gonchenko *et al.*, 2013b]. *Assume*

$$(\gamma - \alpha)(\alpha - \beta + \gamma) > 0. \quad (5)$$

Then map (4) has a pseudohyperbolic Lorenz-like attractor for all ε from an open, adjoining to $\varepsilon = 0$, subregion of $\{\varepsilon_1 > 0, \varepsilon_1 + \varepsilon_3 > 0, |\varepsilon_2 - \varepsilon_1 - \varepsilon_3| \leq L(\varepsilon_1^2 + \varepsilon_3^2)\}$ with some $L > 0$.

Example for which the hypothesis of the lemma holds is given by the map

$$\bar{x} = y, \quad \bar{y} = z, \quad \bar{z} = M + Bx + Cy - z^2, \quad (6)$$

for which a discrete Lorenz attractor was found in [Gonchenko *et al.*, 2005] for an open domain of the parameters (M, B, C) adjoining to the point $(M = -1/4, B = 1, C = 1)$. At these values of the parameters, the map has a fixed point $x = y = z = \frac{1}{2}$. After shifting the coordinate origin to this point we have the map in the form

$$\bar{x} = y, \quad \bar{y} = z, \quad \bar{z} = x + y - z - z^2,$$

i.e. the fixed point has multipliers $(-1, -1, +1)$, and $\alpha = 0$, $\beta = 0$, $\gamma = -1$. As we see, condition (5) of the lemma holds. For numerically obtained portraits of Lorenz-like attractors in this map, see Fig. 9.

Another example is given by

$$\begin{aligned} \bar{x} &= y, \\ \bar{y} &= z + \gamma y^2, \end{aligned} \quad (7)$$

$$\bar{z} = M_0 + Bx + M_1y + Az + \delta y^3 + \beta yz.$$

Introduce $z_{\text{new}} = z + \gamma y^2$. Then, map (7) takes the standard Hénon form

$$\begin{aligned} \bar{x} &= y, \\ \bar{y} &= z, \\ \bar{z} &= M_0 + Bx + M_1y + Az - A\gamma y^2 + \gamma z^2 \\ &\quad + \beta yz + (\delta - \beta\gamma)y^3. \end{aligned} \quad (8)$$

Let $x = y = z = x_0$ be a fixed point of map (7), i.e. $M_0 = x_0(1 - B - M_1 - A) - (\delta - \beta\gamma)x_0^3 - (1 + \beta\gamma - A\gamma)x_0^2$. By shifting the coordinate origin to this point, we write the map in form (4):

$$\begin{aligned} \bar{x} &= y, \\ \bar{y} &= z, \\ \bar{z} &= Bx + (M_1 + (\beta - 2A\gamma)x_0 \\ &\quad + 3(\delta - \beta\gamma)x_0^2)y + (A + (\beta + 2\gamma)x_0)z \\ &\quad + \alpha y^2 + \beta yz + \gamma z^2 + \dots \end{aligned}$$

where $\alpha = 3(\delta - \beta\gamma)x_0 - A\gamma$. The fixed point has the multipliers $(-1, -1, +1)$ at $B = 1$, $A + x_0(\beta + 2\gamma) = -1$, $M_1 = 1 + (2A\gamma - \beta)x_0 - 3(\delta - \beta\gamma)x_0^2$. Condition (5) reads as $x_0(3\delta - 2\beta\gamma + 2\gamma^2)[(3\delta - 2\beta\gamma + 2\gamma^2)x_0 - \beta + 2\gamma] < 0$. For every given β , γ and δ one can always find x_0 for which this is fulfilled, provided $\delta \neq \frac{2}{3}\gamma(\gamma - \beta)$ and $\beta \neq 2\gamma$ is fulfilled. Therefore, by Lemma 2, for every fixed β , γ , δ which satisfy these inequalities there is an open region in the space of parameters (M_0, M_1, B, A) which corresponds to the existence of the Lorenz-like attractor. For numerically obtained portraits of Lorenz-like attractors in this map, see Figs. 10 and 11.

Further, discrete Lorenz attractors were found numerically in other models, including systems of nonholonomic mechanics [Gonchenko & Gonchenko, 2013; Gonchenko et al., 2013a] (see Sec. 5). Below we describe the simplest scenarios leading to discrete Lorenz attractors and their “figure-eight” analogues.

3.2. Discrete attractors of “Lorenz-like” and “figure-eight” shapes

In this section, we describe a basic scenario of transition to chaos in three-dimensional maps, which is different from the Shilnikov scenario of Sec. 2. Here, the first bifurcation that determines the future shape of the strange attractor is the period-doubling bifurcation (for the spiral attractor, a similar role is played by the Andronov–Hopf bifurcation).

Consider a one parameter family f_μ of three-dimensional orientable diffeomorphisms and assume that for the values of μ from some interval I the diffeomorphism f_μ has an absorbing domain \mathcal{D}_μ .

Let μ_1 and μ_2 be certain values from I such that $\mu_1 < \mu_2$. Assume that at $\mu \leq \mu_1$ the forward orbit of every point in \mathcal{D}_μ tend to a stable fixed point O_μ . Assume that at $\mu = \mu_1$ the point O_μ undergoes a soft (supercritical) *period-doubling bifurcation*. As a result, a stable period-2 orbit $P_\mu = (p_1, p_2)$, where $f_\mu(p_1) = p_2$ and $f_\mu(p_2) = p_1$, is born from O_μ at $\mu > \mu_1$, and the point O_μ becomes a saddle. We denote its multipliers as $\lambda_1, \lambda_2, \lambda_3$, where $\lambda_1 < -1$, and $|\lambda_3| < |\lambda_2| < 1$. Note that we have here two cases: $\lambda_2 < 0, \lambda_3 > 0$ and $\lambda_2 > 0, \lambda_3 < 0$.

Let the saddle point O_μ have a transverse homoclinic orbit at $\mu > \mu_2$. Then, the maximal attractor A_μ in the absorbing domain \mathcal{D}_μ contains the nontrivial hyperbolic set associated with this homoclinic, i.e. we may speak about a quasiattractor (if there are no obvious stable periodic orbits in it). There are two distinct possibilities for the shape of this attractor, which mainly depends on the signs of the stable multipliers (see Fig. 7). Recall that the unstable multiplier λ_1 of O_μ is negative, therefore its unstable separatrices (the two components into which O_μ divides its unstable manifold) are mapped to each other by f_μ . Thus, the homoclinic orbit belongs to both of these separatrices, i.e. they both intersect the stable manifold $W^s(O_\mu)$. Typically, the homoclinic intersection does not belong to the strong stable manifold $W^{ss}(O_\mu)$ which is tangent to the eigenspace that corresponds to the nonleading multiplier λ_3 . The manifold $W^{ss}(O_\mu)$ divides $W^s(O_\mu)$ into two parts. These parts are invariant with respect to f_μ if the leading stable multiplier λ_2 is positive, and they are taken to each other by f_μ if $\lambda_2 < 0$. Thus, we have two cases:

- (1) if $\lambda_2 > 0$, then both unstable separatrices of O_μ can intersect $W^s(O_\mu)$ on one side from $W^{ss}(O_\mu)$ — we say that the attractor A_μ has a *Lorenz-like* shape in this case;
- (2) if $\lambda_2 < 0$, then the unstable separatrices must intersect $W^s(O_\mu)$ on both sides from $W^{ss}(O_\mu)$ — we say that A_μ has a *figure-eight* shape.

Note that the loss of stability of the period-2 orbit P_μ does not need to be correlated with the emergence of the homoclinics to the fixed point O_μ . In fact, many different variants are possible. For example, a cascade of period-doublings may continue and the transition to chaos may precede the creation of homoclinics to O_μ (this is typical if

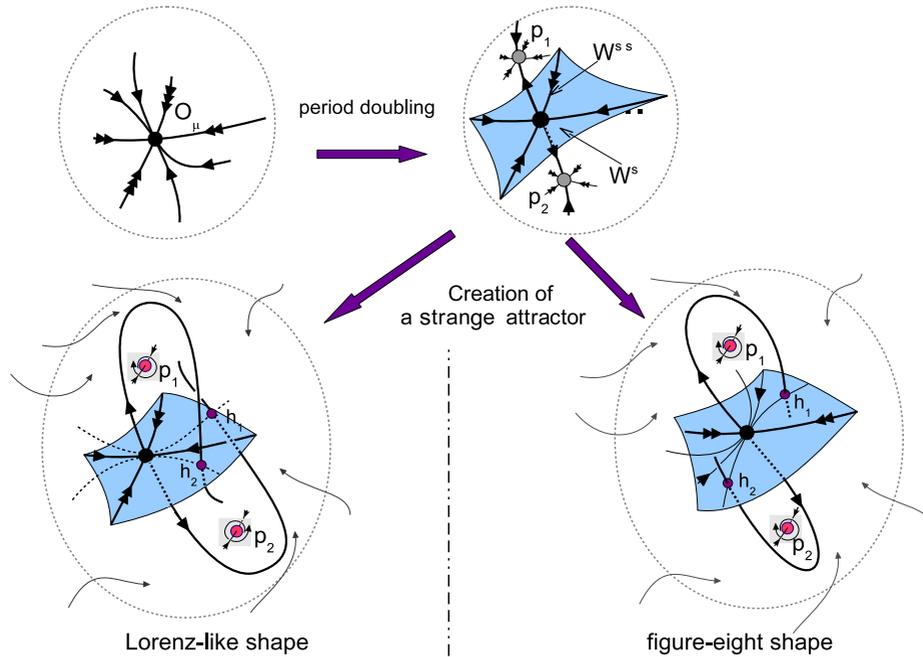


Fig. 7. Two shapes of the attractor.

the Jacobian is small and the map is close to one-dimensional). For nonsmall Jacobians, the period-2 orbit may undergo an Andronov–Hopf bifurcation. A supercritical bifurcation leads to the birth of a stable invariant curve with two closed connected components, see Fig. 8. At the further growth of μ this curve can get destroyed and transformed to a “homoclinic structure” involving O_μ . If the Andronov–Hopf bifurcation is subcritical, then a saddle closed period-2 curve merges with P_μ . The homoclinics to O_μ can already exist in this case, so the period-2 curve is formed at the fringes of the homoclinic structure. Depending on the situation, the two-dimensional stable manifold of this curve may serve as a barrier that separates the attraction domains of the period-2 orbit P_μ and the Lorenz-like attractor that contains O_μ , or there may be no such attractor separate from P_μ (this happens when the closure of the unstable manifold of O_μ contains P_μ ; then we should speak about the Lorenz-like attractor only after P_μ loses stability). Similar scenarios (where the period-2 orbit is replaced by a pair of symmetric equilibria and the period-2 closed curve is replaced by a pair of symmetric limit cycles) are known to lead to the onset of the Lorenz attractor in the Lorenz model (with subcritical Andronov–Hopf) [Afraimovich *et al.*, 1977; Shilnikov, 1980; Barrio *et al.*, 2012] and

the Shimizu–Morioka model (with supercritical Andronov–Hopf) [Shilnikov, 1986, 1993]; see Fig. 8. Therefore, the above described transition.

Fixed Point \Rightarrow Period-2 Orbit \Rightarrow Stable/Saddle Period-2 Curve \Rightarrow discrete Lorenz-like Attractor should be typical for Poincaré maps for small periodic perturbations of these systems and, hence, for arbitrary maps near the moment of bifurcations of periodic orbits with multipliers $(-1, -1, +1)$ (see e.g. Lemma 2 in the previous section).

As we mentioned in the previous section, the Lorenz-like and figure-eight attractors of three-dimensional maps can be true strange attractors, provided they satisfy the pseudohyperbolicity property. The pseudohyperbolicity should be verified at each point of the absorbing domain. In particular, at the fixed point O_μ this property requires that the saddle value $\sigma = |\lambda_1 \lambda_2|$ is greater than 1 (as at least some two-dimensional areas must be expanded by the linearization of the map at the fixed point). This is a necessary condition that is easiest to check; if it is not satisfied, then stable periodic orbits will be born from homoclinic tangencies to O_μ , i.e. the attractor A_μ will be a quasiattractor. However, this condition does not need to be sufficient. One may also numerically estimate Lyapunov exponents at some randomly chosen orbit in A_μ . The pseudohyperbolicity requires the positivity of the sum of

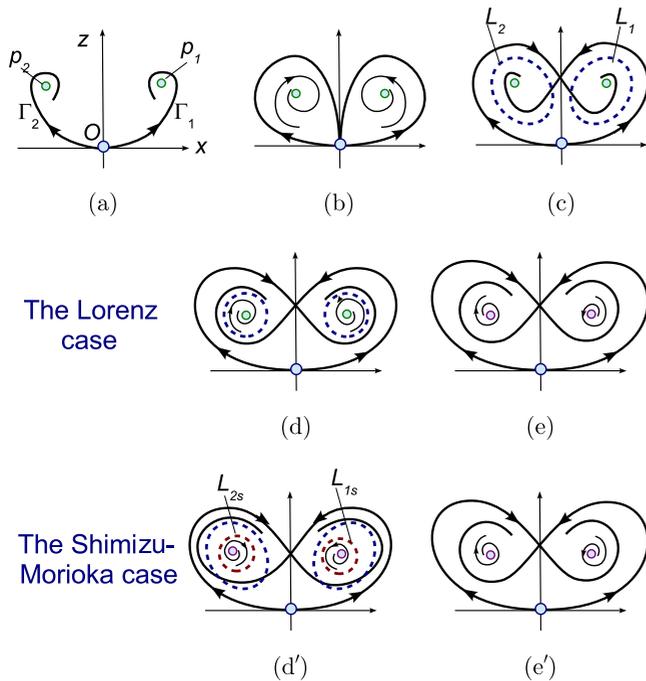


Fig. 8. A sketch of the Poincaré map for a small time-periodic perturbation of the Lorenz or Shimizu–Morioka models. (a) After a period-doubling, the fixed point becomes a saddle, and a stable period-2 orbit (p_1, p_2) is created, (b) creation of a thin homoclinic butterfly structure, (c) a saddle period-2 closed curve (L_1, L_2) detaches from the butterfly, (d) the Lorenz-like attractor gets separated from the stable orbit (p_1, p_2) , (e) the orbit (p_1, p_2) becomes unstable (subcritical Andronov–Hopf bifurcation), (d′) the Lorenz-like attractor gets separated from the stable period-2 closed curve born from (p_1, p_2) at a supercritical Andronov–Hopf bifurcation and (e′) the stable and saddle curves of period 2 get destroyed.

the two largest Lyapunov exponents. Again, this is not yet a sufficient condition for the true chaoticity of the attractor, even if the orbit appears to be dense in A_μ : one also needs to verify that the angle between the invariant subspace corresponding to the two largest Lyapunov exponents and the subspace that corresponds to the rest of Lyapunov exponents stays bounded away from zero. The most robust approach to prove hyperbolicity is, of course, based on Lemma 1. In the examples below we do not go into such depths in the verification of the pseudohyperbolicity. However, we do the simple checks of the saddle value and Lyapunov exponents. Also, the similarity of the shape of our discrete Lorenz-like attractors with the classical Lorenz attractor is often very high, therefore we are quite certain these attractors are pseudohyperbolic, hence truly chaotic.

4. Numerical Experiments with Hénon-Like Maps

We now present numerics that illustrates the theory above. We consider, first, map (6) with the Jacobian $B = 0.7$ (i.e. the dissipation is weak enough). In Fig. 9 the corresponding phase portraits (numerical iterations of a single initial condition) are shown for fixed $M_2 = 0.85$ and varying M_1 . The transition to a Lorenz-like attractor proceeds in the following steps: the orbit of period 2 [Fig. 9(a)] gives rise to a stable two-component closed invariant curve [Fig. 9(b)], which then gets destroyed by a “collision” with a saddle two-component invariant curve that was formed from a homoclinic butterfly to the saddle fixed point, and a Lorenz-like attractor is formed [Figs. 9(c) and 9(d)]. This scenario is similar to what one should observe in a periodically perturbed Morioka–Shimizu system.

As M_1 grows, the attractor grows in size and evolves into a strange attractor “without holes” [Fig. 9(f)]. It reminds of the attractor of Lorenz model after the absorption of the saddle-foci. This effect is related to a creation of a heterodimensional cycle where the one-dimensional unstable manifold of the saddle fixed point intersects with the one-dimensional stable manifold of the saddle-focus orbit of period two; this cycle is analogous to the “Bykov contour” of the Lorenz model [Bykov, 1978, 1980, 1993]. This bifurcation (e.g. in the Lorenz model [Afraimovich et al., 1980; Petrovskaya & Yudovich, 1980; Bykov & Shilnikov, 1989; Barrio et al., 2012] and in the Shimizu–Morioka model [Shilnikov, 1986, 1993; Shilnikov et al., 1993]) leads to the creation of stable periodic orbits. By analogy, we expect that the same is true for the “no hole” attractor in the Henon map (6), i.e. it loses the pseudohyperbolic structure and becomes a quasiattractor.

Next, we show results of numerical simulations of map (7). We start with the case $\beta = -3, \delta = -3, \gamma = 1, M_0 = -2$. At $M_1 = 4, B = 1, A = 0$ this map has a fixed point at $x = y = z = 1$ with the multipliers of $(-1, -1, +1)$. It is easy to check that condition (5) is satisfied, so we can expect the Lorenz-like attractor for parameter values close to these. We choose $B = 0.7, A = 0.1$, and vary M_1 from 4.345 down to 4.265. The results are shown in Fig. 10. The attractor forms in the way similar to the previous case. The destruction of the attractor [Figs. 10(d)–10(f)] proceeds via formation of a lacuna where a stable closed invariant curve emerges [Fig. 10(e)]

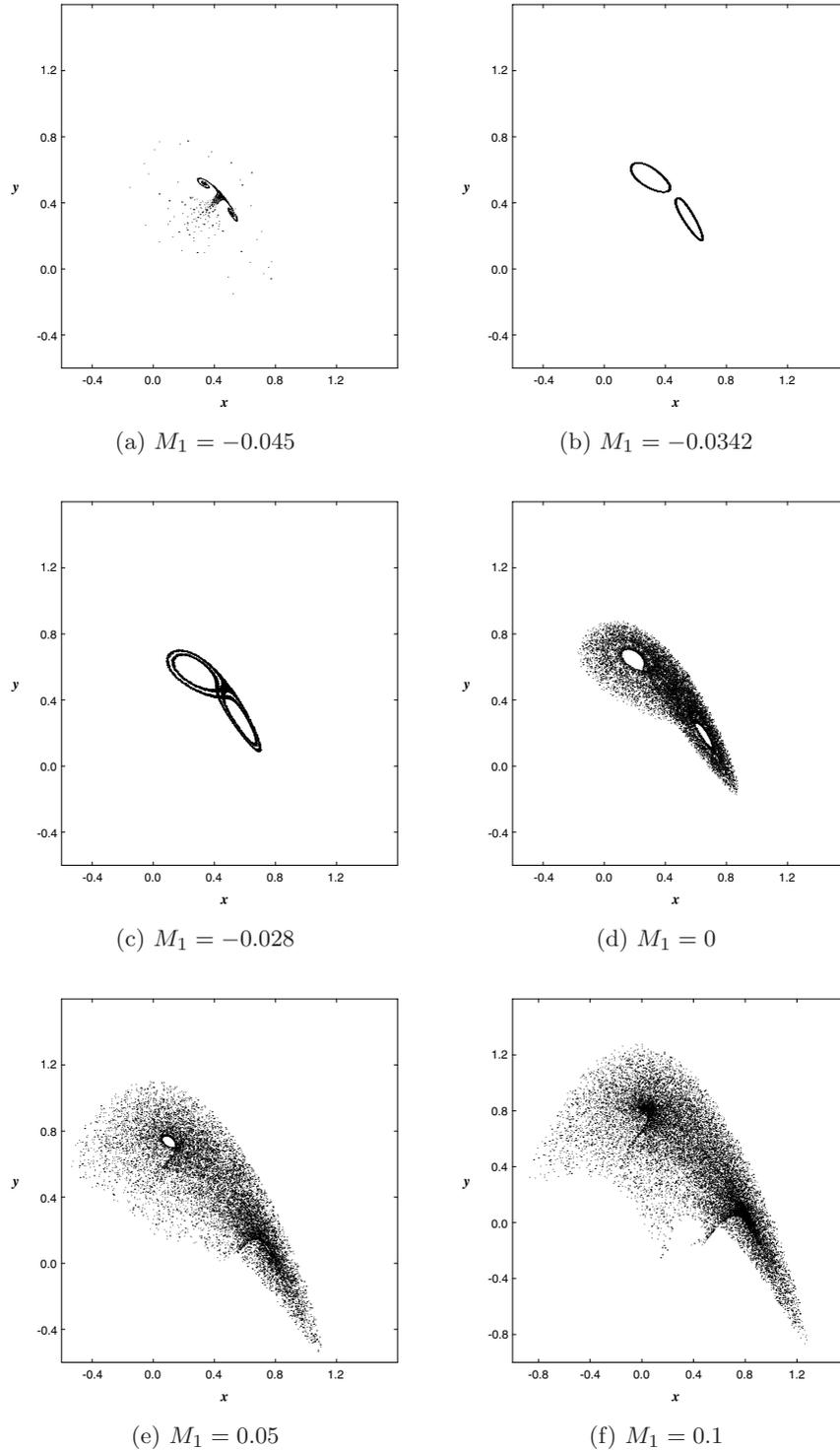


Fig. 9. Evolution of attractors in the 3D Hénon map (6) with $B = 0.7$, $M_2 = 0.85$, as M_1 varies: (a) period-2 point, (b) closed curve of period 2 and (c)–(f) strange attractors.

which next breaks-down and forms a strange attractor [Fig. 10(f)] of the “wriggled” shape typical for the “torus-chaos” quasiattractor [Afraimovich & Shilnikov, 1974b; Curry & Yorke, 1978; Aronson *et al.*, 1982; Afraimovich & Shilnikov, 1983b]. Note

that the route of the destruction of the Lorenz-like attractor via formation of a lacuna, which we see in these figures, reminds one of the scenarios of the disappearance of the Lorenz attractor that was described in [Afraimovich *et al.*, 1982] and

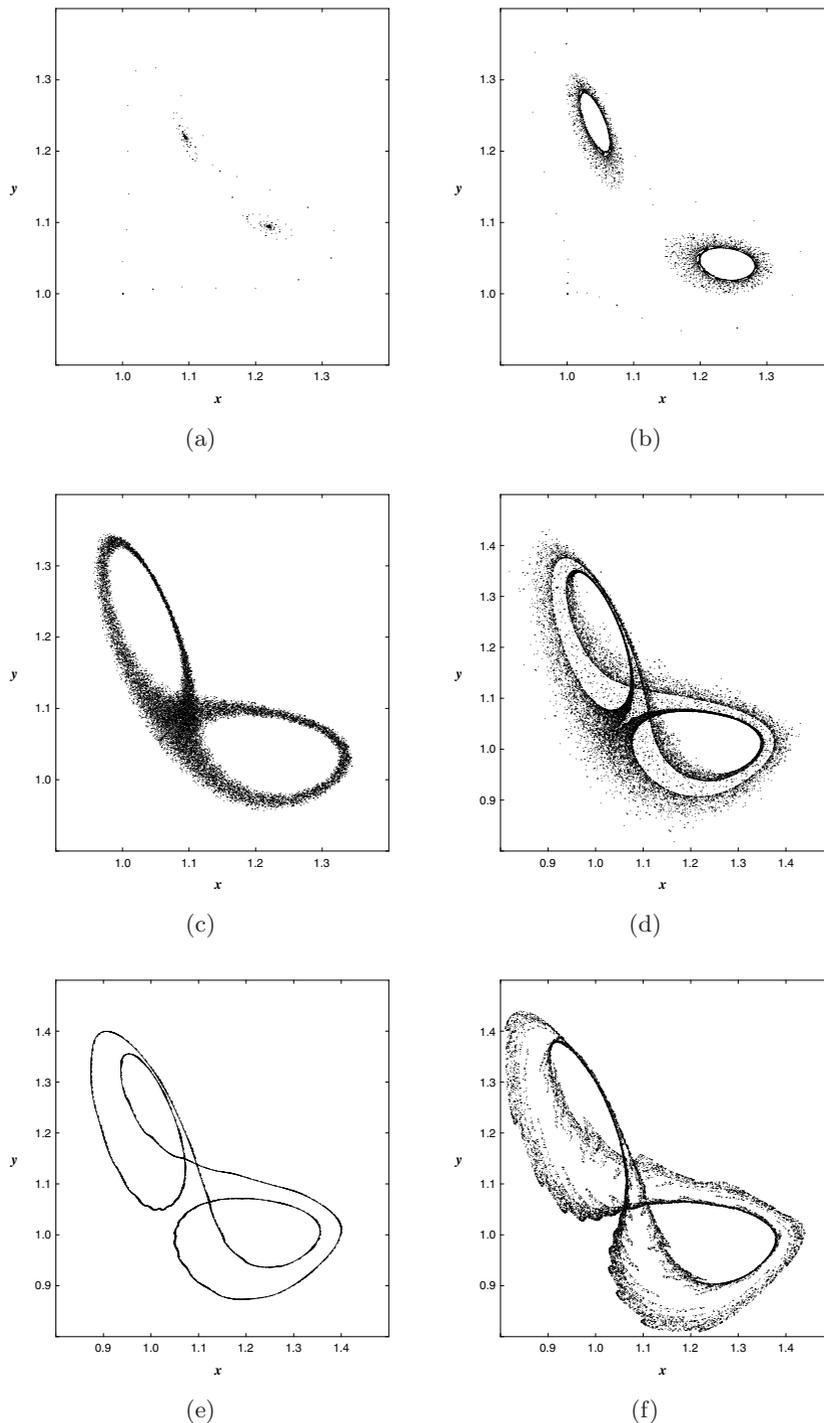


Fig. 10. Creation and destruction of the Lorenz-like attractor in map (7) at $\beta = -3$, $\delta = -3$, $\gamma = 1$, $M_0 = -2$, $B = 0.7$, $A = 0.1$. (a) Cycle of period 2 (after period doubling), (b) closed curve of period 2 (after Andronov–Hopf bifurcation of the cycle), one can also see the location of the saddle curve of period 2 [cf. Figs. 8(d') and 8(e')], (c) Lorenz-like attractor, (d) Lorenz-like attractor with a lacuna, (e) and (f) bifurcation stages after the destruction of the attractor (stable invariant curve and torus-chaos).

was also discovered in the Shimizu–Morioka model [Shilnikov, 1986, 1993; Shilnikov *et al.*, 1993].

Another case corresponds to $\beta = 2$, $\delta = 1/3$, $\gamma = 0$. A fixed point at $x = y = z = 1$ has

multipliers $(-1, -1, +1)$ at $M_0 = -2$, $M_1 = -2$, $B = 1$, $A = -3$. Again, condition (5) is satisfied. Numerics was performed at $M_0 = 3.67$, $B = 0.7$, $A = -3.1$, with M_1 varying from -2.5345 to

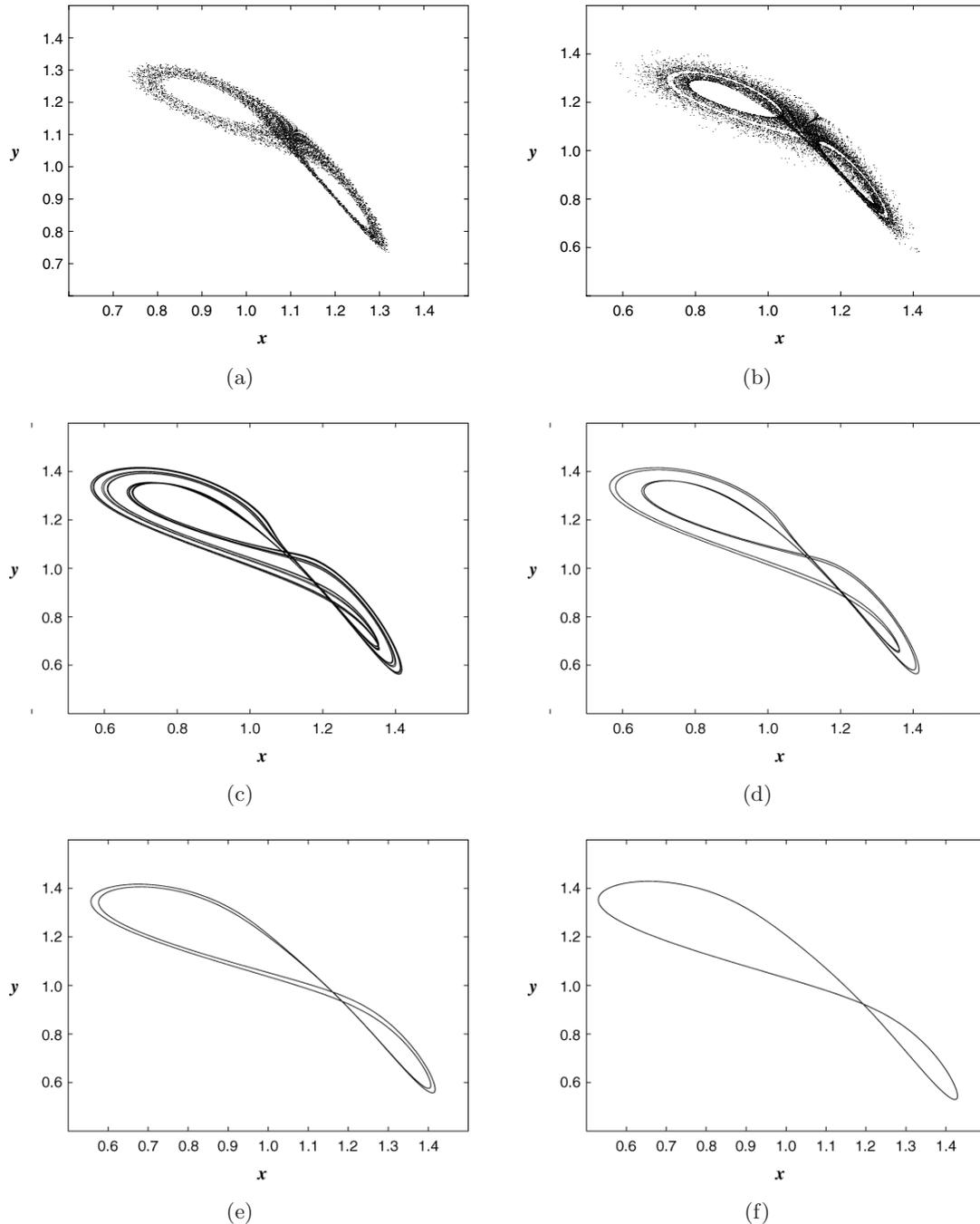


Fig. 11. Plots of attractors of map (7) for $\beta = 2$, $\delta = 1/3$, $\gamma = 0$, $M_0 = 3.67$, $B = 0.7$, $A = -3.1$ as M_1 varies. (a) and (b) As in Fig. 10, (c) strange quasiattractor and (d)–(f) stable closed invariant curves.

–2.505. The results are shown in Fig. 11. The first stages [Figs. 11(a) and 11(b)] on the route to the Lorenz-like attractor [Fig. 11(b)] are the same here as in the previous cases. The destruction of the attractor proceeds via the formation of a lacuna where a stable invariant curve emerges [Fig. 11(d)], which then gives place to a strange quasiattractor [Fig. 11(c)]. These stages are as seen in Fig. 11.

However, the quasiattractor has now a different structure, and unravels via a backward cascade of torus-doubling bifurcations [Figs. 11(d)–11(f)]. The last invariant curve disappears by colliding with a saddle invariant curve at a saddle-node bifurcation at $M_1 \sim -2.501$. See [Chenciner, 1985; Los, 1989; Broer *et al.*, 1990; Braaksma *et al.*, 1990; Anishenko & Nikolaev, 2005] for the theory of the

saddle-node and doubling bifurcations for invariant curves. Numerous examples of such bifurcations in three-dimensional diffeomorphisms can be found in [Vitolo, 2003].

It is curious that, despite our numerics being performed for parameter values sufficiently far from the bifurcation of a fixed point with the multipliers $(-1, -1, 1)$, the bifurcation scenarios are quite similar to those one should have in its normal form, i.e. in the Shimizu–Morioka model with a small periodic forcing. Namely, the Lorenz attractor in the Shimizu–Morioka model transforms into a strange quasiattractor in a variety of ways [Shilnikov, 1986, 1993; Shilnikov *et al.*, 1993], depending on the choice of a path in the parameter plane, and this variety does include an absorption of saddle-foci like in Fig. 9, or formation of a lacuna with a consequent boundary crisis of the Lorenz attractor and emergence of a quasiattractor which may be accompanied by period-doubling cascades or not, like in Figs. 10 and 11. The differences between the destruction of the Lorenz-like attractor in our maps and the destruction of the Lorenz attractor in the model flow are still visible (mainly due to the effects of loss of smoothness and breakdown of invariant curves), but they do not seem to play a major role.

5. Discrete Lorenz-Like and Figure-Eight Attractors in Models of Nonholonomic Mechanics

In this section, we show how the strange attractors described in Sec. 3.2 emerge in the dynamics of rigid bodies moving on a plane without slipping. This means that we consider a nonholonomic model of motion for which the contact point of the body has zero velocity, i.e. $\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r} = 0$, where \mathbf{r} is the vector from the center of mass C to the contact point, \mathbf{v} is the velocity of C and $\boldsymbol{\omega}$ is the angular velocity. By introducing a coordinate frame rigidly rotating with the body the equations of motion can be written in the form [Borisov & Mamaev, 2003]:

$$\begin{aligned} \dot{\mathbf{M}} &= \mathbf{M} \times \boldsymbol{\omega} + m\dot{\mathbf{r}} \times (\boldsymbol{\omega} \times \mathbf{r}) + m\mathbf{g}\mathbf{r} \times \boldsymbol{\gamma}, \\ \dot{\boldsymbol{\gamma}} &= \boldsymbol{\gamma} \times \boldsymbol{\omega}, \\ \mathbf{M} &= [\mathbf{J} + m(\mathbf{r}, \mathbf{r})\mathbf{I} - m\mathbf{r} \cdot \mathbf{r}^T] \cdot \boldsymbol{\omega}, \\ \boldsymbol{\gamma} &= -\frac{\nabla F(\mathbf{r})}{\|\nabla F(\mathbf{r})\|}, \end{aligned} \tag{9}$$

where \mathbf{M} is the angular momentum with respect to the contact point, $\boldsymbol{\gamma}$ is the unit vector normal to the surface of the body at this point (all the vectors are taken in the rotating frame), F is the function which defines the shape of the body such that $F(\mathbf{r}) = 0$ is the equation of its surface, $m\mathbf{g}$ is the value of the gravity force, \mathbf{J} is the inertia tensor, \mathbf{I} is the 3×3 identity matrix and (\cdot) means the matrix product. We choose the axes of the rotating coordinate frame to coincide with the principal axes of inertia, i.e. $\mathbf{J} = \text{diag}(J_1, J_2, J_3)$.

Equation (9) admits two conserved quantities, the energy integral $E = \frac{1}{2}(\mathbf{M}, \boldsymbol{\omega}) - m\mathbf{g}(\mathbf{r}, \boldsymbol{\gamma})$ and $(\boldsymbol{\gamma}, \boldsymbol{\gamma}) = 1$. By restricting system (9) to a constant energy level, we obtain a four-dimensional system of differential equations. By choosing an appropriate cross-section, we obtain a three-dimensional Poincaré map which depends on the value of energy E . Below we study two different examples of how attractors of this map evolve as E changes.

5.1. Discrete Lorenz attractors in a Celtic stone dynamics

A Celtic stone is a rigid body such that one of its inertia axes is vertical and the two others are rotated by an angle δ with respect to the horizontal geometrical axes. Namely, we consider a Celtic stone in the shape of elliptic paraboloid, i.e.

$$F(\mathbf{r}^*) = \frac{1}{2} \left(\frac{r_1^{*2}}{a_1} + \frac{r_2^{*2}}{a_2} \right) - (r_3^* + h) = 0,$$

where a_1 and a_2 are the principal radii of curvature at the paraboloid vertex $(0, 0, -h)$ and

$$\mathbf{r}^* = \begin{pmatrix} \cos \delta & \sin \delta & 0 \\ -\sin \delta & \cos \delta & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{r}.$$

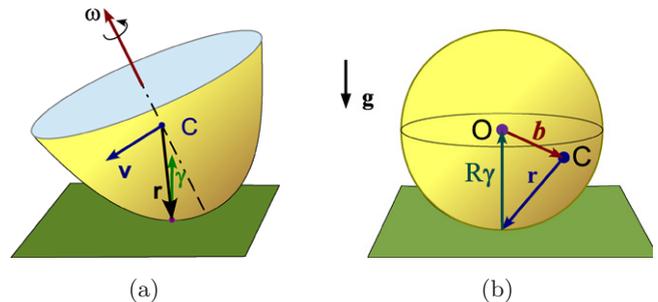


Fig. 12. (a) Celtic stone and (b) unbalanced ball.

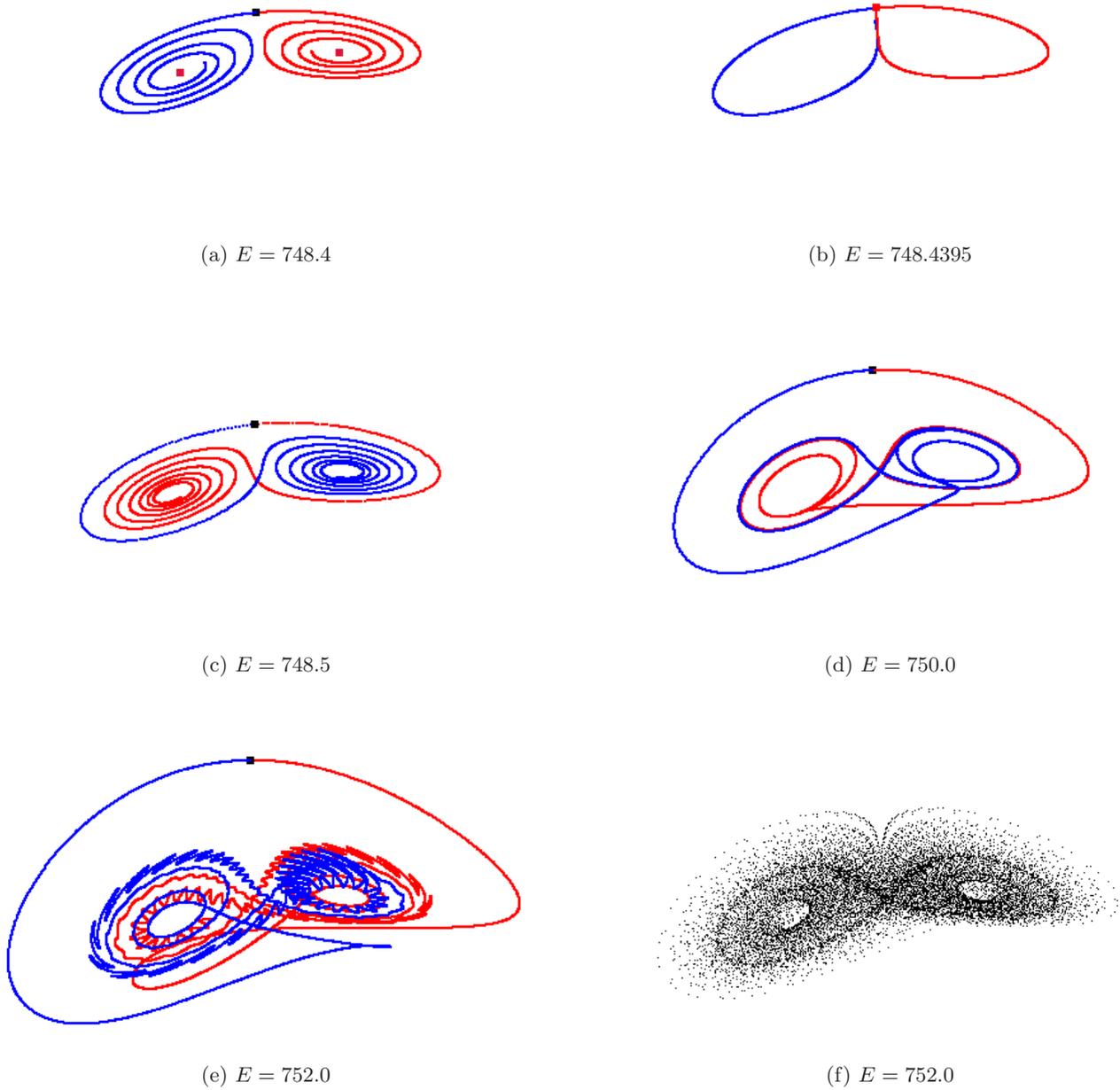
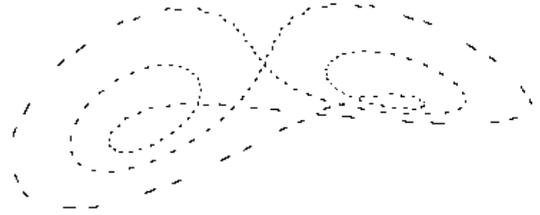


Fig. 13. The main stages of evolution of the Lorenz-like attractor in the Poincaré map for the Celtic stone: (f)–(i) show iterations of a single point and (a)–(e) show the unstable manifold of the saddle fixed point O .



(g) $E = 754.0$



(h) $E = 755.0$



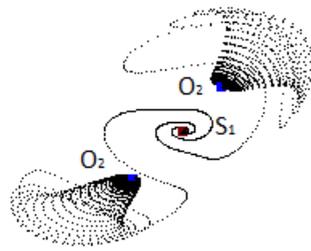
(i) $E = 765.0$

Fig. 13. (Continued)

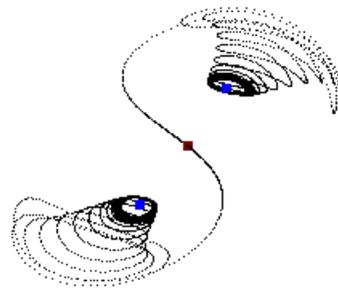
We take $J_1 = 2$, $J_2 = 6$, $J_3 = 7$, $m = 1$, $g = 100$, $a_1 = 9$, $a_2 = 4$, $h = 1$, $\delta = 0.485$. Figure 13 illustrates the evolution of the attractor of the Poincaré map as the energy E grows from $E = 748$ to $E = 765$.

Initially the attractor is a stable fixed point O . At $E \sim 747.61$ this point undergoes a period-doubling bifurcation and becomes a saddle; the stable orbit $P = (p_1, p_2)$ of period two becomes an attractor, Fig. 13(a). At $E = E_2 = 748.4395$, a homoclinic butterfly of the unstable manifold of the saddle O has been formed, Fig. 13(b); as E grows, this homoclinic structure gives rise to a saddle periodic curve $L = (L_1, L_2)$ of two components, L_1 that surrounds the point p_1 and L_2 that surrounds p_2 . At the same time, the unstable manifold of O tends to the stable periodic orbit P , Fig. 13(c). At $E \sim 748.97$ (not shown in the figure) the separatrices touch the stable manifold of the curve L and

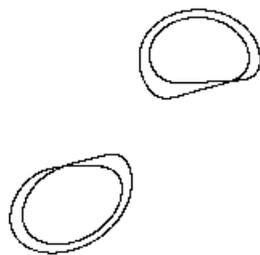
then leave it, after which the discrete Lorenz-like attractor is formed. Almost immediately after that, at $E \sim 748.98$, the period-2 orbit P loses stability at a subcritical torus bifurcation: the saddle periodic closed curve L merges with P , the cycle becomes a saddle and the curve disappears. The discrete Lorenz-like attractor is shown in Figs. 13(d)–13(g). Note that the scenario of the attractor formation is very similar to that in the Lorenz system subject to a small periodic perturbation [see Figs. 8(a)–8(e)]. Note also that close to the moment of the attractor creation [Fig. 13(d)] the behavior of the unstable manifold of the fixed point O is quite similar to the behavior of the separatrices of the saddle equilibrium state in the Lorenz model (here we have a difference with the Henon-like maps described in Sec. 4 where the transition to the Lorenz-like attractor was similar to that in the Shimizu–Morioka model).



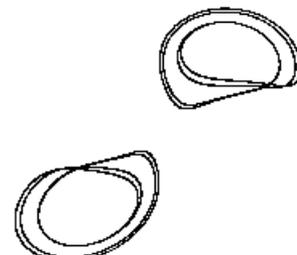
(a) $E = 455$



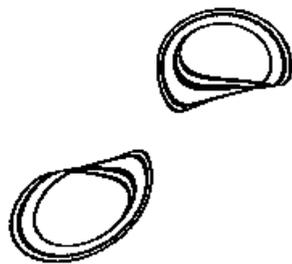
(b) $E = 455$



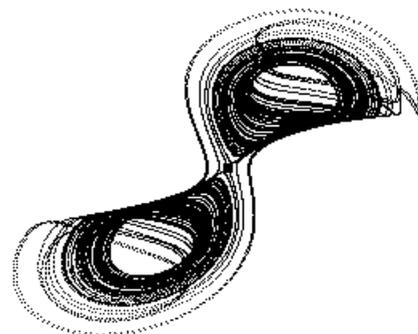
(c) $E = 457.904$



(d) $E = 457.910$



(e) $E = 457.911$



(f) $E = 457.913$

Fig. 14. The main stages of the evolution to the figure-eight attractor.

As E grows the unstable manifold starts forming visible wriggles [see Fig. 13(e)], so the dynamics of the discrete Lorenz-like attractor is no longer “flow-like”, even though it still looks quite similar to the classical Lorenz attractor [see Figs. 13(f) and 13(g)]. In order to check the pseudohyperbolicity of the attractor, we computed the multipliers of the saddle fixed point O at $E = 752.0$: $\lambda_1 = -1.312$, $\lambda_2 = 0.996$, $\lambda_3 = -0.664$; the spectrum of Lyapunov exponents for a randomly chosen trajectory is $\Lambda_1 = 0.0248$; $\Lambda_3 = -0.2445$, $0.00007 < \Lambda_2 < 0.00015$. Evidently, the necessary conditions for area expansion, $|\lambda_1 \lambda_2| > 1$ and $\Lambda_1 + \Lambda_2 > 0$, are fulfilled, so we, probably, have a true strange attractor here.

Figures 13(h) and 13(i) show the destruction of the discrete Lorenz attractor. As E grows, a stable invariant curve is formed in a lacuna [Fig. 13(h)]; later, the invariant curve gets destroyed and we see a characteristic shape of the torus-chaos quasiattractor [Fig. 13(i)]. The latter disappears at $E > 790$ and the orbits tend to a new stable regime, a spiral attractor, is observed in [Gonchenko et al., 2012b; Borisov et al., 2012].

5.2. Figure-eight attractor in the dynamics of the unbalanced ball

A model of an unbalanced ball (a ball with displaced center of gravity) rolling on the plane is given by Eqs. (9) with $F(\mathbf{r}) = (\mathbf{r} - \mathbf{b})^2 - R^2$, where \mathbf{b} is the vector of the displacement of the center of mass from the geometric center of the ball; R is the ball’s radius. We choose the following parameters: $J_1 = 2$, $J_2 = 6$, $J_3 = 7$, $m = 1$, $g = 100$, $R = 3$, $b_1 = 1$, $b_2 = 1.5$, $b_3 = 1.9$. A figure-eight, seemingly pseudohyperbolic attractor was numerically found in this model in [Borisov et al., 2014].¹ Figure 14 shows the development of the attractor of the Poincaré map in the model as the energy E varies from $E = 455.0$ to $E = 457.913$.

At first, for $E_1 \simeq 417.5 < E < E_2 \simeq 455.95$ [Fig. 14(a)] the attractor is a period-2 orbit (O_1, O_2) that emerges at $E = E_1$ along with a saddle orbit $S = (s_1, s_2)$ as a result of a saddle-node bifurcation. Simultaneously, the system has a saddle fixed point S_1 : this point, a saddle-focus then a saddle,

has a two-dimensional unstable manifold; then at $E = E_3 \simeq 456.15$, the fixed point becomes a saddle with one-dimensional unstable manifold as a result of a subcritical period-doubling bifurcation when the saddle orbit (s_1, s_2) merges to S_1 . At $E = E_2 \simeq 455.95$ the orbit (O_1, O_2) loses the stability at a supercritical Andronov–Hopf bifurcation and a stable period-2 closed curve appears. Thus, at $E > E_3$ the one-dimensional unstable separatrices of the saddle fixed point S_1 (with multipliers $\lambda_1 < -1$, $|\lambda_{2,3}| < 1$ and $\lambda_2 \lambda_3 < 0$) wind up onto a stable closed curve of period-2, Fig. 14(b). Next, several doublings of the invariant curve take place, see Figs. 14(c)–14(e). The further growth of E leads to a figure-eight attractor, Fig. 14(f).

Note that at $E = 457.913$, the fixed point S_1 has the multipliers $\lambda_1 \simeq -1.00907$, $\lambda_2 \simeq -0.99732$, $\lambda_3 \simeq 0.98885$. Thus, the area-expansion conditions $|\lambda_1 \lambda_2| > 1$ is fulfilled. Moreover, the Lyapunov exponents for a random trajectory in the attractor are as follows: $\Lambda_1 \simeq 0.00063$, $\Lambda_2 \simeq -0.00003$, $\Lambda_3 \simeq -0.00492$, which gives $\Lambda_1 + \Lambda_2 > 0$ and hints the pseudohyperbolicity.

Acknowledgments

The research presented in Secs. 4 and 5 was supported solely by a grant RSF (Project No. 14-12-00811). The rest of the research was supported by grants RFBR 13-01-00589, 13-01-97028-r-povoljje, and 14-01-344, and a grant of the Russian Ministry of Education and Science (agreement 2013 No. 02.B.49.21.003 of August 27). A. Gonchenko and A. Kazakov are supported by basic funding of Russian Ministry of Education and Science, D. Turaev supported by the Royal Society and the Leverhulme Trust.

References

- Afraimovich, V. S. & Shilnikov, L. P. [1974a] “On small periodic perturbations of autonomous systems,” *Soviet Math. Dokl.* **15**, 206–211.
 Afraimovich, V. S. & Shilnikov, L. P. [1974b] “On some global bifurcations connected with the disappearance of saddle-node fixed point,” *Soviet Math. Dokl.* **15**, 1761–1765.

¹A model of an unbalanced rubber ball (i.e. the unbalanced ball that moves without spinning) was considered in [Kazakov, 2013]. The additional nonholonomic constraint (no spinning) reduces the dimension of the problem, i.e. the Poincaré map becomes two-dimensional. Still the dynamics of the system remains very complex. In particular, coexisting strange attractors and repellers, as well as mixed dynamics [Gonchenko et al., 1997; Delshams et al., 2013] were found.

- Afraimovich, V. S. & Shilnikov, L. P. [1977] “The ring principle in problems of interaction between two self-oscillating systems,” *J. Appl. Math. Mech.* **41**, 632–641.
- Afraimovich, V. S., Bykov, V. V. & Shilnikov, L. P. [1977] “The origin and structure of the Lorenz attractor,” *Soviet Phys. Dokl.* **22**, 253–255.
- Afraimovich, V. S., Bykov, V. V. & Shilnikov, L. P. [1980] “On the existence of stable periodic orbits in the Lorenz model,” *Russ. Math. Surv.* **35**, 164–165.
- Afraimovich, V. S., Bykov, V. V. & Shilnikov, L. P. [1982] “On attracting structurally unstable limit sets of Lorenz attractor type,” *Trans. Mosc. Math. Soc.* **44**, 153–216.
- Afraimovich, V. S. & Shilnikov, L. P. [1983a] “Strange attractors and quasiattractors,” *Nonlinear Dynamics and Turbulence*, eds. Barenblatt, G. I., Iooss, G. & Joseph, D. (Pitman Advanced Publishing Program, Boston).
- Afraimovich, V. S. & Shilnikov, L. P. [1983b] “On invariant two-dimensional tori, their breakdown and stochasticity,” *Methods of the Qualitative Theory of Differential Equations, Gorky*, pp. 3–26 (English translation [1991] *Amer. Math. Soc. Transl.* **149**, 201–212).
- Afraimovich, V. S. & Vozovoi, L. P. [1988] “Mechanism of the generation of a two-dimensional torus upon loss of stability of an equilibrium state,” *Soviet Phys. Dokl.* **33**, 720–723.
- Afraimovich, V. S. & Vozovoi, L. P. [1989] “The mechanism of the hard appearance of a two-frequency oscillation mode in the case of Andronov–Hopf reverse bifurcation,” *J. Appl. Math. Mech.* **53**, 24–28.
- Anishenko, V. S. & Nikolaev, S. M. [2005] “Bifurcation doubling of the two-dimension torus,” *Letters to JTPH* **31**, 88–94.
- Arneodo, A., Couillet, P. & Tresser, C. [1981] “Possible new strange attractors with spiral structure,” *Commun. Math. Phys.* **79**, 573–579.
- Arneodo, A., Couillet, P. H. & Spiegel, E. A. [1985] “The dynamics of triple convection,” *Geophys. Astrophys. Fluid Dyn.* **31**, 1–48.
- Arnold, V. I. [1977] “Loss of stability of self-oscillations close to resonance and versal deformations of equivariant vector fields,” *Funct. Anal. Appl.* **11**, 85–92.
- Aronson, D. G., Chory, M. A., McGehee, R. P. & Hall, G. R. [1982] “Bifurcation from an invariant circle for two-parameter families of maps of the plane,” *Commun. Math. Phys.* **83**, 303–354.
- Bamon, R., Kiwi, J. & Rivera, J. [2005] “Wild Lorenz like attractors,” Preprint, arXiv:math/0508045.
- Barrio, R., Shilnikov, A. & Shilnikov, L. [2012] “Kneadings, symbolic dynamics, and painting Lorenz chaos,” *Int. J. Bifurcation and Chaos* **22**, 1230016.
- Belyakov, L. A. [1980] “A case of the generation of a periodic orbit motion with homoclinic curves,” *Math. Notes* **15**, 336–341.
- Belykh, V., Belykh, I. & Mosekilde, E. [2005] “The hyperbolic Plykin attractor can exist in neuron models,” *Int. J. Bifurcation and Chaos* **15**, 3567–3578.
- Benedicks, M. & Carleson, L. [1991] “The dynamics of the Henon map,” *Ann. Math.* **133**, 73–169.
- Bonatti, C., Diaz, L. & Viana, M. [2005] *Dynamics Beyond Uniform Hyperbolicity: A Global Geometric and Probabilistic Perspective*, Encyclopaedia of Mathematical Sciences, Vol. 102 (Springer).
- Borisov, A. V. & Mamaev, I. S. [2003] “Strange attractors in rattleback dynamics,” *Physics-Uspexhi* **46**, 393–403.
- Borisov, A. V., Jalnine, A. Yu., Kuznetsov, S. P., Sataev, I. R. & Sedova, J. V. [2012] “Dynamical phenomena occurring due to phase volume compression in non-holonomic model of the rattleback,” *Reg. Chaot. Dyn.* **17**, 512–532.
- Borisov, A. V., Kazakov, A. O. & Sataev, I. R. [2014] “Regular and chaotic phenomena in the nonholonomic model of the unbalanced ball rolling on a plane,” *Reg. Chaot. Dyn.* **18**, 512–532.
- Bosh, M. & Simo, C. [1993] “Attractors in a Shilnikov–Hopf scenario and a related one-dimensional map,” *Physica D* **62**, 217–229.
- Braaksma, B. L. J., Broer, H. W. & Huitema, G. B. [1990] “Toward a quasi-periodic bifurcation theory,” *Memoirs AMS* **83**, 83–175.
- Broer, H. W., Huitema, G. B. & Takens, F. [1990] “Unfoldings of quasi-periodic tori,” *Memoirs AMS* **83**, 1–82.
- Bykov, V. V. [1978] “On the structure of a neighborhood of a separatrix contour with a saddle-focus,” *Methods of the Qualitative Theory of Differential Equations, Gorky*, pp. 3–32.
- Bykov, V. V. [1980] “On bifurcations of dynamical systems close to systems with a separatrix contour containing a saddle-focus,” *Methods of the Qualitative Theory of Differential Equations, Gorky*, pp. 44–72.
- Bykov, V. V. & Shilnikov, A. L. [1989] “On the boundaries of the domain of existence of the Lorenz attractor,” *Methods of the Qualitative Theory of Differential Equations, Gorky*, pp. 151–159 (English translation [1992] in *Selecta Math. Soviet.* **11**, 375–382).
- Bykov, V. V. [1993] “The bifurcations of separatrix contours and chaos,” *Physica D* **62**, 290–299.
- Chenciner, A. [1985] “Bifurcations de points fixes elliptiques. I. Courbes invariantes,” *IHES Publ. Math.* **61**, 67–127; [1985] “II. Orbites periodiques et ensembles de Cantor invariants,” *Invent. Math.* **80**, 81–106; [1987] “III. Orbites periodiques de “petites” periodes

- et elimination resonante des couples de courbes invariantes,” *IHES Publ. Math.* **66**, 5–91.
- Curry, J. H. & Yorke, J. A. [1978] “A transition from Hopf bifurcation to chaos: Computer experiments with maps in R^2 ,” *The Structure of Attractors in Dynamical Systems*, eds. Martin, J. C., Markley, N. G. & Perrizo, W., Lecture Notes Math., Vol. 668 (Springer, Berlin), pp. 48–66.
- Delshams, A., Gonchenko, S. V., Gonchenko, V. S., Lazaro, J. T. & Stenkin, O. [2013] “Abundance of attracting, repelling and elliptic periodic orbits in two-dimensional reversible maps,” *Nonlinearity* **26**, 1–33.
- Gavrillov, N. K. [1977] “On bifurcations of periodic orbits near inner resonance 1:3,” *Investigations on Stability and the Theory of Oscillations. Yaroslavl*, pp. 192–199.
- Gonchenko, S. V., Turaev, D. V. & Shilnikov, L. P. [1993] “Dynamical phenomena in multi-dimensional systems with a non-rough Poincaré homoclinic curve,” *Russ. Acad. Sci. Dokl. Math.* **47**, 410–415.
- Gonchenko, S. V., Shilnikov, L. P. & Turaev, D. V. [1996] “Dynamical phenomena in systems with structurally unstable Poincaré homoclinic orbits,” *Chaos* **6**, 15–31.
- Gonchenko, S. V., Shilnikov, L. P. & Turaev, D. V. [1997] “On Newhouse regions of two-dimensional diffeomorphisms close to a diffeomorphism with a nontransversal heteroclinic cycle,” *Proc. Steklov Inst. Math.* **216**, 70–118.
- Gonchenko, S. V., Ovsyannikov, I. I., Simo, C. & Turaev, D. [2005] “Three-dimensional Henon-like maps and wild Lorenz-like attractors,” *Int. J. Bifurcation and Chaos* **15**, 3493–3508.
- Gonchenko, S. V., Shilnikov, L. P. & Turaev, D. V. [2008] “On dynamical properties of multidimensional diffeomorphisms from Newhouse regions. I,” *Nonlinearity* **21**, 923–972.
- Gonchenko, A. S., Gonchenko, S. V. & Kazakov, A. O. [2012a] “On new aspects of chaotic dynamics of Celtic stone,” *Rus. Nonlin. Dyn.* **8**, 507–518.
- Gonchenko, A. S., Gonchenko, S. V. & Shilnikov, L. P. [2012b] “Towards scenarios of chaos appearance in three-dimensional maps,” *Rus. Nonlin. Dyn.* **8**, 3–28.
- Gonchenko, S. V., Ovsyannikov, I. I. & Turaev, D. [2012c] “On the effect of invisibility of stable periodic orbits at homoclinic bifurcations,” *Physica D* **241**, 1115–1122.
- Gonchenko, A. S. & Gonchenko, S. V. [2013] “On existence of Lorenz-like attractors in a nonholonomic model of a Celtic stone,” *Rus. Nonlin. Dyn.* **9**, 77–89.
- Gonchenko, S. V., Gonchenko, A. S. & Kazakov, A. O. [2013a] “Richness of chaotic dynamics in nonholonomic models of a Celtic stone,” *Reg. Chaot. Dyn.* **15**, 521–538.
- Gonchenko, S. V., Gonchenko, A. S., Ovsyannikov, I. I. & Turaev, D. V. [2013b] “Examples of Lorenz-like attractors in Henon-like Maps,” *Math. Model. Nat. Phenom.* **8**, 32–54.
- Guckenheimer, J. [1976] “A strange, strange attractor,” *The Hopf Bifurcation Theorem and Its Applications*, eds. Marsden, J. & McCracken, M. (Springer-Verlag), pp. 368–381.
- Guckenheimer, J. & Williams, R. F. [1979] “Structural stability of Lorenz attractors,” *IHES Publ. Math.* **50**, 59–72.
- Kazakov, A. O. [2013] “Strange attractors and mixed dynamics in the problem of an unbalanced rubber ball rolling on a plane,” *Reg. Chaot. Dyn.* **15**, 508–520.
- Khibnik, A. I., Roose, D. & Chua, L. O. [1993] “On periodic orbits and homoclinic bifurcations in Chua’s circuit with smooth nonlinearity,” *Int. J. Bifurcation and Chaos* **3**, 363–384.
- Kuznetsov, Yu. A. [1998] *Elements of Applied Bifurcation Theory* (Springer-Verlag, NY–Berlin, Heidelberg).
- Los, J. E. [1989] “Non-normally hyperbolic invariant curves for maps in R^3 and doubling bifurcation,” *Nonlinearity* **2**, 149–174.
- Mora, L. & Viana, M. [1993] “Abundance of strange attractors,” *Acta Math.* **171**, 1–71.
- Newhouse, S. E. [1974] “Diffeomorphisms with infinitely many sinks,” *Topology* **13**, 9–18.
- Ovsyannikov, I. M. & Shilnikov, L. P. [1987] “On systems with a saddle-focus homoclinic curve,” *Math. USSR Sbornik* **58**, 91–102.
- Petrovskaya, N. V. & Yudovich, V. I. [1980] “Homoclinic loops of the Saltzman–Lorenz system,” *Methods of the Qualitative Theory of Differential Equations, Gorky*, pp. 73–83.
- Shilnikov, L. P. [1965] “A case of the existence of a denumerate set of periodic motions,” *Soviet Math. Dokl.* **6**, 163–166.
- Shilnikov, L. P. [1967] “On a Poincaré–Birkhoff problem,” *Math. USSR Sbornik* **3**, 91–102.
- Shilnikov, L. P. [1970] “A contribution to the problem of the structure of an extended neighbourhood of a rough equilibrium state of saddle-focus type,” *Math. USSR Sbornik* **10**, 91–102.
- Shilnikov, L. P. [1980] “Bifurcation theory and the Lorenz model,” Addition 1 to the book *Bifurcation of Cycle Birth and Its Applications*, eds. Marsden, J. & Mac-Cracken, M. (Moscow, Mir), pp. 317–335.
- Shilnikov, A. L. [1986] “Bifurcation and chaos in the Morioka–Shimizu system,” *Methods of the Qualitative Theory of Differential Equations, Gorky*, pp. 180–193 (English translation [1991] *Selecta Math. Soviet.* **10**, 105–117); [1989] II. *Methods of Qualitative Theory and Theory of Bifurcations, Gorky*, pp. 130–138.

- Shilnikov, L. P. [1986] “Bifurcation theory and turbulence,” *Methods of the Qualitative Theory of Differential Equations, Gorky*, pp. 150–163 (English translation [1991] *Selecta Math. Soviet.* **10**, 43–53).
- Shilnikov, A. L. & Shilnikov, L. P. [1991] “On the non-symmetric Lorenz model,” *Int. J. Bifurcation and Chaos* **1**, 773–776.
- Shilnikov, A. L. [1993] “On bifurcations of the Lorenz attractor in the Shimizu–Morioka model,” *Physica D* **62**, 338–346.
- Shilnikov, A. L., Shilnikov, L. P. & Turaev, D. V. [1993] “Normal forms and Lorenz attractors,” *Int. J. Bifurcation and Chaos* **3**, 1123–1139.
- Shilnikov, L. P., Shilnikov, A. L., Turaev, D. V. & Chua, L. O. [1998] *Methods of Qualitative Theory in Nonlinear Dynamics, Part I* [2001] *Part II* (World Scientific, Singapore).
- Turaev, D. V. [1996] “On dimension of non-local bifurcational problems,” *Int. J. Bifurcation and Chaos* **6**, 919–948.
- Turaev, D. V. & Shilnikov, L. P. [1998] “An example of a wild strange attractor,” *Sb. Math.* **189**, 291–314.
- Turaev, D. V. & Shilnikov, L. P. [2008] “Pseudo-hyperbolicity and the problem on periodic perturbations of Lorenz-like attractors,” *Russ. Dokl. Math.* **467**, 23–27.
- Ures, R. [1995] “On the approximation of Hénon-like attractors by homoclinic tangencies,” *Ergod. Th. Dyn. Syst.* **15**, 1223–1229.
- Vitolo, R. [2003] “Bifurcations of attractors in 3D diffeomorphisms: A study in experimental mathematics,” Doctoral thesis, University of Groningen Press.
- Williams, R. F. [1977] “The structure of Lorenz attractors,” *Lect. Notes Math.* **615**, 94–112.