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A Proof of Shilnikov's Theorem for C^1-Smooth Dynamical Systems

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Abstract. Dynamical systems with a homoclinic loop to a saddle equilibrium state are considered. Andronov and Leontovich showed (see [**1, 3**]) that a generic bifurcation of a two-dimensional C^1-smooth dynamical system with a homoclinic loop leads to the appearance of a unique periodic orbit. Shilnikov [**14, 15, 18**] proved that in the case of dynamical systems of sufficiently high smoothness, this result holds true in the multidimensional setting if some additional conditions are satisfied. In the present paper we give a proof of the Shilnikov theorem for dynamical systems in C^1.

1. Main theorem

Let us consider a family of C^1-smooth vector fields X_μ on an $(n+1)$-dimensional manifold. We assume that the vector field X_μ and its first derivatives depend on μ continuously. Let the following hold.

(A) *The system* X_μ *has a saddle equilibrium state* O, *and the roots* $\lambda_n, \dots, \lambda_1, \gamma$ *of the characteristic equation of the linearized system at the point* O *for* $\mu = 0$ *satisfy the inequalities* $\operatorname{Re}\lambda_n \leqslant \cdots \leqslant \operatorname{Re}\lambda_1 < 0 < \gamma$.

Thus, we can introduce local coordinates (x, y) ($x \in \mathbb{R}^n$, $y \in \mathbb{R}^1$) in a small neighborhood of O so that the system X_μ takes the following form near O for $\mu = 0$

$$\begin{cases} \dot{x} = \Lambda x + \dots, \\ \dot{y} = \gamma y + \dots . \end{cases} \tag{1.1}$$

Here Λ is an $(n \times n)$-matrix with the eigenvalues $\{\lambda_1, \dots, \lambda_n\}$; the dots stand for nonlinearities.

The unstable manifold W^u of O is one-dimensional (it is tangent to the y-axis at O) and consists of three orbits: the point O itself and two *separatrices* leaving O in opposite directions. The stable manifold W^s is n-dimensional; it divides a small neighborhood of the equilibrium into two parts: U^+ and U^- (see Figure 1). Assume that

(B) *for* $\mu = 0$ *one of the separatrices* Γ *is homoclinic to* O, *i.e.*, $\Gamma \subset (W^s \cap W^u)$.

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Without loss of generality we assume that the separatrix Γ leaves the point O towards the region U^+ (i.e., towards positive y, see Figure 1).

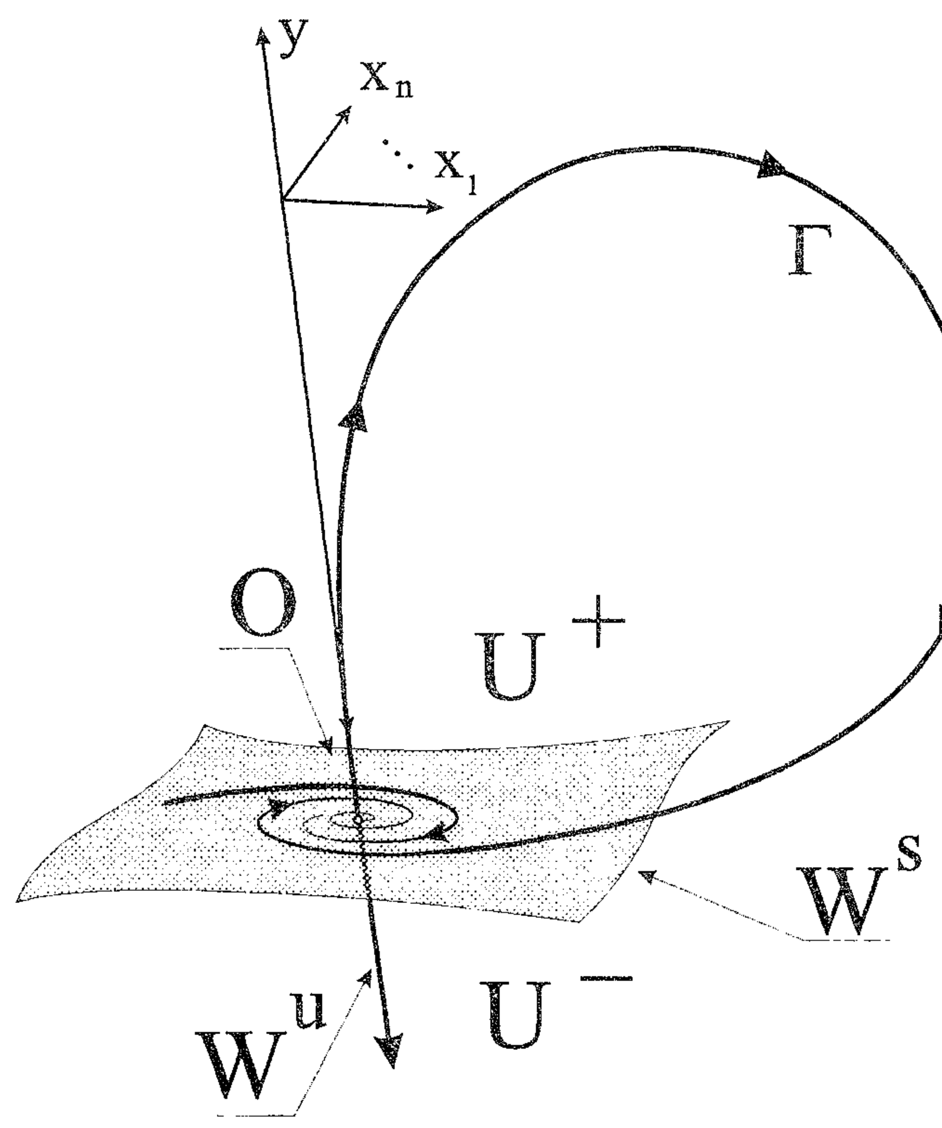

FIGURE 1. The system X_0 has a homoclinic orbit Γ to the saddle equilibrium O. The stable manifold W^s divides a small neighborhood of O into two regions: U^+ and U^-.

We consider the behavior of orbits in a small neighborhood U of the homoclinic loop $\mathcal{L} = O \cup \Gamma$.

For systems on the plane ($n = 1$) this problem was completely solved by Andronov and Leontovich [**1, 2, 3**] (see also [4]). In particular, it was shown that if the saddle value $\sigma = \lambda_1 + \gamma$ is nonzero, then bifurcations of the homoclinic loop produce only one periodic orbit. Thus, the bifurcation of such a homoclinic loop was proved to be one of the four main bifurcations of the birth of a limit cycle on a plane.

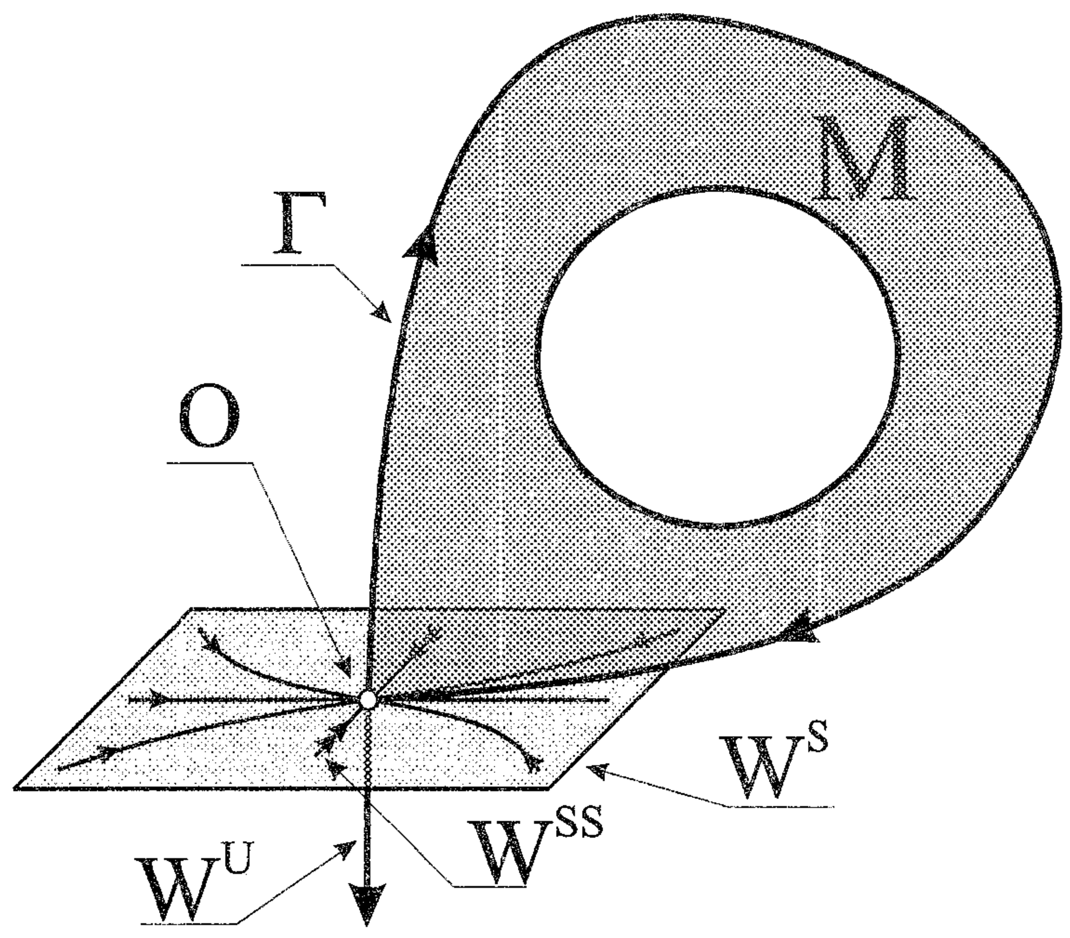

FIGURE 2. A two-dimensional invariant manifold M exists near the homoclinic loop $\mathcal{L} = O \cup \Gamma$ if and only if the leading eigenvalue λ_1 is real and simple, the loop does not lie in the strong stable manifold W^{ss} and some additional transversality conditions are fulfilled.

A similar multidimensional problem was considered by Shilnikov [**15**]. From the modern point of view, one should immediately obtain a result similar to the two-dimensional one in the case where a smooth, normally-hyperbolic two-dimensional invariant manifold exists near the homoclinic loop (see Figure 2). However, the existence of such a manifold requires some extra conditions. First, the negative eigenvalue λ_1 nearest to the imaginary axis must be real and simple. The orbit Γ should not lie in the strong stable submanifold W^{ss} that corresponds to the eigenvalues $\lambda_n, \ldots, \lambda_2$. Moreover, some transversality conditions must be satisfied by the flow map near Γ (see [**21, 23, 11, 12, 7**], [**10**] (this also includes the PDE case), [**13**]).

In fact, the existence of a two-dimensional invariant manifold is not so relevant to the dynamics near a homoclinic loop. It was a remarkable discovery of Shilnikov [**16, 19**] that if the characteristic exponents at the point O satisfy a condition which reads in our case as $\operatorname{Im}\lambda_1 \neq 0$, $-\operatorname{Re}\lambda_1 < \gamma$, then generically there exist nontrivial hyperbolic sets in a small neighborhood of the loop. In other words, the dynamics near a homoclinic loop to a saddle-focus with positive saddle value is quite opposite to that in dimension two. As of today, the Shilnikov homoclinic loop is a model of chaotic behavior, which is very simple to describe and which has a very complicated dynamics.

On the other hand, in the case of a negative saddle value, i.e., if

(C) $\sigma = \operatorname{Re}\lambda_1 + \gamma < 0$,

the bifurcation of the homoclinic loop leads to the appearance of only one stable periodic orbit, exactly as for the systems on the plane, no matter what the equilibrium state O is—a saddle or a saddle-focus [**15**].

In the present paper we give a proof of the corresponding result for C^1-smooth systems. In order to describe bifurcations of X_μ, we introduce the small parameter μ as described below. Namely, we suppose that

(D) *the separatrix* Γ *does not belong to* W^s *if* $\mu \neq 0$.

It follows from continuity with respect to μ that after leaving a small neighborhood of O, the separatrix Γ for $\mu \neq 0$ stays close to the locus of the homoclinic loop $\mathcal{L}$ until it enters the small neighborhood of O once again. Without loss of generality we assume that Γ enters U^+ at $\mu > 0$ and U^- at $\mu < 0$.

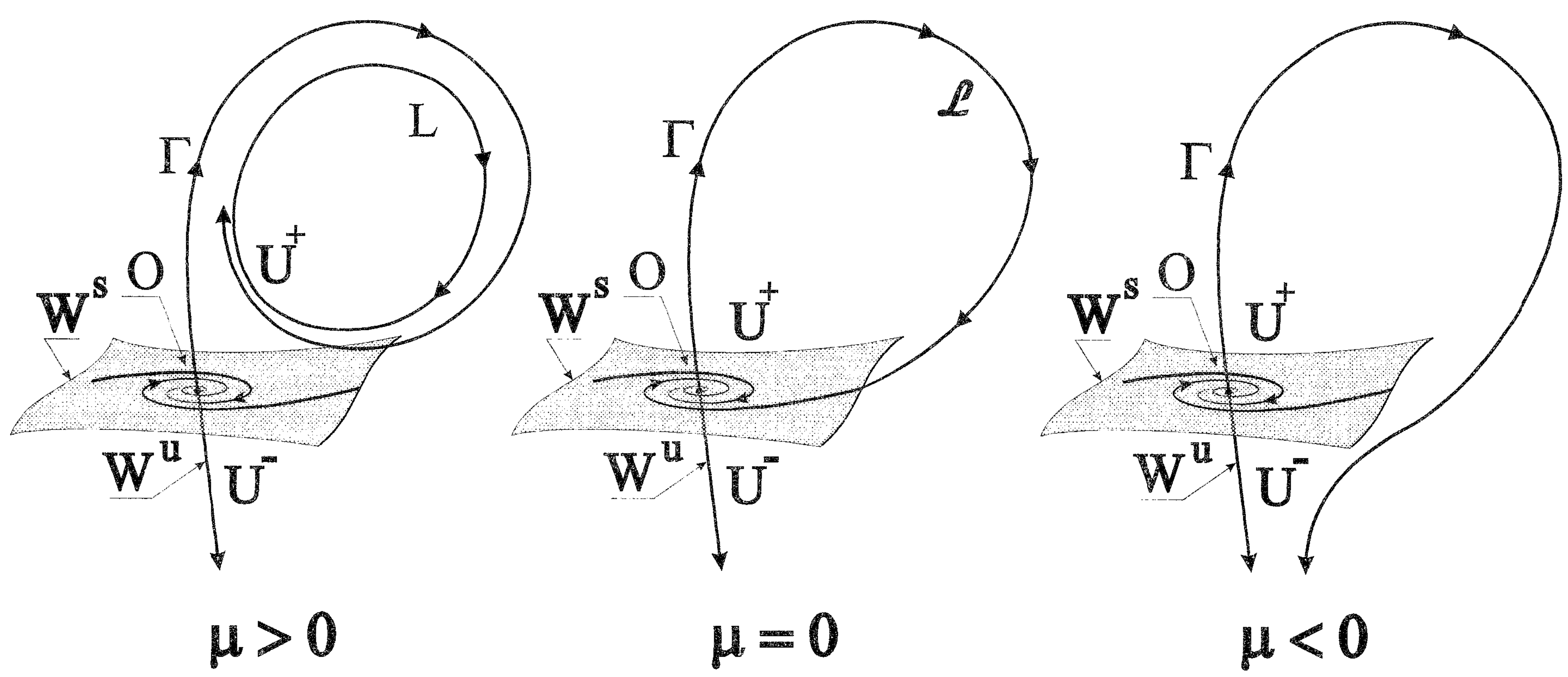

FIGURE 3. At $\mu > 0$, a stable periodic orbit L is born from the loop $\mathcal{L}$ ($\mu = 0$). All the orbits (except for those tending to O) leave at $\mu < 0$.

THEOREM 1.1 (see Figure 3). *If conditions* (A)–(D) *are fulfilled, then there exists a small neighborhood* U *of the homoclinic loop such that at all small* $\mu > 0$ *the* *system has a unique periodic orbit L, which is stable and, in particular, the separatrix Γ tends to L as $t \to +\infty$. The other orbits in U that do not lie in W^s either tend to L or leave U in finite time. For $\mu = 0$ the periodic orbit becomes a homoclinic loop (which may attract some orbits of $U \setminus W^s$, the other orbits leaving U). For all small $\mu < 0$ all orbits of $U \setminus W^s$ leave U in finite time.*

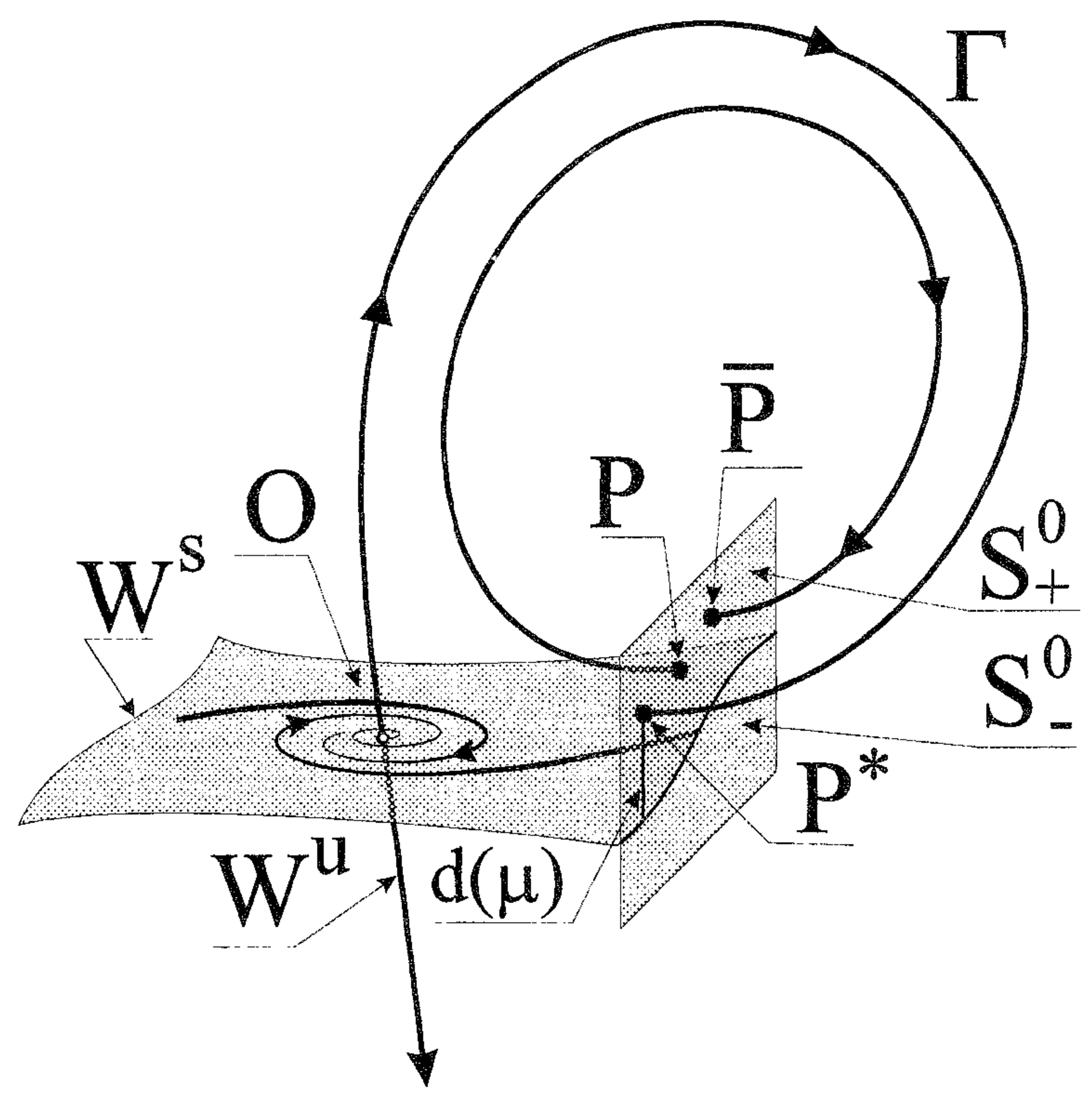

FIGURE 4. The Poincaré map $T\colon S^0_+ \to S^0$ is defined near $S^0 \cap W^s$. The image $T(W^s \cap S^0) = P^*_\mu$ is defined by continuity. The point $P^*_\mu = \Gamma \cap S^0$ lies at the distance $|d(\mu)|$ from $S^0 \cap W^s$.

PROOF. We follow the lines of the original proof in [**15**]. Take a small cross-section S^0 of the stable manifold W^s so as to intersect the homoclinic loop at $\mu = 0$. The stable manifold of O divides S^0 into two regions: $S^0_+ = S^0 \cap U_+$ and $S^0_- = S^0 \cap U_-$ (i.e., S^0_+ lies above W^s; see Figure 4). Let P^*_μ be the intersection point $\Gamma \cap S^0$. For $\mu = 0$ the separatrix Γ forms a homoclinic loop, so $P^*_0 \in \{S^0 \cap W^s\}$. Thus, the intersection point exists for all small μ. Let $d(\mu)$ be the distance from P^*_μ to $W^s \cap S^0$, taken with the positive sign when $P^*_\mu \in S^0_+$ and the negative one when $P^*_\mu \in S^0_-$. By assumption (D), the sign of $d(\mu)$ coincides with the sign of μ (Figure 4).

An orbit which starts at a point $P \in S^0_+$ goes near the stable manifold in a small neighborhood of O and then leaves the neighborhood staying close to the separatrix Γ. If μ is sufficiently small, then moving along Γ, such an orbit intersects S^0 again at some point $\bar{P}$ near the point P^*_μ. Thus, the Poincaré map $T\colon P \mapsto \bar{P}$ is defined on S^0_+ in a neighborhood of W^s. On $W^s \cap S^0$ the map T is defined by continuity: $T(W^s \cap S^0) = P^*_\mu$. The orbits which start on S^0_- leave a small neighborhood of O close to the other separatrix and, therefore, they leave the neighborhood U of the homoclinic loop under consideration. Thus, the Poincaré map T is not defined on S^0_-.

Shilnikov proved in [**15**] that if the saddle value σ (see (C)) is negative, then the map T is *strongly contracting* for small μ (i.e., $\operatorname{dist}(TP_1, TP_2) \leqslant K \operatorname{dist}(P_1, P_2)$, where the contraction factor K tends uniformly to zero as both P_1, P_2 tend to $W^s \cap S^0$). Then, he artificially defines the map T on S^0_-. We shall do the same, assuming, say, that $TP \equiv P^*_\mu$ at $P \in S^0_-$. This extended map is also contracting (with the same factor K). In particular, at $\mu = 0$, this map takes a small neighborhood of the point P^*_0 into itself. The same, obviously, holds for all small μ. Thus, the Banach principle gives the existence of a unique fixed point; moreover, this point attracts iterations (by the map T extended onto all S^0) of every initial point on S^0.

For $\mu \leqslant 0$ the fixed point is, by definition, the point P^*_μ. Since it lies in the region $S^0_- \cup (W^s \cap S^0)$ where the Poincaré map is not defined, no periodic orbit corresponds to this point; it is a homoclinic loop at $\mu = 0$ or just a fake at $\mu < 0$.

For $\mu > 0$, the fixed point is the limit of the iterations of the point P^*_μ. This point is the image of the line $W^s \cap S^0$ and it lies at the distance $d(\mu)$ from this line. Therefore, due to the contraction, all the iterations of this point (and their limit, the fixed point) lie in the ball of radius $K(1-K)^{-1}d(\mu)$ with center at P^*_μ. If μ is sufficiently small, one can assume that $K < 1/2$ and in this case the radius is less than $d(\mu)$. Thus, for $\mu > 0$ the fixed point of the extended map belongs to the region S^0_+. Hence, it is a fixed point of the true Poincaré map and there exists the corresponding periodic orbit of the system.

All this is in a complete correspondence with the statement of the theorem. The key point in the proof is to show that the Poincaré map is strongly contracting. For this, computations explicitly involving second derivatives of the right-hand sides of the system were used in [**15**]. Below (Sections 2 and 3) we prove the contraction in the case of minimal smoothness (C^1), by using Shilnikov's boundary value problem method discussed in [**17**]. □

At first glance, the passage from, say, C^2 to C^1 is an insignificant step. However, dynamical systems of low smoothness appear naturally when studying high-dimensional systems reduced to a normally hyperbolic invariant manifold (say, to the inertial manifold, or to a nonlocal center manifold as in the example below). The smoothness of such a manifold, and therefore the smoothness of the reduced system, does not correlate with the smoothness of the original system. In particular, the conditions for the existence of a C^2-smooth invariant manifold are much more restrictive than for a C^1 one. Thus, the study of the bifurcational problems in the case of the smallest possible smoothness may be crucial for a rigorous description of the high-dimensional dynamics.

As an example, consider a C^1-version of the result of Shilnikov [**18**]: a generalization of Theorem 1.1 to the case where the dimension of the unstable manifold of O is larger than one. Namely, let X_μ be a continuous family of C^1-smooth dynamical systems on an $(n+m)$-dimensional manifold. Let us modify conditions (A), (B) in the following way.

(A′) *The system X_μ has a hyperbolic equilibrium state O, and the characteristic exponents $\lambda_n, \dots, \lambda_1, \gamma, \gamma_2, \dots, \gamma_m$ at the point O for $\mu = 0$ satisfy the following condition*: $\operatorname{Re}\lambda_n \leqslant \cdots \leqslant \operatorname{Re}\lambda_1 < 0 < \gamma < \operatorname{Re}\gamma_2 \leqslant \cdots \leqslant \operatorname{Re}\gamma_m$.

(B′) *For $\mu = 0$ there exists a homoclinic orbit Γ, i.e., $\Gamma \subset (W^s \cap W^u)$.*

The conditions (C), (D) remain unchanged.

In this case the dimension of the unstable manifold W^u is equal to m and, moreover, there exists an $(m-1)$-dimensional strong unstable invariant submanifold $W^{uu} \subset W^u$. The characterizing feature of W^{uu} is that all orbits in it are tangent to the linear subspace that corresponds to the eigenvalues $\gamma_2, \dots, \gamma_m$, while all orbits of $W^u \setminus W^{uu}$ are tangent to the eigendirection corresponding to the *leading* eigenvalue γ. Let us assume the following condition.

(E) *The homoclinic orbit Γ does not belong to W^{uu}* (see Figure 5).

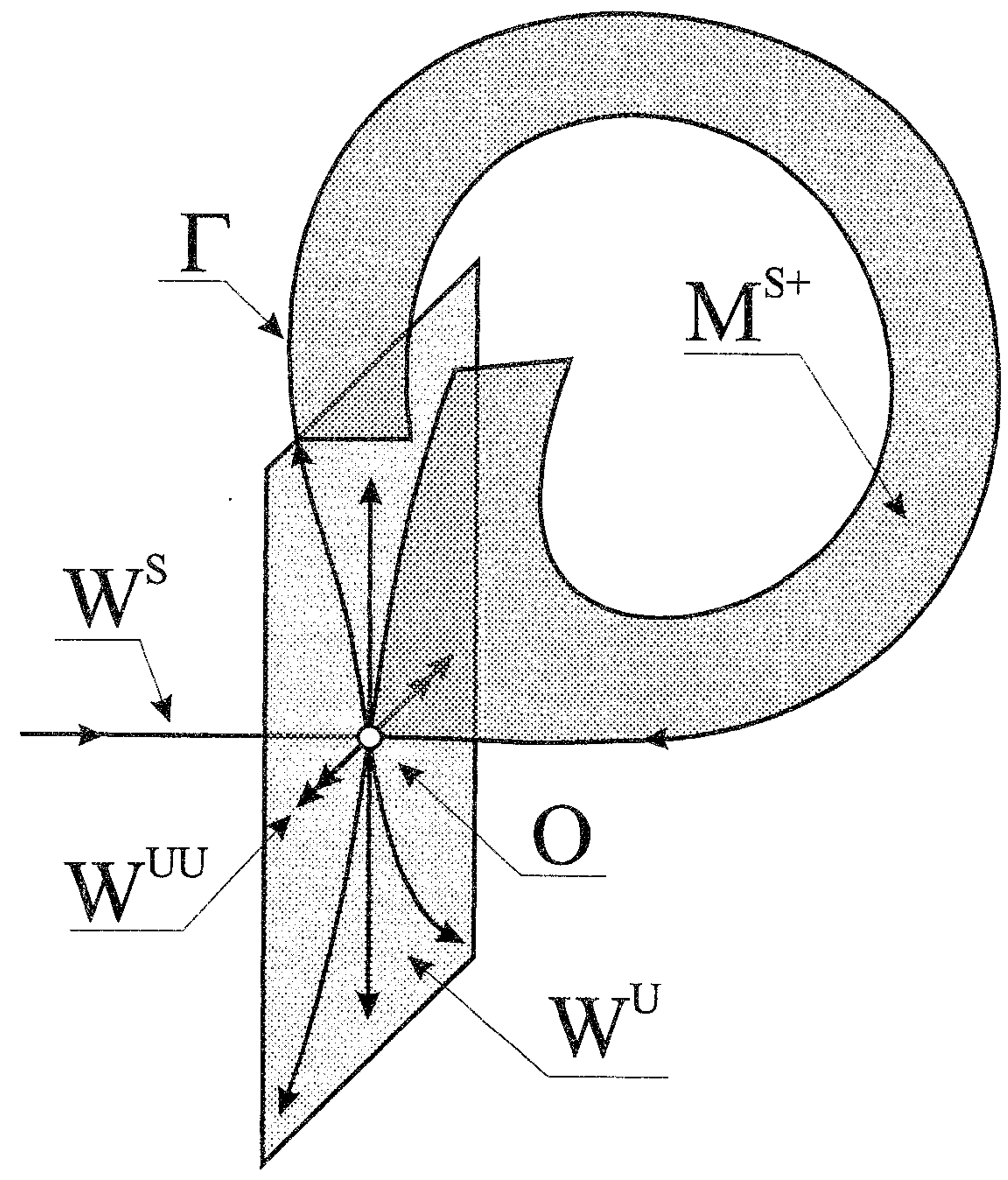

FIGURE 5. The orbit Γ does not lie in the strong unstable submanifold W^{uu}. The extended stable manifold M^{s+} is transverse to the unstable manifold W^u.

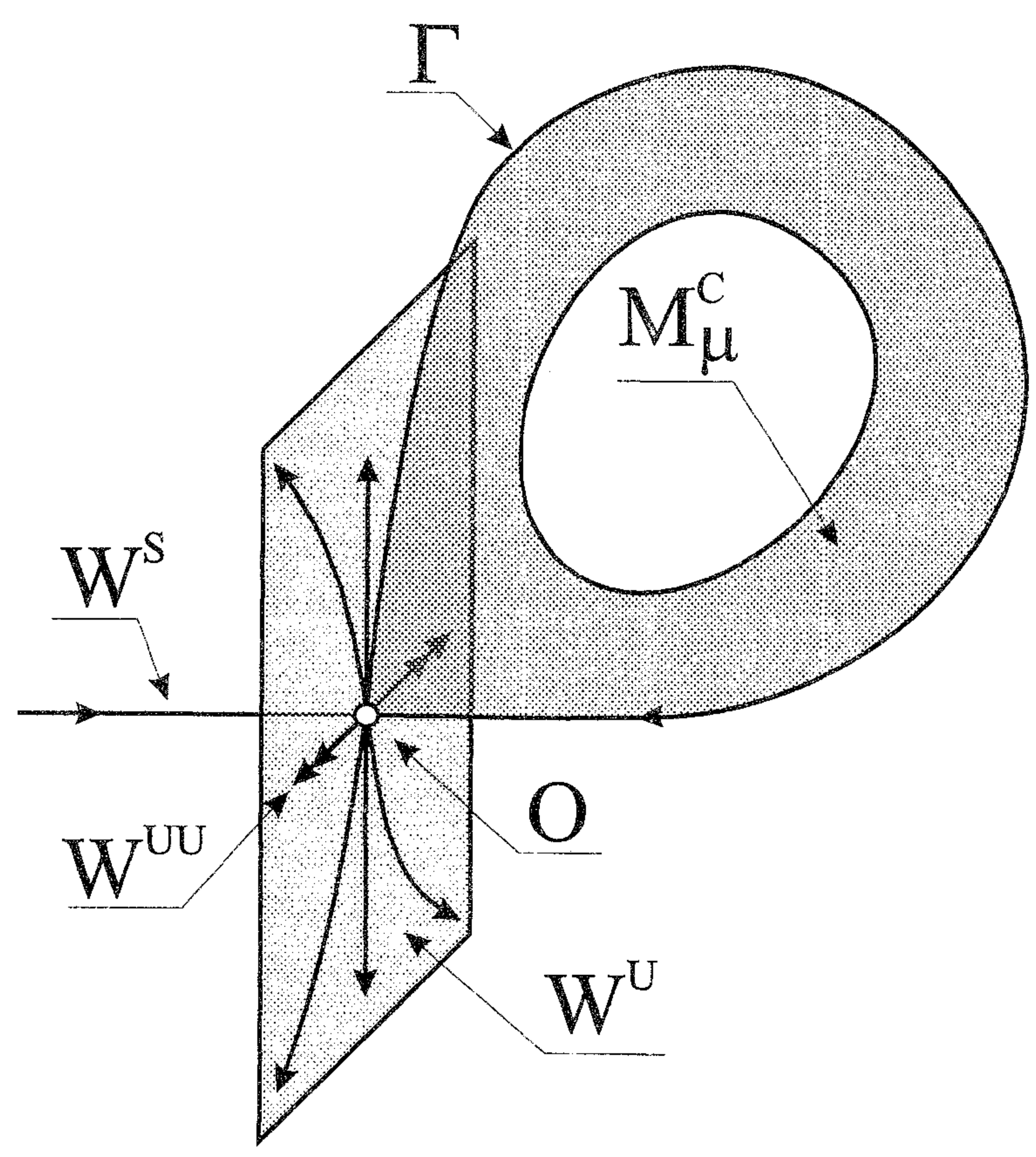

FIGURE 6. There exists an $(n+1)$-dimensional C^1-smooth center invariant manifold M^c_μ if conditions (A′), (B′), (E), and (F) are fulfilled.

The next assumption is necessary [**23**] to ensure the presence of an $(n+1)$-dimensional global invariant manifold (as well as condition (E)). Denote by $E^{s+} \subset R^{n+1}$ the invariant subspace of the system X_0 linearized at the point O, corresponding to the eigenvalues $\lambda_n, \dots, \lambda_1, \gamma$. It is well known (see for instance [**5**]) that there exists an invariant C^1-smooth manifold M^{s+} tangent to E^{s+} at O (see Figure 5). The manifold M^{s+} contains W^s. It is not uniquely defined, but any two such submanifolds have the same tangent at each point of W^s. We require the following condition to be fulfilled.

(F) *The manifold M^{s+} is transverse to the manifold W^u at each point of* Γ (see Figure 5).