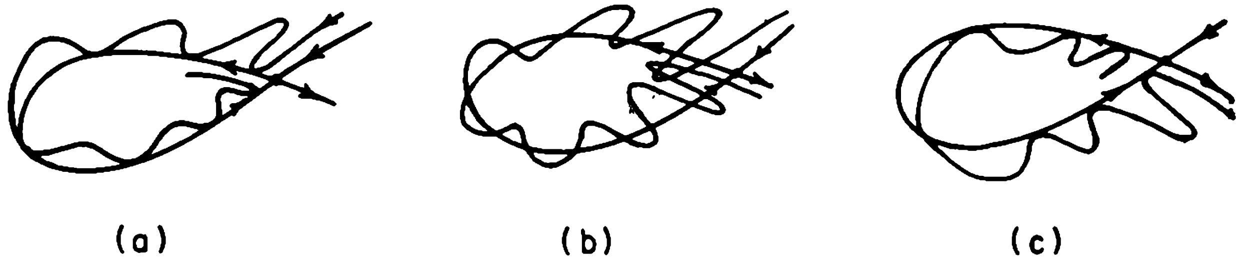

Figure 4.4.3. A generic perturbation of the degenerate homoclinic bifurcation for the map P_ε^μ.

Much of this book, including the rest of this chapter and Chapters 5 and 6, will be concerned with the detection and analysis of transverse homoclinic points of two-dimensional maps, the bifurcations leading to them, and the complicated dynamics resulting from them.

4.5. Melnikov's Method: Perturbations of Planar Homoclinic Orbits

In this and the next section, we develop a method which enables us to study the Poincaré map for time-periodic systems of the form

$$\dot{x} = f(x) + \varepsilon g(x, t); \qquad x = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2, \tag{4.5.1}$$

where g is of (fixed) period T in t. Equivalently, we have the suspended system:

$$\left.\begin{aligned} \dot{x} &= f(x) + \varepsilon g(x, \theta) \\ \dot{\theta} &= 1 \end{aligned}\right\}; \qquad (x, \theta) \in \mathbb{R}^2 \times S^1.$$

Here $f(x)$ is a Hamiltonian vector field defined on $\mathbb{R}^2$ or some subset thereof, and $\varepsilon g(x, t)$ is a small perturbation which need not be Hamiltonian itself. Many physical problems, such as the buckled beam of Section 2.2, can be expressed in the form (4.5.1), but we must admit that our main reason for studying systems of this type is that they present one of the few cases in which *global* information on specific systems can be obtained analytically. We also remark that, after a time scale change $t \to \varepsilon t$, the averaged system of equation (4.1.3) also falls into this class, but as we point out in Section 4.7, there are difficulties in applying the Melnikov analysis directly to averaged systems, since the period of g is then $\mathcal{O}(1/\varepsilon)$ (cf. equation (4.4.1)). Various extensions to the method are outlined in Section 4.8, which enable it to be applied to a wider class of systems.

The basic ideas to be introduced here are due to Melnikov [1963]. More recently Chow *et al.* [1980] have obtained similar results using alternative methods, and Holmes and Marsden [1981, 1982a, b, 1983a] have applied the method to certain infinite-dimensional flows arising from partial differential equations and to multidegree of freedom autonomous Hamiltonian systems. The basic idea is to make use of the globally computable solutions of the unperturbed *integrable* system in the computation of perturbed solutions. To do this we must first ensure that the perturbation calculations are uniformly valid on arbitrarily long or semi-infinite time intervals.

First we make our assumptions precise. We consider systems of the form (4.5.1) where

$$f = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}, \qquad g = \begin{pmatrix} g_1(x, t) \\ g_2(x, t) \end{pmatrix}$$

are sufficiently smooth (C^r, $r \geq 2$) and bounded on bounded sets, and g is T-periodic in t. For simplicity, we assume that the unperturbed system is Hamiltonian with $f_1 = \partial H/\partial v$, $f_2 = -\partial H/\partial u$. (The non-Hamiltonian case is considered by Melnikov [1963] and Holmes [1980b], cf. Exercise 4.5.1.) In general we shall restrict ourselves to a bounded region $D \subset \mathbb{R}^2$ of the phase space. Our specific assumptions on the unperturbed flow are:

A1 For $\varepsilon = 0$ (4.5.1) possesses a homoclinic orbit $q^0(t)$, to a hyperbolic saddle point p_0.

A2 Let $\Gamma^0 = \{q^0(t) | t \in \mathbb{R}\} \cup \{p_0\}$. The interior of Γ^0 is filled with a continuous family of periodic orbits $q^\alpha(t)$, $\alpha \in (-1, 0)$. Letting $d(x, \Gamma^0) = \inf_{q \in \Gamma^0} |x - q|$ we have $\lim_{\alpha \to 0} \sup_{t \in \mathbb{R}} d(q^\alpha(t), \Gamma^0) = 0$.

A3 Let $h_\alpha = H(q^\alpha(t))$ and T_α be the period of $q^\alpha(t)$. Then T_α is a differentiable function of h_α and $dT_\alpha/dh_\alpha > 0$ inside Γ^0.

We note that A2 and A3 imply that $T_\alpha \to \infty$ monotonically as $\alpha \to 0$. We illustrate the situation in Figure 4.5.1.

Many of the results to follow can be proved under less restrictive assumptions. In particular, for this section we only require A1. In what follows we outline the main ideas of the proof but omit some details. See Greenspan [1981] for more information.

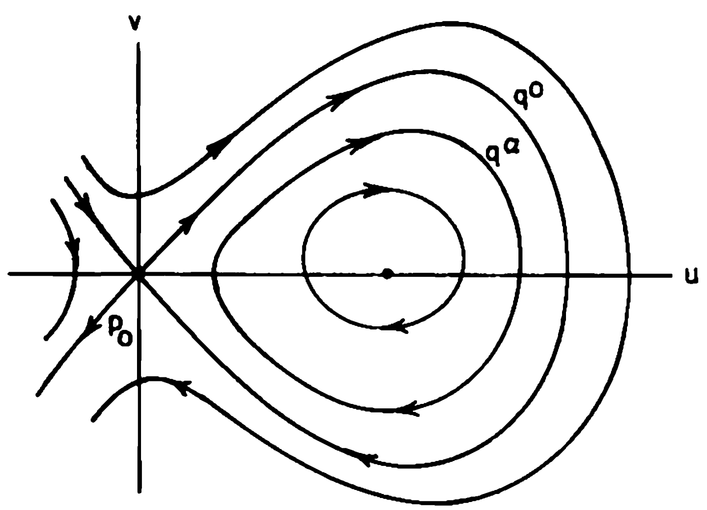

Figure 4.5 1. The unperturbed system.

As in Section 4.1 we define a Poincaré map $P_\varepsilon^{t_0}: \Sigma^{t_0} \to \Sigma^{t_0}$, where $\Sigma^{t_0} = \{(x, t) | t = t_0 \in [0, T]\} \subset \mathbb{R}^2 \times S^1$ is the global cross section at time t_0 for the suspended autonomous flow of (4.5.1). Note that we will need to vary the "section time" t_0 in what follows.

Assumption A1 immediately implies that the unperturbed Poincaré map $P_0^{t_0}$ has a hyperbolic saddle point p_0 and that the closed curve $\Gamma^0 = W^u(p_0) \cap W^s(p_0)$ is filled with nontransverse homoclinic points for $P_0^{t_0}$.* We expect this highly degenerate structure to break under the perturbation $\varepsilon g(x, t)$, and perhaps to yield transverse homoclinic orbits or no homoclinic points at all. The goal of this section is the development of a method to determine what happens in specific cases. In particular, this method will enable us to prove the existence of transverse homoclinic points and homoclinic bifurcations in important physical examples, and hence, using the results of Chapter 5, to prove that horseshoes and chaotic motions occur. This is one of the few analytical methods available for the detection and study of chaotic motions.

We start with two basic perturbation results:

Lemma 4.5.1. *Under the above assumptions, for ε sufficiently small,* (4.5.1) *has a unique hyperbolic periodic orbit $\gamma_\varepsilon^0(t) = p_0 + \mathcal{O}(\varepsilon)$. Correspondingly, the Poincaré map $P_\varepsilon^{t_0}$ has a unique hyperbolic saddle point $p_\varepsilon^{t_0} = p_0 + \mathcal{O}(\varepsilon)$.*

PROOF. This is a straightforward application of the implicit function theorem, our assumptions implying that $DP_0^{t_0}(p_0)$ does not contain 1 in its spectrum and hence that $\mathrm{Id} - DP_0^{t_0}(p_0)$ is invertible and there is a smooth curve of fixed points $(p_\varepsilon^{t_0}, \varepsilon)$ in (x, ε) space passing through $(p_0, 0)$. □

Lemma 4.5.2. *The local stable and unstable manifolds $W_{\mathrm{loc}}^s(\gamma_\varepsilon)$, $W_{\mathrm{loc}}^u(\gamma_\varepsilon)$ of the perturbed periodic orbit are C^r-close to those of the unperturbed periodic orbit $p_0 \times S^1$. Moreover, orbits $q_\varepsilon^s(t, t_0)$, $q_\varepsilon^u(t, t_0)$ lying in $W^s(\gamma_\varepsilon)$, $W^u(\gamma_\varepsilon)$ and based on Σ^{t_0} can be expressed as follows, with uniform validity in the indicated time intervals.*

$$\begin{aligned} q_\varepsilon^s(t, t_0) &= q^0(t - t_0) + \varepsilon q_1^s(t, t_0) + \mathcal{O}(\varepsilon^2), \qquad t \in [t_0, \infty); \\ q_\varepsilon^u(t, t_0) &= q^0(t - t_0) + \varepsilon q_1^u(t, t_0) + \mathcal{O}(\varepsilon^2), \qquad t \in (-\infty, t_0]. \end{aligned} \tag{4.5.2}$$

PROOF. The existence of the perturbed manifolds follows from invariant manifold theory (cf. Nitecki [1971], Hartman [1973], or Hirsch *et al.* [1977]). As in the proof of the averaging theorem, we fix a ν-neighborhood $(0 \leq \varepsilon \ll \nu \ll 1)$ U_ν of p_0 inside which the local perturbed manifolds and their tangent spaces are ε-close to those of the unperturbed flow (or Poincaré map). A standard Gronwall estimate shows that perturbed orbits starting within $\mathcal{O}(\varepsilon)$ of $q^0(0)$ remain within $\mathcal{O}(\varepsilon)$ of $q^0(t - t_0)$ for finite times and hence that

* Here the subscript 0 implies that we set $\varepsilon = 0$ in (4.5.1).

one can follow any such orbit from an arbitrary point near $q^0(0)$ on Γ^0 *outside* U_v to the boundary of U_v, at, say $t = t_1$. Once in U_v, if the perturbed orbit q_ε^s is selected to lie in $W^s(\gamma_\varepsilon)$, then its behavior is governed by the exponential contraction associated with the linearized system. Moreover, the perturbation theory of invariant manifolds implies that

$$|q_\varepsilon^s(t_1, t_0) - q^0(t_1 - t_0)| = \mathcal{O}(\varepsilon),$$

since the perturbed manifold is C^r-close to the unperturbed manifold. Straightforward estimates then show that

$$|q_\varepsilon^s(t, t_0) - q^0(t - t_0)| = \mathcal{O}(\varepsilon)$$

for $t \in (t_1, \infty)$. Reversing time, one obtains a similar result for $q_\varepsilon^u(t, t_0)$ (cf. Figure 4.1.1). □

Sanders [1980, 1982] was the first to work out the details of the asymptotics of solutions in the perturbed manifolds.

This lemma implies that solutions lying in the stable manifold are *uniformly* approximated, for $t \geq 0$, by the solution q_1^s of the first variational equation:

$$\dot{q}_1^s(t, t_0) = Df(q^0(t - t_0))q_1^s(t, t_0) + g(q^0(t - t_0), t). \qquad (4.5.3)$$

A similar expression holds for $q_1^u(t, t_0)$ with $t \leq t_0$. We can thus use regular perturbation theory to approximate solutions in the stable and unstable manifolds of the perturbed system. Note that the initial time, t_0, appears explicitly, since solutions of the perturbed systems are not invariant under arbitrary translations in time ((4.5.1) is nonautonomous for $\varepsilon \neq 0$).

We next define the *separation of the manifolds* $W^u(p_\varepsilon^{t_0})$, $W^s(p_\varepsilon^{t_0})$ *on the section* Σ^{t_0} *at the point* $q^0(0)$ as

$$d(t_0) = q_\varepsilon^u(t_0) - q_\varepsilon^s(t_0), \qquad (4.5.4)$$

where $q_\varepsilon^u(t_0) \stackrel{\text{def}}{=} q_\varepsilon^u(t_0, t_0)$, $q_\varepsilon^s(t_0) \stackrel{\text{def}}{=} q_\varepsilon^s(t_0, t_0)$ are the unique points on $W^u(p_\varepsilon^{t_0})$, $W^s(p_\varepsilon^{t_0})$ "closest" to $p_\varepsilon^{t_0}$ and lying on the normal

$$f^\perp(q^0(0)) = (-f_2(q^0(0)), f_1(q^0(0)))^T$$

to Γ^0 at $q^0(0)$. The C^r closeness of the manifolds to Γ^0, and Lemma 4.5.2, then imply that

$$d(t_0) = \varepsilon \frac{f(q^0(0)) \wedge (q_1^u(t_0) - q_1^s(t_0))}{|f(q^0(0))|} + \mathcal{O}(\varepsilon^2). \qquad (4.5.5)$$

Here the wedge product is defined by $a \wedge b = a_1 b_2 - a_2 b_1$ and $f \wedge (q_1^u - q_1^s)$ is the projection of $q_1^u - q_1^s$ onto $f^\perp$; cf. Figure 4.5.2. Finally, we define the *Melnikov function*

$$M(t_0) = \int_{-\infty}^{\infty} f(q^0(t - t_0)) \wedge g(q^0(t - t_0), t)\, dt. \qquad (4.5.6)$$

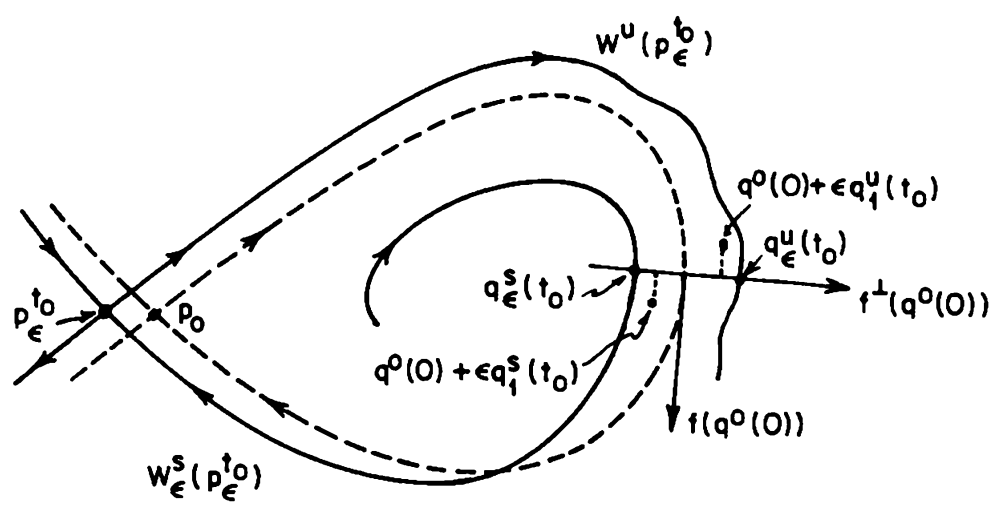

Figure 4.5.2. The perturbed manifolds and the distance function.

Theorem 4.5.3. *If $M(t_0)$ has simple zeros and is independent of ε, then, for $\varepsilon > 0$ sufficiently small, $W^u(p_\varepsilon^{t_0})$ and $W^s(p_\varepsilon^{t_0})$ intersect transversely. If $M(t_0)$ remains away from zero then $W^u(p_\varepsilon^{t_0}) \cap W^s(p_\varepsilon^{t_0}) = \varnothing$.*

Remark. This result is important, since it permits us to test for the existence of transverse homoclinic orbits in specific differential equations. As we show in Chapter 5, the presence of such orbits implies, via the Smale–Birkhoff theorem, that some iterate $(P_\varepsilon^{t_0})^N$ of the Poincaré map has an invariant hyperbolic set: a Smale horseshoe. As noted in Section 2.4, a horseshoe contains a countable infinity of (unstable) periodic orbits, an uncountable set of bounded, nonperiodic orbits, and a dense orbit. The sensitive dependence on initial conditions which it engenders in the flow of the differential equation is of great practical interest.

PROOF. Consider the time-dependent distance function

$$\begin{aligned}\Delta(t, t_0) &= f(q^0(t - t_0)) \wedge (q_1^u(t, t_0) - q_1^s(t, t_0)) \\ &\stackrel{\text{def}}{=} \Delta^u(t, t_0) - \Delta^s(t, t_0),\end{aligned} \tag{4.5.7}$$

and note that $d(t_0) = \varepsilon\Delta(t_0, t_0)/|f(q^0(0))| + \mathcal{O}(\varepsilon^2)$, from (4.5.5). We compute the derivative

$$\begin{aligned}\frac{d}{dt}\Delta^s(t, t_0) = &Df(q^0(t - t_0))\dot{q}^0(t - t_0) \wedge q_1^s(t, t_0) \\ &+ f(q^0(t - t_0)) \wedge \dot{q}_1^s(t, t_0).\end{aligned}$$

Using (4.5.3) and the fact that $\dot{q}^0 = f(q^0)$, this yields

$$\begin{aligned}\dot{\Delta}^s &= Df(q^0)f(q^0) \wedge q_1^s + f(q^0) \wedge (Df(q^0)q_1^s + g(q^0, t)) \\ &= \operatorname{trace} Df(q^0)\Delta^s + f(q^0) \wedge g(q^0, t).\end{aligned} \tag{4.5.8}$$

But, since f is Hamiltonian, trace $Df \equiv 0$, and integrating (4.5.8) from t_0 to ∞ we have

$$\Delta^s(\infty, t_0) - \Delta^s(t_0, t_0) = \int_{t_0}^{\infty} f(q^0(t - t_0)) \wedge g(q^0(t - t_0), t)\, dt. \tag{4.5.9}$$

However, $\Delta^s(\infty, t_0) = \lim_{t\to\infty} f(q^0(t - t_0)) \wedge q_1^s(t, t_0)$ and $\lim_{t\to\infty} q^0(t - t_0) = p_0$, so that $\lim_{t\to\infty} f(q^0(t - t_0)) = 0$ while $q_1^s(t, t_0)$ is bounded, from Lemma 4.5.2. Thus $\Delta^s(\infty, t_0) = 0$ and (4.5.9) gives us a formula for $\Delta^s(t_0, t_0)$. A similar calculation gives

$$\Delta^u(t_0, t_0) = \int_{-\infty}^{t_0} f(q_0(t - t_0)) \wedge g(q^0(t - t_0), t)\, dt, \tag{4.5.10}$$

and addition of (4.5.9) and (4.5.10) and use of (4.5.5) yields

$$d(t_0) = \frac{\varepsilon M(t_0)}{|f(q^0(0))|} + \mathcal{O}(\varepsilon^2). \tag{4.5.11}$$

Since $|f(q^0(0))| = 0(1)$, $M(t_0)$ provides a good measure of the separation of the manifolds at $q^0(0)$ on Σ^{t_0}. We recall that the vector $f^\perp(q^0(0))$ and its base point $q^0(0)$ are fixed on the section Σ^{t_0} and that, as t^0 varies, Σ^{t_0} sweeps around $\mathbb{R}^2 \times S^1$. Thus, if $M(t_0)$ oscillates about zero with maxima and minima independent of ε, then, from (4.5.4)–(4.5.5), $q^u(t_0)$ and $q^s(t_0)$ must change their orientation with respect to $f^\perp(q^0(0))$ as t^0 varies. We require that M be independent of ε to ensure that ε can be chosen sufficiently small so that the $\mathcal{O}(\varepsilon^2)$ error in (4.5.11) is dominated by the term $\varepsilon M/|f(q^0)|$. (In Section 4.7 we shall see that M can depend upon ε in certain cases, and this leads to considerable difficulties.)

This implies that there must be a time $t_0 = \tau$ such that $q_\varepsilon^s(\tau) = q_\varepsilon^u(\tau)$ and we have a homoclinic point $q \in W^s(p_\varepsilon^\tau) \cap W^u(p_\varepsilon^\tau)$. But since all the Poincaré maps P^{t_0} are equivalent, $W^s(p_\varepsilon^{t_0})$ and $W^u(p_\varepsilon^{t_0})$ must intersect for all $t_0 \in [0, T]$. Moreover, if the zeros are simple ($dM/dt_0 \neq 0$), then it follows that the intersections are transversal. Conversely, if no zeros exist, then $q_\varepsilon^u(t_0)$ and $q_\varepsilon^s(t_0)$ retain the same orientation and hence the manifolds do not intersect. □

Remarks. 1. We note that $M(t_0)$ is T-periodic in t_0, as it should be, since the maps $P_\varepsilon^{t_0}$ and $P_\varepsilon^{t_0+T}$ are identical, and thus $d(t_0) = d(t_0 + T)$. In computing $M(t_0)$ we are effectively standing at a fixed point $q^0(0)$ on a moving cross section Σ^{t_0} and watching the perturbed manifolds oscillate as t_0 varies. In his analysis, Greenspan [1981] fixes the section and moves the base point $q^0(0)$ along the unperturbed loop Γ^0. However, the two approaches are equivalent, since the perturbed solutions are dominated by the one-parameter family of unperturbed orbits $q^0(t - t_0)$ lying in Γ^0 for which translations in time (t_0) and along Γ^0 are indistinguishable.

2. If the perturbation g is derived from a (time-dependent) Hamiltonian function $G(u, v)$: $g_1 = \partial G/\partial v$, $g_2 = -\partial G/\partial u$; then we have

$$M(t_0) = \int_{-\infty}^{\infty} \{H(q^0(t - t_0)), G(q^0(t - t_0), t)\}\, dt, \tag{4.5.12}$$

where $\{H, G\}$ denotes the Poisson bracket (Goldstein [1980]):

$$\{H, G\} = \frac{\partial H}{\partial u}\frac{\partial G}{\partial v} - \frac{\partial H}{\partial v}\frac{\partial G}{\partial u}. \tag{4.5.13}$$

This is a natural formula if one recalls that the first variation of the unperturbed Hamiltonian energy H will be obtained by integrating its evolution equation,

$$\dot{H} = \{H, G\}, \tag{4.5.14}$$

along the unperturbed orbit $q^0(t - t_0)$; cf. Arnold [1964].

3. If $g = g(x)$ is not explicitly time dependent, then we have, using Green's theorem,

$$\begin{aligned}\int_{-\infty}^{\infty} f(q^0(t - t_0)) \wedge g(q^0(t - t_0))\, dt &= \int_{-\infty}^{\infty} (f_1 g_2 - f_2 g_1)\, dt \\ &= \int_{-\infty}^{\infty} (g_2(u^0, v^0)\dot{u}^0 - g_1(u^0, v^0)\dot{v}^0)\, dt \\ &= \int_{\Gamma^0} g_2(u, v)\, du - g_1(u, v)\, dv \\ &= \int_{\operatorname{int}\Gamma^0} \operatorname{trace} Dg(x)\, dx. \end{aligned} \tag{4.5.15}$$

Thus the formula obtained in Andronov *et al.* [1971] is a special (planar) case of the more general Melnikov function which describes the "splitting" of the perturbed saddle separatrices. (Also see Section 6.1.)

4. We end by noting that the change of variables $t \to t + t_0$ puts the Melnikov integral (4.5.6) into the form

$$M(t_0) = \int_{-\infty}^{\infty} f(q^0(t)) \wedge g(q^0(t), t + t_0)\, dt, \tag{4.5.16}$$

which is often more convenient in calculations.

EXERCISE 4.5.1. Suppose that assumption A1 made above holds, but that $\dot{x} = f(x)$ is not Hamiltonian, so that trace $Df \neq 0$. Derive a Melnikov function for this case.

We now turn to the case in which the perturbation $g = g(x, t; \mu)$ depends upon parameters $\mu \in \mathbb{R}^k$. For simplicity we take $k = 1$.

Theorem 4.5.4. *Consider the parametrized family* $\dot{x} = f(x) + \varepsilon g(x, t; \mu)$, $\mu \in \mathbb{R}$ *and let hypotheses* A1–A3 *hold. Suppose that the Melnikov function* $M(t_0, \mu)$ *has a (quadratic) zero* $M(\tau, \mu_b) = (\partial M/\partial t_0)(\tau, \mu_b) = 0$ *but* $(\partial^2 M/\partial t_0^2)(\tau, \mu_b) \neq 0$ *and* $(\partial M/\partial \mu)(\tau, \mu_b) \neq 0$. *Then* $\mu_B = \mu_b + \mathcal{O}(\varepsilon)$ *is a bifurcation value for which quadratic homoclinic tangencies occur in the family of systems.*

PROOF. By hypothesis, using (4.5.5), we have

$$d(t_0, \mu) = \varepsilon\{\alpha(\mu - \mu_b) + \beta(t_0 - \tau)^2\} + \mathcal{O}(\varepsilon|\mu - \mu_b|^2) + \mathcal{O}(\varepsilon^2), \quad (4.5.17)$$

where we have expanded in a Taylor series about $(t_0, \mu) = (\tau, \mu_b)$, and α, β are finite constants. Taking ε sufficiently small we find that $d(t_0, \mu)$ has a quadratic zero with respect to t_0 for some μ_B near μ_b, and hence that $W^u(p_\varepsilon^\tau)$, $W^s(p_\varepsilon^\tau)$ have a quadratic tangency near $q^0(0)$ on Σ^τ. □

This result is important, since it permits us to verify in specific examples one of the hypotheses of Newhouse's [1979] theorem on wild hyperbolic sets (see Chapter 6).

As an example, we apply the results of this section to the Duffing equation with negative linear stiffness and weak sinusoidal forcing and damping, introduced in Chapter 2. Written in the form (4.5.1), we have

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= u - u^3 + \varepsilon(\gamma \cos \omega t - \delta v), \end{aligned} \quad (4.5.18)$$

where the force amplitude γ, frequency ω, and the damping δ are variable parameters and ε is a small (scaling) parameter. For $\varepsilon = 0$ the system has centers at $(u, v) = (\pm 1, 0)$ and a hyperbolic saddle at $(0, 0)$. The level set

$$H(u, v) = \frac{v^2}{2} - \frac{u^2}{2} + \frac{u^4}{4} = 0 \quad (4.5.19)$$

is composed of two homoclinic orbits, Γ_+^0, Γ_-^0 and the point $p_0 = (0, 0)$. The unperturbed homoclinic orbits based at $q_\pm^0(0) = (\pm\sqrt{2}, 0)$ are given by

$$\begin{aligned} q_+^0(t) &= (\sqrt{2} \operatorname{sech} t, -\sqrt{2} \operatorname{sech} t \tanh t), \\ q_-^0(t) &= -q_+^0(t). \end{aligned} \quad (4.5.20)$$

We will compute the Melnikov function for q_+^0 (the computation for q_-^0 is identical). Using the form (4.5.16), we have

$$\begin{aligned} M(t_0) &= \int_{-\infty}^{\infty} v^0(t)[\gamma \cos \omega(t + t_0) - \delta v^0(t)]\, dt \\ &= -\sqrt{2}\gamma \int_{-\infty}^{\infty} \operatorname{sech} t \tanh t \cos \omega(t + t_0)\, dt \\ &\quad - 2\delta \int_{-\infty}^{\infty} \operatorname{sech}^2 t \tanh^2 t\, dt. \end{aligned} \quad (4.5.21)$$

The integrals can be evaluated (the first by the method of residues) to yield

$$M(t_0; \gamma, \delta, \omega) = -\frac{4\delta}{3} + \sqrt{2}\gamma\pi\omega \operatorname{sech}\left(\frac{\pi\omega}{2}\right) \sin \omega t_0. \quad (4.5.22)$$

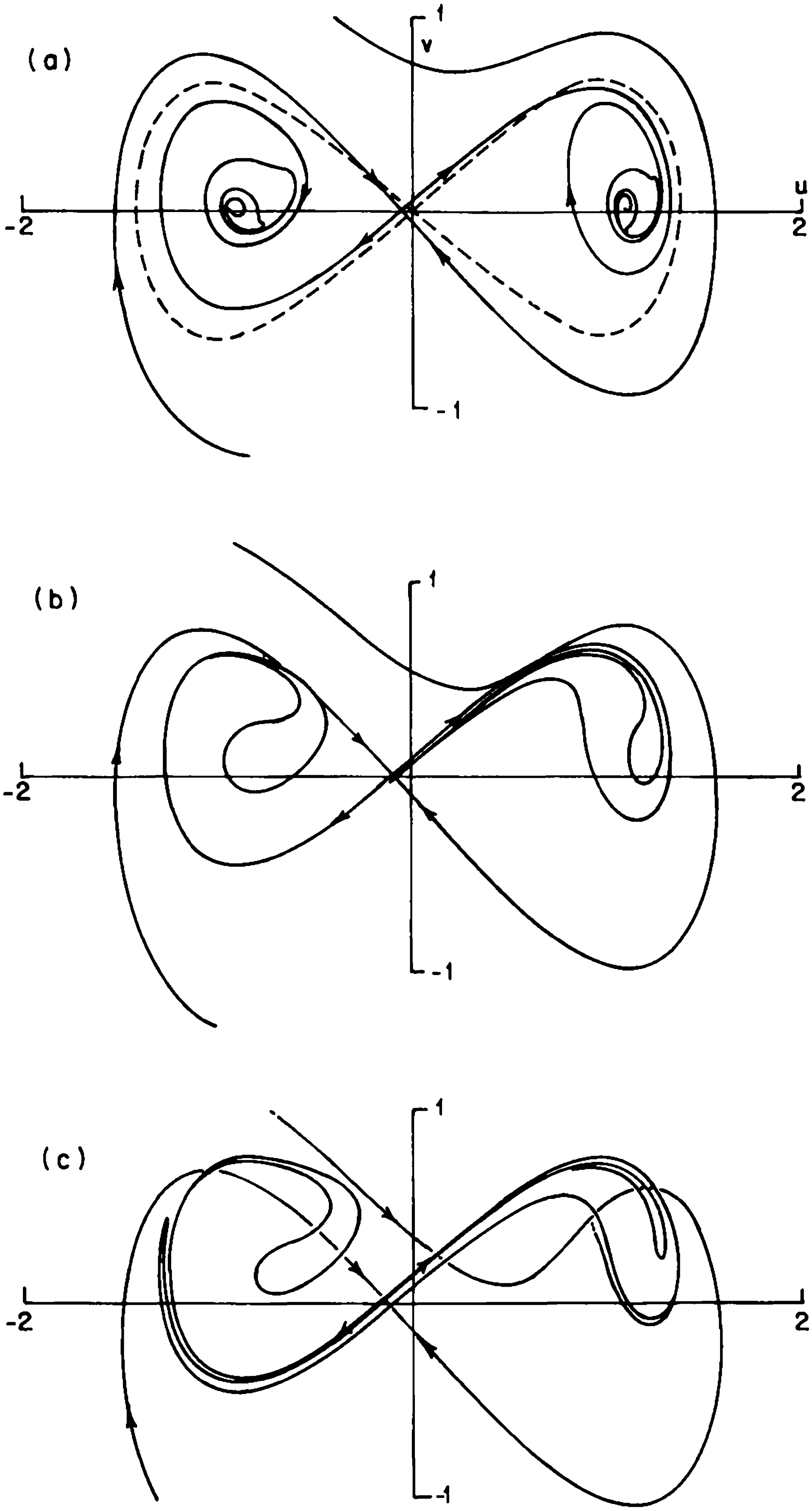

Figure 4.5.3. Poincaré maps for the Duffing equation, showing stable and unstable manifolds of the saddle point near (0, 0), $\omega = 1.0$, $\varepsilon\delta = 0.25$. (a) $\varepsilon\gamma = 0.11$; (b) $\varepsilon\gamma = 0.19$; (c) $\varepsilon\gamma = 0.30$.

If we define

$$R^0(\omega) = \frac{4\cosh(\pi\omega/2)}{3\sqrt{2}\pi\omega}, \tag{4.5.23}$$

then it follows from Theorem 4.5.3 that if $\gamma/\delta > R^0(\omega)$, $W^s(p_\varepsilon)$ intersects $W^u(p_\varepsilon)$ for ε sufficiently small, and if $\gamma/\delta < R^0(\omega)$, $W^s(p_\varepsilon) \cap W^u(p_\varepsilon) = \varnothing$. Moreover, since $M(t_0; \gamma. \delta. \omega)$ has quadratic zeros when $\gamma/\delta = R^0(\omega)$, Theorem 4.5.4 implies that there is a bifurcation curve in the γ, δ plane for each fixed ω, tangent to $\gamma = R^0(\omega)\delta$ at $\gamma = \delta = 0$, on which quadratic homoclinic tangencies occur. (To use Theorem 4.5.4 directly, we fix ω and δ, for example, and vary γ.) We show some Poincaré maps of equation (4.5.17) in Figure 4.5.3. These were computed numerically by Ueda [1981a]. The unperturbed double homoclinic loop $\Gamma_+^0 \cup \{0, 0\} \cup \Gamma_-^0$ is also shown for reference on Figure 4.5.3(a). Note that the (first) tangency appears to occur about $\varepsilon\gamma = 0.19$, in comparison with a theoretical value of 0.188 from (4.5.23)! We will continue our study of this example in the next section.

Exercise 4.5.2. Use the Melnikov method to compute bifurcation curves near which quadratic homoclinic tangencies occur for the plane pendulum with mixed constant and oscillating torque excitation (the undamped sine-Gordon equation):

$$\dot{\theta} = v,$$

$$\dot{v} = -\sin\theta + \varepsilon(\alpha + \gamma\cos t).$$

Exercise 4.5.3. Show that the Hamiltonian system with time-dependent Hamiltonian perturbation

$$H(p, q; t) = \frac{p^2 + q^2}{2} - \frac{q^3}{3} + \frac{\varepsilon q^2 \cos t}{2}$$

has transverse homoclinic orbits for all $\varepsilon \neq 0$, small.

4.6. Melnikov's Method: Perturbations of Hamiltonian Systems and Subharmonic Orbits

We continue to assume that the system (4.5.1) satisfies hypotheses A1–A3 of Section 4.5 and now turn to the family of periodic orbits $q^\alpha(t)$ lying within Γ^0. We wish to know if any of these will be preserved under the perturbation $\varepsilon g(x, t)$. Once more we start with a perturbation lemma:

Lemma 4.6.1. *Let $q^\alpha(t - t_0)$ be a periodic orbit of the unperturbed system based on Σ^{t_0}, with period T_α. Then there exists a perturbed orbit $q_\varepsilon^\alpha(t, t_0)$, not necessarily periodic, which can be expressed as*

$$q_\varepsilon^\alpha(t, t_0) = q^\alpha(t - t_0) + \varepsilon q_1^\alpha(t, t_0) + \mathcal{O}(\varepsilon^2), \tag{4.6.1}$$

uniformly in $t \in [t_0, t_0 + T_\alpha]$, for ε sufficiently small and all $\alpha \in (-1, 0)$.