Imperial College London

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BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2016

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Bifurcation Theory

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BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2016

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

M3PA24/M4PA24/M53PA24

Bifurcation Theory

Date: examdate Time: examtime

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Consider a system of differential equations in R^4. How many stable equilibria and periodic orbits can be born at the following bifurcations:
(i) An equilibrium state with the eigenvalues of the linearisation matrix equal to $-1 \pm i$, -0.5, 0 and the Lyapunov coefficients $l_2 = l_3 = 0$, $l_4 \neq 0$?
(ii) A periodic orbit with the multipliers -2, -0.5, 1?
(iii) A periodic orbit with the multipliers -1, -0.5, 0.5 and the first Lyapunov coefficient negative?
(iv) A periodic orbit with the multipliers 0.5 and $e^{\pm 2\pi i \omega}$ with irrational ω and the first Lyapunov coefficient negative?
(v) A homoclinic loop to an equilibrium with the eigenvalues of the linearisation matrix equal to $-2 \pm i$, -2, 1?

2. (i) Find the Taylor expansion at zero, up to the second order terms, for the center manifold in the following system of differential equation:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = 3x^2 - z^2, \\ \dot{z} = -2z + x^2 + z^2, \end{cases}$$

(ii) Find the Taylor expansion at zero, up to the third order terms, for the restriction of the following map to the center manifold:

$$\begin{cases} \bar{x} = -x + 2x^2 + 3xy, \\ \bar{y} = 3y - x^2. \end{cases}$$

3. Compute the first Lyapunov coefficient for the zero equilibria of the following systems:

$$\text{(i)} \qquad \begin{cases} \dot{x} = y, \\ \dot{y} = -x + 3x^2 + 2xy, \end{cases}$$

and

$$\text{(ii)} \qquad \begin{cases} \dot{x} = y + x^2 + xy - 2y^2, \\ \dot{y} = -x - y^2 - xy + 2x^2, \end{cases}$$

4. Consider the following map on the interval $[0, 1]$:

$$\bar{x} = f_a(x) = a(x - x^3)$$

with a positive parameter a.
(i) For which values of a the map has a fixed point with the multiplier $+1$ or -1?
(ii) For which values of a the map has a stable fixed point?
(iii) For which values of a the map has an orbit of period 2?

5. Consider a two-parameter family of two-dimensional maps which have a fixed point with multipliers $(1+\mu)e^{\pm i\omega}$ where the parameter μ varies near 0 and ω near $\omega_0 = 2\pi/5$.

(i) By counting resonant terms, show that the normal form for such map is given by

$$\bar{z} = (1+\mu)e^{\pm i\omega}[z(1+(L+i\Omega)|z|^2) + A(z^*)^4 + O(|z|^5)],$$

where z is a complex variable, z^* is complex-conjugate to z, and $A = ae^{i\psi}$, L, and Ω are constants.

(ii) Assume that the first Lyapunov coefficient satisfies $L < 0$. By scaling z we can always make $L = -1$ in this case. In the polar coordinates $z = re^{i\phi}$ the normal form recasts as

$$\bar{r} = (1+\mu)r(1-r^2+ar^3\cos(5\varphi-\psi)+O(r^4)), \qquad \bar{\phi} = \phi+\frac{2\pi}{5}+\delta+\Omega r^2-ar^3\sin(5\varphi-\psi)+O(r^4)),$$

where $\delta = \omega - 2\pi/5$ is a small parameter; you do not need to verify this formula. We know that the condition $L < 0$ implies that a closed invariant curve is born from the fixed point at small $\mu > 0$. The invariant curve attracts all orbits from a small neighbourhood of the fixed point, independent of μ and δ. It can be shown that the curve has an equation $r = f(\varphi)$ where f is a smooth, positive, periodic function of ϕ. Show that

$$f = \sqrt{\mu} + O(\mu).$$

(iii) Show that in the (μ, ω)-plane near the origin there exists a region corresponding to the existence of orbits of period 5 and that the boundaries of this region are tangent to the line $\delta + \Omega\mu = 0$.

Solutions:

Question 1 (seen similar). Consider a system of differential equations in R^4. How many stable equilibria and periodic orbits can be born at the following bifurcations:

(i, 4 points) An equilibrium state with the eigenvalues of the linearisation matrix equal to $-1 \pm i$, -0.5, 0 and the Lyapunov coefficients $l_2 = l_3 = 0$, $l_4 \neq 0$?

There is only one eigenvalue on the imaginary axis, so $dim W^c = 1$, so no periodic orbits here. As $l_4 \neq 0$, we may have 4 equilibria born, 2 of them stable (since the other eigenvalues have negative real parts).

(ii, 4 points) A periodic orbit with the multipliers -2, -0.5, 1?

There are multipliers outside the unit circle, so no stable periodic orbits can be born.

(iii, 4 points) A periodic orbit with the multipliers -1, -0.5, 0.5 and the first Lyapunov coefficient negative?

This is a subcritical period-doubling bifurcation, hence one periodic orbit is born.

(iv, 4 points) A periodic orbit with the multipliers 0.5 and $e^{\pm 2\pi i \omega}$ with irrational ω and the first Lyapunov coefficient negative?

A stable invariant torus is born at this bifurcation; the flow on this torus can be smoothly conjugate to a linear rotation if the rotation number is made Diophantine. After that, infinitely many stable periodic orbits can be created on the invariant torus.

(v, 4 points) A homoclinic loop to an equilibrium with the eigenvalues of the linearisation matrix equal to $-2 \pm i$, -2, 1?

The saddle value is negative, so just 1 stable periodic orbit is born.

Question 2 (partly seen similar) (i, 10 points) Find the Taylor expansion at zero, up to the second order terms, for the center manifold in the following system of differential equation:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = 3x^2 - z^2, \\ \dot{z} = -2z + 2x^2 + z^2. \end{cases}$$

(ii, 10 points) Find the Taylor expansion at zero, up to the third order terms, for the restriction of the following map to the center manifold:

$$\begin{cases} \bar{x} = -x + 2x^2 + 3xy, \\ \bar{y} = 3y - x^2. \end{cases}$$

(i) The linearisation matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ has a double zero eigenvalue (algebraic multiplicity 2 and geometric multiplicity 1) and one eigenvalue equal to -2. So, the center manifold is 2-dimensional and tangent to the two-dimensional invariant subspace corresponding to the zero eigenvalue, i.e. to the plane $z = 0$. Let us kill the x^2 term in the equation for $\dot{z}$: after the transformation

$$z_{new} = z + ax^2 + bxy + cy^2,$$

we find

$$\frac{d}{dt}z_{new} + 2z_{new} = \frac{dz}{dt} + 2z + 2ax^2 + 2bxy + 2cy^2 + 2ax\frac{dx}{dt} + b\frac{dx}{dt}y + bx\frac{dy}{dt} + 2cy\frac{dy}{dt} =$$
$$= 2x^2 + 2ax^2 + 2bxy + 2cy^2 + 2axy + by^2 + O(z^2 + |x|^3 + |y|^3).$$

Take $a = -1$, $b = 1$, $c = -1/2$, then

$$\frac{d}{dt}z_{new} = -2z_{new} + O(z_{new}^2 + |x|^3 + |y|^3).$$

It follows that in the new coordinates the center manifold satisfies $z_{new} = O(x|^3 + |y|^3)$, which gives, in the old coordinates

$$z = x^2 - xy + y^2/2 + O(x|^3 + |y|^3).$$

(ii) The linearisation matrix is $\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$. The center manifold is 1-dimensional and tangent to $y = 0$. Do the coordinate transformation $y_{new} = y + ax^2$. We have

$$\bar{y}_{new} - 3y_{new} = -x^2 + ax^2 - 3ax^2 + O(|x|^3 + |y|^3).$$

If we take $a = -1/2$, then the quadratic term will be killed, so the center manifold satisfies $y_{new} = O(|x|^3 + |y|^3)$, which gives, in the old coordinates, This gives the map on the center manifold in the form

$$\bar{x} = -x + 2x^2 + 3x^3/2 + O(x^4).$$

Question 3 (seen similar).
Compute the first Lyapunov coefficient for the zero equilibria of the following systems:

$$(\text{i}, 10\text{points}) \qquad \begin{cases} \dot{x} = y, \\ \dot{y} = -x + 3x^2 + 2xy, \end{cases}$$

and

$$(\text{ii}, 10\text{points}) \qquad \begin{cases} \dot{x} = y + x^2 + xy - 2y^2, \\ \dot{y} = -x - y^2 - xy + 2x^2, \end{cases}$$

(i) Let $z = x - iy$, so $x = (z + z^*)/2$, $y = i(z - z^*)/2$. The system takes the form

$$\dot{z} = iz - 3i(z + z^*)^2/4 + (z^2 - (z^*)^2)/2 = iz + z^2(2 - 3i)/4 - 3izz^*/2 - (z^*)^2(2 + 3i)/4.$$

We use the formula

$$L_1 = \frac{1}{2\omega^2}\mathrm{Re}(iab)$$

where ω is the coefficient of iz and a,b are the coefficients of z^2 and zz^*, respectively. This gives

$$L_1 = \frac{3}{8}.$$

(ii) The equation is time-reversible (it does not change if we change $t \to -t,\ x \to y,\ y \to x$), so all Lyapunov coefficients are zero.

Question 4 (seen similar). Consider the following map on the interval $[0, 1]$:

$$\bar{x} = f_a(x) = a(x - x^3)$$

with a positive parameter a.
(i, 5 points) For which values of a the map has a fixed point with the multiplier $+1$ or -1?
The fixed points are $O_1 : x = 0$ and $O_2 : x = \sqrt{1 - 1/a}$ (the point O_2 exists at $a > 1$). The multiplier equals to $\lambda = a(1 - 3x^2)$, so $\lambda = 1$ corresponds to $a = 1$ (point O_1) and $\lambda = -1$ corresponds to $a = 2$ (point O_2).

(ii, 7 points) For which values of a the map has a stable fixed point?

If $|\lambda| < 1$, the fixed point is stable, so O_1 is stable at $a < 1$ and O_2 is stable at $a < 2$. The stability at $\lambda = 1$ is determined by the sign of the 3d derivative while the stability at $\lambda = -1$ is determined

$a = 1$. The Schwartzian $S = f'''/f' - \frac{3}{2}(f''/f')^2 = -6/(1-3x^2) - \frac{3}{2}(6x)^2/(1-3x^2)^2 = -6\frac{1+6x^2}{(1-3x^2)^2}$ is negative for all x, so O_2 is stable at $a = 2$. Thus, a stable fixed point exists at $a \leq 2$.

(iii, 8 points) For which values of a the map has an orbit of period 2?

An orbit of period 2 is born at the period-doubling bifurcation of the point O_2 at $a = 2$; since the Schwartzian is negative, the domain of existence corresponds to $a > 2$. The period-2 orbit (x_1, x_2) can disappear only if its multiplier becomes equal to 1 - this would mean the following system of equations has a solution with $x_1 \neq x_2$:

$$x_1 = f_a(x_2), \qquad x_2 = f_a(x_1), \qquad f'(x_1)f'(x_2) = 1.$$

In our case we have

$$x_1 = a(x_2 - x_2^3), \qquad x_2 = a(x_1 - x_1^3), \qquad a^2(1-3x_1^2)(1-3x_2^2) = 1.$$

By taking the sum and difference and product of the first two equations, we find

$$x_1 - x_2 + a(x_1 - x_2)(1 - x_1^2 - x_1x_2 - x_2^2) = 0 \implies a = \frac{1}{x_1^2 + x_2^2 + x_1x_2 - 1},$$

and

$$x_1 + x_2 = a(x_1 + x_2)(1 - x_1^2 + x_1x_2 - x_2^2) \implies a = \frac{1}{1 - x_1^2 - x_2^2 + x_1x_2},$$

so

$$x_1^2 + x_2^2 = 1.$$

By dividing the 3d equation to the product of the first two, we obtain

$$(1-3x_1^2)(1-3x_2^2) = (1-x_1^2)(1-x_2^2) \implies x_1^2x_2^2 = (x_1^2 + x_2^2)/4 = 1/4.$$

The only possible solution is $x_1^2 = x_2^2 = 1/2$, but this contradicts the condition $x_1 \neq x_2$. Thus, the point of period 2 exists for all $a > 2$.

Question 5.

Consider a two-parameter family of two-dimensional maps which have a fixed point with multipliers $(1+\mu)e^{\pm i\omega}$ where the parameter μ varies near 0 and ω near $\omega_0 = 2\pi/5$.

(i, 6 points, partly seen) By counting resonant terms, show that the normal form for such map is given by

$$\bar{z} = (1+\mu)e^{\pm i\omega}[z(1 + (L + i\Omega)|z|^2) + A(z^*)^4 + O(|z|^5)],$$

where z is a complex variable, z^* is complex-conjugate to z, and $A = ae^{i\psi}$, L, and Ω are constants.

The term $z^m(z^*)^n$ is present in the normal form if the resonance condition $e^{2\pi i/5} = e^{2\pi i/5(m-n)}$ is satisfied. This condition gives where s is an arbitrary integer. There are only two non-negative integer solutions with $2 \le m+n \le 4$: $m = 2$, $n = 1$ ($s = 0$), and $m = 0$, $n = 4$ ($s = -1$). This gives the required normal form.

(ii, 7 points, partly seen) Assume that the first Lyapunov coefficient satisfies $L < 0$. By scaling z we can always make $L = -1$ in this case. In the polar coordinates $z = re^{i\phi}$ the normal form recasts as

$$\bar{r} = (1+\mu)r(1-r^2+ar^3\cos(5\varphi-\psi)+O(r^4)), \qquad \bar{\phi} = \phi+\frac{2\pi}{5}+\delta+\Omega r^2-ar^3\sin(5\varphi-\psi)+O(r^4)),$$

where $\delta = \omega - 2\pi/5$ is a small parameter; you do not need to verify this formula. We know that the condition $L < 0$ implies that a closed invariant curve is born from the fixed point at small $\mu > 0$. The invariant curve attracts all orbits from a small neighbourhood of the fixed point, independent of μ and δ. It can be shown that the curve has an equation $r = f(\varphi)$ where f is a smooth, positive, periodic function of ϕ. Show that

$$f = \sqrt{\mu} + O(\mu).$$

It is enough to show that the annulus

$$|r - \sqrt{\mu}| \le 2a\mu$$

is invariant with respect to μ - the invariant curve attracts all orbits from this annulus, so it must lie in this annulus. The rest is a straightforward computation: at the outer boundary of the annulus we have

$$\frac{\bar{r}}{r} \le (1+\mu)(1-r^2+2ar^3) \le (1+\mu)(1-\mu-4a\mu\sqrt{\mu}+2a\mu\sqrt{\mu}+O(\mu^2)) < 1,$$

and at the inner boundary

$$\frac{\bar{r}}{r} \ge (1+\mu)(1-r^2-2ar^3) \le (1+\mu)(1-\mu+4a\mu\sqrt{\mu}-2a\mu\sqrt{\mu}+O(\mu^2)) > 1.$$

(iii, 7 points, unseen) Show that in the (μ, ω)-plane near the origin there exists a region corresponding to the existence of orbits of period 5 and that the boundaries of this region are tangent to the line $\delta + \Omega\mu = 0$.

The restriction of the map onto the invariant curve is

$$\phi = \phi + \frac{2\pi}{5} + \delta + \Omega\mu + O(\mu^{3/2}),$$

so the rotation number is larger than $2\pi/5$ at $\delta > -\Omega\mu + O(\mu^{3/2})$ and smaller than $2\pi/5$ at $\delta < -\Omega\mu + O(\mu^{3/2})$. As the rotation number depends continuously on parameters, the region of existence of the points of period 5 is indeed tangent to the line $\delta = -\Omega\mu$.

Before printing the exam some details have to be entered in the source code near the beginning They look like this:

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