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BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2011

M3A23/M4A23

Dynamical Systems

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BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2011

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

M3A23/M4A23

Dynamical Systems

Date: examdate Time: examtime

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

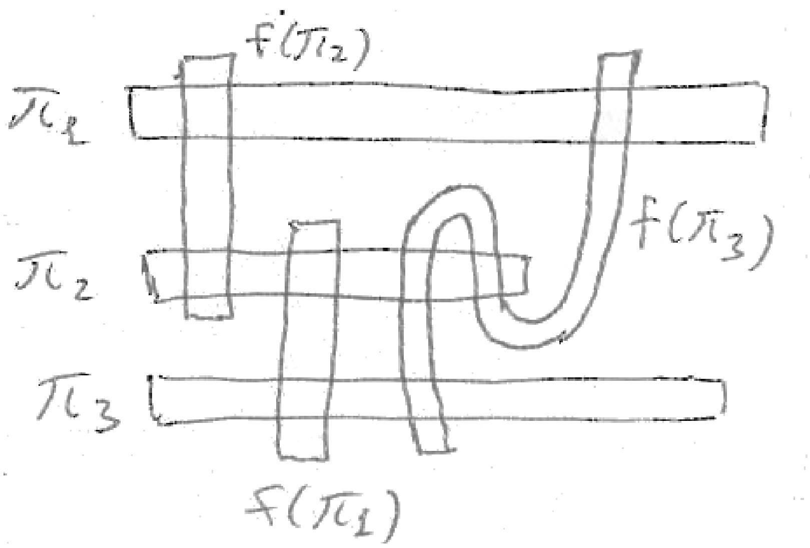
Calculators may not be used.

1. Consider the map $T : (x, y) \mapsto (\bar{x}, \bar{y})$ where

$$\bar{x} = y, \qquad \bar{y} = 400 + 2y - y^2 - x^2/2.$$

Prove that T has positive topological entropy.

2. Let a map f have a hyperbolic set Λ be defined by the Markov partition shown in the figure.



a) Find the topological entropy of $f|_\Lambda$.
b) Show that $f|_\Lambda$ is not topologically conjugate to the Smale horseshow.
c) Are periodic points dense in Λ?
d) Is there an orbit of f which is dense in Λ?
e) Let P_n be the number of points of period n in Λ. Find $\liminf_{n\to+\infty} \frac{\ln P_n}{n}$ and $\limsup_{n\to+\infty} \frac{\ln P_n}{n}$.

3. Prove chaotic behaviour in the system

$$\dot{x} = y, \qquad \dot{y} = x - 8x^3 + \varepsilon \cos t$$

for all small $\varepsilon \neq 0$. Hint: you may use that $\displaystyle\int_{-\infty}^{+\infty} \frac{\cos t \, dt}{e^t + e^{-t}} = \frac{\pi}{e^{\pi/2} + e^{-\pi/2}}$.

4. Prove that the rotation number of a degree-1 homeomorphism f of a circle is rational if and only if f has a periodic point.

Solutions

1. (20 points) Consider the map $T : (x,y) \mapsto (\bar{x},\bar{y})$ where

$$\bar{x} = y, \qquad \bar{y} = 400 + 2y - y^2 - x^2/2.$$

Prove that T has positive topological entropy.

Solution: Write the map in the cross-form $T^\times$:

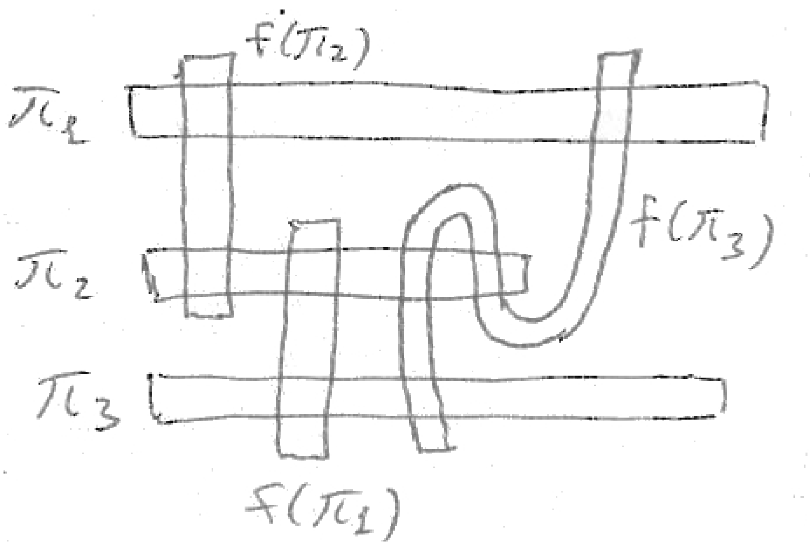
$$\bar{x} = y = f_\pm(x,\bar{y}) = 1 \pm \sqrt{401 - x^2/2 - \bar{y}}.$$

We have two branches, defined by the functions f_+ and f_-. Each branch of $T^\times$ takes the square $\Pi : \{-20 \le x \le 22,\ -20 \le y \le 22\}$ into itself.

On this square, we have

$$\|\frac{\partial f_\pm}{\partial x}\| + \|\frac{\partial f_\pm}{\partial \bar{y}}\| \le \frac{|x|+1}{2\sqrt{401 - x^2/2 - \bar{y}}} \le \frac{22+1}{2\sqrt{401 - 22^2/2 - 22}} = \frac{23}{2\sqrt{137}} < 1.$$

Therefore, both branches of $T^\times$ are contracting, hence the set Λ of all orbits of the original map T which never leave Π is in one-to-one correspondence with the set of all bi-infinite sequences of two symbols. Therefore, the topological entropy of $T|_\Lambda = \ln 2 > 0$.



2. Let a map f have a hyperbolic set Λ be defined by the Markov partition shown in the figure.
a) (5 points) Find the topological entropy of $f|_\Lambda$.

Solution: The map $f|_\Lambda$ is topologically conjugate to the shift map on the set of all bi-infinite paths along the edges of the graph G defined by the transition matrix $N = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$.
The characteristic equation is

$$-det \begin{pmatrix} -\lambda & 1 & 1 \\ 1 & 1-\lambda & 0 \\ 1 & 2 & 1-\lambda \end{pmatrix} = \lambda(\lambda^2 - 2\lambda - 1).$$

The eigenvalues are $\lambda_1 = 1 + \sqrt{2}$, $\lambda_2 = -1 + \sqrt{2}$, $\lambda_3 = 0$. The topological entropy

$$h = \ln \lambda_1 = \ln(1 + \sqrt{2}).$$

b) (5 points) Show that $f|_\Lambda$ is not topologically conjugate to the Smale horseshow.

Solution: The topological entropy of the Smale horseshoe is $\ln 2 \neq \ln(1 + \sqrt{2}) = h(f|_\Lambda)$. Since the topological entropy is an invariant of the topological conjugacy, we obtain that the Smale horseshoe is not topologically conjugate to $f|_\Lambda$.

c) (5 points) Are periodic points dense in Λ?

Solution. The graph G is transitive (for any two vertices there is a path that connects them), therefore periodic points are dense in Λ.

d) (5 points) Is there an orbit of f which is dense in Λ?

Solution. Since the graph G is transitive, there is a dense orbit in Λ.

e) (5 points) Let P_n be the number of points of period n in Λ. Find $\liminf_{n\to+\infty} \frac{\ln P_n}{n}$ and $\limsup_{n\to+\infty} \frac{\ln P_n}{n}$.

Solution. $P_n = tr(N^n) = \lambda_1^n + \lambda_2^n + \lambda_3^n = (1+\sqrt{2})^n + (\sqrt{2}-1)^n$. Since $(\sqrt{2}-1)^n = o((1+\sqrt{2})^n)$, we find

$$\liminf_{n\to+\infty} \frac{\ln P_n}{n} = \limsup_{n\to+\infty} \frac{\ln P_n}{n} = \ln(1+\sqrt{2}).$$

3. (20 points) Prove chaotic behaviour in the system

$$\dot{x} = y, \qquad \dot{y} = x - 8x^3 + \varepsilon \cos t$$

for all small $\varepsilon \neq 0$. Hint: you may use that $\displaystyle\int_{-\infty}^{+\infty} \frac{\cos t\, dt}{e^t + e^{-t}} = \frac{\pi}{e^{\pi/2} + e^{-\pi/2}}$.

Solution. At $\varepsilon = 0$ the system is Hamiltonian with the Hamilton function $H = \frac{y^2}{2} - \frac{x^2}{2} + 2x^4$. The zero level of H contains a homoclinic loop $\{y = \pm x\sqrt{1-4x^2}, x \in (0, \frac{1}{2}]\}$ to a saddle at $(0,0)$. The equation of motion on the loop is

$$\frac{dx}{dt} = y = \pm x\sqrt{1-4x^2},$$

which gives the homoclinic solution

$$x_h(t) = \frac{1}{e^t + e^{-t}}\ .$$

The Melnikov function is then given by

$$M(\theta) = \int_{-\infty}^{+\infty} \frac{\partial H}{\partial y}(x_h(t), y_h(t))\ \cos(t+\theta)\, dt =$$

$$= \int_{-\infty}^{+\infty} \dot{x}_h(t)\ \cos(t+\theta)\, dt = \int_{-\infty}^{+\infty} x_h(t)\ \sin(t+\theta)\, dt = \int_{-\infty}^{+\infty} \frac{\sin(t+\theta)\, dt}{e^t + e^{-t}}.$$

Since sinus is an odd function, $M(0) = 0$. This is a simple root of M: indeed $M'(0) = \displaystyle\int_{-\infty}^{+\infty} \frac{\cos t\, dt}{e^t + e^{-t}} = \frac{\pi}{e^{\pi/2} + e^{-\pi/2}} \neq 0$. Thus, this root corresponds to a transverse homoclinic orbit at all small $\varepsilon \neq 0$, hence chaos.

4. (20 points) Prove that the rotation number of a degree-1 homeomorphism f of a circle is rational if and only if f has a periodic point.

Solution. If x is a periodic point of f, then $f^n(x) = x + m$ for some integer n and m. Therefore, $f^{kn}(x) = x + km$ for any integer k. It follows that

$$\lim_{k\to+\infty} \frac{f^{kn}(x)}{kn} = \frac{m}{n},$$

i.e. the rotation number of f is rational.

Conversely, assume the rotation number ρ of f is rational: $\rho = m/n$ for some integer m and n. Consider the map $g = f^n - m$. We have $g^k(x) = f^{nk}(x) - mk$ for any integer k, therefore the rotation number of g is zero:

$$\lim_{k\to+\infty} \frac{g^k(x)}{k} = n \lim_{k\to+\infty} \frac{f^{nk}(x)}{nk} - m = n\rho(f) - m = 0.$$

Let us show that g has a fixed point. If this is not the case, then either $g(x) < x$ for all x, or $g(x) > x$ for all x. Thus, $M = \min |g(x) - x| > 0$. We obtain either

$$g(x) \geq x + M \ \text{ for all } \ x \Longrightarrow \ g^k(x) \geq x + kM \quad \text{for all } \ k \Longrightarrow \lim_{k\to+\infty} \frac{g^k(x)}{k} \geq M > 0,$$

or

$$g(x) \leq x - M \ \text{ for all } \ x \Longrightarrow \ g^k(x) \leq x - kM \quad \text{for all } \ k \Longrightarrow \lim_{k\to+\infty} \frac{g^k(x)}{k} \leq -M < 0,$$

a contradiction. Therefore, g must have a fixed point x:

$$g(x) = x \ \Longrightarrow \ f^n(x) = x + m,$$

i.e. f must have a periodic point.