

Lecture I

In essence, **Dynamical Systems** is a science which studies differential equations. A differential equation here is the equation

$$\dot{x}(t) = f(x(t), t)$$

where f is a given function, and $x(t)$ is an unknown function. We think about t as time, and the set of numbers $x \in R^n$ is supposed to describe the state of a certain system. So, a solution of the given differential equation represents the dependence of the state of the system on time, which means the differential equation under consideration describes how the state of the system varies with time, i.e. it describes the dynamics of the system. Mathematically, the term “dynamical system” has a specific meaning of a one-parameter family of transformations of the state space, and this family must satisfy a group property. The precise definition will be given later. Formally, a dynamical system defined in this way does not need to be generated by differential equations. Still, differential equations provide the basic example of dynamical systems.

The main idea of the dynamical systems approach to differential equations is that instead of looking for closed formulas that would solve these equations (which is a hopeless task in most cases), we try to find out which qualitative properties the solutions possess. One of such properties is given by the following fundamental theorem.

Theorem 1.1. *Given any $x_0 \in R^n$ and $t_0 \in R^1$ such that the function f is continuous in a small neighbourhood of (x_0, t_0) and has a derivative $f'_x := \partial f / \partial x$ which is also continuous in this neighbourhood, there exists a unique solution to the differential equation*

$$\dot{x} = f(x, t) \tag{1}$$

which satisfies the initial condition

$$x(t_0) = x_0. \tag{2}$$

Denote this solution as $x(t) = X(t; x_0, t_0)$, then the function X is smooth, i.e. it has continuous derivatives with respect to (t, x_0, t_0) .

Problem (1),(2) is called *Cauchy problem*. The existence and uniqueness of the solution to the Cauchy problem is quite natural: once we know $x(t_0)$ from (2), we can compute $x'(t_0)$ by plugging $x = x_0, t = t_0$ into the right-hand side of (1), then we can compute $x''(t_0)$ by differentiating (1) with respect to time and plugging the already obtained values of $x(t_0)$ and $x'(t_0)$ into the resulting expression, and so on - we can inductively compute the values of all the derivatives of $x(t)$ at $t = t_0$, i.e. equations (1), (2) define the Taylor expansion of the sought solution $x(t)$ completely. However, this argument alone does not prove Theorem 1: we can compute the Taylor expansion only when the function f is infinitely differentiable, and even in this case the analyticity of the solution $x(t)$ (necessary in order to define $x(t)$ uniquely by its Taylor expansion) is not a priori true.

The standard way to prove Theorem 1.1 is to rewrite the Cauchy problem (1),(2) as the following integral equation:

$$x(t) = x_0 + \int_{t_0}^t f(x(s), s) ds. \quad (3)$$

Obviously, any continuous solution to (3) solves the Cauchy problem (1),(2), and vice versa. Then, one considers a sequence of *Picard iterations*. This is the sequence of functions $x^{(n)}(t)$, $n = 0, 1, \dots$, defined by the rule

$$\begin{aligned} x^{(0)}(t) &= x_0 = \text{const}, \\ x^{(n+1)}(t) &= x_0 + \int_{t_0}^t f(x^{(n)}(s), s) ds. \end{aligned} \quad (4)$$

One shows that this sequence converges *uniformly* on a certain small interval $[t_0 - \delta, t_0 + \delta]$. Since the convergence is uniform, it follows that the limit function $x^*(t) = \lim_{n \rightarrow +\infty} x^{(n)}(t)$ is continuous, and, by taking the limit in the iteration formula (4), it satisfies (3), i.e. we obtain the sought solution of the Cauchy problem. Moreover, the iterations converge to x^* uniformly with all derivatives, which proves the smoothness of x^* with respect to t_0 and x_0 . Instead of implementing this approach directly, we will prove the existence, uniqueness and smoothness of the solution to (3) by referring to a generalization of the above described construction, *Banach contraction mapping principle*, which we briefly discuss below.

Metric and Banach spaces, contraction mapping principle, smoothness

A set Y is called a *metric space* if for any two elements y_1 and y_2 from Y a *distance* $\rho(y_1, y_2)$ is defined, which is a non-negative real-valued function which satisfies

$$\begin{aligned}\rho(y_1, y_2) = 0 &\iff y_1 = y_2, \\ \rho(y_1, y_2) &= \rho(y_2, y_1), \\ \rho(y_1, y_2) &\leq \rho(y_1, y_3) + \rho(y_2, y_3) \quad \text{for every } y_1, y_2, y_3.\end{aligned}$$

A metric space Y is called *complete* if any Cauchy sequence in Y has a limit in Y . Namely, if a sequence $y_n \in Y$ satisfies the Cauchy property: for every $\varepsilon > 0$ there exists $N(\varepsilon)$ such that $\rho(y_{n_1}, y_{n_2}) < \varepsilon$ for every $n_1 \geq N(\varepsilon), n_2 \geq N(\varepsilon)$, then it has a limit $y^* \in Y$, i.e. for every $\varepsilon > 0$ there exists $M(\varepsilon)$ such that if $n \geq M(\varepsilon)$, then $\rho(y_n, y^*) < \varepsilon$.

For example, R^n is a complete metric space (for any distance function equivalent to $\rho(y^{(1)}, y^{(2)}) = \sqrt{\sum_{j=1}^n (y_j^{(1)} - y_j^{(2)})^2}$; two distance functions on the same set are equivalent if convergence in terms of one distance function implies convergence in terms of the other one, and vice versa). Importantly, any *closed* subset of a complete metric space is also a complete metric space.

Let Y and Z be metric spaces. Consider the set \hat{Y} of all continuous bounded functions $Z \rightarrow Y$. Define the distance between elements of \hat{Y} as follows: for two functions $\hat{y}_1(z)$ and $\hat{y}_2(z)$ the distance between them is given by

$$\hat{\rho}(\hat{y}_1, \hat{y}_2) = \sup_{z \in Z} \rho(\hat{y}_1(z), \hat{y}_2(z))$$

(this is called a metric of uniform convergence on \hat{Y}). It is an easy exercise to check that if Y is complete, then \hat{Y} with the metric of uniform convergence is also a complete metric space. Our basic example is the space of continuous functions on a closed interval.

A map T which takes a metric space Y into itself is called *contracting* if there exists a constant $q < 1$ such that for every pair of points y_1, y_2 from Y we have

$$\rho(Ty_1, Ty_2) \leq q\rho(y_1, y_2). \quad (5)$$

In particular, $\rho(Ty_1, Ty_2) \rightarrow 0$ if $\rho(y_1, y_2) \rightarrow 0$, i.e. a contracting map T is always continuous.

Theorem (Banach Principle). *Any contracting map T of a complete metric space Y has a unique fixed point $y^* \in Y$:*

$$Ty^* = y^*,$$

and

$$\lim_{n \rightarrow +\infty} T^n y_0 = y^*$$

For any $y_0 \in Y$.

Proof. Take any $y_0 \in Y$ and define $y_n = T^n y_0$. By (5), we have

$$\rho(y_{n+1}, y_n) \leq q^n \rho(y_1, y_0),$$

hence, for every $k \geq 0$

$$\rho(y_{n+k}, y_n) \leq \sum_{j=n}^{n+k-1} \rho(y_{j+1}, y_j) \leq Cq^n,$$

where the constant $C = \rho(y_0, y_1)/(1 - q)$ is independent of n . Since $q < 1$, it follows that y_n is a Cauchy sequence, therefore it has a limit, y^* , by virtue of completeness of the space Y . We have $y_n \rightarrow y^*$; therefore $y_{n+1} = Ty_n \rightarrow Ty^*$ by continuity of T . Since the sequences y_{n+1} and y_n must have the same limit, it follows that $y^* = Ty^*$, i.e. the existence of the sought fixed point is proven. The uniqueness is obvious: if both y^* and y^{**} are fixed points of T , then

$$\rho(y^*, y^{**}) \leq q\rho(y^*, y^{**}) \implies \rho(y^*, y^{**}) = 0 \implies y^* = y^{**}.$$

We will be further interested in the question of how the fixed point of the contracting map depends on parameters. Let a map T depend on a parameter μ . Namely, let M be a metric space, and $T : Y \times M \rightarrow Y$. Let T be continuous with respect to μ at some $\mu_0 \in M$ which means $T(y, \mu) \rightarrow T(y, \mu_0)$ as $\mu \rightarrow \mu_0$, for every $y \in Y$. Let T be *uniformly contracting* in a neighbourhood of μ_0 , which means that for all $\mu \in M$ which are close enough to μ_0 the map T is contracting on Y and the contraction coefficient q in (5) can be chosen independent from μ (and bounded away from 1).

Theorem (on continuous dependence). *The fixed point of the uniformly contracting map T is continuous with respect to μ at $\mu = \mu_0$.*

Proof. Let $y^*(\mu)$ be the fixed point of T at the given value of μ , i.e.

$$T(y^*(\mu), \mu) = y^*(\mu).$$

We have

$$\begin{aligned} \rho(y^*(\mu), y^*(\mu_0)) &= \rho(T(y^*(\mu), \mu), T(y^*(\mu_0), \mu_0)) \leq \\ &\leq \rho(T(y^*(\mu), \mu), T(y^*(\mu_0), \mu)) + \rho(T(y^*(\mu_0), \mu), T(y^*(\mu_0), \mu_0)) \leq \\ &\leq q \rho(y^*(\mu), y^*(\mu_0)) + \rho(T(y^*(\mu_0), \mu), T(y^*(\mu_0), \mu_0)), \end{aligned}$$

which implies

$$\rho(y^*(\mu), y^*(\mu_0)) \leq \frac{1}{1-q} \rho(T(y^*(\mu_0), \mu), T(y^*(\mu_0), \mu_0)). \quad (6)$$

The continuity of T at $\mu = \mu_0$ implies that the right-hand side of (6) tends to zero as $\mu \rightarrow \mu_0$, hence $\rho(y^*(\mu), y^*(\mu_0)) \rightarrow 0$ as well.

Note that if the map T is uniformly continuous with respect to μ on $Y \times M$, then estimate (6) implies that the fixed point depends on μ uniformly continuously too.

In order to study the smooth dependence on parameters, we need to have a linear structure on Y . Recall the definitions. A normed linear space is a linear space Y with a norm defined on it. The norm is a real-valued positive function $\|\cdot\|$ such that

$$\|y\| = 0 \iff y = 0,$$

$$\|y_1 + y_2\| \leq \|y_1\| + \|y_2\| \quad \text{for every } y_1 \in Y, y_2 \in Y,$$

$$\|\lambda y\| = |\lambda| \|y\| \quad \text{for every } y \in Y, \lambda \in \mathbb{R}^1.$$

The norm defines a metric $\rho(y_1, y_2) = \|y_2 - y_1\|$, so a linear normed space is always a metric space. A normed linear space is called *Banach space* when it is complete. Note that any closed subspace of a Banach space is a complete metric space. An example of a finite-dimensional Banach space is \mathbb{R}^n for any $n \geq 1$; a space of bounded continuous functions $\mathbb{R}^1 \rightarrow \mathbb{R}^n$ with the uniform norm is an example of an infinite-dimensional Banach space.

If Y and Z are normed linear spaces, then the norm of a linear operator $A : Y \rightarrow Z$ is defined as

$$\|A\| = \sup_{y \in Y, y \neq 0} \frac{\|Ay\|_Z}{\|y\|_Y}.$$

Obviously,

$$\|AB\| \leq \|A\| \|B\|.$$

A bounded linear operator A is the *derivative* (the Frechet derivative) of a map $T : Y \rightarrow Z$ at a point y_0 if

$$T(y_0 + \Delta y) = T(y_0) + A\Delta y + o(\|\Delta y\|)$$

for all small $\Delta y \in Y$. Such operator A , when exists, is uniquely defined. We will denote the derivative as $T'(y_0)$, or $\frac{\partial T}{\partial y}(y_0)$. It is easy to see that when Y and Z are finite-dimensional, then such defined derivative T' is the matrix of partial derivatives of T (the Jacobian matrix).

Two properties of the derivative are important for us. One is the chain rule:

$$[T(G(y))]' = T'(G(y)) G'(y),$$

the other is the following inequality: given a convex subset Y of a normed linear space,

$$\|T(y_1) - T(y_2)\| \leq \sup_{y \in Y} \|T'(y)\| \|y_1 - y_2\| \quad \text{for every } y_1 \in Y, y_2 \in Y. \quad (7)$$

Note that the convexity of Y is important in the last inequality. The convexity means that if $y_1 \in Y$, $y_2 \in Y$, then the entire straight-line segment $\{sy_1 + (1-s)y_2 | s \in [0,1]\}$ that connects y_1 and y_2 also lie in Y . Thus, one can consider the function $H(s) = T(sy_1 + (1-s)y_2)$ defined on the segment $[0, 1]$. Note that $H(1) = T(y_1)$ and $H(0) = T(y_2)$. Since H is a function of one variable, one has

$$H(1) - H(0) = \int_0^1 H'(s)ds,$$

which immediately implies

$$\|H(1) - H(0)\| \leq \sup_{s \in [0,1]} \|H'(s)\|.$$

Since, by the chain rule, $H'(s) = T'(sy_1 + (1-s)y_2)(y_1 - y_2)$, and, by convexity, $sy_1 + (1-s)y_2 \in Y$ for all $s \in [0, 1]$ the sought inequality (7) follows.

We will call a map T defined on a subset Y *smooth* on Y if both the map itself and its Frechet derivative $T'(y)$ is uniformly continuous and uniformly bounded on Y . In what follows Y is a closed and convex subset of a Banach space. By virtue of (7), if a smooth map $T : Y \rightarrow Y$ satisfies

$$\sup_{y \in Y} \left\| \frac{\partial T}{\partial y} \right\| = q < 1, \quad (8)$$

then this map is contracting (hence it has a unique fixed point in Y). Now, let the smooth contracting map T depend smoothly on a parameter μ . Namely, we assume that μ belongs to a convex subset M of some Banach space (in our further examples μ runs a ball M in R^n), and that our map T is a smooth map $Y \times M \rightarrow Y$. We also assume that the contraction constant q in (8) is uniformly smaller than 1 for all $\mu \in M$.

Theorem (on smooth dependence). *The fixed point $y^*(\mu)$ of the smooth contracting map T depends smoothly on the parameter μ .*

Proof. Take any value of $\mu \in M$ and add a small increment $\Delta\mu$ to μ . Let $y = y^*(\mu)$ and $y + \Delta y = y^*(\mu + \Delta\mu)$. We have

$$T(y + \Delta y, \mu + \Delta\mu) = T(y, \mu) + \Delta y. \quad (9)$$

Hence,

$$\begin{aligned} \|\Delta y\| &\leq \|T(y + \Delta y, \mu + \Delta\mu) - T(y, \mu + \Delta\mu)\| + \|T(y, \mu + \Delta\mu) - T(y, \mu)\| \leq \\ &\leq \sup_{(y, \mu) \in Y \times M} \left\| \frac{\partial T}{\partial y} \right\| \|\Delta y\| + \sup_{(y, \mu) \in Y \times M} \left\| \frac{\partial T}{\partial \mu} \right\| \|\Delta\mu\|, \end{aligned}$$

which, by (8), gives

$$\|\Delta y\| \leq K \|\Delta\mu\| \quad (10)$$

where $K = \frac{1}{1-q} \sup \left\| \frac{\partial T}{\partial \mu} \right\| < \infty$. Now, using the definition of the derivative, we rewrite (9) as

$$\Delta y = \frac{\partial T}{\partial y} \Delta y + \frac{\partial T}{\partial \mu} \Delta\mu + o(\|\Delta y\| + \|\Delta\mu\|).$$

Since $o(\|\Delta y\|) = o(\|\Delta\mu\|)$ by (10), we obtain

$$(Id - \frac{\partial T}{\partial y}) \Delta y = \frac{\partial T}{\partial \mu} \Delta\mu + o(\|\Delta\mu\|),$$

hence

$$\Delta y = \left[Id - \frac{\partial T}{\partial y} \right]^{-1} \frac{\partial T}{\partial \mu} \Delta \mu + o(\|\Delta \mu\|),$$

which means that y^* indeed has a derivative with respect to μ , and this derivative is given by

$$\frac{\partial y^*}{\partial \mu} = \left[Id - \frac{\partial T}{\partial y}(y^*(\mu), \mu) \right]^{-1} \frac{\partial T}{\partial \mu}(y^*(\mu), \mu). \quad (11)$$

The inverse to $\left[Id - \frac{\partial T}{\partial y} \right]$ in this formula can be found by the rule

$$\left[Id - \frac{\partial T}{\partial y} \right]^{-1} = \sum_{k=0}^{\infty} \left(\frac{\partial T}{\partial y} \right)^k$$

(the series in the right-hand side converges uniformly since $\|(\frac{\partial T}{\partial y})^k\| \leq q^k$ by (8)). The required uniform continuity and uniform boundedness of $\frac{\partial y^*}{\partial \mu}$ follows from the uniform continuity and uniform boundedness of $\left[Id - \frac{\partial T}{\partial y} \right]^{-1}$ and $\frac{\partial T}{\partial \mu}$ (and from the uniform continuity of $y^*(\mu)$ - see remark after the theorem on continuous dependence).

Note that if the contracting map T is C^r -smooth with respect to (y, μ) (i.e. its higher order derivatives exist up to the order r and are uniformly continuous and uniformly bounded), then the fixed point y^* depends on μ also C^r -smoothly. One proves this (and also finds the higher order derivatives of y^*) just by differentiating formula (11) with respect to μ .

A corollary of the contraction mapping principle is given by the following

Theorem (on implicit function). *Let Y, M, Z be Banach spaces and the map $F : Y \times M \rightarrow Z$ be C^r -smooth ($r \geq 1$) in a small neighbourhood of $(y_0, \mu_0) \in Y \times M$. Let $F(y_0, \mu_0) = 0$ and let the inverse to $\frac{\partial F}{\partial y}(y_0, \mu_0)$ exist and be bounded. Then, for all μ sufficiently close to μ_0 there exists a uniquely defined $y^*(\mu)$ such that*

$$F(y^*(\mu), \mu) = 0. \quad (12)$$

The function $y^(\mu)$ is C^r -smooth.*

Proof. Rewrite equation (14) as

$$y = y - D_0^{-1}F(y, \mu), \quad (13)$$

where we denote $D_0 = \frac{\partial F}{\partial y}(y_0, \mu_0)$. We will prove the theorem by showing that the right-hand side of (46) defines, for all μ close to μ_0 , a contracting map T of a small ball with the center at y_0 . We have

$$\frac{\partial T}{\partial y} = Id - D_0^{-1} \frac{\partial F}{\partial y} = -D_0^{-1} \left(\frac{\partial F}{\partial y}(y, \mu) - \frac{\partial F}{\partial y}(y_0, \mu_0) \right).$$

Since D_0^{-1} is bounded and $\frac{\partial F}{\partial y}$ is continuous, it follows that

$$\frac{\partial T}{\partial y}(y, \mu) \rightarrow 0 \quad \text{as} \quad (y, \mu) \rightarrow (y_0, \mu_0),$$

i.e. there exists $\delta_0 > 0$ such that

$$\sup_{\|\mu - \mu_0\| \leq \delta_0, \|y - y_0\| \leq \delta_0} \left\| \frac{\partial T}{\partial y}(y, \mu) \right\| = q < 1.$$

Thus, given any $\delta \leq \delta_0$, the map T is contracting on $B_\delta = \{\|y - y_0\| \leq \delta\}$. The ball B_δ is a closed convex subset of a Banach space (the space Y), so we are left to show that $T(B_\delta) \subset B_\delta$ for all sufficiently small δ and all μ sufficiently close to μ_0 . In order to do this, expand

$$F(y, \mu) = F(y_0, \mu_0) + D_0(y - y_0) + \frac{\partial F}{\partial \mu}(y_0, \mu_0)(\mu - \mu_0) + o(\|y - y_0\|) + o(\|\mu - \mu_0\|).$$

This gives us, by (46), that

$$\|T(y, \mu) - y_0\| \leq \left\| \frac{\partial F}{\partial \mu}(y_0, \mu_0) \right\| \|\mu - \mu_0\| + o(\|\mu - \mu_0\|) + o(\|y - y_0\|)$$

(recall that $F(y_0, \mu_0) = 0$). When $y \in B_\delta$, this gives us

$$\|T(y, \mu) - y_0\| \leq K\|\mu - \mu_0\| + o(\delta)$$

for some $K < \infty$. Thus, if δ and $\|\mu - \mu_0\|$ are small enough, we have $\|T(y, \mu) - y_0\| \leq \delta$, i.e. $T(y, \mu) \in B_\delta$, as was required.

Note that applying this theorem to a function $F(y, \mu) = \mu - G(y)$ gives us the following

Theorem (on inverse function). *Let Y, M be Banach spaces and the map $G : Y \times M$ be C^r -smooth ($r \geq 1$) in a small neighbourhood of $y_0 \in Y$. Let the inverse to $G'(y_0)$ exist and be bounded. Then, for all μ sufficiently close to $\mu_0 = G(y_0)$ there exists a uniquely defined $y^*(\mu)$ such that*

$$G(y^*(\mu)) = \mu. \tag{14}$$

The function $y^(\mu)$ is C^r -smooth.*

Lecture II

Now we can return to the proof of Theorem 1.1 on the existence and uniqueness of the solution to Cauchy problem

$$\dot{x} = f(x, t), \quad x(t_0) = x_0. \quad (15)$$

We rewrite this problem as the integral equation

$$x(t) = x_0 + \int_{t_0}^t f(x(s), s) ds. \quad (16)$$

By Banach principle, in order to prove the existence and uniqueness of the solution to this equation, it is enough to prove that the operator T which maps a function $x(s)_{s \in [t_0 - \delta, t_0 + \delta]}$ to the function

$$\bar{x}(t) = x_0 + \int_{t_0}^t f(x(s), s) ds \quad (17)$$

is contracting on the appropriate function space. Recall that both f and f'_x are continuous in a neighbourhood of the point (x_0, t_0) by the condition of Theorem 1, therefore they are bounded in any sufficiently small ball with the center at this point. Choose a sufficiently small δ_0 and denote

$$M = \sup_{\|x-x_0\| \leq \delta_0, |t-t_0| \leq \delta_0} \|f(x, t)\|, \quad (18)$$

$$L = \sup_{\|x-x_0\| \leq \delta_0, |t-t_0| \leq \delta_0} \left\| \frac{\partial f}{\partial x}(x, t) \right\|. \quad (19)$$

Take any $\delta > 0$ which satisfy

$$\delta \leq \delta_0, \quad M\delta \leq \delta_0, \quad L\delta \leq q < 1. \quad (20)$$

Let Y be the space of continuous functions $x(s)_{s \in [t_0 - \delta, t_0 + \delta]}$ which satisfy

$$\|x(s) - x_0\| \leq M\delta \quad \text{for all } s \in [t_0 - \delta, t_0 + \delta]. \quad (21)$$

We equip the space Y with the uniform norm. Obviously, Y is a closed and convex subset of the Banach space of all continuous functions on the interval $[t_0 - \delta, t_0 + \delta]$.

We will next show that the map T takes the space Y into itself and that T is contracting on Y . If $x(s)$ is any function from Y , then $\|x(s) - x_0\| \leq \delta_0$ by (21),(20) and, therefore, $\|f(x, s)\| \leq M$ by (18), for all $s \in [t_0 - \delta, t_0 + \delta]$. It follows (see (17) that

$$\|\bar{x}(t) - x_0\| \leq \int_{t_0}^t \|f(x(s), s)\| ds \leq M(t - t_0) \leq M\delta \quad \text{for all } |t - t_0| \leq \delta,$$

which means that the function \bar{x} (the image of the function x by T) belongs to the space Y as well. Thus, we have shown that $T(Y) \subset Y$. In order to prove contraction, compute the derivative of T with respect to the function x . By (17), if we take any function $x(s)$ and give it an increment $\Delta x(s)$, then the image \bar{x} gets an increment

$$\Delta \bar{x}(t) = \int_{t_0}^t \frac{\partial f}{\partial x}(x(s), s) \Delta x(s) ds + o(\|\Delta x\|).$$

This means that the derivative T'_x is the operator

$$\Delta x \mapsto \int_{t_0}^t \frac{\partial f}{\partial x}(x(s), s) \Delta x(s) ds.$$

Since

$$\begin{aligned} \sup_{|t-t_0| \leq \delta} \left\| \int_{t_0}^t \frac{\partial f}{\partial x}(x(s), s) \Delta x(s) ds \right\| &\leq \sup_{|t-t_0| \leq \delta} \int_{t_0}^t \left\| \frac{\partial f}{\partial x}(x(s), s) \right\| \|\Delta x(s)\| ds \leq \\ &\leq L\delta \sup_{|s-t_0| \leq \delta} \|\Delta x(s)\| \leq q \sup_{|s-t_0| \leq \delta} \|\Delta x(s)\| \end{aligned}$$

(see (19),(20)), we find that $\|T'_x\| \leq q < 1$ indeed, which proves the existence and uniqueness of the solution to integral equation (16) (hence, to the original Cauchy problem (15)). The smoothness of the solution with respect to (x_0, t_0) follows from the smoothness of the contracting map T . Note that if f is of class C^r , then the map T is C^r as well, hence the solution to the Cauchy problem is C^r with respect to (t, t_0, x_0) , $r \geq 1$. It also follows that if f is continuous or smooth with respect to some parameter μ , then the solution of the Cauchy problem depends on μ continuously or smoothly, respectively.

Theorem 1.1. gives us the existence of the solution $x(t)$ only on a small time interval $[t_0 - \delta, t_0 + \delta]$. However, if at the point $(t_0 + \delta, x(t_0 + \delta))$ the conditions of the theorem (the continuity of f and f'_x in a neighbourhood of the point) are still fulfilled, one can consider $t_0 + \delta$ as the new t_0 , and $x(t_0 + \delta)$ as the new x_0 , apply the theorem to the new t_0 and x_0 and, thus, continue the solution further, to larger values of time, in a unique way. One can, therefore, consider the *maximal* time interval I on which the solution to Cauchy problem (15) is defined uniquely, and ask whether this interval is finite or infinite, and if it is finite from right or left, then what happens at the corresponding end point. If the maximal interval ends at a finite point $t = a$, i.e. the solution $x(t)$ does not continue in a unique way beyond this point, then one possibility is that

$$\lim_{t \rightarrow a} x(t) = \infty.$$

If this is not the case, then at *every* limit point of $x(t)$ at $t = a$ the conditions of Theorem 1.1 must be violated, i.e. either f or f'_x must be discontinuous in a neighbourhood of this point. Indeed, assume the contrary. If x^* is a partial limit point of $x(t)$ at $t = a$, this means that for any neighbourhood U of the point $(t = a, x = x^*)$ there exists a sequence of the values $t_n \rightarrow a$ for which $(t_n, x(t_n)) \in U$. If the conditions of Theorem 1.1 hold at (a, x^*) , then we can take the neighbourhood U such that f and f'_x were continuous and bounded everywhere in U , i.e. we take U to be a box of radius δ_0 such that (18),(19) hold (with $t_0 = a$ and $x_0 = x^*$). Then for any initial point in U the solution can be continued uniquely at least for time δ , which depends on the constants δ_0, M, L only (see (20)), hence its is independent on the point in U . It follows that if we start with t_n sufficiently close to $t = a$, then the solution can be continued beyond $t = a$, a contradiction.

As we see, if f and f'_x are continuous for all $(x, t) \in R^n \times R^1$, then the only possibility for the maximal interval to end at a *finite* time corresponds to $x(t)$ becoming infinite at the end point. In other words, the solution in this case continues uniquely as long as it remains bounded. One of the methods which is useful for determining whether the solution stays finite or goes to infinity (without actually finding the solution) is discussed next.

Comparison principle

The method is based on the three following theorems (versions of the so-called comparison principle).

Theorem 2.1. *Suppose a function $u(t) : [a, b] \rightarrow R^1$ satisfies the inequality*

$$\frac{du}{dt} \leq f(u(t), t) \quad \text{for all } t \in [a, b] \quad (22)$$

for some continuous function $f : R^1 \times [a, b] \rightarrow R^1$, and a function $v(t) : [a, b] \rightarrow R^1$ satisfies the inequality

$$\frac{dv}{dt} > f(v(t), t) \quad \text{for all } t \in [a, b]. \quad (23)$$

Assume $v(a) > u(a)$. Then $v(t) > u(t)$ for all $t \in [a, b]$.

Proof. If, on the contrary, $u(t) - v(t) \geq 0$ at some $t \in [a, b]$, then, by continuity of the functions u and v , it follows from $u(a) - v(a) < 0$ that there exists $t^* > a$ such that $u(t^*) = v(t^*)$ and $u(t) < v(t)$ at $t < t^*$. It follows that $\frac{d(u-v)}{dt}(t^*) \geq 0$, hence $f(u(t^*), t^*) > f(v(t^*), t^*)$, which is impossible as $u(t^*) = v(t^*)$.

An immediate consequence of Theorem 2.1 is the following

Theorem 2.2. *Let the functions $u(t)$ and $v(t)$ with values in R^1 solve, at $t \in [a, b]$, the equations*

$$\frac{du}{dt} = g(u, t), \quad \frac{dv}{dt} = f(v, t)$$

for some continuous functions g and f such that

$$g(x, t) < f(x, t) \quad \text{for all } (x, t) \in R^1 \times [a, b].$$

Assume $v(a) > u(a)$. Then $v(t) > u(t)$ for all $t \in [a, b]$.

Note that in order to formulate a similar statement where the strict inequality $v(a) > u(a)$ on the initial conditions is replaced by a non-strict one,

$v(a) \geq u(a)$, we need more restrictions to be imposed on the right-hand side of the differential equations and inequalities under consideration. Namely, we have the following

Theorem 2.3. *Let $f(x, t)$ and $h(x, t)$ be continuous functions $R^1 \times [a, b] \rightarrow R^1$, with continuous derivatives f'_x and h'_x . Let*

$$h(x, t) \leq f(x, t) \quad \text{for all } (x, t) \in R^1 \times [a, b].$$

Let $v(t)$ and $w(t)$ solve the equations

$$\frac{dv}{dt} = f(v, t) \quad \text{and} \quad \frac{dw}{dt} = h(w, t)$$

with the initial conditions such that

$$w(a) \leq v(a).$$

Let $u(t)$ satisfy

$$h(u, t) \leq \frac{du}{dt} \leq f(u, t) \quad \text{for all } t \in [a, b].$$

Assume $w(a) \leq u(a) \leq v(a)$. Then $w(t) \leq u(t) \leq v(t)$ for all $t \in [a, b]$.

Proof. Let $v_\varepsilon(t)$ be the solution of the Cauchy problem

$$\frac{dv_\varepsilon}{dt} = f(v_\varepsilon, t) + \varepsilon, \quad v_\varepsilon(a) = v(a) + \varepsilon \tag{24}$$

for $\varepsilon > 0$. As $\frac{dv_\varepsilon}{dt} > f(v_\varepsilon, t)$ and $v_\varepsilon(a) > u(a)$, Theorem 2.1 implies that $v_\varepsilon(t) > u(t)$ for all $t \in [a, b]$. By taking the limit $\varepsilon \rightarrow 0$, we find that $v(t) \geq u(t)$ for all $t \in [a, b]$, as required (in order to be able to take this limit, we need a continuity of the solution of Cauchy problem (24) with respect to the parameter ε ; this continuity is guaranteed since both f and f'_x are continuous, i.e. the equation satisfies the conditions of Theorem 1.1). The inequality $w(t) \leq u(t)$ is proved in the same way.

Let us consider a simple example of the use of the comparison principle. Let $y(t)$ be the solution of the Cauchy problem

$$\dot{y} = y^2 + t^2, \quad y(0) = 1. \quad (25)$$

There is no explicit formula for the solutions of this equation. However, since $\dot{y} \geq y^2$, we can conclude from Theorem 2.3 that $y(t) \geq u(t)$ at $t \geq 0$, where

$$\dot{u} = u^2, \quad u(0) = 1.$$

This is easily solved as $u(t) = \frac{1}{1-t}$, so we find that

$$y(t) \geq \frac{1}{t-1}.$$

As we see, the solution of (25) is unbounded: it must tend to $+\infty$ as $t \rightarrow t^*$ for some $t^* \leq 1$. In order to find a lower bound on t^* , note that at $t \leq 1$ we have

$$\begin{aligned} \frac{dy}{dt} \leq y^2 + 1 &\implies \int_1^{y(t)} \frac{dy}{y^2 + 1} \leq t \implies \arctan(y) - \arctan(1) \leq t \implies \\ &\implies y \leq \tan\left(t + \frac{\pi}{4}\right), \end{aligned}$$

i.e. $y(t)$ remains bounded at least for $t < \frac{\pi}{4}$. Thus, we can estimate the end of the maximal existence interval as

$$t^* \in \left[\frac{\pi}{4}, 1\right].$$

Sublinear right-hand side, globally defined solutions

In fact, it is quite rare that the solutions of systems of differential equations are defined for all $t \in (-\infty, +\infty)$. A class of systems for which this is the case is given by the systems with *sublinear* right-hand side. Namely, we will call a function $f(x, t)$ which maps $R^n \times R^1 \rightarrow R^1$ sublinear if there exist a continuous functions $\alpha(t)$ and $\beta(t)$ such that

$$\|f(x, t)\| \leq \alpha(t)\|x\| + \beta(t) \quad \text{for all } x \text{ and } t.$$

Note that linear systems

$$\frac{dx}{dt} = A(t)x + b(t)$$

also belong to the class of sublinear systems we just defined.

Theorem 2.4. *Let a sublinear function f be continuous along with f'_x for all $x \in R^n$ and $t \in R^1$. Then any solution of*

$$\dot{x} = f(x, t)$$

exists for all $t \in (-\infty, +\infty)$.

Proof. Let $x(t)$ be a solution, denote $y(t) = x^2(t) = \|x(t)\|^2$. We have

$$\frac{dy}{dt} = 2x(t) \cdot \frac{dx}{dt} \leq 2\|x(t)\| \left\| \frac{dx}{dt} \right\| \leq 2\alpha(t)\|x\|^2 + 2\beta(t)\|x(t)\| \leq (2\alpha(t) + \beta(t))\|x\|^2 + \beta(t),$$

so

$$\frac{dy}{dt} \leq (2\alpha(t) + \beta(t))y + \beta(t).$$

By Theorem 2.3, $y(t) \leq u(t)$ for all $t \geq 0$, where u is a solution of the linear equation

$$\dot{u} = (2\alpha(t) + \beta(t))u + \beta(t).$$

This solution is given by

$$u(t) = e^{\int_0^t (2\alpha(s) + \beta(s)) ds} \left[u(0) + \int_0^t \beta(s) e^{-\int_0^s (2\alpha(\sigma) + \beta(\sigma)) d\sigma} ds \right],$$

and it is well-defined and stays finite for all t , hence $y(t) = \|x(t)\|^2$ cannot become infinite at a finite value of $t \geq 0$. This proves that the solution $x(t)$ continues for all $t \geq 0$. In order to prove that it continues for all $t \leq 0$ as well, define $z(t) = x(-t)^2$. We have

$$\begin{aligned} \frac{dz}{dt} &= -2x(-t) \cdot x'(-t) \leq 2\|x(-t)\| \|x'(-t)\| \leq 2\alpha(t)\|x(-t)\|^2 + 2\beta(t)\|x(-t)\| \leq \\ &\leq (2\alpha(t) + \beta(t))\|x(-t)\|^2 + \beta(t), \end{aligned}$$

so

$$\frac{dz}{dt} \leq (2\alpha(t) + \beta(t))z + \beta(t),$$

and the finiteness of $z(t) = x(-t)^2$ for all $t \geq 0$ (hence the existence of $x(t)$ for all $t \leq 0$) follows by the comparison with the solution of a linear equation, as before.

Let $X(t; t_0, x_0)$ be the (uniquely defined) solution of the Cauchy problem $x(t_0) = x_0$ for a given system of sublinear differential equations, or any other differential equations for which the solutions are defined globally, for all $t \in (-\infty, +\infty)$. As we have shown, for every t_1 and t_2 the map $U_{t_1 \rightarrow t_2} : x_0 \mapsto X(t_2; t_1, x_0)$ is smooth. Moreover, the smooth map $U_{t_2 \rightarrow t_1}$ is inverse to $U_{t_1 \rightarrow t_2}$. Indeed, $U_{t_1 \rightarrow t_2}$ takes the point x_0 at $t = t_1$ and maps it to the point of the corresponding solution at $t = t_2$, while $U_{t_2 \rightarrow t_1}$ maps this latter point to the point on the same solution at $t = t_1$, i.e. to the same point x_0 : by uniqueness, there is only one solution that passes through the point $X(t_2; t_1, x_0)$ at $t = t_2$, so this solution must equal to x_0 at $t = t_1$, by the definition of the function X . Thus, each map $U_{t_1 \rightarrow t_2} : R^n \rightarrow R^n$ is smooth with a smooth inverse, i.e. it is a *diffeomorphism* of R^n or, simply, a smooth coordinate transformation.

We may use this to introduce new coordinates $y(t, x) = U_{t \rightarrow 0}(x)$. By the definition, if $x(t)$ is a solution of the system of differential equations under consideration, then $y(t, x(t))$ is the initial value for this solution at time $t = 0$. Thus, $y(t, x(t))$ does not depend on time, hence $\frac{dy}{dt} = 0$. We have proved the following

Theorem 2.5. *Given any C^r -smooth system of differential equations*

$$\dot{x} = f(x, t), \quad x \in R^n,$$

whose all solutions are defined for all $t \in (-\infty, +\infty)$, there exists a C^r -smooth, time-dependent change of variable x which brings the system to the trivial form

$$\dot{y} = 0.$$

In a sense, this theorem gives a general formula for solutions of all such differential equations. Of course, it is too general to be useful for actual solving the equations. However, it is a valid mathematical fact, that all such equations are equivalent up to a change of coordinates. In particular, all

linear equations in R^n are equivalent from this point of view. It is intuitively clear, that the theory should not stop here (for example, we know that different linear equations may show quite different behaviour). Obviously, in order to be able to distinguish between different types of behaviour of solutions of differential equations, one has to restrict possible types of allowed coordinate transformations. There is no universal recipe of how to do this for a general system of differential equations; one should adopt different approaches for different classes of equations. A first idea coming to mind is to allow only time-independent coordinate transformations. This is indeed a proper approach when we consider the so-called *autonomous* differential equations. Their study is the main subject of Dynamical Systems theory, and we will focus on this class for the most of the rest of this course.

Lecture III

A system of differential equations is called *autonomous*, if its right-hand side does not depend on time:

$$\dot{x} = f(x). \quad (26)$$

Here $x \in R^n$; the point x is often called a phase point, and the set of all phase points (i.e. R^n in our case) is called the phase space. An orbit x_t of a point x_0 is the solution of system (26) which passes through the point x_0 at $t = 0$. We will further always assume that f is a smooth function, so there is a unique orbit for each point $x_0 \in R^n$ (by the existence and uniqueness theorem). Clearly, if x_t is the orbit of x_0 , then x_{t-t_0} is a solution of the smooth autonomous system (26) that corresponds to $x = x_0$ at $t = t_0$, i.e. if we know all the orbits of (26), then we know all the solutions of it.

For a fixed t , the map $X_t : x_0 \mapsto x_t$ is called the time- t shift map of the system. In our notations $x_t = X_t(x_0)$ (e.g. $x_0 = X_0(x_0)$). By uniqueness of solutions, $x_t = X_{t-s}(x_s)$ for any s (for which x_s is defined), i.e. $x_t = X_t(x_0) = X_{t-s}(x_s) = X_{t-s}(X_s(x_0))$. In other words, the time shift maps of the autonomous system satisfy the group property: for all s and t

$$X_{t-s} \circ X_s = X_t \quad (27)$$

in the region of of the phase space where the maps X_s and X_t are defined. The family of maps X_t (parameterised by the time) is called the flow of the system.

In general, a one-parameter group of maps is called a dynamical system; if a dynamical system is parameterised by a continuous parameter, i.e. it is isomorphic to the additive group R^1 , then such dynamical system is also called a flow. Such defined flow does not need to be generated by a smooth system of differential equations, however the autonomous differential equations whose solutions are defined for all $t \in R^1$ provide the main example. Often, a system of differential equations does not have solutions defined for all $t \in R^1$. In this case the time shifts do not, strictly speaking, form a well-defined group (as the domains of definition of the maps X_t do not coincide for different t in this case, so the composition of such maps is not always defined). In such situation, one should use the term flow with caution, bearing in mind that the group property (27) is satisfied only for some subregions of

the phase space (which may depend on s and t). It may also happen that even if the solutions are not globally defined for all $t \in \mathbb{R}^1$, still every orbit is defined for all $t \geq 0$ (e.g. Theorem 3.5 gives examples of such behaviour). In this case the domain of definition of each map X_t with $t \geq 0$ is the whole phase space, so such maps form a semigroup according to (27), and one may call the family $\{X_t\}_{t \geq 0}$ a semiflow. Importantly, the maps X_t are all smooth (by the smooth dependence on initial conditions), and their inverse (the maps X_{-t}) are also smooth, so the maps X_t are diffeomorphisms; we say that the smooth autonomous system of differential equations generates a smooth flow.

As an example, consider the equation

$$\dot{x} = x$$

with $x \in \mathbb{R}^1$. The time- t map $x_0 \mapsto e^t x_0$ is for each $t \in \mathbb{R}^1$ well-defined (it is a linear map whose domain is the whole phase space). So this differential equation generates a proper flow. Another example:

$$\dot{x} = x^2.$$

The time- t map is given by $x_0 \mapsto \frac{x_0}{1 - tx_0}$ and is defined at $tx_0 < 1$, i.e. its domain of definition is not the whole phase space if $t \neq 0$. In a similar example,

$$\dot{x} = -x^3,$$

the time- t map is given by $x_0 \mapsto \frac{x_0}{\sqrt{1 + 2tx_0^2}}$ and is defined at $2tx_0^2 > -1$.

In particular, at $t \geq 0$ each time- t map is defined for all $x_0 \in \mathbb{R}^1$, i.e. these maps form a semiflow.

Another key concept in the analysis of autonomous differential equations is the notion of a *phase curve*. If x_t is an orbit of system (26), then the curve which x_t draws in the phase space as t runs the maximal existence interval of x_t is called a phase curve. By construction, for each point x_0 in the phase space there is a unique phase curve of (26) which passes through this point, i.e. the phase space is foliated by the phase curves. This is a smooth foliation (because of the smooth dependence of the curves on the initial conditions). The vector $f(x)$ is, by (26), tangent to the phase curve that passes through the point x . Thus, the problem of solving autonomous differential equation (26) can be interpreted as follows: given a smooth vector field $f(x)$ find the

foliation with one-dimensional leaves such that the leaves are tangent to the vector field everywhere (except for the points where $f = 0$). We note that a phase curve can correspond to two opposite directions of the time flow; to avoid this ambiguity, in drawing pictures of the phase curves (the “phase portraits”) the direction of the phase curve (the direction of the tangent vector f) is denoted by arrows.

The simplest type of a phase curve is an *equilibrium state*. Namely, if the phase curve corresponds to a stationary solution $x_t = x^* = \text{const}$, then this curve consists of only one point, x^* . Such points are called equilibria. Obviously, a point x^* is an equilibrium state of system (26) if and only if $f(x^*) = 0$.

An important type of phase curves is given by *closed phase curves*. These correspond to periodic solutions of (26). Namely, a phase curve that corresponds a solution x_t is closed if and only if $x_{t_0} = x_{t_0+T}$ for some t_0 and $T \neq 0$. By uniqueness, the orbit of x_{t_0} coincides with the orbit of x_{t_0+T} in this case, i.e. we have $x_t = x_{t+T}$ for all t , which means periodicity of x_t . Recall that phase curves cannot have self-intersections, i.e. the closed phase curve is a simple closed curve (a homeomorphic image of a circle).

Example. For a system

$$\dot{x}_1 = -x_2, \quad \dot{x}_2 = x_1$$

the point $x_1 = x_2 = 0$ is an equilibrium state, while all the other phase curves are closed: these are the circles $x_1^2 + x_2^2 = \text{const}$ which correspond to the solutions $(x_1 = A \cos t + B \sin t, x_2 = A \sin t - B \cos t)$ with $A^2 + B^2 \neq 0$.

Given an autonomous system of differential equations the ultimate goal is to build its “phase portrait”, i.e. to create a picture which captures the behaviour and mutual position of all the system’s phase curves. This problem is generally solvable for systems in R^2 and R^1 only, in higher dimensions it admits a reasonable solution only for some special cases. Even the question on how to define rigorously the notion of the phase portrait is unresolved in R^n for $n \geq 3$. In R^2 , the phase portrait can be defined constructively, as the so-called “scheme” by Leontovich and Maier. Namely, one indicates all isolated equilibria and isolated closed phase curves, and the so-called special phase curves asymptotic to equilibria. This set of curves separates the phase plane into a number of cells, and one draws one orbit in each cell. Unless the system is very degenerate, the resulting phase portrait gives a complete

description of the behaviour of the system on a plane, as it is shown in the classical book by Andronov, Gordon, Leontovich and Maier. In higher dimensions we do not know exactly how to provide a complete description of system's behaviour. Therefore, there is no constructive definition of the phase portrait for systems in R^n with $n \geq 3$. Instead of defining the phase portrait itself, one defines what does it mean that two systems have the same phase portraits. This introduces a certain equivalence relation on the set of systems (then, formally, the term "phase portrait" can just refer to the corresponding equivalence class). The natural classical equivalence relation goes back to the work by Andronov and Pontryagin: two systems are *topologically equivalent* if there exists a homeomorphism of the phase space which takes the phase curves of one system to the phase curves of the other. This is a very important notion, as it allows to discuss the qualitative features of the behaviour of systems of differential equations in formal mathematical terms, i.e. properly formulate and prove theorems, etc.. However, from the very beginning it was clear that the notion of topological equivalence does not exactly captures our intuitive idea of qualitative similarity between the dynamic behaviour. For example, systems

$$\begin{cases} \dot{x}_1 = -x_1 \\ \dot{x}_2 = -x_2 \end{cases} \quad \text{and} \quad \begin{cases} \dot{x}_1 = -x_1 - x_2 \\ \dot{x}_2 = x_1 - x_2 \end{cases}$$

are topologically equivalent although their phase portraits look quite different (the phase curves for the first system are rays of the straight lines convergent to equilibrium state at zero, while the phase curves of the second system are spirals). In other words, the topological equivalence relation is too weak in order to distinguish the oscillatory and monotone behaviour in this example. On the other hand, systems

$$\begin{cases} \dot{\phi}_1 = \sqrt{2} \\ \dot{\phi}_2 = 1 \end{cases} \quad \text{and} \quad \begin{cases} \dot{\phi}_1 = \sqrt{3} \\ \dot{\phi}_2 = 1 \end{cases}$$

are not topologically equivalent if we consider them as systems on a torus (i.e. if we identify the points $(\phi_1 + m, \phi_2 + n)$ for all integer m and n), even though there is no obvious difference in the phase portraits (each phase curve forms a dense subset of the torus, for both systems). So, the topological equivalence is, probably, too strong for this example. These examples are well understood and do not create significant problems. In general, however, it occurs that for many types of systems with chaotic behaviour in R^n ($n \geq 3$)

the topological equivalence relation is too strong to provide a comprehensible classification. Moreover, no reasonable weaker form of the equivalence relation is known for such systems.

A trivial example of topological equivalence is given by the following

Theorem 3.1. *Let $f : R^n \rightarrow R^n$ be smooth and $\lambda : R^n \rightarrow R^1$ be smooth and everywhere positive. Then the two systems in R^n*

$$\dot{x} = f(x) \quad \text{and} \quad \dot{x} = \lambda(x)f(x)$$

are topologically equivalent.

Proof. We prove that the two systems have exactly the same phase curves, i.e. the sought homeomorphism is just the identity map. The phase curve $x = x(t)$ of the first system will be the phase curve of the second system if and only if the function $x(s(t))$ satisfies the second system for some time reparameterisation $s(t)$. By plugging $x(s(t))$ into the second system, we obtain

$$\dot{x}(s) \frac{ds}{dt} = \lambda(x(s))f(x(s)).$$

Since $x(s)$ solves the first system, i.e. $\dot{x}(s) = f(x(s))$, we find that we may take as the sought $s(t)$ the solution of

$$\frac{ds}{dt} = \lambda(x(s)).$$

Since λ is positive scalar, we can write

$$t = \int_0^s \frac{ds}{\lambda(x(s))},$$

which gives us t as a monotonically increasing function of s , so $s(t)$ is defined as the inverse of this function.

As we see, in the geometrical approach to autonomous differential equations based on the notions of phase curves and topological equivalence we ignore the exact pace of time along the phase curves (we only retain the information about its direction). Therefore, in this approach we do not, strictly speaking, consider a system of differential equations as a dynamical system

(the two systems in Theorem 3.1 generate different flows even though they have exactly the same phase portraits). In fact, both views on the solutions of differential equations (as a smooth flow, and as a smooth foliation tangent to a given vector field) should be used in parallel.

In fact, there is rarely a point in stressing the distinction between the orbits of the system (which are functions of t) and the phase curves of it (which are curves in R^n , the graphs of the orbits). When there is no confusion, we will use the term “orbit” (also “trajectory”) to mean the corresponding phase curve as well. It is also typical to say “set of orbits” meaning the set of all the points of these orbits.

Limit sets, invariant sets, absorbing domains, attractors, Lyapunov functions

A central notion of the dynamical system theory is that of a *limit set* (an ω -limit set) of a trajectory. Typically, given an initial condition x_0 , we are not concerned with the transient process (i.e. with a behaviour on an initial finite time interval, which does not repeat itself). We want to know to which sustainable behaviour the given initial conditions will lead in the future, i.e. what is the limit of the orbit x_t as $t \rightarrow +\infty$. If x_t tends to a certain constant,

$$\lim_{t \rightarrow +\infty} x_t = x^*,$$

then $\frac{dx}{dt} = f(x_t) \rightarrow f(x^*)$ (by the continuity of f). Moreover, as $\frac{dx}{dt}$ cannot stay bounded away from zero (otherwise x_t would not have a limit), it follows that $f(x^*) = 0$, i.e. if x_t has a limit then it is necessarily an equilibrium state.

This observation allows us to give a complete characterisation of autonomous equations in R^1 .

Theorem 3.2. *Consider an equation*

$$\dot{x} = f(x)$$

where $x \in R^1$. Let $a < b$ be such that $f(a) = f(b) = 0$ and $f(x) \neq 0$ for all $x \in (a, b)$. If $f(x) > 0$ on (a, b) , then $x_t \rightarrow b$ as $t \rightarrow +\infty$ and $x_t \rightarrow a$ as $t \rightarrow -\infty$ for every $x_0 \in (a, b)$. If $f(x) < 0$ on (a, b) , then $x_t \rightarrow a$ as $t \rightarrow +\infty$

and $x_t \rightarrow b$ as $t \rightarrow -\infty$ for every $x_0 \in (a, b)$.

Proof. The points a and b are equilibria of the system. Since different phase curves cannot intersect, an orbit x_t starting in (a, b) cannot get to these equilibria at a finite time, i.e. the orbit can never leave the interval (a, b) and must stay there for all times from $-\infty$ to $+\infty$. It follows that $\frac{dx_t}{dt} (= f(x_t))$ remains non-zero for all times, hence x_t is a monotonous function of t . It is also a bounded function, as we just have shown. Thus it must have a limit both as $t \rightarrow +\infty$ and $t \rightarrow -\infty$. These limits must be equilibria of the system, i.e. one of them must be a and the other is b . Which is which, that depends on whether x_t is a decreasing or increasing function of x : if $f(x_t) > 0$, then x_t is increasing, so $\lim_{t \rightarrow +\infty} x_t > \lim_{t \rightarrow -\infty} x_t$, hence $\lim_{t \rightarrow +\infty} x_t = b$ and $\lim_{t \rightarrow -\infty} x_t = a$, and if $f(x_t) < 0$, then x_t is decreasing, so $\lim_{t \rightarrow +\infty} x_t < \lim_{t \rightarrow -\infty} x_t$, hence $\lim_{t \rightarrow +\infty} x_t = a$ and $\lim_{t \rightarrow -\infty} x_t = b$.

We have a similar behaviour if f keeps a constant sign on an infinite interval: if $f > 0$ on this interval, then any trajectory in the interval tends to its right end as t grows and to its left end as t decreases, and if $f < 0$ on this interval, then any trajectory in the interval tends to its right end as t decreases and to its left end as t increases; the difference with the case of a finite interval is that $x = \pm\infty$ can be achieved at a finite moment of time t . The above theorem says us that the behaviour of equations in R^1 is determined completely by the equilibrium states (the zeros of the right-hand side f) and by the sign of f in the intervals between the equilibria: every trajectory tends to the nearest equilibrium from the side determined by the side of $f(x_0)$.

In higher dimensions in general, it is quite often that an orbit has more than a single (partial) limit point.

Definition. If $x_{t_m} \rightarrow x^*$ for some sequence of time values $t_m \rightarrow +\infty$, then x^* is called an ω -limit point of the trajectory x_t or, what is the same, an ω -limit point of the initial point x_0 . The union of all limit points of a given trajectory (or of a given initial condition) is called its ω -limit set. Similarly, α -limit points and the α -limit set are defined, by taking the limit $t_m \rightarrow -\infty$.

As every bounded sequence in R^n has at least one partial limit point,

it follows that every bounded trajectory has a non-empty ω -limit set. For examples of ω - and α -limit sets consider the system

$$\dot{x}_1 = -x_2 + x_1(1 - x_1^2 - x_2^2), \quad \dot{x}_2 = x_1 + x_2(1 - x_1^2 - x_2^2). \quad (28)$$

Introduce polar coordinates $x_1 = r \cos \phi$, $x_2 = r \sin \phi$. The system takes the form

$$\dot{r} = r(1 - r^2), \quad \dot{\phi} = -1. \quad (29)$$

As the first equation is independent on ϕ , one shows (e.g. by Theorem 3.2) that $r(t) \rightarrow +\infty$ as $t \rightarrow +\infty$, for any $r_0 \neq 0$. Thus, the ω -limit set for any point $x_0 \neq 0$ is the curve $r = 1$ (every point on this curve is a limit point, as $\dot{\phi} \neq 0$, i.e. when the point x_t approaches $r = 1$ it rotates with a non-zero velocity, hence it regularly visits a neighbourhood of every point on the curve $r = 1$). The ω -limit set of the equilibrium state $x = 0$ is the equilibrium state itself. If we reverse the direction of time, we notice that $r(t) \rightarrow \infty$ as $t \rightarrow -\infty$ for every $r_0 > 1$. If $r_0 < 1$, then $r(t) \rightarrow 0$ as $t \rightarrow -\infty$. Therefore, the α -limit set of every point x_0 inside the circle $r = 1$ consists of a single point $x = 0$, while this set is empty (or we may say that the α -limit set is infinity) if $r_0 > 1$. For the points on the circle $r = 1$ their α -limit set is the circle itself.

Theorem 3.3. *Let x_t be a trajectory which does not tend to infinity as $t \rightarrow +\infty$. Then its ω -limit set $\Omega(x_0)$ is non-empty and closed. The set $\Omega(x_0)$ consists of entire phase curves (i.e. with every point $x_0^* \in \Omega(x_0)$ the phase curve of x_0^* lies in $\Omega(x_0)$ entirely), and either it is connected, or each of its connected components are unbounded. For a bounded trajectory x_t the ω -limit set is always connected.*

Proof. Since the infinity is not a limit of x_t , it follows that there exists a bounded sequence x_{t_m} for some $t_m \rightarrow +\infty$. Every partial limit point of this sequence belongs to $\Omega(x_0)$, so $\Omega(x_0) \neq \emptyset$. To prove the closeness of $\Omega(x_0)$, we need to show that if x_k^* is a sequence of ω -limit points of x_t , and $x_k^* \rightarrow x^{**}$ as $k \rightarrow +\infty$, then the point x^{**} is also an ω -limit point of x_t . In fact, this is obvious: as x_k^* is an ω -limit point of x_t , it follows that for any $\varepsilon > 0$ and $T > 0$ there exists $t_{k,\varepsilon}$ such that $\|x_k^* - x_{t_{k,\varepsilon}}\| < \varepsilon$ and $t_{k,\varepsilon} > T$. As x^{**} is a limit point of x_k^* , then for any $\varepsilon > 0$ there exists k such that $\|x^{**} - x_k^*\| < \varepsilon$. Hence, for any $\varepsilon > 0$, we have $\|x^{**} - x_{t_{k,\varepsilon}}\| < 2\varepsilon$, i.e. x^{**} is an ω -limit point of x_t indeed.

Let X_t denotes the time- t shift map. If $x_0^* \in \Omega(x_0)$, then $x_{t_m} \rightarrow x_0^*$ for some tending to plus infinity sequence t_m . By continuity of the map X_t for any t for which this map is defined we have $X_t(x_{t_m}) \rightarrow X_t(x_0^*)$, i.e. for any point $x_t^* = X_t(x_0^*)$ on the phase curve of x_0^* we have $x_{t+t_m} \rightarrow x_t^*$. This means that $x_t^* \in \Omega(x_0)$, i.e. the entire phase curve of x_0^* lies in $\Omega(x_0)$ indeed.

To prove the claim about the connected components of $\Omega(x_0)$, let us assume its converse, i.e. let $\Omega(x_0)$ has at least two connected components, A and B , and let A be bounded. It follows that there is an open bounded set $C \supset A$ such that the boundary of C does not intersect Ω and the closure of C does not intersect B . Let $a \in A$ and $b \in B$. Since both a and b are ω -limit sets of x_t , it follows that there exist two tending to plus infinity sequences, $t_{m,1}$ and $t_{m,2}$, such that $t_{m+1,1} > t_{m,2} > t_{m,1}$ for all m , and $x_{t_{m,1}} \rightarrow a$ and $x_{t_{m,2}} \rightarrow b$. Since a lies in the interior of C and b lies in the interior of the complement to C , it follows that $x_{t_{m,1}} \in C$ and $x_{t_{m,2}} \notin C$ for all m large enough. Hence, the arc of the phase curve $x = x_t$ between the points $x_{t_{m,1}}$ and $x_{t_{m,2}}$ must intersect the boundary of C at some $t_m^* \in (t_{m,1}, t_{m,2})$ for each m large enough. Thus, we have found a tending to infinite sequence of points $x_{t_m^*}$ which belong to the boundary of C . Since C is bounded, its boundary is bounded as well, so it is a closed bounded set. Hence, the sequence $x_{t_m^*}$ has a partial limit point in this set. This point is an ω -limit point of x_t , but this contradicts to the assumption that the boundary of C does not intersect the ω -limit set of x_t .

In the same way one proves the same statement for the α -limit sets. By the theorem above, ω - (and α -) limit sets give examples of *invariant* sets, namely the sets which consist of entire trajectories. A general invariant set is not necessarily built of ω - and α - limit sets, however if an invariant set is closed, it must contain, along with each its point, both the ω - and α - limit sets of this point.

By the definition, invariant sets are invariant with respect to the time-shift maps X_t , i.e. they are taken by such maps into itself. As we mentioned, the maps X_t may be not everywhere defined (for some initial conditions the trajectory may go to infinity at a finite time, so beyond this time the time-shift map is undefined). So a proper formulation of the invariance of the set Λ with respect to time-shift maps should be as follows: $X_t(\Lambda \cap \text{Dom}(X_t)) \subseteq \Lambda$ for every $t \in R^1$, where we denote as $\text{Dom}(X_t)$ the domain of definition of the map X_t . The importance of the invariant sets in general is that once an invariant set Λ is known, one can restrict the system on the set Λ , namely

consider the behaviour of the flow maps X_t on the set Λ only (this may be much simpler than studying the flow in the whole phase space). An invariant set, usually, does not need to be a smooth manifold. However, invariant smooth manifolds can exist, and restriction of the system on such a manifold can be defined explicitly, as follows. Let one be able to decompose the phase variables x into two groups of variables, $x = (u, v)$, so the invariant manifold Λ is written as $u = \varphi(v)$ for some smooth function φ (so the value of the v -variables defines the value of the u -variables in a unique way for every point on Λ , i.e. the points on Λ are completely defined by the v -variables alone; one says that the u -variables are enslaved by the v -variables). Let the original system take the form

$$\dot{u} = g(u, v), \quad \dot{v} = h(u, v) \quad (30)$$

in these variables. We are given that Λ is invariant, so $u_t = \varphi(v_t)$ for all times if the starting point belongs to Λ . Thus, for any solution of our system which starts on Λ we have

$$\dot{v} = h(\varphi(v), v). \quad (31)$$

This is the sought restriction of the system on Λ . As the right-hand side is smooth, this equation defines the evolution of the v -variables completely, hence it carries a complete information on the behaviour of the trajectories that lie in Λ . As the number of the v -variables (the dimension of the v -space) is smaller than the number of the x -variables (which equals the dimension of the v -space plus the dimension of the u -space), the system on Λ is simpler than the original system.

It is easy to check whether a given smooth manifold is invariant or not. Let a set Λ be defined by the equation $F(x) = 0$, where F is a smooth function $R^n \rightarrow R^k$. If

$$\text{rank} \left(\frac{\partial F}{\partial x} \right) = k \quad (32)$$

everywhere on Λ , then this equation defines an $(n-k)$ -dimensional smooth manifold in R^n . In particular, if F is a scalar function $R^n \rightarrow R^1$, then this condition reads as

$$F'(x) \neq 0 \quad \text{for all } x \text{ such that } F(x) = 0.$$

If Λ is invariant for the system

$$\dot{x} = f(x),$$

then for any $x_0 \in \Lambda$ we have $x_t \in \Lambda$ for all t , i.e. $F(x_t)$ is identically zero for all t . Therefore, $\frac{d}{dt}F(x_t)_{t=0} = 0$ for each $x_0 \in \Lambda$, which reads as

$$\frac{\partial F}{\partial x} \cdot f(x) = 0 \quad \text{for all } x \text{ such that } F(x) = 0. \quad (33)$$

This condition is, thus, necessary for the invariance of the set Λ . Let us show that if (32) holds, then condition (33) is also a sufficient condition for the invariance of the manifold Λ . Indeed, take any point $x_0 \in \Lambda$. By (32), one can introduce variables $(w, v) = x$, $w \in R^k$, $v \in R^{n-k}$ such that near $x_0 = (w_0, v_0)$

$$\det \left(\frac{\partial F(u, v)}{\partial w} \right) \neq 0.$$

By virtue of the implicit function theorem, it follows that the equation $F(w, v) = 0$ can be uniquely resolved with respect to the w variables near the point x_0 , i.e. the manifold Λ near x_0 can be written in the form

$$w = \varphi(v)$$

for some smooth function $\varphi : R^{n-k} \rightarrow R^k$ defined in a small neighbourhood of v_0 . Choose $u = w - \varphi(v)$ as a new variable instead of w . In the variables (u, v) the manifold Λ is simply given by the equation

$$u = 0.$$

As the function $F(x)$ must vanish on Λ , we have

$$F(0, v) \equiv 0.$$

By differentiating this identity with respect to v , we find

$$\frac{\partial F}{\partial v}(0, v) \equiv 0.$$

Hence condition (32) in this coordinates transforms into

$$\det \frac{\partial F}{\partial u}(0, v) \neq 0.$$

If we write the system in the form (30), i.e. $f = \begin{pmatrix} g \\ h \end{pmatrix}$, then condition (33) transforms into

$$g(0, v) \equiv 0.$$

Hence, if we take the uniquely defined solution v_t of the equation

$$\dot{v} = h(0, v)$$

with the initial condition v_0 , then the function $x_t = (0, v_t)$ will solve equation (30) with the initial condition $x_0 = (0, v_0)$, i.e. it is the uniquely defined solution of our system with the initial condition x_0 . As we see, u stays zero on this solution as time changes, which means that the solution does not leave Λ , i.e. the manifold Λ is invariant indeed.

We stress again that an invariant set does not need to be a smooth manifold. An important example of invariant sets (which can have a very complicated nature) is given by *attractors*. The notion of attractor is parallel to the notion of ω -limit set. While an ω -limit set indicates the asymptotic regime (as $t \rightarrow +\infty$) which corresponds to a given initial condition, an attractor is supposed to capture the asymptotic behaviour of the system as a whole or, at least, for a sufficiently representative set of initial conditions. No definition of attractor exists which would express this idea in an entirely satisfactory way. Attractors we consider here (or the so-called maximal attractors) are defined relative to a choice of the so-called *absorbing domain*. The latter is defined as follows.

Let an open, bounded region $U \in R^n$ have the following property: there exists $T > 0$ such that for each point x_0 in the closure $cl(U)$ of the domain U the forward trajectory x_t of this point lies in U (i.e. strictly inside it) for all $t \geq T$. In other words,

$$X_t(cl(U)) \subset U \quad \text{for all } t \geq T.$$

Then U is called an absorbing domain. Note that it is important in this definition that the time T can be chosen the same for each x_0 .

Examples. Consider a system $\dot{x} = -x$ with $x \in R^n$. Any open ball $U = \{\|x\| < R\}$ is an absorbing domain. Here $cl(U) = \{\|x\| \leq R\}$. As the time T one can take any strictly positive number. Indeed, given any $T > 0$ we have $e^{-T} < 1$. By solving our system we find $x_T = e^{-T}x_0$, so if $x_0 \in cl(U)$, i.e. if $\|x_0\| \leq R$, then $\|x_T\| = e^{-T}\|x_0\| < R$, i.e. $x_T \in U$.

In the next example, every trajectory enters a connected bounded open region V and stays there forever after, however V is not an absorbing domain,

since the time T after which all the trajectories that start in $cl(V)$ will remain in V is different for different trajectories and is unbounded. The system of this example is given by

$$\dot{u} = -u((u-1)^2 + v^2), \quad \dot{v} = -v((u-1)^2 + v^2).$$

It differs from the system in the previous example by a time change, i.e. by the multiplication of the right-hand side to a scalar factor $(u-1)^2 + v^2$. This factor is positive everywhere except for the point $O_1(u=1, v=0)$. Thus, the phase curves of the system that do not pass through the point O_1 are the same as in the system

$$\dot{u} = -u, \quad \dot{v} = -v,$$

i.e. the trajectories lie in the straight lines $u/v = const$, and all except for those with the initial condition at $(v=0, u \geq 1)$ tend to the equilibrium state $O(0,0)$ as $t \rightarrow +\infty$. The trajectory $(v=0, u > 1)$ tends to the equilibrium state O_1 . Thus, every trajectory tends to an equilibrium O or O_1 as $t \rightarrow +\infty$, so for any region V which contains both the equilibria every forward trajectory eventually enters V and stays there forever. Now, take the open set V such that its boundary intersects the segment $I = \{v=0, 0 < u < 1\}$ and a part of this segment does not lie in V . The segment I is a phase curve of our system, α -limit set is the equilibrium O_1 and the ω -limit set is the equilibrium O . As both the equilibria lie in V , for any initial conditions in a part of I close to O_1 the forward orbit starts inside V , then leaves V , then enters V again and stays there forever (it tends to $O \in V$). As the initial point tends to O_1 , the time of re-entry to V tends to infinity (it is a general fact that as the phase velocity is zero at the equilibrium, the time necessary to leave its neighbourhood tends to infinity as the initial point tends to the equilibrium). Thus, as we mentioned, for such chosen region V there is no chance to choose a common $T > 0$ such that $X_t(cl(V)) \subset V$ for all $t \geq T$.

A fairly general example of an absorbing domain is constructed as follows. Note that if at every point of the boundary of an open bounded region U the vector field of the system is non-zero, is not tangent to the boundary, and looks strictly inside U , then U is an absorbing domain. Indeed, every phase curve of the system is tangent to the vector field, so the phase curves which cross the boundary of U must enter U as time grows. After that they cannot leave. In fact, no trajectory that starts inside U can leave U , as in this case the trajectory must intersect the boundary of U and get out, which would

mean that the tangent vector to the trajectory at the moment of intersection with the boundary of U looks outside U , a contradiction. Thus, we just have shown that $x_T \in U$ for any $x_0 \in cl(U)$ and any $T > 0$, i.e. U is an absorbing domain indeed. For a more formal argument, let us assume that there exists a smooth function $F : R^n \rightarrow R^1$ such that the domain U is given by

$$F(x) < 0,$$

and $F(x) = 0$ is the boundary of U . The condition that the vector field of the system

$$\dot{x} = f(x)$$

looks strictly inside U on the boundary of U reads as

$$\frac{\partial F}{\partial x} \cdot f(x) < 0 \quad \text{for all } x \text{ such that } F(x) = 0. \quad (34)$$

Let an orbit x_t intersect the boundary of U at the time moment t_0 . Then, taking into account $F(x_{t_0}) = 0$ we have

$$F(x_t) = F(x_{t_0}) + \frac{d}{dt}F(x_t)_{t=t_0}t + o(t - t_0) = (t - t_0)\frac{\partial F}{\partial x} \cdot f(x) + o(t - t_0),$$

so $F(x_t) > 0$ for all small $t < t_0$ and $F(x_t) < 0$ for all small $t > t_0$. This implies that no orbit starting at $F(x) < 0$ can get to the region $F(x) > 0$ as time grows, and every orbit starting at the boundary $F(x) = 0$ must enter the region $F(x) < 0$. Thus, $X_T(cl(U)) \in U$ for any $T > 0$, i.e. condition (34) indeed guarantees that U is an absorbing domain. A different condition for the existence of an absorbing domain, based on the use of a Lyapunov function, will be discussed later.

Given an absorbing domain U the associated attractor (or a maximal attractor in U) is defined as

$$A = \bigcap_{m=0}^{\infty} X_{mT}(cl(U)) \quad (35)$$

(where $T > 0$ is the time entering the definition of the absorbing domain).

Theorem 3.4. *The attractor is a non-empty, closed and bounded set. If the absorbing domain U is connected, then A is connected. The attractor satisfies*

$$A = \bigcap_{t \geq 0} X_t(U). \quad (36)$$

Given any $\varepsilon > 0$ there exists $\tau(\varepsilon) > 0$ such that $X_t(U)$ lies in the ε -neighbourhood of A for all $t \geq \tau(\varepsilon)$. In particular, the ω -limit set of every point in $cl(U)$ lies in A . The attractor is an invariant set: if a point belongs to A , then its entire trajectory lies in A for all $t \in (-\infty, +\infty)$. Moreover, every trajectory which entirely (i.e. for all $t \in (-\infty, +\infty)$) lies in $cl(U)$ lies in A , i.e. the attractor is the maximal invariant subset of $cl(U)$.

Proof. As $X_T(cl(U)) \subset U \subset cl(U)$, it follows that all forward iterations $X_{mT} = (X_T)^m$ are well-defined, and $X_{(m+1)T}(cl(U)) = X_{mT}(X_T(cl(U))) \subset X_{mT}(cl(U))$. Thus, the sets $X_{mT}(cl(U))$ form a sequence of nested closed and bounded sets, hence their intersection A is indeed non-empty, bounded and closed. If U is connected, then all these sets in the nested sequence are connected, so A is connected as well in this case. In order to prove (36), note that

$$\bigcap_{t \geq 0} X_t(U) \subseteq \bigcap_{m=0}^{\infty} X_{mT}(cl(U))$$

always, so we only need to show

$$\bigcap_{m=0}^{\infty} X_{mT}(cl(U)) \subseteq \bigcap_{t \geq 0} X_t(U).$$

In words, this means we need to show that if $x \in X_{mT}(cl(U))$ for all $m \geq 0$, then $X_{-t}x \in U$ for all $t \geq 0$. Now, given $t \geq 0$ let $m \geq 0$ be an integer such that $t \leq (m-1)T$. Then, since $X_{-t}x = X_{mT-t}(X_{-mT}x)$ and, by assumption, $X_{-mT}x \in cl(U)$, the inequality $mT - t \geq T$ implies (by the definition of the absorbing domain) that $X_{-t}x \in U$, as required. This proves (36). This formula means $x \in A$ if and only if $X_{-t}x \in U$ for all $t \geq 0$, i.e. A consists of all points whose backward orbits never leave U . Moreover, if $x \in A$, then $X_{-T}x \in U$, which implies $X_t x = X_{t+T}(X_{-T}x) \in U$ for all $t \geq 0$. Thus, the attractor consists of all points whose entire orbit lies in U , for $t \in (-\infty, +\infty)$. This shows that the attractor contains every invariant subset of U . In fact, every invariant subset of $cl(U)$ must lie strictly in U , as $X_T(cl(U)) \subset U$ and the invariant set is invariant with respect to X_T . Hence, the attractor contains every invariant subset of $cl(U)$. As forward orbits of the points in $cl(U)$ stay in U as $t \rightarrow +\infty$, it follows that the ω -limit set of every point of $cl(U)$ lies in $cl(U)$. Since this set is invariant by Theorem 3.3, it proves that it is contained in the attractor. In order to prove a remaining claim of the

theorem, note that

$$X_t(U) = X_{mT}(X_{t-mT}(U)) \subset X_{mT}(U) \quad \text{if } t - mT \geq T.$$

Thus, for any $m \geq 0$ all the sets $X_t(U)$ lie in $X_{mT}(U)$ for all t large enough. Now, take any $\varepsilon > 0$ and let A_ε be the ε -neighbourhood of A . We will prove that $X_t(U) \subset A_\varepsilon$ for all t large enough (and, hence, finish the theorem) if we show that there exists $m \geq 0$ such that $X_{mT}(cl(U)) \subset A_\varepsilon$. In order to prove the latter, assume it is not the case. Then, for every $m \geq 0$ there is a point $x^m \in X_{mT}(cl(U))$ such that $x^m \notin A_\varepsilon$. The sequence x^m is bounded, hence it has a partial limit point, and this limit point stays at a distance of at least ε from A . On the other hand, this limit point belongs to the intersection of all the sets $X_{mT}(cl(U))$ (since these sets are closed and nested), so it must belong to A by (35), a contradiction.

As Theorem 3.4 shows, by restricting the system onto the maximal attractor we will hardly lose any information about the asymptotic behaviour of the orbits entering the absorbing domain under consideration. At the same time, the dimension of the attractor can be much smaller than the dimension of the phase space, hence one hopes that the restriction of the system onto an attractor may greatly simplify the analysis. While the structure of the attractor can be extremely complicated (e.g. beyond human comprehension) in many cases, there are cases when it is sufficiently simple and can be analysed in detail. These cases include systems on a plane (where attractors consist of equilibria, periodic orbits and orbits that connect them) and systems with a global Lyapunov function, where attractors consist of equilibria and connecting orbits only. Before we discuss this, we stress again that the attractor is not defined by the system alone, it depends on the choice of the absorbing domain. For example, in system (28) one can choose any disc $x_1^2 + x_2^2 < R^2$ as an absorbing domain if $R > 1$ (as criterion (34) is fulfilled with the function $F = x_1^2 + x_2^2 - R^2 = r^2 - R^2$, see also (29)). The maximal invariant set inside such domain is the disc $x_1^2 + x_2^2 \leq 1$. This attractor is seemingly too large, e.g. the orbits inside this disc do not attract anything. A more reasonable choice of the absorbing domain would be an annulus $R^2 > x_1^2 + x_2^2 > \rho^2$ with $R > 1 > \rho > 0$. Then the attractor will consist of the only one periodic orbit $x_1^2 + x_2^2 = 1$. This is a better choice of the attractor, as it coincides with the ω -limit sets of all orbits of the system except for one unstable equilibrium at

zero.

A useful tool for establishing the existence of absorbing domains (and analysing the dynamics of systems in general) is given by Lyapunov functions. There can be different versions of them. We start with a *Lyapunov function at infinity*. Let a smooth scalar function $V : R^n \rightarrow R^1$ satisfy the two following properties:

$$V(x) \rightarrow +\infty \quad \text{as} \quad \|x\| \rightarrow +\infty, \quad (37)$$

$$\frac{\partial V}{\partial x} \cdot f(x) \leq 0 \quad \text{for all} \quad \|x\| \geq K \quad \text{for some} \quad K > 0. \quad (38)$$

Then this function is called a Lyapunov function at infinity for the system $\dot{x} = f(x)$.

Theorem 3.5. *If the system $\dot{x} = f(x)$ with a smooth right-hand side $f : R^n \rightarrow R^n$ has a Lyapunov function at infinity, then every forward trajectory is defined and remains bounded for all $t \in [0, +\infty)$.*

Proof. It is enough to prove that the trajectory stays bounded (since the only possibility for a solution to be not defined for all positive t corresponds to $\|x_t\| \rightarrow \infty$ at a finite time). Thus, the proof of the theorem is obtained immediately, as the unboundedness of a trajectory would clearly contradict to (37), (38): by (37) the function $V(x_t)$ must be unbounded if x_t is unbounded, but $V(x_t)$ is, on the other hand, a non-increasing function of time when x_t is large, as

$$\frac{d}{dt}V(x_t) = \frac{\partial V}{\partial x} \cdot \dot{x} = \frac{\partial V}{\partial x} \cdot f(x) \leq 0$$

at $\|x_t\| > K$ by virtue of (38). More formally, note that if a smooth scalar function $\varphi(t)$ is unbounded at $t \geq 0$, then for any R there must exist moment of time such that $\varphi(t) > R$ and $\varphi'(t) > 0$ (if $\varphi'(t) \leq 0$ all the time when $\varphi(t) > R$, then for any time moment t_1 such that $\varphi(t_1) > R$ we obtain for all $t \in [0, t_1]$ that $\varphi(t)$ is nonincreasing and $R < \varphi(t_1) \leq \varphi(t) \leq \varphi(0)$, i.e. φ would be bounded by $\max(R, \varphi(0))$ in this case). If a trajectory x_t is unbounded, then $V(x_t)$ is also unbounded, hence there must be a moment of time for which $\frac{d}{dt}V(x_t) > 0$ and $V(x_t) > R = \max_{\|x\| \leq K} V(x)$. The latter inequality implies $\|x_t\| > K$, hence $\frac{d}{dt}V(x_t) \leq 0$ by (38), a contradiction.

Theorem 3.6. *If the inequality (38) is strict :*

$$\frac{\partial V}{\partial x} \cdot f(x) < 0 \quad \text{for all } \|x\| \geq K \quad \text{for some } K > 0, \quad (39)$$

then there exists $Q \geq 0$ such that every forward orbit of the system must enter the ball $B_Q : \{\|x\| < Q\}$ and remain there forever. Namely, for every initial condition x_0 there exists $T(x_0) \geq 0$ such that $x_t \in B_Q$ for all $t \geq T(x_0)$. Moreover, B_Q is an absorbing domain (i.e. $T(x_0)$ can be taken the same for all $x_0 \in cl(B_R)$).

Proof. First, let us show that any forward orbit must visit the ball B_K , i.e. for any x_0 there exists $T(x_0)$ such that $\|x_t\| < K$ at $t = T(x_0)$. Indeed, if it is not the case, then the orbit stays outside B_K for all positive times, hence its ω -limit set Ω also lies outside B_K (the ω -limit set is non-empty because the orbit does not tend to infinity by Theorem 3.5). By (39), the function $V(x_t)$ is strictly decreasing along every orbit which stays outside B_K :

$$\frac{d}{dt} V(x_t) = \frac{\partial V}{\partial x} \cdot \dot{x} = \frac{\partial V}{\partial x} \cdot f(x) < 0.$$

This function is bounded, so it must have a limit

$$V^* = \lim_{t \rightarrow +\infty} V(x_t).$$

In particular, if $y \in \Omega$, then $y = \lim x_{t_n}$ for some sequence $t_n \rightarrow +\infty$, hence

$$V(y) = \lim V(x_{t_n}) = \lim_{t \rightarrow +\infty} V(x_t) = V^*.$$

So, the value of V is the same ($V = V^*$) for all ω -limit points of x_t . In particular (recall that the ω -limit set Ω is invariant, i.e. it consists of entire trajectories), the function $V(y_t)$ stays constant for any trajectory y_t from Ω . In the other hand, as y_t lies outside B_K , the function $V(y_t)$ must be decreasing strictly, a contradiction.

Now, let us show that every orbit that starts in B_K cannot get outside B_Q where $Q > K$ is chosen such that

$$\min_{\|x\|=Q} V(x) > \max_{\|x\|=K} V(x) \quad (40)$$

(such Q always exists as $V(x) \rightarrow +\infty$ as $\|x\| \rightarrow \infty$). Indeed, if the claim is not true, then we would have an orbit x_t such that for some moments $t_1 < t_2$

$$\|x_{t_1}\| = K, \quad \|x_{t_2}\| = Q \quad \text{and} \quad K < \|x_t\| < Q \quad \text{for all } t \in (t_1, t_2).$$

This is, however, impossible: as $\|x_t\| \geq K$, the function $V(x_t)$ must be strictly decreasing for $t \in (t_1, t_2)$, which would give $V(x_{t_2}) < V(x_{t_1})$, a contradiction to (40).

Thus we have shown that the claim of the theorem is indeed true with $T(x_0) < \infty$ defined as any moment after the orbit x_t enters B_K . To finish the theorem, we must prove that we can take T the same for all $x_0 \in cl(B_Q)$. Note that given any $x_0 \in cl(B_Q)$, if $x_t = X_t(x_0) \in B_K$ for some t , then $X_t(U(x_0)) \subset B_K$ for some sufficiently small neighbourhood U of x_0 (the ball B_K is open, so the point x_t lies at a finite distance from the boundary of B_K , so every point close to x_t also lies inside B_K , and all the points from set $X_t(U(x_0))$ are indeed close to x_t , as $U(x_0)$ is small and X_t is a continuous map for any fixed given t). Thus, we can cover the closed ball $cl(B_Q)$ by open sets $U(x)$ such that for each of these sets there exists a time moment $t(U)$ such that $X_t(U) \subset B_K$. Now recall that the ball $cl(B_Q)$ is compact, so one can choose a finite subcover from any cover of it by open sets. Let $U_i, i = 1, \dots, m$, be this finite subcover. Take $T > \max_{1 \leq i \leq m} t(U_i)$. Since it is the maximum of a finitely many finite numbers, T is finite. By the construction, for any point $x_0 \in cl(B_Q)$ its orbit visited the ball B_K at some moment of time before T . As we have proved above, this implies that none of these orbits can leave B_Q at $t \geq T$, which means that B_Q is an absorbing domain, as required.

As we have seen, the expression $\frac{\partial V}{\partial x} \cdot f(x)$ in condition (39) is the time derivative of $V(x_t)$ if x_t is a solution of the system $\dot{x} = f(x)$. It is important that one can compute derivatives of $V(x_t)$ without explicitly knowing the solution x_t . Thus,

$$\frac{d}{dt}V(x_t) = \frac{\partial V}{\partial x} \cdot f(x), \quad \frac{d^2}{dt^2}V(x_t) = \frac{d}{dt} \left(\frac{\partial V}{\partial x} \Big|_{x=x_t} \cdot f(x_t) \right) = \frac{\partial}{\partial x} \left(\frac{\partial V}{\partial x} \cdot f(x) \right) \cdot f(x), \dots$$

Note also that we do not actually need the strict inequality (39) in Theorem 3.6 to be fulfilled for all x ; we only need the function $V(x_t)$ be strictly increasing. Thus, the claim of Theorem 3.6 holds true, if we change condition (39) to the requirement that the first non-zero time derivative of $V(x_t)$ is strictly negative. Namely, condition (39) in Theorem 3.6 can be replaced by this:

for each x such that $\|x\| > K$ there exists m such that

$$\frac{d^m}{dt^m}V(x) < 0 \quad \text{and} \quad \frac{d^j}{dt^j}V(x) = 0 \quad \text{if} \quad 1 \leq j < m. \quad (41)$$

In the case described by Theorem 3.6, if we take the sufficiently large ball B_Q as an absorbing domain, then the attractor A we obtain will be the maximal bounded invariant set of the system. We may call A the maximal attractor of the system under consideration. It will attract all forward orbits of the system, so if we are able to understand its structure, then we understand the dynamics of the system completely. As we mentioned, this is rarely possible. One of the cases when this problem is solvable is given by the systems which possess a *global Lyapunov function*. Namely, consider a system $\dot{x} = f(x)$ with a smooth right-hand side $f : R^n \rightarrow R^n$. A smooth function V which satisfies condition (37) and, for all x which are not equilibria of the system (i.e. $f(x) \neq 0$), satisfies condition (41) is a global Lyapunov function for the system.

Theorem 3.7. *If a system $\dot{x} = f(x)$ has a global Lyapunov function, then its every orbit is defined and bounded at $t \geq 0$. The ω -limit set of every orbit is the subset of the set of equilibria. If an α -limit set of an orbit is non-empty (i.e. if the orbit does not tend to infinity as $t \rightarrow -\infty$), then its α -limit set is also a subset of the set of equilibria. If all equilibria are isolated, then each ω -limit set or (non-empty) α -limit set consists of only one equilibrium state, i.e. every orbit tends to an equilibrium as $t \rightarrow +\infty$ and either to infinity or to an equilibrium as $t \rightarrow -\infty$. If the set of all equilibria is bounded, then the system has a bounded maximal attractor. The attractor is connected and consists of the equilibria and, possibly, orbits that connect the equilibria.*

Proof. Since the global Lyapunov function is also a Lyapunov function at infinity, Theorem 3.5 guarantees the existence and boundedness of all forward orbits. Condition (41) implies that $V(x_t)$ is a strictly monotone function of time along any orbit x_t which is not an equilibrium. Arguing exactly like in Theorem 3.6 we thus obtain that V must stay constant along any orbit in the ω - or α -limit set of x_t , which immediately imply that each orbit in the ω - or α -limit set of x_t must be an equilibrium. By Theorem 3.3 the ω -limit set must be connected, so it is a connected subset of the set S of the equilibrium states. Hence, if all equilibria are isolated, then any connected component of S is a single point, so the ω -limit set of any orbit consists of only one point in this case; the same for α -limit sets. Finally, if S is bounded, then condition (41) holds everywhere outside a certain ball, i.e. Theorem 3.6. is applied

which guarantees the existence of a connected absorbing domain into which all forward orbits of the system enter eventually. The corresponding attractor A contains all orbits bounded for all $t \in (-\infty, +\infty)$, i.e. it contains the set S , and any other orbit in A must have both ω - and α -limit sets lying in S .

Note that the condition that the equilibria are isolated is not restrictive at all. As we see from the proof, it can be further weakened and replaced by the requirements that every connected subset of the set equilibria consists of a single point. This is true, for example, when the set of equilibria is no more than countable. If the set of equilibria is finite (which is a most typical case, of course), then the condition of the boundedness of this set is also fulfilled. It is very rare, that a system has a continuous set of equilibria, however orbits do not need to tend in this case to one equilibrium each, as the following example shows.

Example. Consider the system

$$\begin{cases} \dot{x}_1 = -x_2(1 - x_1^2 - x_2^2)^2 + x_1(1 - x_1^2 - x_2^2)^3, \\ \dot{x}_2 = x_1(1 - x_1^2 - x_2^2)^2 + x_2(1 - x_1^2 - x_2^2)^3. \end{cases} \quad (42)$$

It has an isolated equilibrium at zero and the whole line $L : x_1^2 + x_2^2 = 1$ filled by equilibria. The function $V = (x_1^2 + x_2^2 - 1)^2$ is a global Lyapunov function:

$$\frac{dV}{dt} = 2(x_1^2 + x_2^2 - 1)(x_1\dot{x}_1 + x_2\dot{x}_2) = -2(x_1^2 + x_2^2)(1 - x_1^2 - x_2^2)^4 < 0$$

outside the set of equilibria. System (42) is obtained from system (28) by multiplication of the right-hand side to the scalar factor $(1 - x_1^2 - x_2^2)^2$. This factor is non-negative at $x_1^2 + x_2^2 \neq 1$, so the phase portraits of the two systems coincide outside the curve L (see Theorem 3.1). Therefore, the ω -limit set of every orbit of system (42) which is not an equilibrium is the whole curve L , like it is for system (28) (this becomes obvious when system (28) is written in the polar coordinates, see (29)).

More examples of Lyapunov functions. A simple example of systems with a global Lyapunov function is provided by the so-called *gradient systems*. These are systems of the form

$$\dot{x} = -V'(x) \quad (43)$$

where $V : R^n \rightarrow R^1$ is at least C^2 -smooth. Obviously, if $V(x) \rightarrow +\infty$ as $\|x\| \rightarrow \infty$ (condition (37)), then it is a global Lyapunov function for system (43):

$$\frac{dV}{dt} = -V'(x)^2 < 0 \quad \text{if} \quad V'(x) \neq 0.$$

A completely different example is given by *Hamiltonian systems*:

$$\begin{cases} \dot{y} = \frac{\partial H(y, p)}{\partial p}, \\ \dot{p} = -\frac{\partial H(y, p)}{\partial y}, \end{cases} \quad (44)$$

where $H(y, p)$ is a given smooth function (at least C^2). Then $H(y, p)$, the Hamiltonian, or the energy of the system, is preserved by the system. Namely,

$$\frac{d}{dt}H(y, p) = \frac{\partial H}{\partial y}\dot{y} + \frac{\partial H}{\partial p}\dot{p} = 0.$$

Thus, H satisfies condition (38), so if

$$H(y, p) \rightarrow +\infty \quad \text{as} \quad \|y\| + \|p\| \rightarrow \infty \quad (45)$$

(condition (37)), then H is a Lyapunov function in the sense of Theorem 3.5. Note that the same energy function H will be a Lyapunov function for the system obtained from (44) by the time reversal (the change of the sign of the right-hand side). Therefore, by Theorem 3.4 (applied to system (44) at $t \geq 0$ and to the time-reversed system at $t \leq 0$) we obtain that condition (45) guarantees the existence and boundedness of solutions of the Hamiltonian system (44) for all $t \in (-\infty, +\infty)$.

For example, any system of the form

$$\begin{cases} \dot{y} = p, \\ \dot{p} = g(y) \end{cases} \quad (46)$$

is Hamiltonian, the energy function here is

$$H = \frac{p^2}{2} + U(y) \quad (47)$$

where $U'(y) = -g(y)$ (note that this is a classical formula for mechanical energy: the first term $\frac{p^2}{2} = \frac{\dot{y}^2}{2}$ is a kinetic energy of a particle of mass 1, and $U(y)$ is the potential energy). As it follows from the explanations above, if the potential $U(y)$ tends to $+\infty$ as $|y| \rightarrow \infty$, then all the solution of system (46) are bounded and globally defined. Note however, that this is not necessarily true if the potential does not grow to infinity. For example, the system on a plane

$$\begin{cases} \dot{y} = p, \\ \dot{p} = y^2, \end{cases} \quad (48)$$

defined by the Hamiltonian $H = \frac{p^2}{2} - \frac{y^3}{3}$ has a solution $y(t) = \frac{6}{(t-1)^2}$ which tends to infinity as $t \rightarrow 1^-$.

By itself, the conservation of energy H means that the level set $H(y, p) = C$ is invariant for the Hamiltonian system (44) for any constant C . Thus, one may reduce the dimension by restricting the system on individual level sets of H . In particular, in the case system (46) is two-dimensional, i.e. $(y, p) \in \mathbb{R}^2$, one finds from (47) that

$$p = \pm \sqrt{2(C - U(y))}$$

with some constant C , which reduces the system to one equation

$$\frac{dy}{dt} = p = \pm \sqrt{2(C - U(y))},$$

which is solved as

$$\pm \int_{y_0}^{y_t} \frac{dy}{\sqrt{2(C - U(y))}} = t - t_0.$$

Returning to the Lyapunov functions, a working method of finding them is just to try something/anything (e.g. a positive definite quadratic form with indeterminate coefficients, or this plus some additional fourth order terms, etc.). Another method is to try to look for a physical interpretation of the system under consideration and guess which function of the state variables can decrease with time due to a relevant physical process, like energy dissipation, entropy increase, etc.. Often, one may try to separate the various terms in the right-hand side into two groups: a ‘‘Hamiltonian core’’, and a ‘‘dissipative perturbation’’ (not necessarily small); then the Hamiltonian of

the Hamiltonian part can be a Lyapunov function for the whole system. Namely, consider a system of the form

$$\begin{cases} \dot{y} = \frac{\partial H(y, p)}{\partial p} + F(y, p), \\ \dot{p} = -\frac{\partial H(y, p)}{\partial y} + G(y, p), \end{cases} \quad (49)$$

where

$$\frac{\partial H}{\partial y} F + \frac{\partial H}{\partial p} G \leq 0 \quad (50)$$

for $\|y\| + \|p\|$ large enough (then we will build a Lyapunov function at infinity) or for all (y, p) (then we will seek for a global Lyapunov function). Condition (50) is equivalent to $\frac{dH}{dt} \leq 0$, so H can be a Lyapunov function if it tends to $+\infty$ as $\|y\| + \|p\|$ grows (see (45)).

For example, consider the system

$$\dot{y} = p, \quad \dot{p} = y - y^3 - p.$$

This is a system of type (49) with $F = 0$, $G = -p$ and $H = \frac{p^2}{2} - \frac{y^2}{2} + \frac{y^4}{4}$. Conditions (45) and (50) are fulfilled. One checks that

$$\frac{dH}{dt} = -p^2 \leq 0.$$

We have

$$\frac{d^2 H}{dt^2} = -2p\dot{p} = -2p(y - y^3 - p),$$

so both $\frac{dH}{dt}$ and $\frac{d^2 H}{dt^2}$ vanish at $p = 0$. The third derivative at $p = 0$ equals to

$$\frac{d^3 H}{dt^3} = -2p \frac{d}{dt}(y - y^3 - p) - 2\dot{p}(y - y^3 - p) = -2(y - y^3)^2,$$

and it is strictly negative outside the equilibria at $y = 0$ or $y = \pm 1$. Thus, condition (41) is fulfilled everywhere except for the equilibrium states $O_1(0, 0)$, $O_2(-1, 0)$ and $O_3(1, 0)$. Hence, H is a global Lyapunov function. Since the number of equilibria is finite, Theorem 3.7 implies that every forward orbit tends to one of the three equilibria. In fact, one may show that there exists only two orbits (except for the point O_1 itself) that tend to O_1 ,

the rest of the orbits tends to O_2 and O_3 (these two points are asymptotically stable, i.e. each of them serves as the ω -limit set for every initial condition from its neighbourhood). The attractor A of the system contains all three equilibria. Since A must be connected, it also contains orbits that connect O_1 with O_2 and O_3 . In fact, there is exactly two more orbits in A : one tends to O_2 as $t \rightarrow +\infty$ and the other tends to O_3 ; as $t \rightarrow -\infty$, these two orbits tend to O_1 . Clearly the above theorems are not enough for establishing this detailed picture of the attractor – to prove the stability of the points $O_{2,3}$ and to determine the number of orbits that tend to the saddle equilibrium O_1 , one needs to perform a theory of local behaviour near the equilibrium states presented in the next lecture.