Correspondances

La Nature est un temple où de vivants piliers
Laissent parfois sortir de confuses paroles;
L'homme y passe à travers des forêts de symboles
Qui l'observent avec des regards familiers.

Comme de longs échos qui de loin se confondent
Dans une ténébreuse et profonde unité,
Vaste comme la nuit et comme la clarté,
Les parfums, les couleurs et les sons se répondent.

Il est des parfums frais comme des chairs d'enfants,
Doux comme les hautbois, verts comme les prairies,
— Et d'autres, corrompus, riches et triomphants,

Ayant l'expansion des choses infinies,
Comme l'ambre, le musc, le benjoin et l'encens,
Qui chantent les transports de l'esprit et des sens.

CHARLES BAUDELAIRE
Les Fleurs du Mal, 1857

Preface

Introductory Remarks

Problems in dynamics have fascinated physical scientists (and mankind in general) for thousands of years. Notable among such problems are those of celestial mechanics, especially the study of the motions of the bodies in the solar system. Newton's attempts to understand and model their observed motions incorporated Kepler's laws and led to his development of the calculus. With this the study of models of dynamical problems as differential equations began.

In spite of the great elegance and simplicity of such equations, the solution of specific problems proved remarkably difficult and engaged the minds of many of the greatest mechanicians and mathematicians of the eighteenth and nineteenth centuries. While a relatively complete theory was developed for linear ordinary differential equations, nonlinear systems remained largely inaccessible, apart from successful applications of perturbation methods to weakly nonlinear problems. Once more, the most famous and impressive applications came in celestial mechanics.

Analysis remained the favored tool for the study of dynamical problems until Poincaré's work in the late-nineteenth century showed that perturbation methods might not yield correct results in all cases, because the series used in such calculations diverged. Poincaré then went on to marry analysis and geometry in his development of a qualitative approach to the study of differential equations.

The modern methods of qualitative analysis of differential equations have their origins in this work (Poincaré [1880, 1890, 1899]) and in the work of Birkhoff [1927], Liapunov [1949], and others of the Russian School: Andronov and co-workers [1937, 1966, 1971, 1973] and Arnold [1973, 1978, 1982]. In the past 20 years there has been an explosion of research. Smale, in a classic paper [1967], outlined a number of outstanding problems and stimulated much of this work. However, until the mid-1970s the new tools were largely in the hands of pure mathematicians, although a number of potential applications had been sketched, notably by Ruelle and Takens [1971], who suggested the importance of "strange attractors" in the study of turbulence.

Over the past few years applications in solid and structural mechanics as well as fluid mechanics have appeared, and there is now widespread interest in the engineering and applied science communities in strange attractors, chaos, and dynamical systems theory. We have written this book primarily for the members of this community, who do not generally have the necessary mathematical background to go directly to the research literature. We see the book primarily as a "user's guide" to a rapidly growing field of knowledge. Consequently we have selected for discussion only those results which we feel are applicable to physical problems, and have generally excluded proofs of theorems which we do not feel to be illustrative of the applicability. Nor have we given the sharpest or best results in all cases, hoping rather to provide a background on which readers may build by direct reference to the research literature.

This is far from a complete treatise on dynamical systems. While it may irritate some specialists in this field, it will, we hope, lead them in the direction of important applications, while at the same time leading engineers and physical scientists in the direction of exciting and useful "abstract" results. In writing for a mixed audience, we have tried to maintain a balance in our statement of results between mathematical pedantry and readability for those without formal mathematical training. This is perhaps most noticable in the way we define terms. While major new terms are defined in the traditional mathematical fashion, i.e., in a separate paragraph signalled by the word **Definition**, we have defined many of the more familiar terms as they occur in the body of the text, identifying them by *italics*. Thus we formally define *structural stability* on p. 39, while we define *asymptotic stability* (of a fixed point) on p. 3. For the reader's convenience, the index contains references to the terms defined in both manners.

The approach to dynamical systems which we adopt is a geometric one. A quick glance will reveal that this book is liberally sprinkled with illustrations—around 200 of them! Throughout we stress the geometrical and topological properties of solutions of differential equations and iterated maps. However, since we also wish to convey the important analytical underpinning of these illustrations, we feel that the numerous exercises, many of which require nontrivial algebraic manipulations and even computer work, are an essential part of the book. Especially in Chapter 2, the direct experience of watching graphical displays of numerical solutions to the systems of differential equations introduced there is extraordinarily valuable in developing an intuitive feeling for their properties. To help the reader along, we have tried to indicate which exercises are fairly routine applications of theory and which require more substantial effort. However, we warn the reader that, towards the end of the book, and especially in Chapter 7, some of our exercises become reasonable material for Ph.D. theses.

We have chosen to concentrate on applications in nonlinear oscillations for three reasons:

(1) There are many important and interesting problems in that field.
(2) It is a fairly mature subject with many texts available on the classical methods for analysis of such problems: the books of Stoker [1950], Minorsky [1962], Hale [1962], Hayashi [1964], or Nayfeh and Mook [1979] are good representatives. The geometrical analysis of two-dimensional systems (free oscillations) is also well established in the books of Lefschetz [1957] and Andronov and co-workers [1966, 1971, 1973].
(3) Most abstract mathematical examples known in dynamical systems theory occur "naturally" in nonlinear oscillator problems.

In this context, the present book should be seen as an attempt to extend the work of Andronov *et al.* [1966] by one dimension. This aim is not as modest as it might seem: as we shall see, the apparently innocent addition of a (small) periodic forcing term $f(t) = f(t + T)$ to a single degree of freedom nonlinear oscillator,

$$\ddot{x} + g(x, \dot{x}) = 0,$$

to yield the three-dimensional system

$$\ddot{x} + g(x, \dot{x}) = f(t),$$

or

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -g(x, y) + f(\theta), \\ \dot{\theta} &= 1, \end{aligned}$$

can introduce an uncountably infinite set of new phenomena, in addition to the fixed points and limit cycles familiar from the planar theory of nonlinear oscillations. This book is devoted to a partial description and understanding of these phenomena.

A somewhat glib observation, which, however, contains some truth, is that the pure mathematician tends to think of some nice (or nasty) property and then construct a dynamical system whose solutions exhibit it. In contrast, the traditional rôle of the applied mathematician or engineer is to take a given system (perhaps a model that he or she has constructed) and find out what its properties are. We mainly adopt the second viewpoint, but our exposition may sometimes seem schizophrenic, since we are applying ideas of the former group to the problems of the latter group. Moreover, we feel strongly that the properties of specific systems cannot be discovered unless one knows what the possibilities are, and these are often revealed only by the general abstract theory. Practice and theory progress best hand-in-hand.

The Contents of This Book

This book is concerned with the application of methods from dynamical systems and bifurcation theories to the study of nonlinear oscillations. The mathematical models we consider are (fairly small) sets of ordinary differential equations and mappings. Many of the results discussed in this book can be extended to infinite-dimensional evolution systems arising from partial differential equations. However, the main ideas are most easily seen in the finite-dimensional context, and it is here that we shall remain. Almost all the methods we describe also generalize to dynamical systems whose phase spaces are differentiable manifolds, but once more, so as not to burden the reader with technicalities, we restrict our exposition to systems with Euclidean phase spaces. However, in the final section of the last chapter we add a few remarks on partial differential equations.

In Chapter 1 we provide a *review* of basic results in the theory of dynamical systems, covering both ordinary differential equations (flows) and discrete mappings. (We concentrate on diffeomorphisms: smooth invertible maps.) We discuss the connection between diffeomorphisms and flows obtained by their Poincaré maps and end with a review of the relatively complete theory of two-dimensional differential equations. Our discussion moves quickly and is quite cursory in places. However, the bulk of this material has been treated in greater detail from the dynamical systems viewpoint in the books of Hirsch and Smale [1974], Irwin [1980], and Palis and de Melo [1982], and from the oscillations viewpoint in the books of Andronov and his co-workers [1966, 1971, 1973] and we refer the reader to these texts for more details. Here the situation is fairly straightforward and solutions generally behave nicely.

Chapter 2 presents four examples from nonlinear oscillations: the famous oscillators of van der Pol [1927] and Duffing [1918], the Lorenz equations [1963], and a bouncing ball problem. We show that the solutions of these problems can be markedly chaotic and that they seem to possess strange attractors: attracting motions which are neither periodic nor even quasiperiodic. The development of this chapter is not systematic, but it provides a preview of the theory developed in the remainder of the book. We recommend that either the reader skim this chapter to gain a general impression before going on to our systematic development of the theory in later chapters, or read it with a microcomputer at hand, so that he can simulate solutions of the model problems we discuss.

We then retreat from the chaos of these examples to muster our forces. Chapter 3 contains a discussion of the methods of local bifurcation theory for flows and maps, including center manifolds and normal forms. Rather different, less geometrical, and more analytical discussions of local bifurcations can be found in the recent books by Iooss and Joseph [1981] and Chow and Hale [1982].

In Chapter 4 we develop the analytical methods of averaging and perturbation theory for the study of periodically forced nonlinear oscillators, and show that they can yield surprising global results. We end this chapter with a brief discussion of chaos and nonintegrability in Hamiltonian systems and the Kolmogorov–Arnold–Moser theory. More complete introductions to this area can be found in Arnold [1978], Lichtenberg and Lieberman [1982], or, for the more mathematically inclined, Abraham and Marsden [1978].

In Chapter 5 we return to chaos, or rather to the close analysis of geometrically defined two-dimensional maps with complicated invariant sets. The famous horseshoe map of Smale is discussed at length, and the method of symbolic dynamics is described and illustrated. A section on one-dimensional (noninvertible) maps is included, and we return to the specific examples of Chapter 2 to provide additional information and illustrate the analytical methods. We end this chapter with a brief discussion of Liapunov exponents and invariant measures for strange attractors.

In Chapter 6 we discuss global homoclinic and heteroclinic bifurcations, bifurcations of one-dimensional maps, and once more illustrate our results with the examples of Chapter 2. Finally, in our discussion of global bifurcations of two-dimensional maps and wild hyperbolic sets, we arrive squarely at one of the present frontiers of the field. We argue that, while the one-dimensional theory is relatively complete (cf. Collet and Eckmann [1980]), the behavior of two-dimensional diffeomorphisms appears to be considerably more complex and is still incompletely understood. We are consequently unable to complete our analysis of the nonlinear oscillators of van der Pol and Duffing, but we are able to give a clear account of much of their behavior and to show precisely what presently obstructs further analysis.

In the final chapter we show how the global bifurcations, discussed previously, reappear in degenerate local bifurcations, and we end with several more models of physical problems which display these rich and beautiful behaviors.

Throughout the book we continually return to specific examples, and we have tried to illustrate even the most abstract results. In our Appendix we give suggestions for further reading. We make no claims for the completeness of our bibliography. We have, however, tried to include references to the bulk of the papers, monographs, lecture notes, and books which have proved useful to ourselves and our colleagues, but we recognize that our biases probably make this a rather eclectic selection.

We have included a glossary of the more important terminology for the convenience of those readers lacking a formal mathematical training.

Finally, we would especially like to acknowledge the encouragement, advice, and gentle criticisms of Bill Langford, Clark Robinson and David Rod, whose careful readings of the manuscript enabled us to make many corrections and improvements.

Nessen MacGiolla Mhuiris, Xuehai Li, Lloyd Sakazata, Rakesh, Kumarswamy Hebbale, and Pat Hollis suffered through the preparation of this manuscript as students in TAM 776 at Cornell, and pointed out many errors almost as quickly as they were made. Edgar Knobloch, Steve Shaw, and David Whitley also read and commented on the manuscript. The comments of these and many other people have helped us to improve this book, and it only remains for each of us to lay the blame for any remaining errors and omissions squarely on the shoulders of the other.

Barbara Boettcher prepared the illustrations from our rough notes and Dolores Pendell deserves more thanks than we can give for her patient typing and retyping of our almost illegible manuscripts.

Finally, we thank our wives and children for their understanding and patience during the production of this addition to our families.

JOHN GUCKENHEIMER
Santa Cruz, Spring 1983

PHILIP HOLMES
Ithaca, Spring 1983

Preface to the Second Printing

The reprinting of this book some $2\frac{1}{2}$ years after its publication has provided us with the opportunity of correcting many minor typographical errors and a few errors of substance. In particular, errors in Section 6.5 in the study of the Šilnikov return map have been corrected, and we have rewritten parts of Sections 7.4 and 7.5 fairly extensively in the light of recent work by Carr, Chow, Cushman, Hale, Sanders, Zholondek, and others on the number of limit cycles and bifurcations in these unfoldings. In the former case the main result is unaffected, but in the latter case some of our intuitions (as well as the incorrect calculations with which we supported them) have proved wrong. We take some comfort in the fact that our naive assertions stimulated some of the work which disproved them.

Although progress in some areas of applied dynamical systems has been rapid, and significant new developments have occurred since the first printing, we have not seen fit to undertake major revisions of the book at this stage, although we have briefly noted some of the developments which bear directly on topics discussed in the book. These comments appear at the end of the book, directly after the Appendix. A complete revision will perhaps be appropriate 5 or 10 years from now. (Anyone wishing to perform it, please contact us!) In the same spirit, we have not attempted to bring the bibliography up to date, although we have added about 75 references, including those mentioned above. References that were in preprint form at the first printing have been updated in cases where the journal of publication is known. In cases in which the publication date of the journal differs from that of the preprint, the journal date is given at the end of the reference. We note that a useful bibliography due to Shiraiwa [1981] has recently been updated (Shiraiwa [1985]); it contains over 4,400 items.

In preparing the revisions we have benefited from the advice and corrections supplied by many readers, including Marty Golubitsky, Kevin Hockett, Fuhua Ling, Wei-Min Liu, Clark Robinson, Jan Sanders, Steven Shaw, Ed Zehnder, and Zhaoxuan Zhu. Professor Ling, of the Shanghai Jiao Tong University, with the help of his students and of Professor Zhu, of Peking University, has prepared a Chinese translation of this book.

JOHN GUCKENHEIMER
PHILIP HOLMES
Ithaca, Fall 1985

Contents

CHAPTER 1

Introduction: Differential Equations and Dynamical Systems

In this introductory chapter we review some basic topics in the theory of ordinary differential equations from the viewpoint of the global geometrical approach which we develop in this book. After recalling the basic existence and uniqueness theorems, we consider the linear, homogeneous, constant coefficient system and then introduce nonlinear and time-dependent systems and concepts such as the Poincaré map and structural stability. We then review some of the better-known results on two-dimensional autonomous systems and end with a statement and sketch of the proof of Peixoto's theorem, an important result which summarizes much of our knowledge of two-dimensional flows.

In the first two sections our review of basic theory and the linear system $\dot{x} = Ax$ is rapid. We assume that the reader is fairly familiar with this material and with the fundamental notions from analysis used in its derivation. Most standard courses in ordinary differential equations deal with these topics, and the material covered in these sections is treated in detail in the books of Hirsch and Smale [1974] and Arnold [1973], for example. We especially recommend the former text as one of the few elementary introductions to the geometric theory of ordinary differential equations. However, most books on differential equations contains versions of the main results.

1.0. Existence and Uniqueness of Solutions

For the purposes of this book, it is generally sufficient to regard a differential equation as a system

$$\frac{dx}{dt} \stackrel{\text{def}}{=} \dot{x} = f(x), \tag{1.0.1}$$

where $x = x(t) \in \mathbb{R}^n$ is a vector valued function of an independent variable (usually time) and $f: U \to \mathbb{R}^n$ is a smooth function defined on some subset $U \subseteq \mathbb{R}^n$. We say that the *vector field* f generates a *flow* $\phi_t: U \to \mathbb{R}^n$, where $\phi_t(x) = \phi(x, t)$ is a smooth function defined for all x in U and t in some interval $I = (a, b) \subseteq \mathbb{R}$, and ϕ satisfies (1.0.1) in the sense that

$$\frac{d}{dt}(\phi(x, t))|_{t=\tau} = f(\phi(x, \tau)) \tag{1.0.2}$$

for all $x \in U$ and $\tau \in I$. We note that (in its domain of definition) ϕ_t satisfies the group properties (i) $\phi_0 = \text{id}$, and (ii) $\phi_{t+s} = \phi_t \circ \phi_s$. Systems of the form (1.0.1), in which the vector field does not contain time explicitly, are called *autonomous.*

Often we are given an initial condition

$$x(0) = x_0 \in U, \tag{1.0.3}$$

in which case we seek a solution $\phi(x_0, t)$ such that

$$\phi(x_0, 0) = x_0. \tag{1.0.4}$$

(We will also sometimes write such a solution as $x(x_0, t)$, or simply $x(t)$.)

In this case $\phi(x_0, \cdot): I \to \mathbb{R}^n$ defines a *solution curve, trajectory,* or *orbit* of the differential equation (1.0.1) *based at* x_0. Since the vector field of the autonomous system (1.0.1) is invariant with respect to translations in time, solutions based at times $t_0 \neq 0$ can always be translated to $t_0 = 0$.

In classical texts on ordinary differential equations, such as Coddington and Levinson [1955], the stress is on individual solution curves and their properties. Here we shall be more concerned with families of such curves, and hence with the global behavior of the flow $\phi_t: U \to \mathbb{R}^n$ defined for (all) points $x \in U$; see Figure 1.0.1. In particular, the concepts of smooth *invariant manifolds* composed of solution curves, discussed in the books of Hartman [1964] and Hale [1969] will be of importance. We will introduce these ideas in the context of linear systems in the next section.

We will not usually need the more general concept of a dynamical system as a flow on a differentiable manifold M arising from a vector field, regarded as a map

$$f: M \to TM,$$

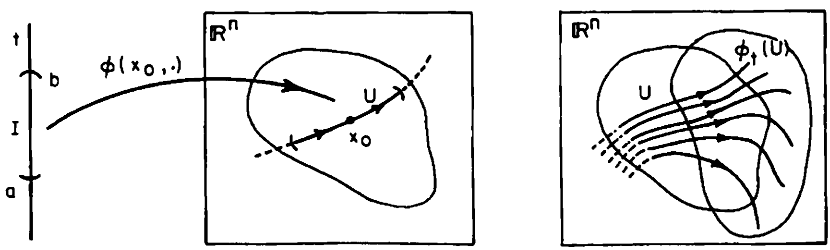


Figure 1.0.1. A solution curve and the flow. (a) The solution curve $\phi_t(x_0)$; (b) the flow ϕ_t.

where TM is the tangent bundle of M. We will therefore not need much from the theory of differential topology. For those interested, Chillingworth's [1976] book provides a good introduction; also see Arnold [1973]. In almost all cases in which we work explicitly with phase spaces which are manifolds, we will have a global coordinate system (a single chart) and we can essentially work in the covering space: i.e., in $\mathbb{R}^n$ modulo some suitable identification, as in the cases of the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and the cylinder $S^1 \times \mathbb{R} = \mathbb{R}^2/\mathbb{Z}$. Such systems typically arise when the vector field f is periodic in (some of) its components. We meet the first such systems in Sections 1.4 and 1.5.

In discussing submanifolds of solutions such as the stable, unstable, and center manifolds, we shall be able to work with copies of real Euclidean spaces defined locally by graphs.

We now state, without proof, the basic local existence and uniqueness theorem (cf. Coddington and Levinson [1955], Hirsch and Smale [1974]):

Theorem 1.0.1. *Let $U \subset \mathbb{R}^n$ be an open subset of real Euclidean space (or of a differentiable manifold M), let $f: U \to \mathbb{R}^n$ be a continuously differentiable (C^1) map and let $x_0 \in U$. Then there is some constant $c > 0$ and a unique solution $\phi(x_0, \cdot): (-c, c) \to U$ satisfying the differential equation $\dot{x} = f(x)$ with initial condition $x(0) = x_0$.*

In fact f need only be (locally) Lipschitz, i.e., $|f(y) - f(x)| \leq K|x - y|$ for some $K < \infty$, where K is called the *Lipschitz constant* for f. Thus we can deal with piecewise linear functions, such as one gets in "stick-slip" friction problems and in the clock problem (cf. Andronov *et al.* [1966], pp. 186ff.).

Intuitively, any solution may leave U after sufficient time. We therefore say that the theorem is only *local*. We can easily construct vector fields $f: U \to \mathbb{R}^n$ such that $x(t)$ leaves any subset $U \subset \mathbb{R}^n$ in a finite time, for example,

$$\dot{x} = 1 + x^2, \tag{1.0.5}$$

which has the general solution $x(t) = \tan(t + c)$. Thus, although there are many equations on non-compact phase spaces (such as $\mathbb{R}^n$) for which solutions do exist globally in time, we cannot assert this in specific cases without further investigation.

Fixed points, also called an *equilibria* or *zeroes*, are an important class of solutions of a differential equation. Fixed points are defined by the vanishing of the vector field $f(x)$: $f(\bar{x}) = 0$. A fixed point $\bar{x}$ is said to be *stable* if a solution $x(t)$ based nearby remains close to $\bar{x}$ for all time, i.e., if for every neighborhood V of $\bar{x}$ in U there is a neighborhood $V_1 \subset V$ such that every solution $x(x_0, t)$ with $x_0 \in V_1$ is defined and lies in V for all $t > 0$. If, in addition, V_1 can be chosen so that $x(t) \to \bar{x}$ as $t \to \infty$ then $\bar{x}$ is said to be *asymptotically stable*. See Figure 1.0.2.

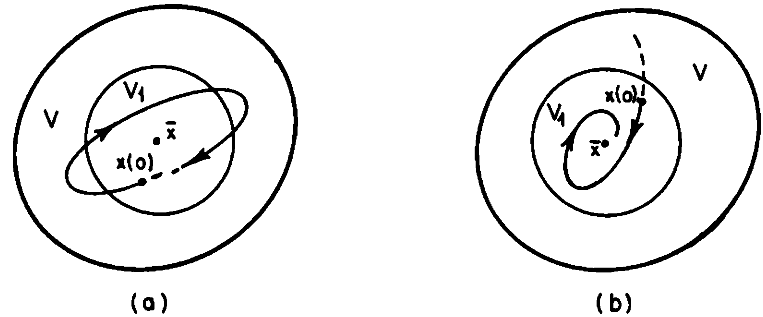


Figure 1.0.2. (a) Stability; (b) asymptotic stability.

EXERCISE 1.0.1. Show that the fixed points of the linear systems

(a) $\dot{x} = y, \dot{y} = -x$;
(b) $\dot{x} = y, \dot{y} = -x - y$,

are both stable. Which one is asymptotically stable? (Routine.)

The type of stability illustrated in Figure 1.0.2(a) is sometimes called *neutral* and is typified by fixed points such as *centers.* Asymptotically stable fixed points are called *sinks.* A fixed point is called *unstable* if it is not stable: *saddle points* and *sources* provide examples of such equilibria. Hirsch and Smale [1974, Chapter 9] give a detailed discussion of stability of fixed points.

The notions of stability defined above are *local* in nature: they concern only the behavior of solutions near the fixed point $\bar{x}$. Even if such solutions remain bounded for all time, other solutions may not exist globally.

EXERCISE 1.0.2. Find the fixed points for the equation $\dot{x} = -x + x^2$ and discuss their stability. Show that this equation has solutions which exist for all time as well as solutions which become unbounded in finite time. (Solving this equation is straightforward, but this interpretation of the behavior of solutions may be new to you.)

Often a Liapunov function approach suffices to show that an energy-like quantity decreases for $|x|$ sufficiently large, so that $x(t)$ remains bounded for all t and all (bounded) initial conditions $x(0)$. Since it is so useful, we outline the method here for completeness. For more details, see Hirsch and Smale [1974, §9.3] or LaSalle and Lefschetz [1961]. The method relies on finding a positive definite function $V: U \to \mathbb{R}$, called the *Liapunov function*, which decreases along solution curves of the differential equation:

Theorem 1.0.2 (Hirsch and Smale [1974], pp. 192ff.). *Let $\bar{x}$ be a fixed point for* (1.0.1) *and $V: W \to \mathbb{R}$ be a differentiable function defined on some neighborhood $W \subseteq U$ of $\bar{x}$ such that:*

(i) $V(\bar{x}) = 0$ *and* $V(x) > 0$ *if* $x \neq \bar{x}$; *and*
(ii) $\dot{V}(x) \leq 0$ *in* $W - \{\bar{x}\}$.

Then $\bar{x}$ is stable. Moreover, if

(iii) $\dot{V}(x) < 0$ *in* $W - \{\bar{x}\}$;

then $\bar{x}$ is asymptotically stable.

Here

$$\dot{V} = \sum_{j=1}^{n} \frac{\partial V}{\partial x_j} \dot{x}_j = \sum_{j=1}^{n} \frac{\partial V}{\partial x_j} f_j(x)$$

is the derivative of V along solution curves of (1.0.1).

If we can choose $W(=U) = \mathbb{R}^n$ in case (iii), then x is said to be *globally asymptotically stable*, and we can conclude that all solutions remain bounded and in fact approach $\bar{x}$ as $t \to \infty$. Thus the stability of equilibria and boundedness of solutions can be tested without actually solving the differential equation. There are, however, no general methods for finding suitable Liapunov functions, although in mechanical problems the energy is often a good candidate.

EXAMPLE. Consider the motion of a particle of mass m attached to a spring of stiffness $k(x + x^3)$, $k > 0$, where x is displacement. The differential equation governing the system is

$$m\ddot{x} + k(x + x^3) = 0, \tag{1.0.6}$$

or, letting $\dot{x} = y$

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\frac{k}{m}(x + x^3). \end{aligned} \tag{1.0.7}$$

The associated total energy of the system is

$$E(x, y) = \frac{my^2}{2} + k\left(\frac{x^2}{2} + \frac{x^4}{4}\right). \tag{1.0.8}$$

We note that $E(x, y)$ provides a Liapunov function for (1.0.7), since $E(0, 0) = 0$ at the (unique) equilibrium $(x, y) = (0, 0)$ and $E(x, y) > 0$ for $(x, y) \neq (0, 0)$. Moreover, we have

$$\begin{aligned} \dot{E} &= my\dot{y} + k(x + x^3)\dot{x} \\ &= -ky(x + x^3) + k(x + x^3)y \equiv 0; \end{aligned} \tag{1.0.9}$$

thus $(x, y) = (0, 0)$ is (neutrally) stable. If we add some damping $\alpha > 0$, to the system, so that the equation of motion becomes

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\frac{k}{m}(x + x^3) - \alpha y, \end{aligned} \tag{1.0.10}$$

then the same Liapunov function yields

$$\dot{E} = -\alpha m y^2, \tag{1.0.11}$$

which is negative for all $(x, y) \neq (0, 0)$ except on the x-axis. We therefore modify the Liapunov function slightly to

$$V(x, y) = \frac{my^2}{2} + k\left(\frac{x^2}{2} + \frac{x^4}{4}\right) + \beta\left(xy + \frac{\alpha x^2}{2}\right), \tag{1.0.12}$$

so that

$$\begin{aligned} \dot{V} &= my\dot{y} + k(x + x^3)\dot{x} + \beta(\dot{x}y + x\dot{y} + \alpha x\dot{x}) \\ &= (my + \beta x)\left(-\frac{k}{m}(x + x^3) - \alpha y\right) + k(x + x^3)y + \beta y^2 + \alpha\beta xy \\ &= -\beta\frac{k}{m}(x^2 + x^4) - (\alpha m - \beta)y^2. \end{aligned} \tag{1.0.13}$$

If we choose β sufficiently small, V remains positive definite and $\dot{V}$ is strictly negative for *all* $(x, y) \neq (0, 0)$. Thus $(0, 0)$ is globally asymptotically stable for $\alpha > 0$.

In differentiating V along solution curves we are trying to verify that all solutions cross the level curves of V "inwards." A sketch of the level curves of E and the modified function V for this example show that those of V are slightly tilted, so that the vector field is nowhere tangent to them, whereas, even with damping present, the vector field is tangent to E = constant on $y = 0$ (Figure 1.0.3).

EXERCISE 1.0.3. Using the Liapunov function $V = \frac{1}{2}(x^2 + \sigma y^2 + \sigma z^2)$, obtain conditions on σ, ρ, and β sufficient for global asymptotic stability of the origin $(x, y, z) = (0, 0, 0)$ in the Lorenz equations

$$\dot{x} = \sigma(y - x); \qquad \dot{y} = \rho x - y - xz; \qquad \dot{z} = -\beta z + xy; \qquad \sigma, \beta > 0.$$

Are your conditions also necessary?

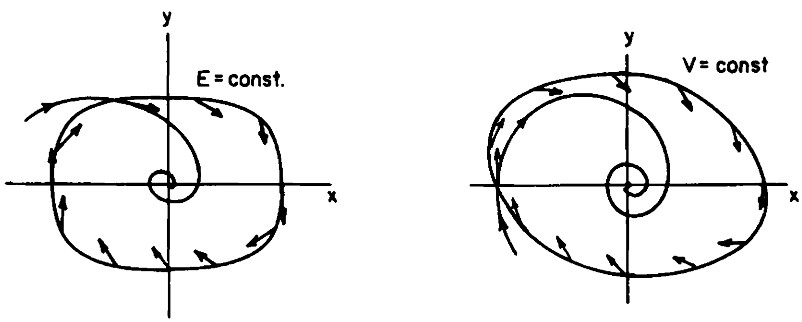


Figure 1.0.3. Level curves of the Liapunov functions E and V and the vector field of equations (1.0.10).

For problems with multiple equilibria, local Liapunov functions can be sought, or one can attempt to find a compact hypersurface $S \subset \mathbb{R}^n$ such that the vector field is directed everywhere inward on S. If such a surface exists, then any solution starting on or inside S can never leave the interior of S and thus must remain bounded for all time. We use this approach in several examples later in this book.

The local existence theorem (Theorem 1.0.1) becomes global in all cases when we work on *compact* manifolds M instead of open spaces like $\mathbb{R}^n$:

Theorem 1.0.3 (Chillingworth [1976], pp. 187–188). *The differential equation $\dot{x} = f(x)$, $x \in M$, with M compact, and $f \in C^1$, has solution curves defined for all $t \in \mathbb{R}$.*

Thus flows on spheres and tori are globally defined, since there is no way in which solutions can escape from such manifolds.

The local theorem can be extended to show that solutions depend in a "nice" way on initial conditions (cf. Coddington and Levinson [1955], Hirsch and Smale [1974]):

Theorem 1.0.4. *Let $U \subseteq \mathbb{R}^n$ be open and suppose $f: U \to \mathbb{R}^n$ has a Lipschitz constant K. Let $y(t)$, $z(t)$ be solutions to $\dot{x} = f(x)$ on the closed interval $[t_0, t_1]$. Then, for all $t \in [t_0, t_1]$,*

$$|y(t) - z(t)| \leq |y(t_0) - z(t_0)| e^{K(t-t_0)}.$$

We note that this continuous dependence does not preclude the exponentially fast separation of solutions typical of the chaotic flows to be encountered in subsequent chapters, cf. Figure 1.0.4.

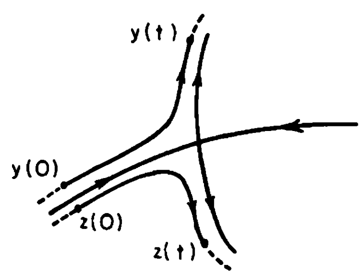


Figure 1.0.4. Exponential separation of neighboring solutions near a saddle point.

Exercise 1.0.4. Which of the following systems give rise to globally defined flows?

(a) $\dot{x} = x$, $x \in \mathbb{R}$;
(b) $\dot{x} = x^2$, $x \in \mathbb{R}$;
(c) $\dot{x} = 2 + \cos x$, $x \in \mathbb{R}$;
(d) $\dot{x} = \cos^2 x$, $x \in S^1$;
(e) $\dot{x} = -x^3$, $x \in \mathbb{R}$;
(f) $\dot{x} = Ax$, $x \in \mathbb{R}^n$, where A is an $n \times n$ constant matrix.

(You can integrate all of these directly, but will need linear algebra, reviewed in the next section, for the last one.)

EXERCISE 1.0.5. Show that $\dot{x} = x^{2/3}$ does not have unique solutions for all initial points $x(0)$. Under what conditions are solutions unique? (This example is an old favorite in classical texts on differential equations.)

1.1. The Linear System $\dot{x} = Ax$

We first review some features of the linear system

$$\frac{dx}{dt} \stackrel{\text{def}}{=} \dot{x} = Ax, \qquad x \in \mathbb{R}^n, \tag{1.1.1}$$

where A is an $n \times n$ matrix with constant coefficients. For more information and background see a standard introductory text on differential equations such as Braun [1978]; for a more detailed review of the linear algebra from the viewpoint of dynamical systems theory, Hirsch and Smale [1974] or Arnold [1973] are recommended.

By a solution of (1.1.1) we mean a vector valued function $x(x_0, t)$ depending on time t and the initial condition

$$x(0) = x_0; \tag{1.1.2}$$

$x(x_0, t)$ is thus a solution of the initial value problem (1.1.1)–(1.1.2). In terms of the flow ϕ_t, we have $x(x_0, t) \equiv \phi_t(x_0)$. Theorem 1.0.4 guarantees that the solution $x(x_0, t)$ of the linear system is defined for all $t \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$. Note that such *global* existence in time does not generally hold for nonlinear systems, as we have already seen. However, no such problems occur for (1.1.1), the solution of which is given by

$$x(x_0, t) = e^{tA}x_0, \tag{1.1.3}$$

where e^{tA} is the $n \times n$ matrix obtained by exponentiating A. We will see how e^{tA} can be calculated most conveniently in a moment, but first note that it is defined by the convergent series

$$e^{tA} = \left[I + tA + \frac{t^2}{2!}A^2 + \cdots + \frac{t^n}{n!}A^n + \cdots\right]. \tag{1.1.4}$$

We leave it to the reader to make use of (1.1.4) to prove that (1.1.3) does indeed solve (1.1.1)–(1.1.2).

A *general* solution to (1.1.1) can be obtained by linear superposition of n linearly independent solutions $\{x^1(t), \ldots, x^n(t)\}$:

$$x(t) = \sum_{j=1}^{n} c_j x^j(t), \tag{1.1.5}$$

where the n unknown constants c_j are to be determined by initial conditions.

If A has n linearly independent eigenvectors v^j, $j = 1, \dots, n$, then we may take as a basis for the space of solutions the vector valued functions

$$x^j(t) = e^{\lambda_j t} v^j, \tag{1.1.6}$$

where λ_j is the eigenvalue associated with v^j. For complex eigenvalues without multiplicity, λ_j, $\bar{\lambda}_j = \alpha_j \pm i\beta_j$, having eigenvectors $v^R \pm iv^I$, we may take

$$\begin{aligned} x^j &= e^{\alpha_j t}(v^R \cos \beta t - v^I \sin \beta t), \\ x^{j+1} &= e^{\alpha_j t}(v^R \sin \beta t + v^I \cos \beta t), \end{aligned} \tag{1.1.7}$$

as the associated pair of (real) linearly independent solutions. When there are repeated eigenvalues and less than n eigenvectors, then one generates the *generalized* eigenvectors as described by Braun [1978], for example. Again one obtains a set of n linearly independent solutions. We denote the *fundamental solution matrix* having these n solutions for its columns as

$$X(t) = [x^1(t), \dots, x^n(t)]. \tag{1.1.8}$$

The columns $x^j(t)$, $j = 1, \dots, n$ of $X(t)$ form a basis for the space of solutions of (1.1.1). It is easy to show that

$$e^{tA} = X(t)X(0)^{-1}; \tag{1.1.9}$$

we again leave the proof as an exercise.

Exercise 1.1.1. Find e^{tA} for

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then solve $\dot{x} = Ax$ for initial conditions

$$x_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}.$$

What do you notice about the last two solutions? Look carefully at the geometry of the solutions and eigenspaces.

Equation (1.1.1) may also be solved by first finding an invertible transformation T which diagonalizes A or at least puts it into Jordan normal form (if there are repeated eigenvalues). Equation (1.1.1) becomes

$$\dot{y} = Jy, \tag{1.1.10}$$

where $J = T^{-1}AT$ and $x = Ty$. Equation (1.1.10) is easy to work with, but since the columns of T are the (generalized) eigenvectors of A, just as much work is required as in the former method. The exponential e^{tA} may be computed as

$$e^{tA} = Te^{tJ}T^{-1} \tag{1.1.11}$$

(cf. Hirsch and Smale [1974], pp. 84–87), where exponentials are evaluated for the three 2×2 Jordan form matrices:

$$
\begin{aligned}
A &= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, & e^{tA} &= \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}; \\
A &= \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}, & e^{tA} &= e^{\alpha t}\begin{bmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{bmatrix}; \\
A &= \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}, & e^{tA} &= e^{\lambda t}\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}.
\end{aligned}
\tag{1.1.12}
$$

We also note that if v^j is an eigenvector belonging to a real eigenvalue λ_j of A, then v^j is also an eigenvector belonging to the eigenvalue e^{λ_j} of e^A. Moreover, if $\operatorname{span}\{\operatorname{Re}(v^j), \operatorname{Im}(v^j)\}$ is an eigenspace belonging to a complex conjugate pair λ_j, $\bar{\lambda}_j$ of eigenvalues, then it is also an eigenspace belonging to e^{λ_j}, $e^{\bar{\lambda}_j}$.

1.2. Flows and Invariant Subspaces

The matrix e^{tA} can be regarded as a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$: given any point x_0 in $\mathbb{R}^n$, $x(x_0, t) = e^{tA}x_0$ is the point at which the solution based at x_0 lies after time t. The operator e^{tA} hence contains *global* information on the set of *all* solutions of (1.1.1), since the formula (1.1.3) holds for all points $x_0 \in \mathbb{R}^n$. As in the general case, described in Section 1.0, we say that e^{tA} defines a *flow* on $\mathbb{R}^n$ and that this flow (or "phase flow") is *generated by* the vector field Ax defined on $\mathbb{R}^n$: e^{tA} is our first specific example of a flow ϕ_t.

The flow $e^{tA}: \mathbb{R}^n \to \mathbb{R}^n$ can be thought of as the set of all solutions to (1.1.1). In this set certain solutions play a special role; those which lie in the linear subspaces spanned by the eigenvectors. These subspaces are *invariant* under e^{tA}, in particular, if v^j is a (real) eigenvector of A, and hence of e^{tA}, then a solution based at a point $c_j v^j \in \mathbb{R}^n$ remains on $\operatorname{span}\{v^j\}$ for all time; in fact

$$x(cv^j, t) = cv^j e^{\lambda_j t}. \tag{1.2.1}$$

Similarly, the (two-dimensional) subspace spanned by $\operatorname{Re}\{v^j\}$, $\operatorname{Im}\{v^j\}$, when v^j is a complex eigenvector, is invariant under e^{tA}. In short, the eigenspaces of A are invariant subspaces for the flow. It is worth returning to Exercise 1.1.1 in the light of this discussion.

We divide the subspaces spanned by the eigenvectors into three classes:

$$
\begin{aligned}
&\text{the } \textit{stable subspace}, \ E^s = \operatorname{span}\{v^1, \ldots, v^{n_s}\}, \\
&\text{the } \textit{unstable subspace}, \ E^u = \operatorname{span}\{u^1, \ldots, u^{n_u}\}, \\
&\text{the } \textit{center subspace}, \ E^c = \operatorname{span}\{w^1, \ldots, w^{n_c}\},
\end{aligned}
$$

where $v^1, \ldots, v^{n_s}$ are the n_s (generalized) eigenvectors whose eigenvalues have negative real parts, $u^1, \ldots, u^{n_u}$ are the n_u (generalized) eigenvectors whose eigenvalues have positive real parts and $w^1, \ldots, w^{n_c}$ are those whose eigenvalues have zero real parts. Of course, $n_s + n_c + n_u = n$. The names reflect the facts that solutions lying on E^s are characterized by exponential decay (either monotonic or oscillatory), those lying in E^u by exponential growth, and those lying in E^c by neither. In the absence of multiple eigenvalues, these latter either oscillate at constant amplitude (if $\lambda, \bar{\lambda} = \pm i\beta$) or remain constant (if $\lambda = 0$). A schematic picture appears in Figure 1.2.1, with two specific examples.

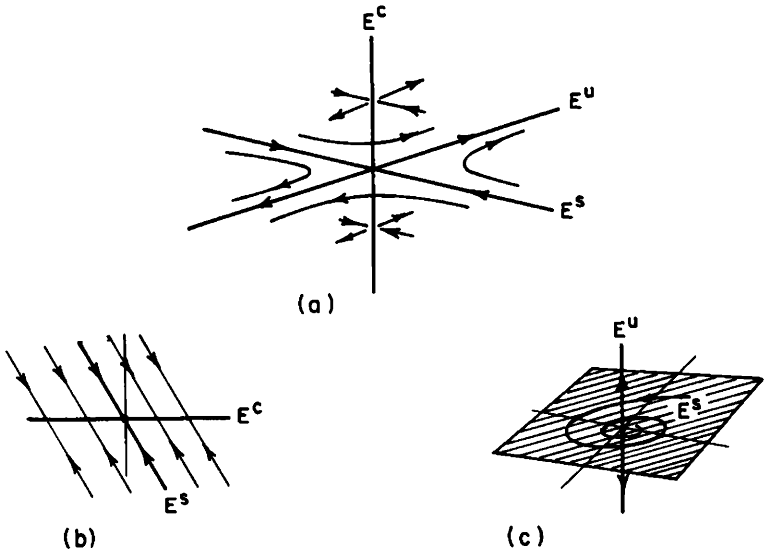


Figure 1.2.1. Invariant subspaces. (a) The three subspaces; (b)

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -4 \end{bmatrix},$$

($E^s = \text{span}(1, -4)$, $E^c = \text{span}(1, 0)$, $E^u = \varnothing$); (c)

$$A = \begin{bmatrix} -1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

($E^s = \text{span}\{(1, 0, 0), (1, 1, 0)\}$, $E^c = \varnothing$, $E^u = (0, 0, 1)$).

When there are multiple eigenvalues for which algebraic and geometric multiplicities differ, then one may have growth of solutions in E^c, as the following exercise demonstrates:

Exercise 1.2.1. Find general solutions for the linear system $\dot{x} = Ax$, $x \in \mathbb{R}^2$ with

(a) $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; (b) $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

For more information on the flow e^{tA}, and for a complete classification of two- and three-dimensional systems, the reader is referred to Hirsch and Smale [1974] or Arnold [1973].

1.3. The Nonlinear System $\dot{x} = f(x)$

We must start by admitting that almost nothing beyond general statements can be made about most nonlinear systems. In the remainder of this book we will meet some of the delights and horrors of such systems, but the reader must bear in mind that the line of attack we develop in this text is only one, and that any other tool in the workshop of applied mathematics, including numerical integration, perturbation methods, and asymptotic analysis, can and should be brought to bear on a specific problem.

We recall that the basic existence–uniqueness theorem for ordinary differential equations, given in Section 1.0, implies that, for smooth functions* $f(x)$, the solution to the initial value problem

$$\dot{x} = f(x); \qquad x \in \mathbb{R}^n, \qquad x(0) = x_0 \tag{1.3.1}$$

is defined at least in some neighborhood $t \in (-c, c)$ of $t = 0$. Thus a *local* flow $\phi_t: \mathbb{R}^n \to \mathbb{R}^n$ is defined by $\phi_t(x_0) = x(t, x_0)$ in a manner analogous to that in the linear case, although of course we cannot give a general formula like e^{tA}.

A good place to start the study of the nonlinear system $\dot{x} = f(x)$ is by finding the *zeros* of f or the *fixed points* of (1.3.1). These are also referred to as *zeros, equilibria,* or *stationary solutions.* Even this may be a formidable task, although in most of our examples it will not be. Suppose then that we have a fixed point $\bar{x}$, so that $f(\bar{x}) = 0$, and we wish to characterize the behavior of solutions near $\bar{x}$. We do this by linearizing (1.3.1) at $\bar{x}$, that is, by studying the linear system

$$\dot{\xi} = Df(\bar{x})\xi, \qquad \xi \in \mathbb{R}^n, \tag{1.3.2}$$

where $Df = [\partial f_i/\partial x_j]$ is the Jacobian matrix of first partial derivatives of the function $f = (f_1(x_1, \ldots, x_n), f_2(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n))^T$ (T denotes transpose), and $x = \bar{x} + \xi$, $|\xi| \ll 1$. Since (1.3.2) is just a linear system of the form (1.1.1), we can do this easily. In particular, the linearized flow map $D\phi_t(\bar{x})\xi$ arising from (1.3.1) at a fixed point $\bar{x}$ is obtained from (1.3.2) by integration:

$$D\phi_t(\bar{x})\xi = e^{tDf(\bar{x})}\xi. \tag{1.3.3}$$

The important question is, what can we say about the solutions of (1.3.1) based on our knowledge of (1.3.2)? The answer is provided by two fundamental results of dynamical systems theory which we give below, and may be

* Throughout this book by *smooth* we generally mean C^∞, unless stated otherwise. We note that we do not always concentrate upon obtaining optimal smoothness in our results.

summed up by saying that local behavior (for $|\xi|$ small) does carry over in certain "nice" cases.

Theorem 1.3.1 (Hartman–Grobman). *If $Df(\bar{x})$ has no zero or purely imaginary eigenvalues then there is a homeomorphism h defined on some neighborhood U of $\bar{x}$ in $\mathbb{R}^n$ locally taking orbits of the nonlinear flow ϕ_t of* (1.3.1), *to those of the linear flow $e^{tDf(\bar{x})}$ of* (1.3.2). *The homeomorphism preserves the sense of orbits and can also be chosen to preserve parametrization by time.*

A more delicate situation in which the nonlinear and linear flows are related via *diffeomorphisms* (Sternberg's theorem) requires certain nonresonance conditions among the eigenvalues of $Df(\bar{x})$. We shall not consider this here, but see the discussion of normal forms in Chapter 3.

When $Df(\bar{x})$ has no eigenvalues with zero real part, $\bar{x}$ is called a *hyperbolic* or *nondegenerate* fixed point and the asymptotic behavior of solutions near it (and hence its stability type) is determined by the linearization. If any one of the eigenvalues has zero real part, then stability cannot be determined by linearization, as the example

$$\ddot{x} + \varepsilon x^2\dot{x} + x = 0 \tag{1.3.4}$$

shows. Rewritten as a system (with $x_1 = x$, $x_2 = \dot{x}$),

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} - \varepsilon\begin{pmatrix}0\\ x_1^2x_2\end{pmatrix}, \tag{1.3.5}$$

we find eigenvalues $\lambda, \bar{\lambda} = \pm i$. However, unless $\varepsilon = 0$, the fixed point $(x_1, x_2) = (0, 0)$ is not a center, as in the linear system, but a nonhyperbolic or weak attracting spiral sink if $\varepsilon > 0$, and a repelling source if $\varepsilon < 0$.

Exercise 1.3.1. Verify that $(x_1, x_2) = (0, 0)$ is globally asymptotically stable for (1.3.5) when $\varepsilon > 0$. (Use a Liapunov function approach, cf. equation (1.0.10).)

Before the next result we need a couple of definitions. We define the *local stable and unstable manifolds* of $\bar{x}$, $W^s_{\text{loc}}(\bar{x})$, $W^u_{\text{loc}}(\bar{x})$ as follows

$$\begin{aligned} W^s_{\text{loc}}(\bar{x}) &= \{x \in U \mid \phi_t(x) \to \bar{x} \text{ as } t \to \infty, \text{ and } \phi_t(x) \in U \text{ for all } t \geq 0\},\\ W^u_{\text{loc}}(\bar{x}) &= \{x \in U \mid \phi_t(x) \to \bar{x} \text{ as } t \to -\infty, \text{ and } \phi_t(x) \in U \text{ for all } t \leq 0\}, \end{aligned} \tag{1.3.6}$$

where $U \subset \mathbb{R}^n$ is a neighborhood of the fixed point $\bar{x}$. The invariant manifolds W^s_{loc} and W^u_{loc} provide nonlinear analogues of the flat stable and unstable eigenspaces E^s, E^u of the linear problem (1.3.2). The next result tells us that W^s_{loc} and W^u_{loc} are in fact tangent to E^s, E^u at $\bar{x}$.

Theorem 1.3.2 (Stable Manifold Theorem for a Fixed Point). *Suppose that $\dot{x} = f(x)$ has a hyperbolic fixed point $\bar{x}$. Then there exist local stable and unstable manifolds $W^s_{\text{loc}}(\bar{x})$, $W^u_{\text{loc}}(\bar{x})$, of the same dimensions n_s, n_u as those of* *the eigenspaces E^s, E^u of the linearized system* (1.3.2), *and tangent to E^s, E^u at $\bar{x}$. $W^s_{loc}(\bar{x})$, $W^u_{loc}(\bar{x})$ are as smooth as the function f.*

For proofs of these two theorems see, for example, Hartman [1964] and Carr [1981], or, for a more modern treatment, Nitecki [1971], Shub [1978], or Irwin [1980]. Hirsch *et al.* [1977] contains a more general result. The two results may be illustrated as in Figure 1.3.1.

Note that we have not yet said anything about a center manifold, tangent to E^c at $\bar{x}$, and have, in fact, confined ourselves to hyperbolic cases in which E^c does not exist. We shall consider nonhyperbolic cases later when we deal with bifurcation theory in Chapter 3.

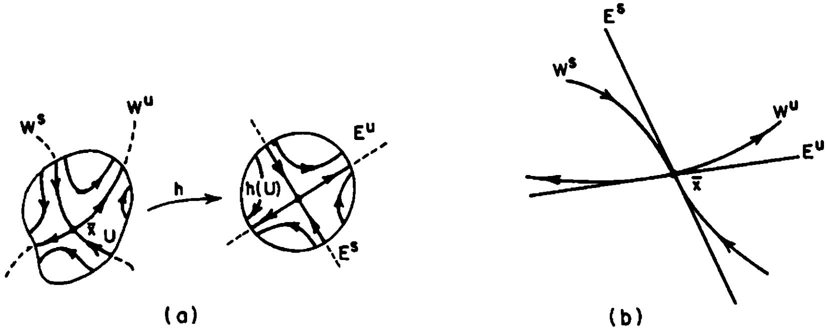


Figure 1.3.1. Linearization and invariant subspaces. (a) Hartman's theorem; (b) local stable and unstable manifolds.

The local invariant manifolds W^s_{loc}, W^u_{loc} have global analogues W^s, W^u, obtained by letting points in W^s_{loc} flow backwards in time and those in W^u_{loc} flow forwards:

$$\begin{aligned} W^s(\bar{x}) &= \bigcup_{t \le 0} \phi_t(W^s_{loc}(\bar{x})), \\ W^u(x) &= \bigcup_{t \ge 0} \phi_t(W^u_{loc}(\bar{x})). \end{aligned} \tag{1.3.7}$$

Existence and uniqueness of solutions of (1.3.1) ensure that two stable (or unstable) manifolds of distinct fixed points $\bar{x}^1$, $\bar{x}^2$ cannot intersect, nor can $W^s(\bar{x})$ (or $W^u(\bar{x})$) intersect itself. However, intersections of stable and unstable manifolds of distinct fixed points or the same fixed point can occur and, in fact, are a source of much of the complex behavior found in dynamical systems. The global stable and unstable manifolds need not be embedded submanifolds of $\mathbb{R}^n$ since they may wind around in a complex manner, approaching themselves arbitrarily closely. We give an example of a map possessing such a structure in the next section.

To illustrate the ideas of this section, we consider a simple system on the plane:

$$\begin{aligned} \dot{x} &= x, \\ \dot{y} &= -y + x^2, \end{aligned} \tag{1.3.8}$$

which has a unique fixed point at the origin. For the linearized system we have the following invariant subspaces:

$$\begin{aligned} E^s &= \{(x, y) \in \mathbb{R}^2 \mid x = 0\}, \\ E^u &= \{(x, y) \in \mathbb{R}^2 \mid y = 0\}. \end{aligned} \tag{1.3.9}$$

In this case we can integrate the nonlinear system exactly. Rather than obtaining a solution in the form $(x(t), y(t))$, we rewrite (1.3.8) as a (linear) first-order system by eliminating time:

$$\frac{dy}{dx} = \frac{-y}{x} + x. \tag{1.3.10}$$

This can be integrated directly to obtain the family of solution curves

$$y(x) = \frac{x^2}{3} + \frac{c}{x}, \tag{1.3.11}$$

where c is a constant determined by initial conditions. Now Theorem 1.3.1, together with (1.3.9), implies that $W^u_{\text{loc}}(0, 0)$ can be represented as a graph $y = h(x)$ with $h(0) = h'(0) = 0$, since W^u_{loc} is tangent to E^u at $(0, 0)$. Thus $c = 0$ in (1.3.10) and we have

$$W^u(0, 0) = \left\{(x, y) \in \mathbb{R}^2 \mid y = \frac{x^2}{3}\right\}. \tag{1.3.12}$$

Finally, noting that if $x(0) = 0$, then $\dot{x} \equiv 0$, and hence $x(t) \equiv 0$, we see that $W^s(0, 0) \equiv E^s$. Note that, for this example, we have found the global manifolds; see Figure 1.3.2.

EXERCISE 1.3.2. Find and classify the fixed points of the following systems by linearizing about the fixed points (i.e., find eigenvalues and eigenvectors and sketch the local flows). Start by rewriting the second-order equations as first-order systems:

(a) $\ddot{x} + \varepsilon\dot{x} - x + x^3 = 0$;
(b) $\ddot{x} + \sin x = 0$;
(c) $\ddot{x} + \varepsilon\dot{x}^2 + \sin x = 0$;
(d) $\dot{x} = -x + x^2, \dot{y} = x + y$;
(e) $\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = 0$;

(where ε appears let $\varepsilon < 0, \varepsilon = 0, \varepsilon > 0$). Can you calculate (or guess) the global structure of stable and unstable manifolds in any of these cases? (This last part is quite hard if you are not familiar with the tricks outlined later in this chapter.)

It is well known that nonlinear systems possess limit sets other than fixed points; for example, *closed or periodic orbits* frequently occur. A periodic solution is one for which there exists $0 < T < \infty$ such that $x(t) = x(t + T)$ for all t. We consider the stability of such orbits in Section 1.5, but note here that they have stable and unstable manifolds just as do fixed points.

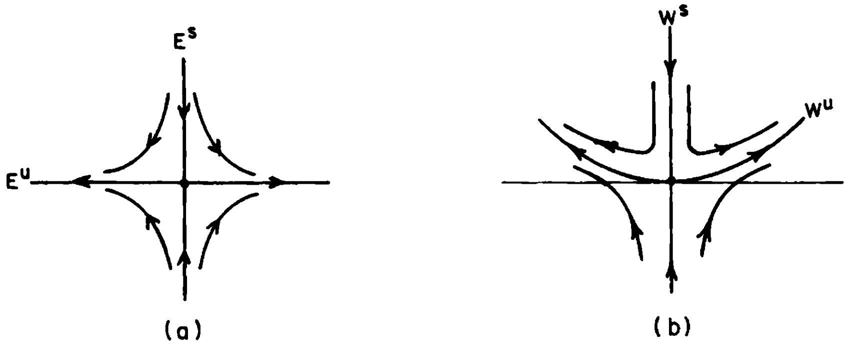


Figure 1.3.2. Stable and unstable manifolds for equation (1.3.8). (a) the linear system; (b) the nonlinear system.

Let γ denote the closed orbit and U be some neighborhood of γ; then we define

$$W^s_{\text{loc}}(\gamma) = \{x \in U \mid |\phi_t(x) - \gamma| \to 0 \text{ as } t \to \infty, \text{ and } \phi_t(x) \in U \text{ for } t \geq 0\},$$
$$W^u_{\text{loc}}(\gamma) = \{x \in U \mid |\phi_t(x) - \gamma| \to 0 \text{ as } t \to -\infty, \text{ and } \phi_t(x) \in U \text{ for } t \leq 0\}.$$

Examples will follow in the sections below.

1.4. Linear and Nonlinear Maps

We have seen how the linear system (1.1.1) gives rise to flow map $e^{tA}: \mathbb{R}^n \to \mathbb{R}^n$, when e^{tA} is an $n \times n$ matrix. For fixed $t = \tau$ let $e^{\tau A} = B$, then B is a constant coefficient matrix and the difference equation

$$x_{n+1} = Bx_n \quad \text{or} \quad x \mapsto Bx, \tag{1.4.1}$$

is a discrete dynamical system obtained from the flow of (1.1.1). Similarly, a nonlinear system and its flow ϕ_t give rise to a nonlinear map

$$x_{n+1} = G(x_n) \quad \text{or} \quad x \mapsto G(x), \tag{1.4.2}$$

where $G = \phi_\tau$ is a nonlinear vector valued function. If the flow ϕ_t is smooth (say r-times continuously differentiable), then G is a smooth map with a smooth inverse: i.e., a *diffeomorphism.* This is one example of the way in which a continuous flow gives rise to a discrete map; a more important one, the Poincaré map, will be considered in Section 1.5.

Diffeomorphisms or discrete dynamical systems can also be studied in their own right and more generally we might also consider noninvertible maps such as

$$x \mapsto x - x^2. \tag{1.4.3}$$

EXERCISE 1.4.1. Show that the map $(x, y) \mapsto (y, bx + dy - y^3)$ is a diffeomorphism for $b \neq 0$ and calculate its inverse. (This map was suggested as an approximation to the Poincaré map of the Duffing equation, cf. Section 2.2 below, and Holmes [1979a].)

An orbit of a linear map $x \to Bx$ is a sequence of points $\{x_i\}_{i=-\infty}^{\infty}$ defined by $x_{i+1} = Bx_i$. Any initial point generates a unique orbit provided that B has no zero eigenvalues.

We define stable, unstable, and center subspaces in a manner analogous to that for linear vector fields:

$$E^s = \text{span}\{n_s \text{ (generalized) eigenvectors whose eigenvalues have modulus} < 1\},$$

$$E^u = \text{span}\{n_u \text{ (generalized) eigenvectors whose eigenvalues have modulus} > 1\},$$

$$E^c = \text{span}\{n_c \text{ (generalized) eigenvectors whose eigenvalues have modulus} = 1\},$$

where the orbits in E^s and E^u are characterized by contraction and expansion, respectively. If there are no multiple eigenvalues, then the contraction and expansion are bounded by geometric series: i.e., there exist constants $c > 0$, $\alpha < 1$ such that, for $n \geq 0$,

$$\begin{aligned} |x_n| &\leq c\alpha^n |x_0| \quad \text{if } x_0 \in E^s, \\ |x_{-n}| &\leq c\alpha^n |x_0| \quad \text{if } x_0 \in E^u. \end{aligned} \tag{1.4.4}$$

If multiple eigenvalues occur, then much as in the case of flows, the contraction (or expansion) need not be exponential, as the following exercise illustrates. However, an exponential bound can still be found if $|\lambda_j| < 1$ for all eigenvalues.

EXERCISE 1.4.2. Compute orbits for

$$x \to \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{bmatrix} x \quad \text{and} \quad x \to \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x$$

and sketch them on the plane. Show that $(0, 0)$ is asymptotically stable in the first case, while it is unstable in the second case, even though $|\lambda| = 1$ (cf. Exercise 1.2.1).

In spite of problems caused by multiplicities, if B has no eigenvalues of unit modulus, the eigenvalues alone serve to determine stability. In this case $x = 0$ is called a *hyperbolic* fixed point and, in general, if $\bar{x}$ is a fixed point for G ($G(\bar{x}) = \bar{x}$) and $DG(\bar{x})$ has no eigenvalues of unit modulus, then $\bar{x}$ is called a hyperbolic fixed point.

There is a theory for diffeomorphisms parallel to that for flows, and in particular the linearization theorem of Hartman–Grobman and the invariant manifold results apply to maps just as the flows (Hartman [1964], Nitecki [1971], Shub [1978]):

Theorem 1.4.1 (Hartman–Grobman). *Let $G: \mathbb{R}^n \to \mathbb{R}^n$ be a (C^1) diffeomorphism with a hyperbolic fixed point $\bar{x}$. Then there exists a homeomorphism h defined on some neighborhood U on $\bar{x}$ such that $h(G(\xi)) = DG(\bar{x})h(\xi)$ for all $\xi \in U$.*

Theorem 1.4.2 (Stable Manifold Theorem for a Fixed Point). *Let $G: \mathbb{R}^n \to \mathbb{R}^n$ be a (C^1) diffeomorphism with a hyperbolic fixed point $\bar{x}$. Then there are local stable and unstable manifolds $W^s_{\text{loc}}(\bar{x})$, $W^u_{\text{loc}}(\bar{x})$, tangent to the eigenspaces $E^s_{\bar{x}}$, $E^u_{\bar{x}}$ of $DG(\bar{x})$ at $\bar{x}$ and of corresponding dimensions. $W^s_{\text{loc}}(\bar{x})$, $W^u_{\text{loc}}(\bar{x})$ are as smooth as the map G.*

Global stable and unstable manifolds are defined as for flows, by taking unions of backward and forward iterates of the local manifolds. We have

$$W^s_{\text{loc}}(\bar{x}) = \{x \in U \mid G^n(x) \to \bar{x} \text{ as } n \to +\infty, \text{ and } G^n(x) \in U, \forall n \geq 0\},$$

$$W^u_{\text{loc}}(\bar{x}) = \{x \in U \mid G^{-n}(x) \to \bar{x} \text{ as } n \to +\infty, \text{ and } G^{-n}(x) \in U, \forall n \geq 0\},$$

and

$$W^s(\bar{x}) = \bigcup_{n \geq 0} G^{-n}(W^s_{\text{loc}}(\bar{x})),$$

$$W^u(\bar{x}) = \bigcup_{n \geq 0} G^{n}(W^u_{\text{loc}}(\bar{x})).$$

The reader should bear in mind, however, that flows and maps differ crucially in that, while the *orbit* or *trajectory* $\phi_t(p)$ of a flow is a *curve* in $\mathbb{R}^n$, the orbit $\{G^n(p)\}$ of a map is a *sequence of points*. Thus, while the invariant manifolds of flows are composed of the unions of solution curves, those of maps are unions of discrete orbit points; see Figure 1.4.1. This distinction will be important later, in the discussion of global behavior.

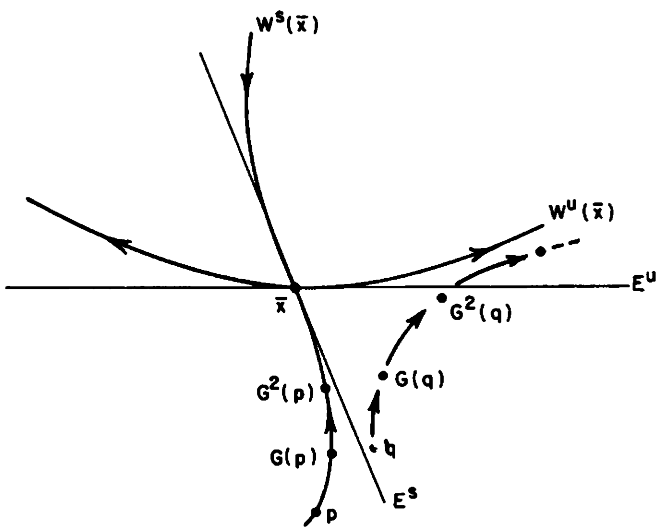
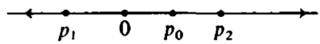
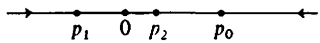
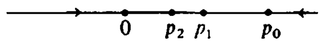
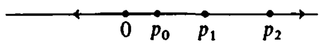
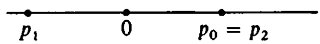
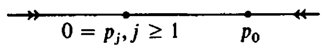
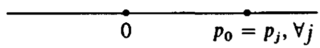


Figure 1.4.1. Invariant manifolds and orbits for a map $G: \mathbb{R}^2 \to \mathbb{R}^2$.

We note that, when we write $G^2(p)$, we mean $G(G(p))$ and, similarly, that $G^n(p)$ means the nth iterate of p under G. Thus, if there is a cycle of k distinct points $p_j = G^j(p_0)$, $j = 0, \dots, k-1$, and $G^k(p_0) = p_0$, we have a *periodic orbit* of period k. The stability of such an orbit is determined by the linearized map $DG^k(p_0)$, or, equivalently $DG^k(p_j)$ for any j. By the chain rule, we have $DG^k(p_0) = DG(G^{k-1}(p_0)) \cdots DG(G(p_0)) \cdot DG(p_0)$.

Much as for flows, the behavior of the linear map (1.4.1) is governed by the eigenvalues and eigenvectors of B. Since maps are rarely dealt with in texts on differential equations or nonlinear oscillations, we include some details here. For a one-dimensional map, where $B = b$ is a scalar and the orbit of a point $\{p_j\}_{j=0}^{\infty}$ is simply given by the geometric sequence $p_j = b^j p_0$, there are four "common" cases and three "unusual" ones listed below in Table 1.4.1. We shall see precisely what we mean by "common" and "unusual" later in this book.

Table 1.4.1. Behavior of the Linear Map $x \to bx$.

Case	Description	Sketch
1. $b < -1$	Orientation reversing source	 p_1 0 p_0 p_2
2. $b \in (-1, 0)$	Orientation reversing sink	 p_1 0 p_2 p_0
3. $b \in (0, 1)$	Orientation preserving sink	 0 p_2 p_1 p_0
4. $b > 1$	Orientation preserving source	 0 p_0 p_1 p_2
5. $b = -1$	Orientation reversing, all points of period 2	 p_1 0 $p_0 = p_2$
6. $b = 0$	All points go to 0 on first iterate (noninvertible)	 $0 = p_j, j \geq 1$ p_0
7. $b = +1$	Orientation preserving, all points fixed	 0 $p_0 = p_j, \forall j$

In general, the stability type of the fixed point $x = 0$ is determined by the magnitude of the eigenvalues of B. If $|\lambda_j| < 1$ for all eigenvalues, then we have a sink; if $|\lambda_j| > 1$ for some eigenvalues and $|\lambda_i| < 1$ for the others: a saddle point, and if $|\lambda_j| > 1$ for all eigenvalues: a source. If $|\lambda_j| = 1$ for any eigenvalues then a norm is preserved in the directions v^j associated with those eigenvalues (unless they are multiple with nontrivial Jordan blocks).

EXERCISE 1.4.3. Develop a classification scheme similar to that of Table 1.4.1 for the two-dimensional map $x \mapsto Bx$;

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

Work in terms of the eigenvalues of B. (Hint: For help see Hsu [1977], or Bernoussou [1977].)

If an even number of eigenvalues have negative real parts, then the map $x \mapsto Bx$ is *orientation preserving*, while if an odd number have negative real parts it reverses orientation. We give some two-dimensional examples (part of the answer to Exercise 1.4.3) in Figure 1.4.2.

To get a feel for the rich and complex behavior possible for nonlinear maps the reader may like to experiment with the following two examples. Solutions may be conveniently obtained on a programmable pocket calculator or a minicomputer:

Exercise 1.4.4. How many fixed and periodic points can you find for the following one-dimensional map and two-dimensional diffeomorphism? Discuss their stability. Let the parameter μ vary over the ranges indicated. Can you find "bifurcation" values of μ at which new periodic points appear?

(a) $x \mapsto \mu x(1 - x)$; $\mu \in [0, 4]$,
(b) $(x, y) \mapsto (y, -\frac{1}{2}x + \mu y - y^3)$; $\mu \in [2, 4]$

(This problem is much harder than it looks. For instance, there are infinitely many periodic points for (a) if $3.7 < \mu \leq 4$. We only expect you to find a few low period ones in each case.)

As a final example of a two-dimensional map with rather rich behavior, consider the simple linear map

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad (x, y) \in T^2 = \mathbb{R}^2/\mathbb{Z}^2, \tag{1.4.5}$$

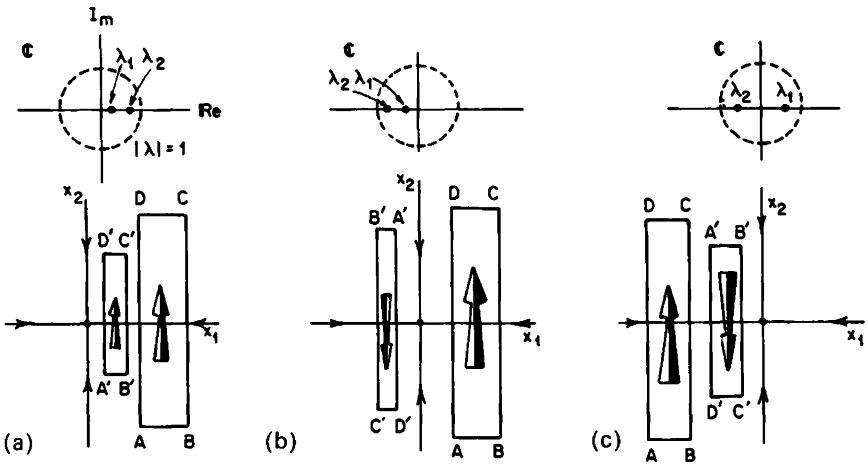


Figure 1.4.2. Orientation preserving (a), (b) and orientation reversing (c) linear maps

$$x \mapsto \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} x.$$

Position of eigenvalues with respect to unit circle in complex plane shown above orbit structures. The oriented rectangle $ABCD$ is mapped to $A'B'C'D'$ in each case.

where the phase space is the two-dimensional torus. On the plane (the covering space) we simply have a saddle point, with eigenvectors $v^{1,2} = (1, (1 \pm \sqrt{5})/2)^T$ belonging to the eigenvalues $\lambda_{1,2} = (3 \pm \sqrt{5})/2$. Since the map is linear, $W^s(0) = E^s$, $W^u(0) = E^u$ and thus span$\{(1, (1 + \sqrt{5})/2)^T\}$ is the unstable manifold and span$\{1, (1 - \sqrt{5})/2)^T\}$ the stable manifold. However, our phase space is the torus, T^2, obtained by identifying points whose coordinates differ by integers. The map is well defined on T^2 since it preserves the periodic lattice. Any point of the unit square $[0, 1) \times [0, 1)$ mapped into another square is translated back into the original square; for example, if $(x, y) = (-1.4, +1.2)$, we set $(x, y) = (0.6, 0.2)$. See Figure 1.4.3. Thus the unstable manifold "runs off the square" at $(2/(1 + \sqrt{5}), 1)$ and reappears, with the same slope, at $(2/(1 + \sqrt{5}), 0)$, to run off at $(1, (\sqrt{5} - 1)/2)$, etc. Since the slopes of W^s and W^u are irrational $((1 \pm \sqrt{5})/2)$ these manifolds are dense in the unit square (or wind densely around the torus). Thus each manifold approaches itself arbitrarily closely, and hence is not an embedded submanifold of T^2.

Exercise 1.4.5. Show that the map of equation (1.4.5) has a countable infinity of periodic points and that the set of such points is dense in T^2. (First show that a point $\bar{x}$ is periodic if and only if both components of $\bar{x}$ are rational numbers with the same denominator.)

Exercise 1.4.6. Describe the set $\Lambda = W^s(0) \cap W^u(0)$ of intersections of the invariant manifolds for the linear map on the torus. What do you think that this implies for the structure of "typical" orbits? (Hint: See Chillingworth [1976], pp. 235–237.)

Arnold and Avez [1968, pp. 5–7] have nice illustrations of the torus map. Also, see Chapter 5 for more information on the invariant sets of such maps.

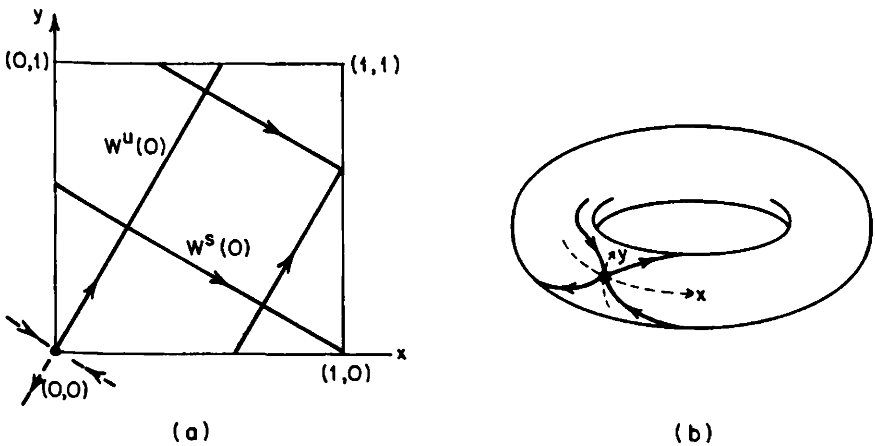


Figure 1.4.3. The linear map on the torus (the hyperbolic toral automorphism). (a) On $\mathbb{R}^2$, the covering space; (b) on T^2.

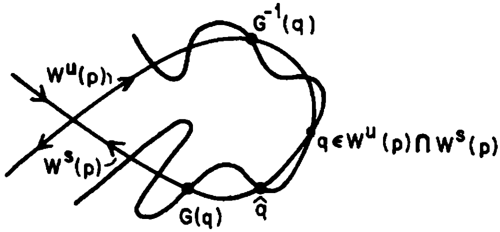


Figure 1.4.4. Homoclinic orbits.

This example might seem rather artificial, but as we shall see, many physically interesting systems have similar properties. In subsequent chapters we shall see that a Poincaré map associated with the forced Duffing equation with negative linear stiffness

$$\ddot{x} + \alpha\dot{x} - x + x^3 = \beta \cos \omega t, \tag{1.4.6}$$

which is a nonlinear diffeomorphism of the plane, possesses a hyperbolic saddle point p whose stable and unstable manifolds intersect transversely somewhat as in the torus map above (cf. Figure 1.4.4). It is fairly easy to see that, if there is one point $q \in W^u(p) \cap W^s(p)$, with $q \neq p$, then, since $G^n(q) \to p$ as $n \to \pm\infty$, and the approach is governed by the linear system for $|q - p|$ small, there must be an infinite set of such *homoclinic* points. Moreover, if the map is orientation preserving (as our Poincaré maps are), then the two homoclinic points q, $G(q)$ must be separated by at least one further point in $W^u(p) \cap W^s(p)$ (marked $\hat{q}$ in Figure 1.4.4). The orbit $\{G^n(q)\}$ of q is called a *homoclinic orbit* and plays an important rôle in the global dynamics of the map G. In particular, the violent winding of the global manifolds $W^u(p)$ and $W^s(p)$ in the neighborhood of p leads to a sensitive dependence of orbits $\{G^n(x_0)\}$ on the initial condition x_0, so that the presence of homoclinic orbits tends to promote erratic behavior. This underlies the chaotic behavior exhibited by the examples of Chapter 2 and in the subject of much of Chapters 5 and 6. If the stable and unstable manifolds $W^s(p_1)$, $W^u(p_2)$ of two distinct fixed points intersect then the resulting orbit is called *heteroclinic*.

EXERCISE 1.4.7. Show that the stable manifold of a saddle point of a two-dimensional map cannot intersect itself.

1.5. Closed Orbits, Poincaré Maps, and Forced Oscillations

In classical texts on differential equations the stability of closed orbits or periodic solutions is discussed in terms of the characteristic or Floquet multipliers. Here we wish to introduce a more geometrical view which is in essence equivalent: the Poincaré map. Since the ideas are so important, we devote a considerable amount of space to familiar examples from forced oscillations.

Let γ be a periodic orbit of some flow ϕ_t in $\mathbb{R}^n$ arising from a nonlinear vector field $f(x)$. We first take a *local cross section* $\Sigma \subset \mathbb{R}^n$, of dimension $n - 1$. The hypersurface Σ need not be planar, but must be chosen so that the flow is everywhere *transverse* to it. This is achieved if $f(x) \cdot n(x) \neq 0$ for all $x \in \Sigma$, where $n(x)$ is the unit normal to Σ at x. Denote the (unique) point where γ intersects Σ by p, and let $U \subseteq \Sigma$ be some neighborhood of p. (If γ has multiple intersections with Σ, then shrink Σ until there is only one intersection.) Then the *first return* or *Poincaré map* $P\colon U \to \Sigma$ is defined for a point $q \in U$ by

$$P(q) = \phi_\tau(q), \tag{1.5.1}$$

where $\tau = \tau(q)$ is the time taken for the orbit $\phi_t(q)$ based at q to first return to Σ. Note that τ generally depends upon q and need not be equal to $T = T(p)$, the period of γ, However, $\tau \to T$ as $q \to p$.

Clearly p is a fixed point for the map P, and it is not difficult to see that the stability of p for P reflects the stability of γ for the flow ϕ_t. In particular, if p is hyperbolic, and $DP(p)$, the linearized map, has n_s eigenvalues with modulus less than one and n_u with modulus greater than one ($n_s + n_u = n - 1$), then dim $W^s(p) = n_s$, and dim $W^u(p) = n_u$ for the map. Since the orbits of P lying in W^s and W^u are formed by intersections of orbits (solution curves) of ϕ_t with Σ, the dimensions of $W^s(\gamma)$ and $W^u(\gamma)$ are each one greater than those for the map. This is most easily seen in the sketches of Figure 1.5.1.

As an example, consider the planar system

$$\begin{aligned} \dot{x} &= x - y - x(x^2 + y^2), \\ \dot{y} &= x + y - y(x^2 + y^2), \end{aligned} \tag{1.5.2}$$

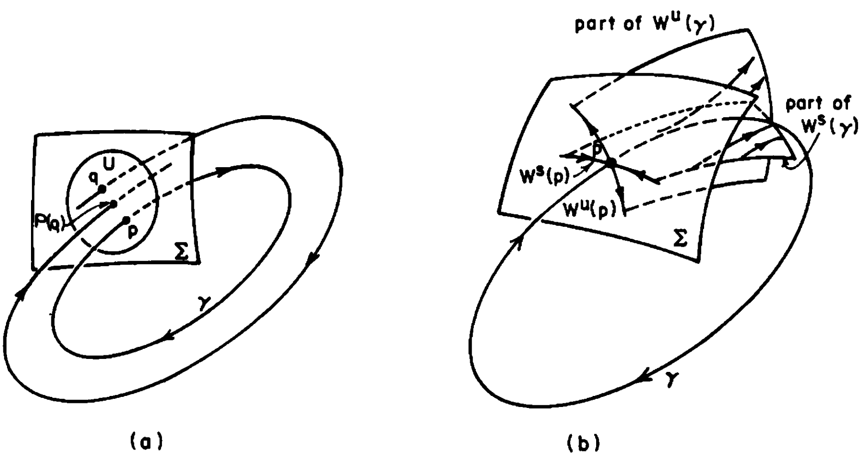


Figure 1.5.1. The Poincaré map. (a) The cross section and the map; (b) a closed orbit.

and take as our cross section

$$\Sigma = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y = 0\}.$$

Transforming (1.5.2) to polar coordinates $r = (x^2 + y^2)^{1/2}$, $\theta = \arctan(y/x)$, we obtain

$$\begin{aligned} \dot{r} &= r(1 - r^2), \\ \dot{\theta} &= 1, \end{aligned} \tag{1.5.3}$$

and the section becomes

$$\Sigma = \{(r, \theta) \in \mathbb{R}^+ \times S^1 \mid r > 0, \theta = 0\}.$$

It is easy to solve (1.5.3) to obtain the global flow

$$\phi_t(r_0, \theta_0) = \left(\left(1 + \left(\frac{1}{r_0^2} - 1\right)e^{-2t}\right)^{-1/2}, t + \theta_0\right).$$

The time of flight τ for any point $q \in \Sigma$ is simply $\tau = 2\pi$, and thus the Poincaré map is given by

$$P(r_0) = \left(1 + \left(\frac{1}{r_0^2} - 1\right)e^{-4\pi}\right)^{-1/2}. \tag{1.5.4}$$

Clearly, P has a fixed point at $r_0 = 1$, reflecting the circular closed orbit of radius 1 of (1.5.3). Here P is a one-dimensional map and its linearization is given by

$$\begin{aligned} DP(1) = \left.\frac{dP}{dr_0}\right|_{r_0 = 1} &= -\frac{1}{2}\left(1 + \left(\frac{1}{r_0^2} - 1\right)e^{-4\pi}\right)^{-3/2} \cdot \left.\left(-\frac{2e^{-4\pi}}{r_0^3}\right)\right|_{r_0 = 1} \\ &= e^{-4\pi} < 1. \end{aligned} \tag{1.5.5}$$

Thus $p = 1$ is a stable fixed point and γ is a stable or attracting closed orbit.

We note that we could have computed $DP(1)$ a little more simply by considering the flow of the vector field (1.5.3) linearized near the closed orbit $r = 1$. Since $(d/dr)(r - r^3) = 1 - 3r^2$, this is

$$\begin{aligned} \dot{\xi} &= -2\xi, \\ \dot{\theta} &= 1, \end{aligned} \tag{1.5.6}$$

with flow

$$D\phi_t(\xi_0, \theta_0) = (e^{-2t}\xi_0, t + \theta_0). \tag{1.5.7}$$

Hence $DP(1) = e^{-2(2\pi)} = e^{-4\pi}$, as above.

To demonstrate the general relationship between Poincaré maps and linearized flows we must review a little Floquet theory (Hartman [1964], §§IV.6, IX.10). Let $\bar{x}(t) = \bar{x}(t + T)$ be a solution lying on the closed orbit γ, based at $x(0) = p \in \Sigma$. Linearizing the differential equation about γ, we obtain the system

$$\dot{\xi} = Df(\bar{x}(t))\xi, \tag{1.5.8}$$

where $Df(\bar{x}(t))$ is an $n \times n$, T-periodic matrix. It can be shown that any fundamental solution matrix of such a T-periodic system can be written in the form

$$X(t) = Z(t)e^{tR}; \qquad Z(t) = Z(t + T), \tag{1.5.9}$$

where X, Z, and R are $n \times n$ matrices (cf. Hartman [1964], p. 60). In particular, we can choose $X(0) = Z(0) = I$, so that

$$X(T) = Z(T)e^{TR} = Z(0)e^{TR} = e^{TR}. \tag{1.5.10}$$

It then follows that the behavior of solutions in the neighborhood of γ is determined by the eigenvalues of the constant matrix e^{TR}. These eigenvalues, $\lambda_1, \ldots, \lambda_n$, are called the *characteristic* (*Floquet*) *multipliers or roots* and the eigenvalues $\mu_1, \ldots, \mu_n$ of R are the *characteristic exponents* of the closed orbit γ. The multiplier associated with perturbations along γ is always unity; let this be λ_n. The moduli of the remaining $n - 1$, if none are unity, determine the stability of γ.

Choosing the basis appropriately, so that the last column of e^{TR} is $(0, \ldots, 0, 1)^T$, the matrix $DP(p)$ of the linearized Poincaré map is simply the $(n - 1) \times (n - 1)$ matrix obtained by deleting the nth row and column of e^{TR}. Then the first $n - 1$ multipliers $\lambda_1, \ldots, \lambda_{n-1}$ are the eigenvalues of the Poincaré map.

Although the matrix R in (1.5.9) is not determined uniquely by the solutions of (1.5.8) (Hartman [1964], p. 60), the eigenvalues of e^{TR} are uniquely determined (e^{TR} can be replaced by any similar matrix $C^{-1}e^{TR}C$). However, to compute these eigenvalues we still need a representation of e^{TR}, and this can only be obtained by actually generating a set of n linearly independent solutions to form $X(t)$. Except in special cases, like the simple example above, this is generally difficult and requires perturbation methods or the use of special functions.

EXERCISE 1.5.1. Repeat the analysis above for the three-dimensional systems obtained by adding the components $\dot{z} = \mu z$ and then $\dot{z} = \mu - z^2$ to (1.5.3): consider $\mu < 0$, $\mu = 0$, and $\mu > 0$. Sketch the stable and unstable manifolds of the periodic orbits in each case. (This is fairly simple.)

EXERCISE 1.5.2. Find the closed orbits of the following system for different values of μ_1 and μ_2: $\dot{r} = r(\mu_1 + \mu_2 r^2 - r^4)$, $\dot{\theta} = 1 - r^2$. Discuss their stability in terms of the Poincaré map. (While the analysis is simple here, since the r and θ equations uncouple, this is a nontrivial example which will reappear in Chapter 7.)

We have seen how a vector field $f(x)$ on $\mathbb{R}^n$ gives rise to a flow map ϕ_t on $\mathbb{R}^n$ and, in the neighborhood of a closed orbit, to a (local) Poincaré map P

on a transversal hypersurface Σ. Another important way in which a flow gives rise to a map is in non-autonomous, periodically forced oscillations. Consider a system

$$\dot{x} = f(x, t); \qquad (x, t) \in \mathbb{R}^n \times \mathbb{R}, \tag{1.5.11}$$

where $f(\cdot, t) = f(\cdot, t + T)$ is periodic of period T in t. System (1.5.11) may be rewritten as an autonomous system at the expense of an increase in dimension by one, if time is included as an explicit state variable:

$$\begin{aligned} \dot{x} &= f(x, \theta), \\ \dot{\theta} &= 1; \qquad (x, \theta) \in \mathbb{R}^n \times S^1. \end{aligned} \tag{1.5.12}$$

The phase space is the manifold $\mathbb{R}^n \times S^1$, where the circular component $S^1 = \mathbb{R} \pmod{T}$ reflects the periodicity of the vector field f in θ. For this problem we can define a *global* cross section

$$\Sigma = \{(x, \theta) \in \mathbb{R}^n \times S^1 \,|\, \theta = \theta_0\}, \tag{1.5.13}$$

since all solutions cross Σ transversely, in view of the component $\dot{\theta} = 1$ of (1.5.12). The Poincaré map $P: \Sigma \to \Sigma$, if it is defined globally, is given by

$$P(x_0) = \pi \cdot \phi_T(x_0, \theta_0), \tag{1.5.14}$$

where $\phi_t: \mathbb{R}^n \times S^1 \to \mathbb{R}^n \times S^1$ is the flow of (1.5.12) and π denotes projection onto the first factor. Note that here the time of flight T is the same for all points $x \in \Sigma$. Alternatively, $P(x_0) = x(x_0, T + \theta_0)$, where $x(x_0, t)$ is the solution of (1.5.12) based at $x(x_0, \theta_0) = x_0$.

The Poincaré map can also be derived as a discrete dynamical system arising from the flow $\psi(x,t)$ of the time-dependent vector field of (1.5.11). Since f is T-periodic, we have $\psi(x, nT) \equiv \psi^n(x, T) \stackrel{\text{def}}{=} \psi_T^n(x)$. The map $P(x_0) = \psi_T(x_0)$ is in this sense another example of a discrete dynamical system of the type considered at the beginning of Section 1.4.

The system

$$\begin{aligned} \dot{x} &= x^2, \\ \dot{\theta} &= 1, \end{aligned} \tag{1.5.15}$$

with solution

$$\phi_t(x_0, \theta_0) = \left(\left(\frac{1}{x_0} - t\right)^{-1}, t + \theta_0\right),$$

and the Poincaré map

$$P(x_0) = \left(\frac{1}{x_0} - 2\pi\right)^{-1}, \qquad x_0 \in (-\infty, 1/2\pi)$$

on $\Sigma = \{(x, \theta) | \theta = 0\}$ shows that P may not be globally defined. Here, trajectories of ϕ_t based at $x_0 \geq 1/2\pi$ approach ∞ at a time $t \leq 2\pi$. However, $P: U \to \Sigma$ is usually defined for some subset $U \subset \Sigma$.

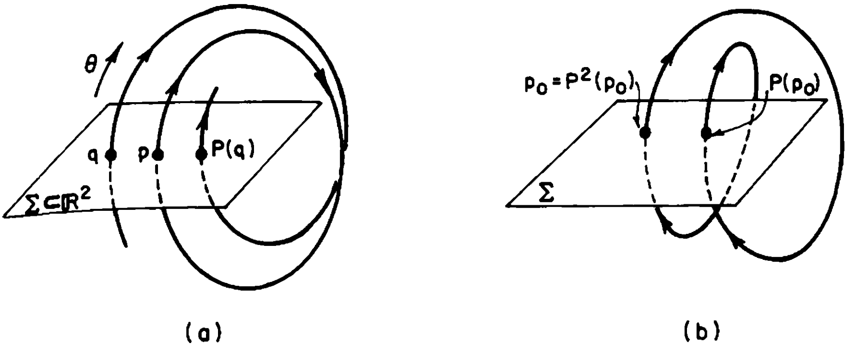


Figure 1.5.2. The Poincaré map for forced oscillations. (a) A periodic orbit of period T and the fixed point $p = P(p)$; (b) a subharmonic of period $2T$.

We illustrate the Poincaré map for forced oscillations in Figure 1.5.2. As in the previous case, it is easy to see that a fixed point p of P corresponds to a periodic orbit of period T for the flow. In addition, a periodic point of period $k > 1$ ($P^k(p) = p$ but $P^j(p) \neq p$ for $1 \leq j \leq k - 1$) corresponds to a subharmonic of period kT. Here P^k means P iterated k times, thus $P^2(p_0) = P(P(p_0))$; etc. This, of course, also applies for the autonomous case discussed earlier. Such periodic points must always come in sets of k: $p_0, \ldots, p_{k-1}$ such that $P(p_i) = p_{i+1}$, $0 \leq i \leq k - 2$ and $p_0 = P(p_{k-1})$.

EXERCISE 1.5.3. (a) Show that the periodic orbits of (1.5.12) can only have periods kT for integers k.
(b) Show that periodic orbits can only have period T if $n = 1$.
(c) Show that the Poincaré map for forced oscillations is orientation preserving.

(Hint: Use uniqueness of solutions in $\mathbb{R}^n \times S^1$.)

Since the definition of the Poincaré map relies on knowledge of the flow of the differential equation, Poincaré maps cannot be computed unless general solutions of these equations are available. However, as we shall see in Chapter 4, perturbation and averaging methods can be used to approximate the map in appropriate cases and valuable information can thus be obtained from the marriage of conventional methods with the geometric approach of dynamical systems theory.

We now consider two examples from the theory of oscillations.

Forced Linear Oscillations

We start with a problem for which a general solution can be found and the Poincaré map computed explicitly. Consider the system

$$\ddot{x} + 2\beta\dot{x} + x = \gamma \cos \omega t; \quad 0 \leq \beta < 1, \tag{1.5.16}$$

or

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2\beta \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \gamma \cos \omega\theta \end{pmatrix}, \qquad (1.5.17)$$
$$\dot{\theta} = 1.$$

Here the forcing is of period $T = 2\pi/\omega$. Since the system is linear, its solution is easily obtained by conventional methods (cf. Braun [1978]):

$$x(t) = e^{-\beta t}(c_1 \cos \omega_d t + c_2 \sin \omega_d t) + A \cos \omega t + B \sin \omega t, \qquad (1.5.18)$$

where $\omega_d = \sqrt{1 - \beta^2}$ is the damped natural frequency and A and B, the coefficients in the particular solution, are given as

$$A = \frac{(1 - \omega^2)\gamma}{[(1 - \omega^2)^2 + 4\beta^2\omega^2]}; \qquad B = \frac{2\beta\omega\gamma}{[(1 - \omega^2)^2 + 4\beta^2\omega^2]}. \qquad (1.5.19)$$

The constants c_1, c_2 are determined by the initial conditions. Letting $x = x_1 = x_{10}$ and $\dot{x} = x_2 = x_{20}$, at $t = 0$, we have

$$\left.\begin{aligned} x(0) &= x_{10} = c_1 + A \\ \dot{x}(0) &= x_{20} = -\beta c_1 + \omega_d c_2 + \omega B \end{aligned}\right\} \Rightarrow \left.\begin{aligned} c_1 &= x_{10} - A, \\ c_2 &= (x_{20} + \beta(x_{10} - A) - \omega B)/\omega_d \end{aligned}\right\}. \qquad (1.5.20)$$

Thus, since $\phi_t(x_{10}, x_{20}, 0)$ is given by (1.5.18) and

$$\begin{aligned} x_2(t) = \dot{x}_1(t) = e^{-\beta t}\{&-\beta(c_1 \cos \omega_d t + c_2 \sin \omega_d t) \\ &+ \omega_d(-c_1 \sin \omega_d t + c_2 \cos \omega_d t)\} \\ &-\omega(A \sin \omega t - B \cos \omega t), \end{aligned}$$

we can compute the Poincaré map explicitly as $\pi \cdot \phi_{2\pi/\omega}(x_{10}, x_{20}, 0)$. In the case of *resonance*, $\omega = \omega_d = \sqrt{1 - \beta^2}$, we obtain

$$P(x_{10}, x_{20}) = ((x_{10} - A)e^{-2\pi\beta/\omega} + A, (x_{20} - \omega B)e^{-2\pi\beta/\omega} + \omega B). \qquad (1.5.21)$$

As expected, the map has an attracting fixed point given by $(x_1, x_2) = (A, \omega B)$ or $c_1 = c_2 = 0$. The map is, of course, linear and since the matrix

$$\begin{bmatrix} \dfrac{\partial P_1}{\partial x_{10}} & \dfrac{\partial P_1}{\partial x_{20}} \\[2ex] \dfrac{\partial P_2}{\partial x_{10}} & \dfrac{\partial P_2}{\partial x_{20}} \end{bmatrix} = \begin{bmatrix} e^{-2\pi\beta/\omega} & 0 \\ 0 & e^{-2\pi\beta/\omega} \end{bmatrix} \qquad (1.5.22)$$

is diagonal with equal eigenvalues, the orbits of P approach $(A, \omega B)$ radially, cf. Figure 1.5.3.

EXERCISE 1.5.4. Compute the Poincaré map for the linear oscillator in the case when $\omega \neq \sqrt{1 - \beta^2} = \omega_d$. What happens when $\beta = 0$ and $\omega = 1$?

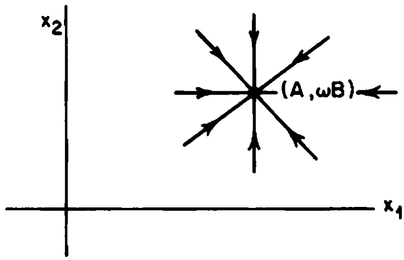


Figure 1.5.3. The Poincaré map of the linear oscillator equation (1.5.16).

Exercise 1.5.5. Consider the forced "negative stiffness" Duffing equation $\ddot{x} + \alpha\dot{x} - x + x^3 = \beta \cos t$, $\alpha > 0$, $\beta \geq 0$. Show that: (a) solutions remain bounded for all time (ϕ_t is globally defined); (b) for $1 \gg \alpha \gg \beta > 0$ there are precisely three periodic orbits of period 2π, one a saddle and the other two attractors. Discuss the structure of stable and unstable manifolds of these periodic orbits by considering the structure of the associated manifolds of the fixed points of the Poincaré map. (This is quite difficult. For (a) you must find a closed curve on which all solutions are directed inward. For (b) you can perturb from the case $\beta = 0$, which is quite simple to analyze.)

The Duffing problem will be taken up in more detail in Chapter 2. As a second example we take a nonlinear system which we linearize about two equilibria.

The Periodically Perturbed Pendulum

The equation of motion of a pendulum with a periodically excited support may be written as a nonlinear Mathieu equation:

$$\ddot{\phi} + (\alpha^2 + \beta \cos t) \sin \phi = 0; \qquad \beta \geq 0, \tag{1.5.23a}$$

or

$$\left.\begin{aligned} \dot{\phi} &= v, \\ \dot{v} &= -(\alpha^2 + \beta \cos \theta) \sin \phi, \\ \dot{\theta} &= 1. \end{aligned}\right\}; \qquad (\phi, v, \theta) \in S^1 \times \mathbb{R} \times S^1. (= \mathbb{R} \times T^2) \tag{1.5.23b}$$

Note that the equilibrium positions $(\phi, v) = (0, 0)$ and $(\pi, 0)$ of the unforced problem ($\beta = 0$) still yield $\dot{\phi} = \dot{v} = 0$ when $\beta \neq 0$. Thus for all β we have periodic orbits given by $(0, 0; \theta(t))$, $(\pi, 0; \theta(t))$ with $\theta(t) = t + t_0$. Linearizing the vector field about these orbits we obtain the linear Mathieu equations

$$\begin{aligned} \dot{\phi} &= v, \\ \dot{v} &= -(\alpha^2 + \beta \cos \theta)\phi, \\ \dot{\theta} &= 1, \end{aligned} \quad \text{and} \quad \begin{aligned} \dot{\phi} &= v, \\ \dot{v} &= (\alpha^2 + \beta \cos \theta)\phi, \\ \dot{\theta} &= 1, \end{aligned} \tag{1.5.24a, b}$$

respectively, where we have retained the same variables (ϕ, v) as for the original nonlinear problem.

We now investigate the stability of these periodic orbits. When $\beta \neq 0$, equations (1.5.24) represent problems in Floquet theory. When $\beta = 0$ we have the simple pendulum and solutions near $\phi = 0$ and $\phi = \pi$ are given, respectively, by the general solutions of the linear oscillator $\ddot{\phi} \pm \alpha^2 \phi = 0$:

$$\begin{pmatrix} \phi \\ v \end{pmatrix} = c_1^0 \begin{pmatrix} \cos \alpha t \\ -\alpha \sin \alpha t \end{pmatrix} + c_2^0 \begin{pmatrix} \sin \alpha t \\ \alpha \cos \alpha t \end{pmatrix}$$

$$\text{and} \quad \begin{pmatrix} \phi \\ v \end{pmatrix} = c_1^\pi \begin{pmatrix} e^{\alpha t} \\ \alpha e^{\alpha t} \end{pmatrix} + c_2^\pi \begin{pmatrix} e^{-\alpha t} \\ -\alpha e^{-\alpha t} \end{pmatrix} \qquad (1.5.25\text{a, b})$$

Letting $(\phi(0), v(0)) = (\phi_0, v_0)$, we find that $c_1^0 = \phi_0$, $c_2^0 = v_0/\alpha$ and

$$c_1^\pi = \frac{\phi_0 + v_0/\alpha}{2}, \qquad c_2^\pi = \frac{\phi_0 - v_0/\alpha}{2}.$$

Integrating these solutions for one period $T = 2\pi$ of the forcing perturbation, we obtain the linearized Poincaré maps

$$DP_0(0, 0)\begin{pmatrix} \phi_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} \phi_0 \cos(2\pi\alpha) + \dfrac{v_0}{\alpha} \sin(2\pi\alpha) \\ -\alpha\phi_0 \sin(2\pi\alpha) + v_0 \cos(2\pi\alpha) \end{pmatrix}, \qquad (1.5.26\text{a})$$

and

$$DP_0(\pi, 0)\begin{pmatrix} \phi_0 \\ v_0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \left(\phi_0 + \dfrac{v_0}{\alpha}\right)e^{2\pi\alpha} + \left(\phi_0 - \dfrac{v_0}{\alpha}\right)e^{-2\pi\alpha} \\ (\alpha\phi_0 + v_0)e^{2\pi\alpha} - (\alpha\phi_0 - v_0)e^{-2\pi\alpha} \end{pmatrix} \qquad (1.5.26\text{b})$$

Thus the linearized operators are

$$DP_0(0, 0) = \begin{bmatrix} \cos(2\pi\alpha) & \dfrac{1}{\alpha} \sin(2\pi\alpha) \\ -\alpha \sin(2\pi\alpha) & \cos(2\pi\alpha) \end{bmatrix}$$

$$\text{and} \quad DP_0(\pi, 0) = \begin{bmatrix} \cosh(2\pi\alpha) & \dfrac{1}{\alpha} \sinh(2\pi\alpha) \\ \alpha \sinh(2\pi\alpha) & \cosh(2\pi\alpha) \end{bmatrix}. \qquad (1.5.27\text{a, b})$$

The eigenvalues of these matrices are

$$\begin{aligned} \lambda_{1,2}^0 &= \cos(2\pi\alpha) \pm i \sin(2\pi\alpha) \quad \text{and} \quad & \lambda_{1,2}^\pi &= \cosh(2\pi\alpha) \pm \sinh(2\pi\alpha) \\ &= e^{i2\pi\alpha}, e^{-i2\pi\alpha}, & &= e^{2\pi\alpha}, e^{-2\pi\alpha}. \end{aligned} \qquad (1.5.28\text{a, b})$$

We conclude that the orbit $(0, 0, \theta(t))$ is neutrally stable, with eigenvalues on the unit circle, and that at $(\pi, 0, \theta(t))$ is of saddle type with one eigenvalue within and one outside the unit circle. Note, however, that when $\alpha = n/2$; $n = 0, 1, 2, \ldots$ the eigenvalues of the neutrally stable orbit are $+1$ (resp. -1) with multiplicity two. We shall return to this below.

We now turn to the more interesting case when $\beta \neq 0$. As is well known, the general solution of (1.5.24) can be written as

$$\begin{pmatrix} \phi(t) \\ v(t) \end{pmatrix} = c_1 x^1(t) + c_2 x^2(t), \tag{1.5.29}$$

where $x^1(t)$ and $x^2(t)$ are two linearly independent solutions. Thus $X(t) = [x^1(t), x^2(t)]$ is a fundamental solution matrix. The linearized Poincaré map can be obtained as

$$DP_\beta = X(2\pi)X^{-1}(0), \tag{1.5.30}$$

since, using (1.5.29) we have

$$\begin{pmatrix} \phi(2\pi) \\ v(2\pi) \end{pmatrix} = X(2\pi)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = X^{-1}(0)\begin{pmatrix} \phi(0) \\ v(0) \end{pmatrix}. \tag{1.5.31}$$

Our problem now becomes one of calculating a pair of linearly independent solutions, a problem solved in many classical textbooks by special functions (Mathieu functions) arising from series solutions, or by perturbation methods (cf. Nayfeh and Mook [1979]). Rather than repeating such analyses, we shall derive an interesting property of the eigenvalues of DP_β and use this to discuss the stability of solutions for $\beta \neq 0$, small. We choose an independent pair of solutions $x^1(t)$, $x^2(t)$ such that

$$x^1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad x^2(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad X(0) = I = X^{-1}(0).$$

Then we have

$$DP_\beta = X(2\pi) = \begin{bmatrix} \phi^1(2\pi) & \phi^2(2\pi) \\ v^1(2\pi) & v^2(2\pi) \end{bmatrix}, \tag{1.5.32}$$

where $X(t) = D\phi_t$, the linearized flow. We claim that the determinant of DP_β (the Wronskian of the solutions x^1, x^2) is unity for our system. To see this, consider the determinant of the linearized flow $D\phi_t$:

$$\begin{aligned} \Delta &= \det(D\phi_t) = \phi^1 v^2 - \phi^2 v^1, \\ \frac{d\Delta}{dt} &= \dot{\phi}^1 v^2 + \phi^1 \dot{v}^2 - \dot{\phi}^2 v^1 - \phi^2 \dot{v}^1 = \phi^1 \dot{v}^2 - \phi^2 \dot{v}^1 \\ &= \phi^1[\pm(\alpha^2 + \beta\cos t)\phi^2] - \phi^2[\pm(\alpha^2 + \beta\cos t)\phi^1] \equiv 0. \end{aligned} \tag{1.5.33}$$

Thus Δ maintains its value. But setting $t = 0$ and using

$$x_1(0) = \begin{pmatrix} \phi^1(0) \\ v^1(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad x_2(0) = \begin{pmatrix} \phi^2(0) \\ v^2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

we have $\Delta = 1$. Therefore det DP_β = det $D\phi_{2\pi} = 1$ also, and the (linearized) Poincaré map is area preserving. The eigenvalues of (1.5.32) are

$$\lambda_{1,2} = a \pm \sqrt{a^2 - 1}, \qquad a = \tfrac{1}{2}(\phi^1(2\pi) + v^2(2\pi)), \tag{1.5.34}$$

and we have

$$\lambda_1 \lambda_2 = 1. \tag{1.5.35}$$

Thus, when $\beta \neq 0$ as well as when $\beta = 0$, the eigenvalues are either complex conjugates with nonzero imaginary parts, real and reciprocal, or multiple and equal to $+1$ or -1.

Now as β increases from zero, the eigenvalues of DP_β vary continuously, starting as those of DP_0, and we therefore have the following result.

Proposition 1.5.1. *The periodic orbit* $(0, 0, \theta(t))$ *is neutrally stable for* $\beta \neq 0$ *and sufficiently small, provided that* $\alpha \neq n/2, n \in \mathbb{Z}$. *The periodic orbit* $(\pi, 0, \theta(t))$ *is of saddle type for* $\beta \neq 0$, *sufficiently small and all* $\alpha \neq 0$.

PROOF. The second assertion is easily proved, since if $\alpha < 0$ then the eigenvalues of $DP_0(\pi, 0)$ are given by $e^{-2\pi\alpha} < 1 < e^{2\pi\alpha}$ and since those of $DP_\beta(\pi, 0)$ vary continuously with β, we can choose a $\beta_0 > 0$ such that for all $0 < \beta < \beta_0$, $DP_\beta(\pi, 0)$ has eigenvalues $1/\lambda_\beta < 1 < \lambda_\beta$. Proof of the first assertion proceeds similarly, except that we must exclude the critical values $\alpha = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, since for those values the eigenvalues of $DP_0(0, 0)$ are ± 1 with multiplicity two, and we cannot know how they will split as β increases from zero. These critical values are, of course, the resonance conditions familiar from texts on linear parametric excitation (cf. Nayfeh and Mook [1979]). □

Note that when $\alpha = \frac{1}{2}, \frac{3}{2}, \ldots$ and $\lambda_{1,2} = -1$ the eigenvalues can split and take the form $-\lambda_\beta < -1 < -1/\lambda_\beta$ $(\lambda_\beta > 0)$ when $\beta \neq 0$. As we shall see in Chapter 3, this bifurcation of the Poincaré map typically involves the appearance of an orbit twice the original period. Here, for example, the instability corresponding to the second subharmonic $(T = 4\pi)$ appears for $\beta > 0$ and $\alpha = \frac{1}{2}$.

Arnold [1978, §25] has a nice treatment of parametric resonance in periodically perturbed Hamiltonian systems. He also works in terms of the Poincaré map.

EXERCISE 1.5.6. Consider the system

$$\dot{\phi} = v,$$
$$\dot{v} = -(\alpha^2 + \beta \cos t)\phi - \gamma v,$$

where the damping parameter $\gamma > 0$ is fixed. Using arguments similar to those above, show that the solution $(\phi, v) = (0, 0)$ is asymptotically stable for *all* α and sufficiently small β. (Hint: First show that det $DP_\beta = e^{-2\pi\gamma}$ in this case.)

1.6. Asymptotic Behavior

Before we can get down to some specific examples of flows and maps, we need a little more technical apparatus. In this section we define various limit sets which represent asymptotic behavior of certain classes of solutions, and in the next section we discuss equivalence relations. While our definitions are fairly general, we concentrate on two-dimensional flows and maps for most of our examples. In Section 8 we shall give a more complete review of two-dimensional flows.

We first define an *invariant set* S for a flow ϕ_t or map G on $\mathbb{R}^n$ as a subset $S \subset \mathbb{R}^n$ such that

$$\phi_t(x) \in S \text{ (or } G(x) \in S) \quad \text{for } x \in S \quad \text{for all } t \in \mathbb{R}. \tag{1.6.1}$$

The stable and unstable manifolds of a fixed point or periodic orbit provide examples of invariant sets. However, the *nonwandering* set is perhaps more important to the study of long-term behavior. We have already seen that fixed points and closed orbits are important in the study of dynamical systems, since they represent stationary or repeatable behavior. A generalization of these sets is the nonwandering set, Ω. A point p is called *nonwandering for the flow* ϕ_t (resp. the map G) if, for any neighborhood U of p, there exists arbitrarily large t (resp. $n > 0$) such that $\phi_t(U) \cap U \neq \varnothing$ (resp. $G^n(U) \cap U \neq \varnothing$). Ω is the set of all such points p. Thus a nonwandering point lies on or near orbits which come back within a specified distance of themselves. Fixed points and periodic orbits are clearly nonwandering. For the damped harmonic oscillator

$$\ddot{x} + \alpha\dot{x} + x = 0, \tag{1.6.2}$$

$(x, \dot{x}) = (0, 0)$ is the only nonwandering point; but for the undamped oscillator

$$\ddot{x} + x = 0, \tag{1.6.3}$$

all points $p \in \mathbb{R}^2$ are nonwandering, since the $(x, \dot{x})$ phase plane is filled with a continuous family of periodic orbits.

EXERCISE 1.6.1. Find the nonwandering sets for the following flows and maps:

(a) $\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = 0$ (van der Pol's equation, $\varepsilon > 0$).
(b) $\dot{\theta} = \mu - \sin\theta$, $\theta \in S^1$ (take $\mu < 1$, $\mu = 1$, and $\mu > 1$).
(c) $\ddot{\theta} + \sin\theta = \frac{1}{2}$, $\theta \in S^1$.

(d) $x \to \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} x$, $x \in T^2$ (see §1.4).

((d) is difficult: try to find a dense orbit; recall that points with rational coordinates are periodic; cf. Exercises 1.4.5–1.4.6.)

Not all invariant sets consist of nonwandering points. For example, the linear map

$$x \mapsto \begin{bmatrix} 1 & 1 \\ 0 & \frac{1}{2} \end{bmatrix} x; \qquad x \in \mathbb{R}^2 \tag{1.6.4}$$

has invariant subspaces (eigenspaces) given by

$$E^s = (2, -1)^T,$$
$$E^c = (1, 0)^T,$$

but points $p \in E^s$ are all wandering (except (0, 0)). The points $q \in E^c$ are, however, nonwandering, since they are all fixed.

Since the set of wandering points is open, Ω is closed, and it must contain the closure of the set of fixed points and periodic orbits. Wandering points correspond to transient behavior, while "long-term" or asymptotic behavior corresponds to orbits of nonwandering points. In particular, the attracting set and attractors will be important. However, before defining an attractor we need another two ideas.

A point p is an *ω-limit point* of x if there are points $\phi_{t_1}(x), \phi_{t_2}(x), \ldots$ on the orbit of x such that $\phi_{t_i}(x) \to p$ and $t_i \to \infty$. A point q is an *α-limit point* if such a sequence exists with $\phi_{t_i}(x) \to q$ and $t_i \to -\infty$. For maps G the t_i are integers. The α- (resp. ω-) limit sets $\alpha(x)$, $\omega(x)$ are the sets of α and ω limit points of x. See Figure 1.6.1.

A closed invariant set $A \subset \mathbb{R}^n$ is called an *attracting set* if there is some neighborhood U of A such that $\phi_t(x) \in U$ for $t \geq 0$ and $\phi_t(x) \to A$ as $t \to \infty$, for all $x \in U$.* The set $\bigcup_{t \leq 0} \phi_t(U)$ is the *domain of attraction* of A (it is, of course, the stable manifold of A). An attracting set ultimately captures all orbits starting in its domain of attraction. A *repelling set* is defined analogously, replacing t by $-t$. Domains of attraction of disjoint attracting sets are necessarily nonintersecting and separated by the stable manifolds of nonattracting sets. See Figure 1.6.2.

In many problems we are able to find a "trapping region," a closed simply connected set $D \subset \mathbb{R}^n$ such that $\phi_t(D) \subset D$ for all $t > 0$. For this, it is sufficient to show that the vector field is directed everywhere inward on the boundary of D. In this case we can define the associated attracting set as

$$A = \bigcap_{t \geq 0} \phi_t(D).$$

For maps, a closed set A is an attracting set if it has some neighborhood U such that $G^n(U) \to A$ as $n \to \infty$. As in the case of flows, if D is a trapping region ($G(U) \subset U$), then the associated attracting set is

$$A = \bigcap_{n \geq 0} G^n(D).$$

In Chapter 2 we shall use this idea in studies of several problems.

* In Chapter 5 we shall relax this definition somewhat

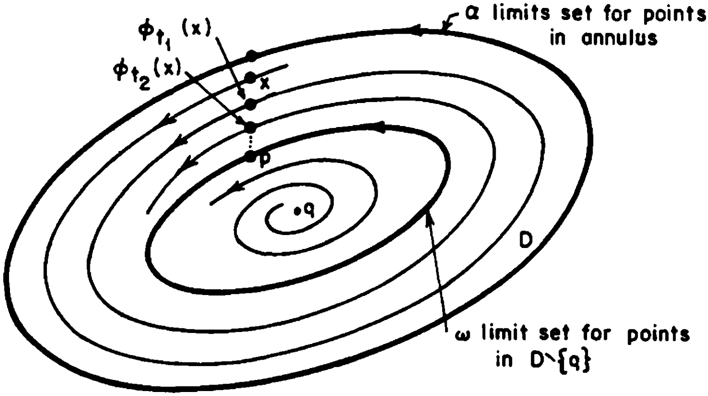


Figure 1.6.1. Examples of α and ω limit sets. D is the open disc bounded by the outer periodic orbit.

EXERCISE 1.6.2. Show that there is a trapping region for the flow of the system

$$\begin{aligned} \dot{x} &= \mu_1 x - x(x^2 + y^2) - xy^2, \\ \dot{y} &= \mu_2 y - y(x^2 + y^2) - yx^2; \end{aligned} \qquad (x, y) \in \mathbb{R}^2,$$

for all finite values of μ_1, μ_2. Find the fixed points and discuss their stability. Show that, for $\mu_1 = \mu_2 > 0$, the line $x = y$ separates two distinct domains of attraction. (Hint: Let D be the closed disc with boundary $x^2 + y^2 = c$ for large c.)

EXERCISE 1.6.3. Show that there is a solid ellipsoid E given by $\rho x^2 + \sigma y^2 + \sigma(z - 2\rho)^2 \leq c < \infty$ such that, for suitable choices of σ, β, $\rho \geq 0$, all solutions of the Lorenz equations

$$\dot{x} = \sigma(y - x); \qquad \dot{y} = \rho x - y - xz; \qquad z = -\beta z + xy;$$

enter E within finite time and thereafter remain in E (cf. Sparrow [1982], Appendix C).

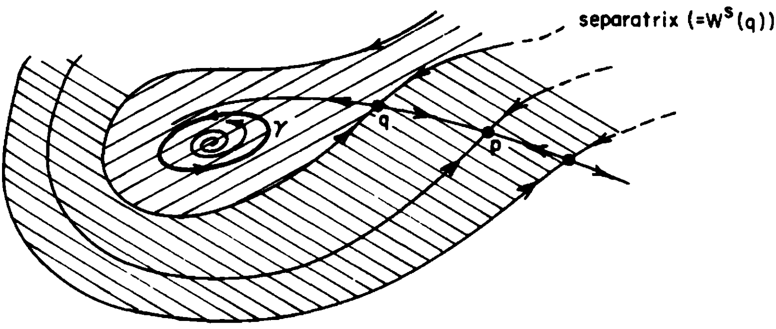


Figure 1.6.2. Domains of attraction: of the closed orbit γ and of the fixed point p

EXERCISE 1.6.4. Consider the system

$$\begin{aligned} \dot{x} &= x - x^3, \\ \dot{y} &= -y; \end{aligned} \qquad (x, y) \in \mathbb{R}^2.$$

Find the nonwandering set, Ω, and the α and ω limit points of typical points $x \in \mathbb{R}^2$. Show that the closed interval $[-1, 1]$ of the x-axis is an attracting set, although most points in it are wandering. Where do you think most orbits will end up?

The last problem should help motivate our working definition of an *attractor* as an *attracting set which contains a dense orbit.* A repellor is defined analogously. Thus, in Exercise 1.6.4 there are two distinct attractors: the points $(\pm 1, 0)$. As we shall see in Chapter 5, it is very difficult to show in examples that a dense orbit exists, and in fact many of the numerically observed "strange attractors" may not be true attractors but merely attracting sets, since they may contain stable periodic orbits. We shall meet the first such examples in Chapter 2.

An example due to Ruelle [1981] shows that, even in one-dimensional flows, attracting sets can be quite complicated. Consider the system

$$\dot{x} = -x^4 \sin\left(\frac{\pi}{x}\right), \tag{1.6.5}$$

which has a countable set of fixed points at $x = 0$ and $\pm 1/n$, $n = 1, 2, \ldots$. The interval $[-1, 1]$ is an attracting set, but it contains a countable set of repelling fixed points at $\pm 1/2n$, $n = 1, 2, \ldots$ and attracting fixed points at $\pm 1/(2n - 1)$, $n = 1, 2, \ldots$, as the reader can check by considering the linearized vector field

$$\left(-4x^3 \sin\left(\frac{\pi}{x}\right) + \pi x^2 \cos\left(\frac{\pi}{x}\right)\right)\bigg|_{x = \pm 1/n} = \frac{\pi}{n^2} \cos n\pi. \tag{1.6.6}$$

However, the fixed point $x = 0$ is itself neither a repellor nor an attractor. Conley [1978] defined "quasiattractors" earlier to cover this type of example.

EXERCISE 1.6.5. Describe the set of fixed points for the map

$$f: x \to |x|^\alpha \cos\left(\ln\left(\frac{1}{|x|}\right)\right), \qquad x \in [-1, 1]$$

for $\alpha < 0$, $\alpha = 0$, and $\alpha > 0$.

A further example may help to illustrate some of the ideas of this section. In the analysis of the weakly forced van der Pol equation, which we shall outline in Section 1 of Chapter 2, the phase portrait shown in Figure 1.6.3 occurs. Clearly, the closed curve $\gamma \cup \{p\}$, including the fixed point p, is an

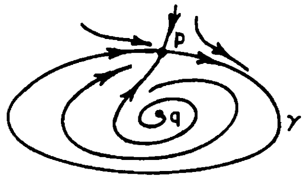


Figure 1.6.3. A planar phase portrait in the averaged van der Pol equation.

attracting set, but the point p is neither an attractor nor a repellor, being simultaneously the α and ω limit point for all points $x \in \gamma$. In fact γ is filled with wandering points and the fixed points p and q are the only components of the nonwandering set. Since there is a dense orbit in $\gamma \cup \{p\}$, our attracting set is in fact an attractor, but it is clear that, in the absence of perturbations, all solutions except those based at q tend towards p from the left as $t \to +\infty$. This example, among others, should warn us that our definition of an attractor may not be the most appropriate for physical applications, and we shall therefore modify it in Chapter 5 in the light of examples arising from physical problems.

This example also illustrates why we include the requirement that $\phi_t(x) \in U$ for all $t \geq 0$, $x \in U$, since there are orbits starting to the right of p which leave a neighborhood of p only to eventually return as $t \to \infty$. The reader should compare this requirement with our definitions of local stable and unstable manifolds in Section 1.3.

Exercise 1.6.6. Show that the circle $r = 1$ is an attracting set for the flow arising from the vector field

$$\dot{r} = r - r^3, \qquad \dot{\theta} = 1 - \cos 2\theta.$$

Which of the equilibrium points are attractors and which repellors? Describe the α and ω limit sets for typical points inside and outside the circle $r = 1$ and in the upper and lower half planes.

Exercise 1.6.7. Construct an example of a two-dimensional flow with an attractor which contains no fixed points or closed orbits. (Hint: Consider linear translation on the torus $T^2 = \mathbb{R}^2/Z^2$ given by the vector field $\dot{\theta} = \alpha$, $\dot{\phi} = \beta$.)

We note that we have not specified that an attractor should be persistent with respect to small perturbations of the vector field or map. While this has been a requirement in many previous definitions, many of the examples which we consider in this book almost certainly do not have such structurally stable attractors. Nonetheless, the idea of structural stability plays an important rôle in dynamical systems theory, and it is to this that we now turn.

1.7. Equivalence Relations and Structural Stability

The idea of a "robust" or "coarse" system—one that retains its qualitative properties under small perturbations or changes to the functions involved in its definition—originated in the work of Andronov and Pontryagin [1937]. A very readable introduction to the special case of planar vector fields may be found in the text on nonlinear oscillations by Andronov *et al.* [1966]. In Chapter 5 we shall question the conventional wisdom that robustness or structural stability is an essential property for models of physical systems, but since the concept has played such a large rôle in the development of dynamical systems theory we discuss it briefly here. We first discuss the idea of perturbations of maps and vector fields.

Given a map $F \in C^r(\mathbb{R}^n)$, we want to specify what is meant by a perturbation G of F. Intuitively, G should be "close to" F, but there are technical issues involved in making a workable definition. We refer the reader to Hirsch [1976] for a full discussion of function spaces and their topologies. Since we have avoided the use of function spaces in this book, we make the following definition which suffices for our discussion of structural stability.

Definition 1.7.1. If $F \in C^r(\mathbb{R}^n)$, $r, k \in \mathbb{Z}^+$, $k \leq r$, and $\varepsilon > 0$, then G is a C^k *perturbation of size* ε if there is a compact set $K \subset \mathbb{R}^n$ such that $F = G$ on the set $\mathbb{R}^n - K$ and for all $(i_1, \ldots, i_n)$ with $i_1 + \cdots i_n = i \leq k$ we have $|(\partial^i/\partial x_1^{i_1} \cdots \partial x_n^{i_n})(F - G)| < \varepsilon$.

We remark that in this definition the functions F and G might be vector fields or maps.

Now that we can discuss the "closeness" of maps or vector fields, we can consider the questions of topological equivalence and structural stability:

Definition 1.7.2. Two C^r maps F, G are C^k *equivalent* or C^k *conjugate* ($k \leq r$) if there exists a C^k homeomorphism h such that $h \circ F = G \circ h$. C^0 equivalence is called *topological equivalence.*

This definition implies that h takes an orbit $\{F^n(x)\}$ to an orbit $\{G^n(x)\}$. The notion of orbit-equivalence is also what we need in the case of vector fields:

Definition 1.7.3. Two C^r vector fields, f, g are said to be C^k *equivalent* ($k \leq r$) if there exists a C^k diffeomorphism h which takes orbits $\phi_t^f(x)$ of f to orbits $\phi_t^g(x)$ of g, preserving senses but not necessarily parametrization by time. If h does preserve parametrization by time, then it is called a *conjugacy.*

The definition of equivalence implies that for any x and t_1, there is a t_2 such that

$$h(\phi_{t_1}^f(x)) = \phi_{t_2}^g(h(x)). \tag{1.7.1}$$

One reason that parametrization by time cannot, in general, be preserved is that the periods of closed orbits in flows can differ.

We now come to the major definition:

Definition 1.7.4. A map $F \in C^r(\mathbb{R}^n)$ (resp. a C^r vector field f) is *structurally stable* if there is an $\varepsilon > 0$ such that all C^1, ε perturbations of F (resp. of f) are topologically equivalent to F (resp. f).

At first sight the use of C^0 equivalence might seem crude and we might be tempted to use C^k equivalence with $k > 0$. This is too strict, however, because it implies that if f and g have fixed points p and $q = h(p)$, then the eigenvalues of the linearized systems $\dot{\xi} = Df(p)\xi$ and $\dot{\eta} = Dg(q)\eta$ must be in the same ratios (we prove this in the appendix to this section). For example, the linear systems

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{1.7.2a}$$

and

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 + \varepsilon \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{1.7.2b}$$

are not C^k orbit equivalent for any $\varepsilon \neq 0$ and $k \geq 1$. In this example the lack of differentiable equivalence is clear, since in the first case solution curves are given by graphs of the form $y = C_1 x$, and in the second by $y = C_2 |x|^{1+\varepsilon}$. Any such pair of curves with $C_1, C_2 \neq 0$ are not diffeomorphic at the origin.

Note that homeomorphic equivalence does not distinguish among nodes, improper nodes, and foci: for example, the two-dimensional linear vector fields with matrices

$$\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -3 & -1 \\ 1 & -3 \end{bmatrix}$$

all have flows which are C^0 equivalent to that of the node with matrix

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1.7.3}$$

However, the C^0 equivalence relation clearly *does* distinguish between sinks, saddles, and sources.

As a further illustration of structural stability of both flows and maps, consider the two-dimensional linear differential equation

$$\dot{x} = Ax, \qquad x \in \mathbb{R}^2, \tag{1.7.4}$$

and the map

$$x \mapsto Bx, \qquad x \in \mathbb{R}^2. \tag{1.7.5}$$

Suppose in the first case that A has no eigenvalues with zero real part, and in the second that B has no eigenvalues of unit modulus. We claim that, if these conditions hold, then both systems are structurally stable.

Consider a small perturbation of (1.7.4):

$$\dot{x} = Ax + \varepsilon f(x), \tag{1.7.6}$$

where f has support in some compact set. Since A is invertible, by the implicit function theorem, the equation

$$Ax + \varepsilon f(x) = 0 \tag{1.7.7}$$

continues to have a unique solution $\bar{x} = 0 + \mathcal{O}(\varepsilon)$ near $x = 0$, for sufficiently small ε. Moreover, since the matrix of the linearized system

$$\dot{\xi} = [A + \varepsilon Df(\bar{x})]\xi$$

has eigenvalues which depend continuously on ε, no eigenvalues can cross the imaginary axis if ε remains small with respect to the magnitude of the real parts of the eigenvalues of A. Thus the perturbed system (1.7.7) has a unique fixed point with eigenspaces and invariant manifolds of the same dimensions as those of the unperturbed system, and which are ε-close locally in position and slope to the unperturbed manifolds. Similar observations apply to the discrete system (1.7.5) and a corresponding small perturbation

$$x \mapsto Bx + \varepsilon g(x). \tag{1.7.8}$$

In both cases the problem is that of finding a homeomorphism which takes orbits of the linear system to those of the perturbed, nonlinear system. Specifically, for the discrete systems, we must prove that there is a homeomorphism h such that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "B"] \arrow[d, "h"] & \mathbb{R}^2 \arrow[d, "h"] \\
\mathbb{R}^2 \arrow[r, "B + \varepsilon g"] & \mathbb{R}^2
\end{tikzcd}$$

For the flow we replace B by $e^{t_1 A}$ and $B + \varepsilon g$ by the flow ϕ_{t_1} generated by the vector field (1.7.7) (they are conjugate in this case).

The proof is now essentially the same as that of Hartman's theorem and the reader is referred to Pugh [1969] or Hartman [1964, Chapter 9].

It should be clear that a vector field (or map) possessing a non-hyperbolic fixed point cannot be structurally stable, since a small perturbation can remove it, if the linearized matrix is noninvertible, having a zero eigenvalue, or turn it into a hyperbolic sink, a saddle, or a source, if the matrix has purely imaginary eigenvalues. Similar observations apply to periodic orbits and we are therefore in a position to state an important requirement for structural stability of flows or maps: *all fixed points and closed orbits must be hyperbolic.* However, as we shall see, this condition alone is not enough to guarantee structural stability, since more subtle, global effects also come into play.

Structurally stable systems have rather "nice" properties in the sense that, if a system is structurally stable, then any sufficiently close system has the same qualitative behavior. However, as we shall see, structurally stable behavior can be extremely complex for flows of dimension ≥ 3 or diffeomorphisms of dimension ≥ 2. Also, it will turn out that structural stability is not even a *generic property*—that is, we can find structurally unstable (and complicated) systems which *remain* unstable under small perturbations, and which, in fact continually change their topological equivalence class as we perturb them. We shall meet our first examples of such systems in Chapter 2.

We have not defined or discussed the notion of generic properties in this section because their definition is formulated in terms of function spaces. Interested readers should consult Chillingworth [1976] or Hirsch and Smale [1974] for introductions to the subject.

Before closing this section we wish to stress that the definition of structural stability is relative to the class of systems we deal with. In our main definition we have allowed all C^1, ε perturbations by C^r vector fields on $\mathbb{R}^n$. If we restrict ourselves to some subset, say all C^r *Hamiltonian* vector fields on $\mathbb{R}^2$, then things are different and we find that the linear system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\omega^2 x, \end{aligned} \qquad \omega \neq 0, \tag{1.7.9}$$

possessing an elliptic center at $(x, y) = (0, 0)$ surrounded by a continuous family of non-hyperbolic closed orbits, *is* stable to small perturbations within this subset. However, the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= 0, \end{aligned} \tag{1.7.10}$$

possessing a degenerate line of fixed points on the x-axis, is not structurally stable, since we can find a Hamiltonian perturbation which yields an isolated fixed point near $(0, 0)$ which is either a center or a hyperbolic saddle point. Of course, both systems are structurally unstable with respect to perturbations by general C^r vector fields. In this book we concentrate on dissipative systems and pay little attention to the special properties of Hamiltonian systems (but see Section 4.8).

EXERCISE 1.7.1. Show that the vector fields $\dot{x} = -x$, $\dot{x} = -4x$, and $\dot{x} = -x^3$ are all C^0 equivalent. Which ones are structurally stable? Find explicitly the homeomorphisms relating their orbits.

EXERCISE 1.7.2. Which of the following systems are structurally stable in the set of all one- (or two-) dimensional systems?

(a) $\dot{x} = x$; (b) $\dot{x} = x^2$;
(c) $\dot{x} = \sin x$; (d) $\ddot{x} = \sin x$;
(e) $\ddot{x} + 2\dot{x} + x = 0$; (f) $\ddot{x} + \dot{x}^2 + x = 0$;
(g) $\ddot{x} + \dot{x} + x^3 = 0$; (h) $\ddot{x} + (x^2 - 1)\dot{x} + x = 0$;
(i) $\dot{\theta} = 1, \dot{\phi} = 2; (\theta, \phi) \in T^2$; (j) $\dot{\theta} = 1, \dot{\phi} = \pi; (\theta, \phi) \in T^2$;
(k) $\dot{\theta} = 2 - \sin\theta, \dot{\phi} = 1; (\theta, \phi) \in T^2$; (l) $\dot{\theta} = 1 - 2\sin\theta, \dot{\phi} = 1; (\theta, \phi) \in T^2$.

(The criteria for determining structural stability of one and two dimensional flows are discussed in the next two sections.)

Appendix to Section 1.7: On C^k Equivalence

Here we show that C^k equivalence, $k \geq 1$, implies that two systems must have the same ratios of eigenvalues when linearized at corresponding fixed points.

If X and Y are C^k orbit equivalent then there is a C^k diffeomorphism $h: \mathbb{R}^n \to \mathbb{R}^n$ such that h "converts" the system $\dot{x} = X(x)$ into $\dot{y} = Y(y)$, i.e., since $y = h(x)$ we have

$$Dh(x)X(x) = \tau(h(x))Y(h(x)), \tag{1.7.11}$$

where $\tau: \mathbb{R}^n \to \mathbb{R}$ is a positive scalar function which allows reparametrization of time.

We now suppose $x = p$ is a fixed point for the flow of X, so that $y = q = h(p)$ is a fixed point for Y. Differentiating (1.7.11) and setting $x = p$, $y = q$, we obtain

$$\begin{aligned} D^2h(x)X(x) + Dh(x)DX(x) = {} & D\tau(h(x))Dh(x)Y(h(x)) \\ & + \tau(h(x))DY(h(x))Dh(x); \end{aligned}$$

since $X(p) = Y(q) = 0$, this gives (1.7.12)

$$Dh(p)DX(p) = \tau(q)DY(q)Dh(p),$$

or

$$DX(p) = \tau(q)Dh(p)^{-1}DY(q)Dh(p). \tag{1.7.13}$$

Thus the two matrices $DX(p)$ and $DY(q)$ are similar, up to a uniform scaling by the constant $\tau(q)$. Hence ratios between eigenvalues are preserved. (If we do not allow reparametrization of time, then $DX(p)$ and $DY(q)$ are strictly similar.)

1.8. Two-Dimensional Flows

In this section we provide a review of some of the theory of two dimensional flows. The Jordan curve theorem, and the fact that solution curves are one dimensional, make the range of solution types on the plane rather limited. The planar system is therefore fairly well understood. However, the reader should realize that we provide only a sampling of the many results here. Andronov and co-workers [1966, 1971, 1973] have well over a thousand pages on the subject and Lefschetz [1957] should also be consulted for more details. Many examples of two-dimensional systems arising in engineering and physics are given by Andronov *et al.* [1966] and in the books of Minorsky [1962], Hayashi [1964], and Nayfeh and Mook [1979]. Here we concentrate on some special classes of systems and give a number of examples, some classical and others less familiar, which will prepare us for the material to follow.

Systems on two manifolds other than $\mathbb{R}^2$ are more complicated and can display surprisingly subtle behavior. In the remainder of this chapter we therefore concentrate on the planar system, although we end with some examples of systems on cylinders and tori.

Suppose that we are given a differential equation

$$\begin{aligned} \dot{x} &= f(x, y), \\ \dot{y} &= g(x, y), \end{aligned} \qquad (x, y) \in U \subseteq \mathbb{R}^2, \tag{1.8.1}$$

where f and g are (sufficiently smooth) functions specified by some physical model. In approaching equation (1.8.1) we normally first seek fixed points, at which $f(x, y) = g(x, y) = 0$. Linearizing (1.8.1) at such a point $(\bar{x}, \bar{y})$, we obtain

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{pmatrix} = \begin{bmatrix} \dfrac{\partial f}{\partial x}(\bar{x}, \bar{y}) & \dfrac{\partial f}{\partial y}(\bar{x}, \bar{y}) \\[2ex] \dfrac{\partial g}{\partial x}(\bar{x}, \bar{y}) & \dfrac{\partial g}{\partial y}(\bar{x}, \bar{y}) \end{bmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \quad \text{or} \quad \dot{\xi} = Df(\bar{x}, \bar{y})\xi. \tag{1.8.2}$$

If the eigenvalues of the matrix $Df(\bar{x}, \bar{y})$ have nonzero real parts, then the solution $\xi(t) = e^{tDf(\bar{x}, \bar{y})}\xi(0)$ of (1.8.2) not only yields local asymptotic behavior, but, by Hartman's theorem and the stable manifold theorem, also provides the local topological structure of the phase portrait. The following exercise shows that the insistence of nonzero eigenvalues is necessary:

EXERCISE 1.8.1. Sketch the phase portraits of the following two nonlinear oscillators and of their linearizations. (You may need to read on to review analytical methods for two-dimensional systems.)

(a) $\ddot{x} + \varepsilon|\dot{x}|\dot{x} + x = 0$; $\varepsilon < 0, \varepsilon = 0, \varepsilon > 0$;
(b) $\ddot{x} + \dot{x} + \varepsilon x^2 = 0$; $\varepsilon < 0, \varepsilon = 0, \varepsilon > 0$.

After locating the fixed points and studying their stability (perhaps using (local) Liapunov functions in the case of nonhyperbolic points), we next wish to ascertain whether (1.8.1) has any periodic orbits. Here the following two results are useful:

Theorem 1.8.1 (The Poincaré–Bendixson Theorem). *A nonempty compact ω- or α-limit set of a planar flow, which contains no fixed points, is a closed orbit.*

For a proof, see Hirsch and Smale [1974, p. 248] or Andronov *et al.* [1966, p. 361].

This result can be used to establish the existence of closed orbits, as in the exercise below:

EXERCISE 1.8.2. Use the Poincaré–Bendixson theorem to prove that the van der Pol oscillator $\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = 0$ has at least one closed orbit. (First find two nested closed curves, C_1, C_2 such that the flow crosses C_1 inward and crosses C_2 outward.)

Proving uniqueness in the above example is considerably harder unless $\varepsilon \ll 1$; see Hirsch and Smale [1974, Chapter 10], for example. However, if the vector field is such that C_1 and C_2 can be chosen to bound a narrow annulus, R, then it may be possible to prove uniqueness by showing that $\partial f/\partial x + \partial g/\partial y$ is everywhere negative (or positive) in R. We give such an example in Section 1 of Chapter 2.

The next result enables us to rule out the occurrence of closed orbits in some cases:

Theorem 1.8.2 (Bendixson's Criterion). *If on a simply connected region $D \subseteq \mathbb{R}^2$ the expression $\partial f/\partial x + \partial g/\partial y$ is not identically zero and does not change sign, then equation* (1.8.1) *has no closed orbits lying entirely in D.*

PROOF. This result is a simple consequence of Green's theorem, for on any solution curve of (1.8.1) we have $dy/dx = g/f$ or, in particular

$$\int_\gamma (f(x, y)\, dy - g(x, y)\, dx) = 0$$

on any closed orbit γ. This implies, via Green's theorem, that

$$\iint_S \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dx\, dy = 0, \tag{1.8.3}$$

where S is the interior of γ. But if $\partial f/\partial x + \partial g/\partial y > 0$ (or <0) on D, then we cannot find a region $S \subseteq D$ such that (1.8.3) holds. Hence there can be no closed orbits entirely in D. □

EXERCISE 1.8.3. Find sufficient conditions for the system $\ddot{x} + \alpha\dot{x} + \beta x + x^2\dot{x} + x^3 = 0$ to have no closed orbits. Are they also necessary?

For a generalization of Bendixson's criterion, Dulac's criterion, see Andronov *et al.* [1966, p. 305].

In addition to fixed points and closed orbits, we have already met examples of other limit sets of two-dimensional flows in Section 1.6. In fact for planar flows, all the possible nonwandering sets fall into three classes (Andronov *et al.* [1966], §VI.2).

(i) Fixed points;
(ii) closed orbits; and
(iii) the unions of fixed points and the trajectories connecting them.

The latter are referred to as *heteroclinic orbits* when they connect distinct points and *homoclinic orbits* when they connect a point to itself. Closed paths formed of heteroclinic orbits are called *homoclinic cycles.* We note that the fixed points contained in such cycles must all be saddle points (if they are hyperbolic), since sinks and sources necessarily have wandering points in their neighborhoods. Some examples of such limit sets are shown in Figure 1.8.1. We will meet specific systems which display almost all of these behaviors later in this book.

EXERCISE 1.8.4. All the examples of planar flows in Figure 1.8.1 are structurally unstable. Why? (Hint: In (a)–(c), try adding a small perturbation near the homoclinic cycles.)

In flows on nonplanar two-dimensional manifolds, such as the torus, limit sets which are neither closed orbits, fixed points, nor homoclinic cycles can arise. In particular, irrational linear flow such as that generated by the vector field

$$\begin{aligned} \dot{\theta} &= 1, \\ \dot{\phi} &= \pi, \end{aligned} \qquad (\theta, \phi) \in T^2; \tag{1.8.4}$$

has a dense orbit and thus *every* point on T^2 is nonwandering. In spite of its apparent artificiality, we shall subsequently see that this example arises naturally in the study of coupled oscillators.

Thus, in two-dimensional flows the global structures of solution curves are generally far richer than those of one-dimensional systems, in which periodic orbits cannot occur and the fixed points are ordered and necessarily connected to their immediate neighbors and only to them. Whether or not such heteroclinic connections are likely to exist in higher-dimensional systems depends upon the relative dimensions of stable and unstable manifolds of neighboring fixed points, but in any case they are generally very difficult to find, unless the system possesses special symmetries or other properties. Our first main example illustrates this point, as well as introducing two important special classes of systems: Hamiltonian and gradient flows. In each type of system, the level curves of a real valued function determine the global structure of the flow.

We consider the example

$$\begin{aligned} \dot{x} &= -\zeta x - \lambda y + xy, \\ \dot{y} &= \lambda x - \zeta y + \tfrac{1}{2}(x^2 - y^2), \end{aligned} \tag{1.8.5}$$

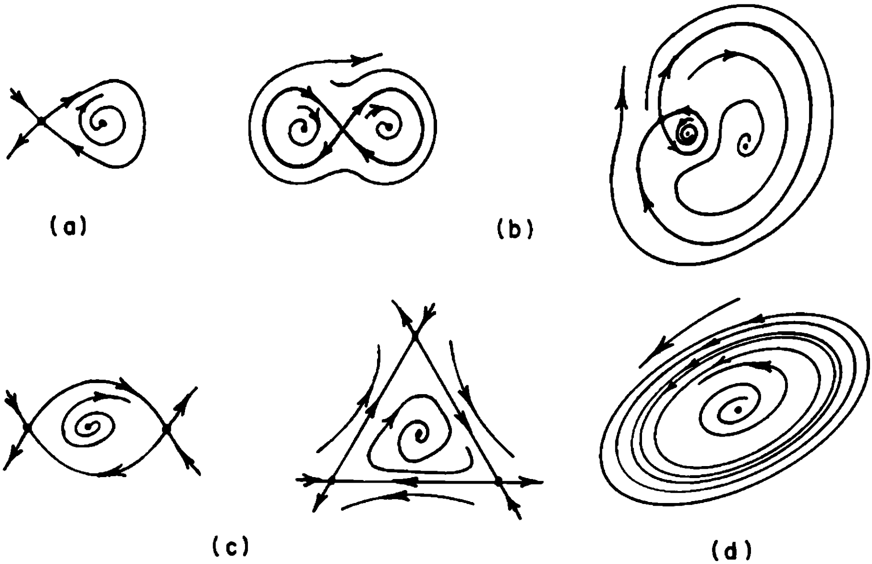


Figure 1.8.1. Some limit sets for flows on the plane. (a) A homoclinic orbit or saddle-loop; (b) double saddle-loops; (c) homoclinic cycles formed of heteroclinic orbits; (d) bands of periodic orbits.

which arose as an averaged system (cf. Chapter 4) in wind induced oscillation studies (Holmes [1979b]). Here $0 \leq \zeta \ll 1$ is a damping factor and λ ($|\lambda| \ll 1$) is a detuning parameter. When $\zeta = 0$, (1.8.5) becomes a *Hamiltonian* system (Goldstein [1980]):

$$\dot{x} = \frac{\partial H}{\partial y}, \qquad \dot{y} = -\frac{\partial H}{\partial x}, \tag{1.8.6}$$

for which the Hamiltonian (energy) function

$$H(x, y) = -\frac{\lambda}{2}(x^2 + y^2) + \frac{1}{2}\left(xy^2 - \frac{x^3}{3}\right) \tag{1.8.7}$$

is a map $H: \mathbb{R}^2 \to \mathbb{R}$. The critical points of H correspond to the fixed points of the flow of Hamilton's equations (1.8.6). Moreover, since

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} = \frac{\partial H}{\partial x}\frac{\partial H}{\partial y} - \frac{\partial H}{\partial y}\frac{\partial H}{\partial x} \equiv 0, \tag{1.8.8}$$

the level curves $H(x, y) =$ constant are solution curves for (1.8.6). Thus for our example the phase portrait may easily be drawn as in Figure 1.8.2. Such a system is said to be integrable, since the solutions, or integral curves, lie along level curves of a smooth function.

Notice that the three saddle-points at $p_3 = (-2\lambda, 0)$ and $p_{2,1} = (\lambda, \pm\sqrt{3}\lambda)$, are connected. The connecting curves $\Gamma_{ij} = W^u(p_i) \cap W^s(p_j)$ are examples of saddle connections or heteroclinic orbits (if such a curve connects a saddle

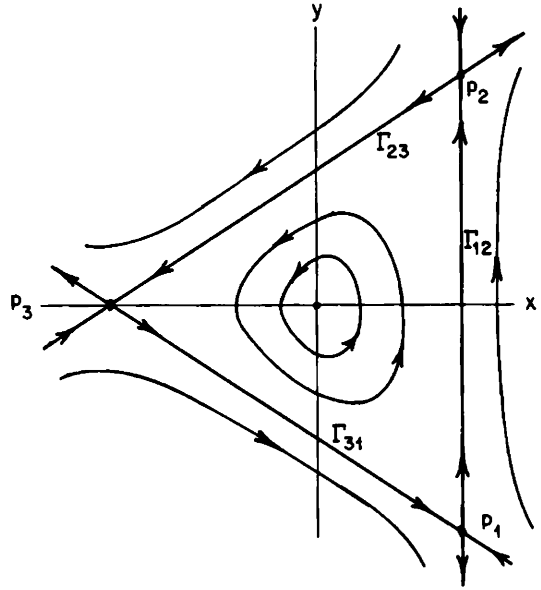


Figure 1.8.2. Phase portrait of equation (1.8.5); $\zeta = 0, \lambda > 0$.

to itself it is referred to as a homoclinic orbit). Here these orbits occur as a result of the Hamiltonian integral constraint, although saddle connections can occur in non-Hamiltonian systems. The reader should check that when $\zeta > 0$ all three connections are broken and the unstable manifolds $W^u(p_i)$ now have components which approach the fixed point at $(x, y) = (0, 0)$ as $t \to +\infty$. This point is then a sink, with eigenvalues $-\zeta \pm i\lambda$. Realizing that the Γ_{ij} are each intersections of two one-dimensional curves, $W^u(p_i)$ and $W^s(p_j)$, in the plane, we would indeed expect such intersections to occur only under special circumstances, and, if they do occur, that they would be broken by arbitrarily small perturbations. Such connections are thus *structurally unstable* in the space of all vector fields on $\mathbb{R}^2$. We shall return to this point later, in sketching the proof of Peixoto's theorem. Note that intersection of $W^u(p_i)$ and $W^s(p_j)$ implies that portions of the two curves are, in fact, *identified*: they cannot merely intersect, as shown in Figure 1.8.3(b), or the solution based at the intersection point q would have two possible futures and pasts, thus violating uniqueness of solutions.

Upon linearizing a planar Hamiltonian system at a fixed point, one finds that trace$(Df) \equiv 0$ and thus all fixed points are either saddles or centers; no sinks or sources can exist. This reflects the more general fact that Hamiltonian flows preserve volume (or, in the two-dimensional case, area); a result known as Liouville's theorem. For more information on this and other results applicable to higher-dimensional Hamiltonian systems, the reader should refer to classical mechanics texts such as Goldstein [1980] or Arnold [1978].

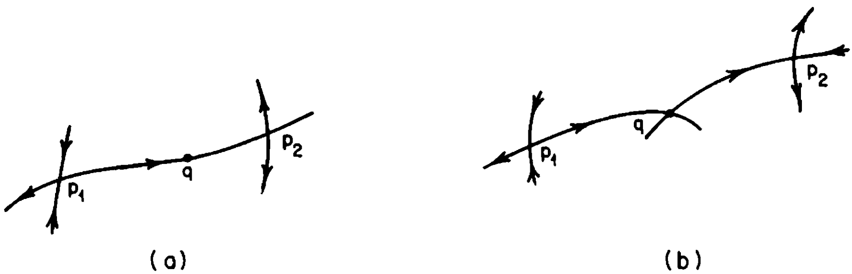


Figure 1.8.3. Heteroclinic points $q \in W^u(p_1) \cap W^s(p_2)$ for flows in $\mathbb{R}^2$. (a) Admissible; (b) not admissible.

EXERCISE 1.8.5. Show that a differential equation of the form $\dot{x} = f(y)$, $\dot{y} = g(x)$ always possesses a first integral $F(y) + G(x)$, the level curves of which are solution curves. Use this fact to study the global solution structure of the system $\dot{x} = -y + y^3$, $\dot{y} = x - x^3$.

A special subset of conservative nonlinear oscillator problems with which we shall be concerned in this book takes the form

$$\ddot{x} + f(x) = 0, \tag{1.8.9}$$

or, as a planar vector field with $y = \dot{x}$:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -f(x). \end{aligned} \tag{1.8.10}$$

Such a Hamiltonian system always possesses a first integral (at least formally):

$$H(x, y) = \frac{y^2}{2} + \int f(x)\,dx \stackrel{\text{def}}{=} \frac{y^2}{2} + V(x), \tag{1.8.11}$$

where $V(x)$ is sometimes called a potential (energy) function, since in mechanical applications it often corresponds to a stored energy (cf. Andronov *et al.* [1966], Marion [1970]).

Any fixed point of (1.8.10) must lie on the x-axis, and correspond to a critical point of $V(x)$. Thus the real valued function $V: \mathbb{R} \to \mathbb{R}$ effectively determines the local form of the vector field and hence the flow near each fixed point. Andronov *et al.* [1966, Chapter 2], Nayfeh and Mook [1979] and others give exhaustive accounts of the various cases. For example, if the critical point of V is nondegenerate (quadratic) then the fixed point is either a hyperbolic saddle or a center, while if the leading term in the Taylor expansion of V is cubic or higher, then the fixed point is degenerate. Here we note that the special structure also enables one to obtain information on the global structure of solution curves, which are simply given by $H(x, y) = c = \text{constant}$, or

$$y = \pm\sqrt{2(c - V(x))}, \tag{1.8.12}$$

and are thus symmetric under reflection about the x-axis. A major consequence of this is that, if there are two saddle points with the same energy level, corresponding to two maxima of $V(x)$, with no higher maximum between them, then they *must* be connected by heteroclinic orbits.

EXERCISE 1.8.6. Find and classify the fixed points and sketch phase portraits for the Hamiltonian systems:

(a) $\ddot{x} + x - x^2 = 0$;
(b) $\ddot{x} + x^2 + x^3 = 0$;
(c) $\ddot{x} + \sin x = 0$;
(d) $\ddot{x} + \sin x = \beta$, $\beta \in (0, 2)$.

Equation (1.8.5) provides an example of another special type of system. When $\lambda = 0$ the system is a *gradient vector field*

$$\dot{x} = -\frac{\partial V}{\partial x}, \qquad \dot{y} = -\frac{\partial V}{\partial y}; \qquad V(x, y) = \frac{\zeta}{2}(x^2 + y^2) + \frac{1}{2}\left(\frac{y^3}{3} - x^2 y\right), \tag{1.8.13}$$

with a sink at $(0, 0)$ (for $\zeta > 0$) and saddles at $(0, -2\zeta)$ and $(\pm\sqrt{3}\zeta, \zeta)$. For a general discussion of such fields see Hirsch and Smale [1974, pp. 199ff.]. We note that the *potential function* $V: \mathbb{R}^2 \to \mathbb{R}$ can be regarded as a Liapunov function. In this way it is possible to show that minima (resp. maxima) of V correspond to sinks (resp. sources) of the n-dimensional system:

$$\dot{x} = -\operatorname{grad} V(x), \tag{1.8.14}$$

for general potential functions $V: \mathbb{R}^n \to \mathbb{R}$. In fact any critical point of V, at which $\operatorname{grad} V = 0$, must be a fixed point of (1.8.14), and the saddle points of V are, of course, saddle points of (1.8.14). Now let $V^{-1}(h)$ be a level (hyper-) surface of V. Then, since for any point $x \in V^{-1}(h)$ at which $\operatorname{grad} V(x) \neq 0$, the vector $-\operatorname{grad} V(x)$ is normal to the tangent to the level surface at x, solution curves of (1.8.14) cross the level surfaces normally and point "downhill" in the direction of decreasing V.

EXERCISE 1.8.7. Show that the nonwandering set of a gradient vector field on $\mathbb{R}^2$ contains only fixed points and that no periodic or homoclinic orbits are possible. (Hint: Use $V(x)$ as a "Liapunov-like" function and modify the usual arguments to work globally.)

EXERCISE 1.8.8. Sketch the phase portraits of equation (1.8.5) for $\zeta = 0$ and $\zeta > 0$, with $\lambda < 0$, $= 0$, > 0. Which cases are structurally stable in the set of all two-dimensional vector fields?

The special properties of gradient vector fields enabled Palis and Smale [1970] to obtain an important general result for such systems in n dimensions:

Theorem 1.8.3. *Gradient systems for which all fixed points are hyperbolic and all intersections of stable and unstable manifolds transversal, are structurally stable.*

For a moment we will step into n dimensions ($n > 2$) to consider this result. There should be no trouble regarding hyperbolicity, but the transversal intersection condition requires some discussion. As we shall see in sketching the proof of Peixoto's theorem, saddle connections for planar systems such as $\Gamma_{12} = W^u(p_1) \cap W^s(p_2)$ of Figure 1.8.1 can be removed (= broken) by small perturbations and are thus structurally unstable. However, suppose that one has a three-dimensional system with a pair of saddle points p_1, p_2 and $\dim W^u(p_1) = \dim W^s(p_2) = 2$. It is now possible for $W^u(p_1)$ and $W^s(p_2)$ to intersect *transversely* on an orbit $\gamma = W^u(p_1) \mp W^s(p_2)$ ($\mp$ = transversal intersection), so that, at any point $q \in \gamma$, the tangent spaces $T_q W^u(p_1)$, $T_q W^s(p_2)$ span $\mathbb{R}^3$ (Figure 1.8.4). If such a transverse heteroclinic orbit exists then it is possible to show that it cannot be removed by an arbitrarily small perturbation, and is thus structurally stable. However, transverse *homoclinic* orbits cannot exist, since $\dim W^u(p) + \dim W^s(p) \leq n$ and for transversality we require $\dim W^u(p_1) + \dim W^s(p_2) > n$ (Figure 1.8.5).

Transverse saddle connections cannot exist at all in two dimensions, since the saddle points have one-dimensional stable and unstable manifolds and a connection $\gamma = W^u(p_1) \cap W^s(p_2)$ is necessarily an open interval on which $W^u(p_1)$ and $W^s(p_2)$ are identified. The tangent space to such a curve at any point $q \in \gamma$ is thus one dimensional (Figure 1.8.3(a)).

Returning to planar flows, we recall a useful result which relates the existence of closed orbits and fixed points. This involves the (Poincaré) index of a fixed point. We start with the general idea of the index. Given a planar flow, we draw a simple closed curve C not passing through any equilibrium points and consider the orientation of the vector field at a point $p = (x, y) \in C$. Letting p traverse C anticlockwise, the vector $(f(x, y), g(x, y))$ rotates continuously and, upon returning to the original position, must have rotated through an angle $2\pi k$ for some integer k. (The angle is also measured anticlockwise.) We call k the *index of the closed curve* C and it can be shown that k is independent of the form of C in the sense that it is determined solely by the character of the fixed points inside C.

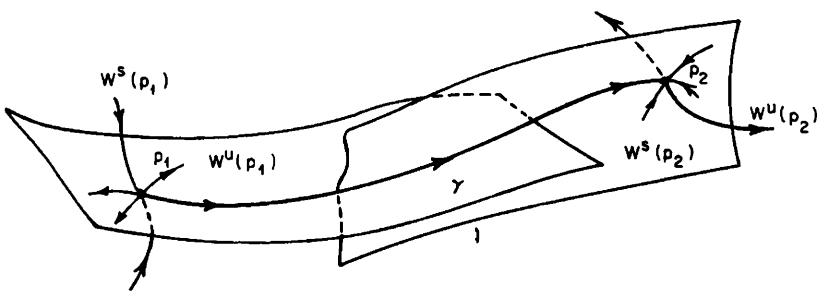


Figure 1.8.4. A transverse heteroclinic orbit in $\mathbb{R}^3$.

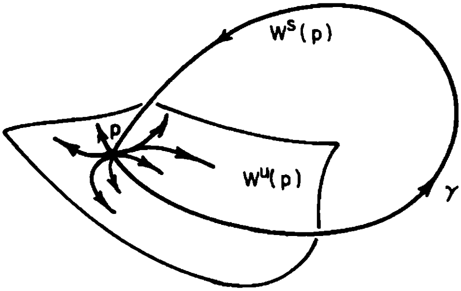


Figure 1.8.5. A nontransverse homoclinic orbit in $\mathbb{R}^3$.

If C is chosen to encircle a single, isolated fixed point, $\bar{x}$, then k is called *the index of* $\bar{x}$. The reader can verify the following statements either by direct examination of the vector fields (cf. Figure 1.8.6) or by evaluation of the curvilinear integral

$$k = \frac{1}{2\pi}\int_C d\left\{\arctan\left(\frac{dy}{dx}\right)\right\} = \frac{1}{2\pi}\int_C d\left\{\arctan\left(\frac{g(x, y)}{f(x, y)}\right)\right\} = \frac{1}{2\pi}\int_C \frac{f\,dg - g\,df}{f^2 + g^2}, \tag{1.8.15}$$

as in Andronov *et al.* [1966, §V.8]:

Proposition 1.8.4.

(i) *The index of a sink, a source or a center is* $+1$.
(ii) *The index of a hyperbolic saddle point is* -1.
(iii) *The index of a closed orbit is* $+1$.
(iv) *The index of a closed curve not containing any fixed points is* 0.
(v) *The index of a closed curve is equal to the sum of the indices of the fixed points within it.*

As a direct corollary to these statements, we find

Corollary 1.8.5. *Inside any closed orbit* γ *there must be at least one fixed point. If there is only one, then it must be a sink or a source. If all the fixed points within* γ *are hyperbolic, then there must be an odd number,* $2n + 1$, *of which* n *are saddles and* $(n + 1)$ *either sinks or sources.*

Degenerate fixed points having indices different from ± 1 are easily constructed. The system

$$\begin{aligned} \dot{x} &= x^2, \\ \dot{y} &= -y, \end{aligned} \tag{1.8.16}$$

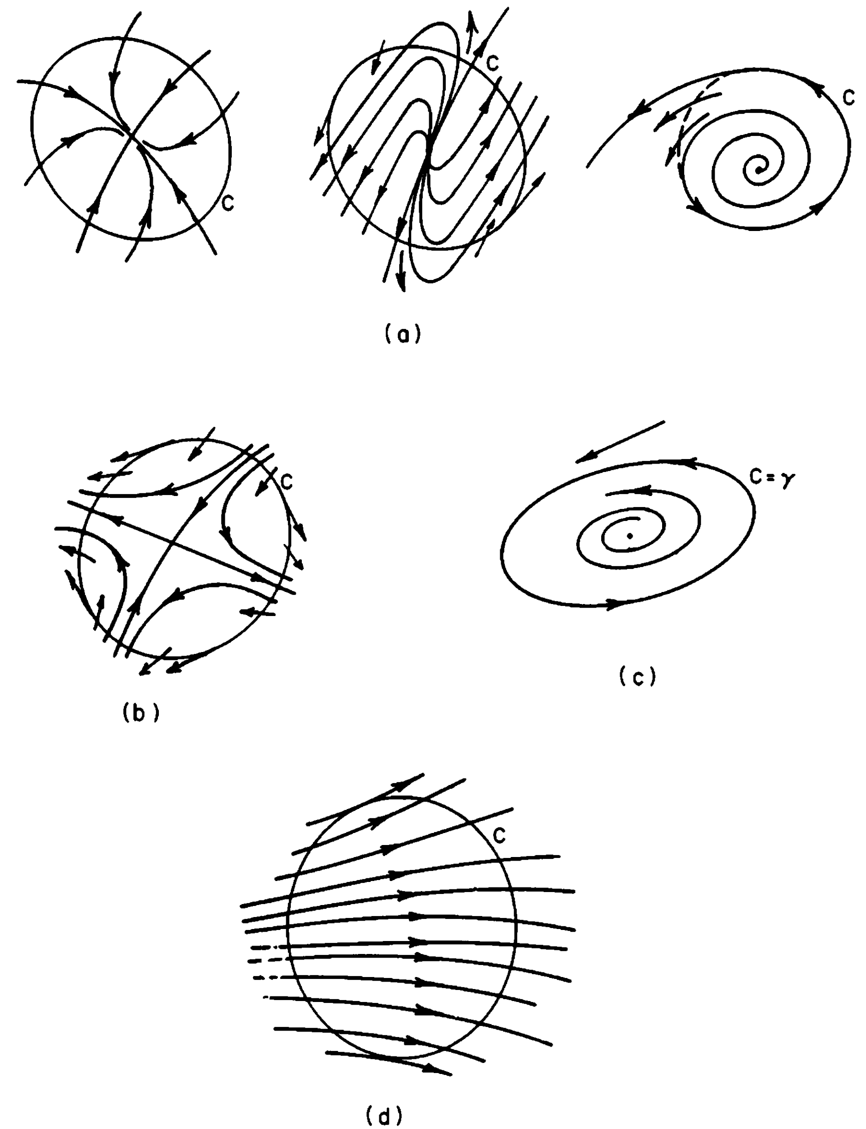
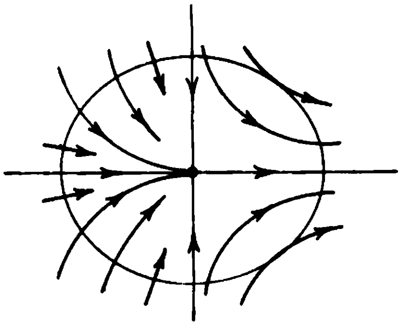


Figure 1.8.6. Indices of fixed points and closed curves. (a) Sinks and sources; (b) a hyperbolic saddle point; (c) closed orbits; (d) C contains no fixed points.

for example, has a degenerate *saddle-node* of index 0 at (0, 0), while that of the system

$$\begin{aligned} \dot{x} &= x^2 - y^2, \\ \dot{y} &= 2xy \end{aligned} \tag{1.8.17}$$

has index 2, cf. Figure 1.8.7.


(a)

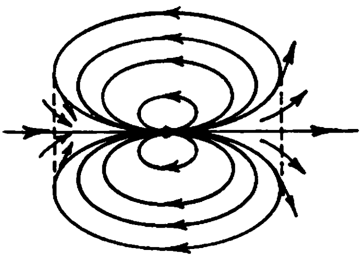

(b)

Figure 1.8.7. Indices of some nonhyperbolic equilibria. (a) The saddle-node; (b) the vector field of (1.8.17).

The following exercise shows how our second example was chosen and analyzed:

EXERCISE 1.8.9. Letting $z = x + iy$, show that the vector fields in the complex plane defined by

$$\dot{z} = z^k \quad \text{and} \quad \dot{z} = \bar{z}^k$$

have unique fixed points at $z = 0((x, y) = (0, 0))$, with indices k and $-k$, respectively. (Here $\bar{z}$ denotes the complex conjugate). (Hint: Write $\dot{x} = \text{Re}(z^k)$, $\dot{y} = \text{Im}(z^k)$ and let $z = re^{i\theta}$.) Sketch the vector fields near such fixed points having indices 3 and -3.

A further simple but useful technique for the global approximation of solution curves is provided by the method of isoclines. Eliminating explicit time dependence from (1.8.1) we obtain the first-order system

$$\frac{dy}{dx} = \frac{g(x, y)}{f(x, y)}. \tag{1.8.18}$$

Neglecting for the moment the fact that (1.8.18) might not be well defined on $f(x, y) = 0$, we seek curves $y = h(x)$ or $x = h(y)$ on which the slope of the vector field $dy/dx = c$ is constant. Such curves are given (perhaps implicitly) by solving the equation

$$g(x, y) = cf(x, y), \tag{1.8.19}$$

and are called *isoclines.*

If a sufficiently close set of isoclines is constructed, then the solutions of (1.8.1) can be sketched fairly accurately. An example will help to illustrate the method:

$$\begin{aligned} \dot{x} &= x^2 - xy, \\ \dot{y} &= -y + x^2. \end{aligned} \tag{1.8.20}$$

We first find the two fixed points at $(x, y) = (0, 0)$ and $(1, 1)$, and ascertain that their linearized matrices are

$$Df(0, 0) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad Df(1, 1) = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}, \tag{1.8.21}$$

with eigenvalues 0, -1, and $\pm i$, respectively. Thus the fixed points are both nonhyperbolic and no conclusions can be drawn from Hartman's theorem. We next note that if $x(0) = 0$ we have $\dot{x} \equiv 0$, and thus the y-axis is an invariant line on which the flow is governed by $\dot{y} = -y$; in fact it is the global stable manifold of $(0, 0)$. The vector field is also vertical on the line $y = x$.

We go on to seek isoclines on which $dy/dx = c \in (-\infty, \infty)$, which are obtained from

$$(-y + x^2) = c(x^2 - xy),$$

or

$$y = \frac{(1 - c)x^2}{1 - cx} \stackrel{\text{def}}{=} h_c(x). \tag{1.8.22}$$

Some of these curves, and the associated directions of the vector field, are sketched in Figure 1.8.8(a). In addition to plotting isoclines, it is sometimes also useful to sketch the vector field on specific lines, such as the line $y = 1$.

While the vectors sketched in Figure 1.8.8(a), together with the knowledge that the y-axis is invariant, give a general indication of the structure of solution curves, detailed information on the local structure near the degenerate fixed points is best obtained by the center manifold methods described in Chapter 3. Application of these techniques, which will follow as exercises and examples in that chapter, show that $(0, 0)$ is the ω limit point for all solutions starting nearby in the left-hand half plane $(x \leq 0)$ and the α limit set for a curve of points nearby with $x > 0$, while use of the Hopf stability formula of Section 3.4, shows that $(1, 1)$ is a (weakly stable) spiral sink. We leave it to the reader to complete the global analysis:

EXERCISE 1.8.10. Prove that (0, 0) is the ω limit point for all points $\{(x, y) \in \mathbb{R}^2 | x \leq 0\}$. Show that, if there are no closed orbits surrounding (1, 1), then (1, 1) is the ω limit point for all points $\{(x, y) \in \mathbb{R}^2 | x > 0\}$. Describe some possible structures involving closed orbits in the right-hand half plane. Can you verify that, in fact, no closed orbits exist? (cf. Figure 1.8.8(b)). (Warning: In numerical integrations, unless very small step sizes are taken, fictitious periodic orbits appear.)

As this example shows, the method of isoclines is rather tedious to apply and often gives incomplete results. Its use is now generally superceded by numerical integration of the system. However, in some cases the idea of

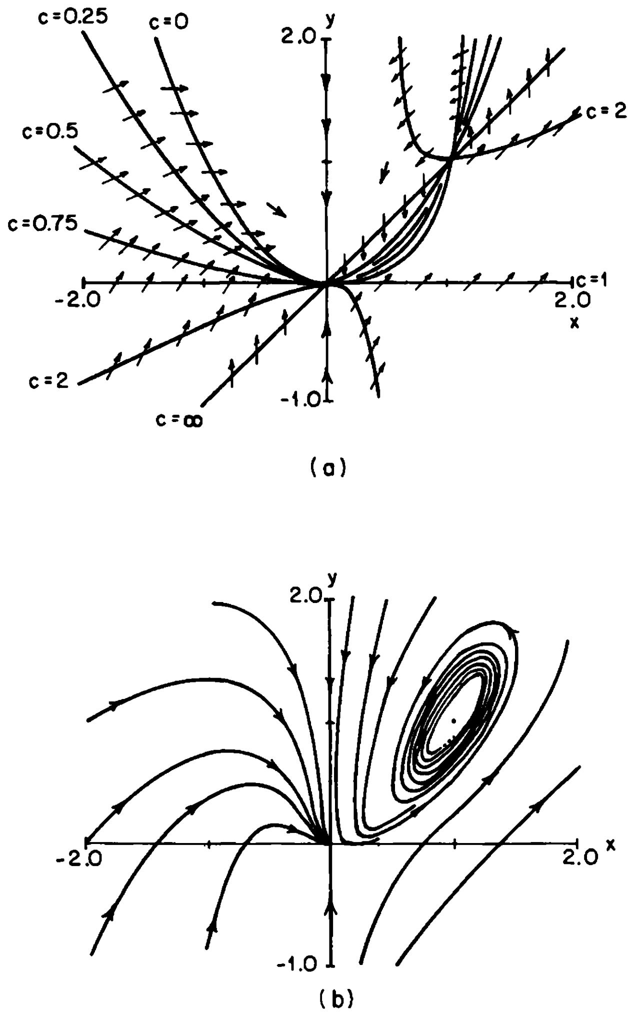


Figure 1.8.8. Isoclines and a partial phase portrait for equation (1.8.20). (a) Isoclines; (b) a numerically computed phase portrait (Runge–Kutta method, step size 0.02).

isoclines and invariant lines can be used to obtain precise information. For example, in the homogeneous cubic system

$$\begin{aligned} \dot{x} &= ax^3 + bxy^2, \\ \dot{y} &= cx^2y + dy^3, \end{aligned} \tag{1.8.23}$$

which we shall meet as the normal form of a degenerate vector field in Chapter 7, we can demonstrate the existence of certain invariant lines in the phase portrait. Clearly the x- and y-axes are invariant, since $\dot{x}(t) \equiv 0$ for all t if $x(0) = 0$ and similarly $\dot{y}(t) \equiv 0$ if $y(0) = 0$. On these lines, the vector fields are simply $\dot{x} = ax^3$ and $\dot{y} = dy^3$, respectively. We claim that, for suitable values of a, b, c, d, further invariant lines passing through the origin can exist. Let them be given by $y = \alpha x$. Then, dividing the two components of (1.8.23) we obtain

$$\frac{dy}{dx} = \frac{y}{x}\frac{(cx^2 + dy^2)}{(ax^2 + by^2)}. \tag{1.8.24}$$

For $y = \alpha x$ to be invariant, we also require that $dy/dx = \alpha$, so that the vector field is everywhere tangent to $y = \alpha x$. Thus, from (1.8.24) we obtain

$$\alpha = \alpha\frac{(cx^2 + d\alpha^2x^2)}{(ax^2 + b\alpha^2x^2)}, \tag{1.8.25}$$

or

$$\alpha^2(d - b) = (a - c),$$

which has the two roots

$$\alpha = \pm\sqrt{(a - c)/(d - b)} \tag{1.8.26}$$

provided that $(a - c)$ and $(d - b)$ have the same sign. The reader should check that on such a line the flow is determined by the one-dimensional system

$$\dot{u} = \left(\frac{ad - bc}{d - b}\right)u^3. \tag{1.8.27}$$

Exercise 1.8.11. Sketch some of the phase portraits for (1.8.23) for various choices of (a, b, c, d). (Refer forward to Section 7.5 if you become discouraged.)

Exercise 1.8.12. Use the method of isoclines to locate the limit cycle of the van der Pol oscillator

$$\ddot{x} + 2(x^2 - 1)\dot{x} + x = 0.$$

We end this section with some examples of systems with nonplanar phase spaces. The first is the inevitable pendulum, which is, in nondimensional variables,

$$\ddot{\theta} + \sin\theta = 0. \tag{1.8.28}$$

This provides a classical example of a system with a nonplanar phase space. The configuration variable $\theta \in [-\pi, \pi)$ is an angle and hence, defining the velocity $\dot{\theta} = v$, the phase space is seen to be the cylinder, and the system becomes

$$\begin{aligned} \dot{\theta} &= v, \\ \dot{v} &= -\sin\theta, \end{aligned} \qquad (\theta, v) \in S^1 \times \mathbb{R}. \tag{1.8.29}$$

This conveniently avoids the embarrassment of an infinite set of distinct equilibrium points at $\theta = \pm n\pi, n = 1, 2, \ldots$, when there are only two physical rest positions at $\theta = 0$ and $\theta = \pi \equiv -\pi$. The phase portrait of (1.8.29) is easily sketched from knowledge of the first integral $H: S^1 \times \mathbb{R} \to \mathbb{R}$:

$$H(\theta, v) = \frac{v^2}{2} + (1 - \cos\theta). \tag{1.8.30}$$

(The constant 1 need not be included, but originates in the physical problem, in which the potential energy $V(\theta)$ is of the form $mgl(1 - \cos\theta)$ for a pendulum of mass m and length l, where θ is measured from the downward vertical, so that $V(0) = 0$.) The phase portrait is, of course, periodic in θ with period 2π (Figure 1.8.9).

The cylindrical phase space is obtained by identifying $\theta = -\pi(AA')$ and $\theta = +\pi(BB')$. A nice sketch appears in Andronov *et al.* [1966, p. 98]. It is important to recognize that orbits such as that marked ab in Figure 1.8.9 are in fact closed orbits which *encircle* the cylinder: such orbits correspond to rotary rather than oscillatory motions of the pendulum, and the two classes of motions are separated by the homoclinic orbits to the saddle point.

Exercise 1.8.13. Sketch the phase portrait for the damped pendulum $\ddot{\theta} + 2\alpha\dot{\theta} + \sin\theta = 0$, $0 \le \alpha \ll 1$, and the pendulum $\ddot{\theta} + \sin\theta = \beta$, subjected to applied torque $\beta > 0$. Consider $\beta < 1$ and $\beta > 1$. Also consider the damped pendulum with torque $\ddot{\theta} + 2\alpha\dot{\theta} + \sin\theta = \beta$. Both undamped systems possess homoclinic and periodic orbits; can the damped systems possess any such orbits? (This is quite difficult. Refer forward to Section 4.6 if you like.)

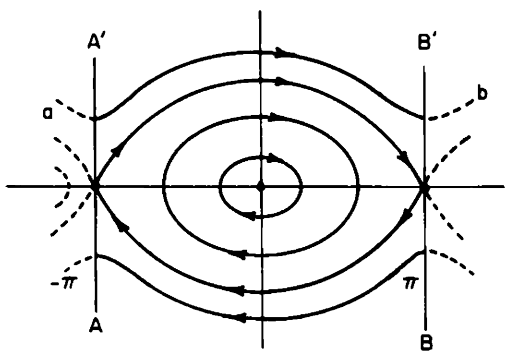


Figure 1.8.9. The phase portrait for the simple pendulum.

In addition to its application in classical mechanics, the system $\ddot{\theta} + 2\alpha\dot{\theta} + \sin\theta = \beta$ provides a discrete (single mode) approximation to the sine-Gordon equation, which is important in physics as a model of wave functions in the superconducting Josephson junction (cf. Levi *et al.* [1978]). We will return to this example in Section 4.6.

A second canonical example of a system defined on a two-dimensional manifold is the flow on a torus $T^2 = S^1 \times S^1$:

$$\begin{aligned}\dot{\theta} &= f(\theta, \phi);\\ \dot{\phi} &= g(\theta, \phi);\end{aligned} \qquad (\theta, \phi) \in T^2; f, g(2\pi)\text{-periodic in } \theta, \phi. \tag{1.8.31}$$

A special and important case is the linear system:

$$\begin{aligned}\dot{\theta} &= a,\\ \dot{\phi} &= b\end{aligned} \tag{1.8.32}$$

As is well known, if a and b are rationally related then one has a continuous family of periodic orbits on T^2, whereas if a/b is irrational, one obtains dense nonperiodic orbits. Since the irrational and rational numbers are each dense in $\mathbb{R}$, either of these two topologically distinct cases can be approximated arbitrarily closely by the other, and hence the system is structurally unstable for *all* values of a, b.

The linear flow (1.8.32) is more important than its special form might suggest, since any (nonlinear) flow on T^2 with no equilibrium points or closed orbits necessarily arises from a vector field C^0 equivalent to (1.8.32) with a/b irrational. We will return to this in Section 6.2.

Flows on two-tori also occur in linear undamped systems with two degrees of freedom, as follows. Consider the system

$$\ddot{x} + \omega_1^2 x = 0, \qquad \ddot{y} + \omega_2^2 y = 0, \tag{1.8.33}$$

where we have written the equations in the canonical (normal mode) coordinates, so that the two modes are uncoupled. The system possesses two independent first integrals:

$$H_1(x, \dot{x}) = \frac{\dot{x}^2}{2} + \frac{\omega_1^2 x^2}{2} = k_1, \qquad H_2(y, \dot{y}) = \frac{\dot{y}^2}{2} + \frac{\omega_2^2 y^2}{2} = k_2, \tag{1.8.34}$$

each of which remains constant as the four-dimensional solution vector $(x(t), \dot{x}(t), y(t), \dot{y}(t))$ evolves with time. The two integral constraints imply that solutions are confined to a two-dimensional torus which is the product of the two ellipses in $(x, \dot{x})$ and $(y, \dot{y})$ spaces given by the constraints. To see this, we perform coordinate changes to action angle variables (Goldstein [1980], Arnold [1978]):

$$\begin{aligned} x &= \sqrt{\frac{2I_1}{\omega_1}}\sin\theta_1, & \dot{x} &= \sqrt{2\omega_1 I_1}\cos\theta_1,\\ y &= \sqrt{\frac{2I_2}{\omega_2}}\sin\theta_2, & \dot{y} &= \sqrt{2\omega_2 I_2}\cos\theta_2,\end{aligned} \tag{1.8.35}$$

to obtain

$$\begin{aligned} \dot{I}_1 &= 0, & \dot{I}_2 &= 0, \\ \dot{\theta}_1 &= \omega_1, & \dot{\theta}_2 &= \omega_2, \end{aligned} \tag{1.8.36}$$

which, for initial conditions $(I_1^0, I_2^0, \theta_1^0, \theta_2^0)$, has the solution

$$\begin{gathered} I_1(t) \equiv I_1^0, \, I_2(t) \equiv I_2^0, \\ \theta_1(t) = \omega_1 t + \theta_1^0, \qquad \theta_2(t) = \omega_2 t + \theta_2^0. \end{gathered} \tag{1.8.37}$$

Thus, the four-dimensional phase space is filled with two-dimensional tori given by $I_1 = I_1^0$, $I_2 = I_2^0$ and each torus carries rational or irrational flow, depending on the ratio ω_1/ω_2. In general, n degree of freedom integrable Hamiltonian systems give rise to flows on n-dimensional tori (cf. Arnold [1978], Goldstein [1980], for examples). The fundamental paper on toral flows by Kolmogorov [1957], which is the basis of the KAM theory, is reprinted in Abraham and Marsden [1978] (cf. Sections 4.8 and 6.2).

EXERCISE 1.8.14. Consider the Hamiltonian system on $\mathbb{R}^4$ with Hamiltonian $H(x, \dot{x}, y, \dot{y}) = \dot{x}^2/2 + \dot{y}^2/2 + x^2/2 + x^3/3 + y^2/2 - y^3/3$. Show that there are two independent integrals and describe the orbit structures of the system. (This represents a special case of the "anti-Hénon–Heiles" system (cf. Hénon and Heiles [1964], Aizawa and Saitô [1972]).)

Flows on two-tori (and, more generally, n-tori) also arise in studies of coupled nonconservative limit cycle oscillators. For example, consider two identical van der Pol oscillators coupled by weak linear interaction, β, with weak de-tuning, δ:

$$\begin{aligned} \ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x &= \beta(y - x), \\ \ddot{y} + \varepsilon(y^2 - 1)\dot{y} + y &= \beta(x - y) - \delta y, \end{aligned} \qquad 0 \le |\delta|, |\beta| \ll \varepsilon \ll 1. \tag{1.8.38}$$

For δ, $\beta = 0$ it is known that each van der Pol oscillator possesses an attracting limit cycle given (approximately) by

$$\begin{aligned} x(t) &= 2\cos(t + \theta_1^0), & \dot{x}(t) &= -2\sin(t + \theta_1^0), \\ y(t) &= 2\cos(t + \theta_2^0), & \dot{y}(t) &= -2\sin(t + \theta_2^0), \end{aligned} \tag{1.8.39}$$

where θ_1^0, θ_2^0 are arbitrary (phase) constants determined by the initial conditions (cf. Section 2.1). The product $S^1 \times S^1$ of the two circles of radius 2 in the $(x, \dot{x})$ and $(y, \dot{y})$ planes is a two-torus $T^2 \subset \mathbb{R}^4$. However, unlike the members of the two-parameter family of two-tori in the Hamiltonian example above, this one is an *attractor*; in fact, nearby orbits approach it exponentially fast and as we shall see, it persists under perturbations. Thus a small perturbation, such as the addition of weak coupling (β, $\delta \ll \varepsilon$), cannot destroy the torus as a whole, which remains an attracting set.

However, since the vector field on the torus may be written

$$\dot{\theta}_1 = -1, \qquad \dot{\theta}_2 = -1, \qquad (\theta_1, \theta_2) \in T^2, \tag{1.8.40}$$

it carries linear (rational) flow, which is structurally unstable. Thus, while the torus as a whole is preserved, the structure of orbits within it changes radically when the oscillators are coupled. Rand and Holmes [1980] have studied (1.8.38) and generalizations of it, and show if $|\beta| > |\delta/2| > 0$, then there are precisely two hyperbolic periodic orbits on the torus, one an attractor and the other a repellor. Such a situation is called 1:1-phase locking or entrainment. Many other studies of phase locking have been carried out, see Nayfeh and Mook [1979] and references therein for examples.

The perturbation schemes employed typically neglect high-order terms (terms of $O(\varepsilon^2)$), but since the coupled flow with two hyperbolic orbits is structurally stable (cf. Peixoto's theorem in the next section), the result is qualitatively correct for the full system, for the addition of the neglected terms constitutes a small perturbation. However, for $|\delta/2| > |\beta|$, the approximate analysis predicts that the flow on the torus will contain no attractors or repellors, but that all orbits will either be periodic or there will be an orbit dense on T^2. This structurally unstable situation tells us nothing (directly) about the true flow on T^2. In fact one can expect a very complex sequence of bifurcations to occur directly after phase locking breaks. We shall discuss such situations in Section 6.2.

EXERCISE 1.8.15. Construct a structurally stable system on T^2 with two closed orbits.

1.9. Peixoto's Theorem for Two-Dimensional Flows

With various examples of two-dimensional flows in mind, we are now ready to state and sketch the proof of Peixoto's theorem, which represents the culmination of much previous work, in particular that of Poincaré [1899] and Andronov and Pontryagin [1937]. Letting $\mathscr{X}^r(M^2)$ denote the set of all C^r vector fields on two-dimensional manifolds, we have

Theorem 1.9.1 (Peixoto [1962]). *A C^r vector field on a compact two-dimensional manifold M^2 is structurally stable if and only if:*

(1) *the number of fixed points and closed orbits is finite and each is hyperbolic;*
(2) *there are no orbits connecting saddle points;*
(3) *the nonwandering set consists of fixed points and periodic orbits alone.*

Moreover, if M^2 is orientable, the set of structurally stable vector fields is open-dense in $\mathscr{X}^r(M^2)$.

One can deal with planar fields provided that there is a compact set $D \subset \mathbb{R}^2$ such that the flow is directed inward (or outward) on the boundary of D, otherwise it is easy to construct systems with countably many fixed points or closed orbits. We also remark that, if the phase space is planar, then conditions (1) and (2) automatically imply that (3) is satisfied, since there are no limit sets possible other than fixed points, closed orbits, and homoclinic cycles, and the latter are excluded by (2).

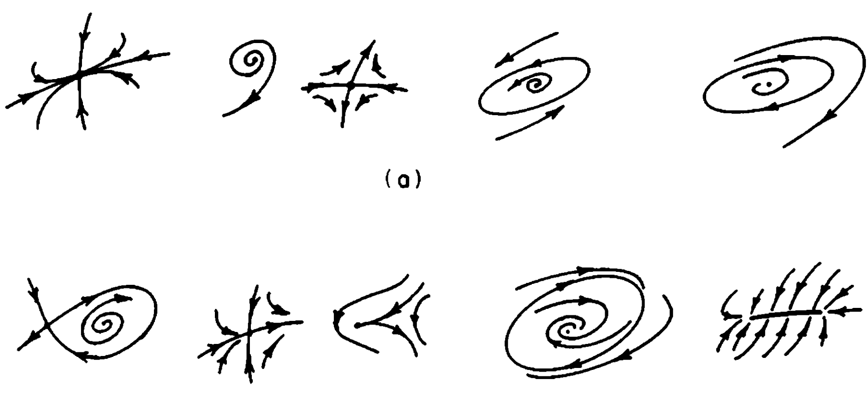
Peixoto's theorem implies that typically a two-dimensional vector field will contain only sinks, saddles, sources, and repelling and attracting closed orbits in its invariant set, Figure 1.9.1(a). Structural stability is a generic property for two-dimensional flows on orientable manifolds. Many of the ingredients of Peixoto's theorem were proved by Andronov and his coworkers (cf. Andronov *et al.* [1966]) in the decades following 1935. Here we shall sketch the proof of the structural stability part.

The first condition (hyperbolicity of fixed points and periodic orbits) follows from a consideration of the linearized flow or of suitable Poincaré maps. It can be shown that the sets of such linear flows and maps contain open dense sets of hyperbolic flows and maps, respectively (cf. Hirsch and Smale [1974], Chapter 7). Thus, if a nonlinear flow contains, say, a non-hyperbolic fixed point, then a small perturbation suffices to render that point hyperbolic; similarly, a hyperbolic fixed point remains hyperbolic under all sufficiently small perturbations.

The second condition is demonstrated as follows. Suppose two saddles p_1, p_2 were connected, so that $W^u(p_1) \cap W^s(p_2) = \Gamma$ for the flow of the vector field

$$\dot{x}_1 = f_1(x_1, x_2),$$
$$\dot{x}_2 = f_2(x_1, x_2).$$

(cf. Figure 1.9.2). We perturb $(f_1(x_1, x_2), f_2(x_1, x_2))$ by addition of a field $(\varepsilon\phi_1(x_1, x_2), \varepsilon\phi_2(x_1, x_2))$ having compact support, vanishing outside some



(a)

(b)

Figure 1.9.1. (a) Some structurally stable nonwandering sets on $\mathbb{R}^2$; (b) some structurally unstable nonwandering sets on $\mathbb{R}^2$.

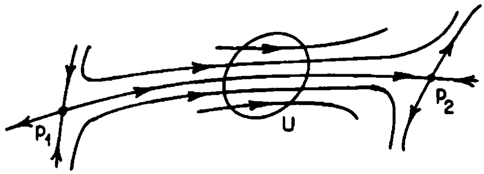


Figure 1.9.2. A saddle connection.

(small) region U chosen to straddle Γ as shown. It is easy to choose the perturbation such that orbits entering U are all "pushed upward" (or downward), causing Γ to break; Figure 1.9.3. Similarly, if the two such manifolds do not intersect, then a sufficiently small perturbation cannot cause them to intersect.

The third condition is necessary to exclude structurally unstable nonwandering sets such as the torus T^2 with irrational flow, as occurs in the linear flow

$$\begin{aligned}\dot{\theta} &= a,\\ \dot{\phi} &= b,\end{aligned} \qquad (\theta, \phi) \in T^2, \tag{1.9.1}$$

with a/b irrational (cf. the discussions in Section 1.8 above). Of course, if one has rational flow on T^2 then the torus is filled with a continuous family of nonhyperbolic closed orbits.

The proof that the set of structurally stable flows on orientable manifolds is open dense is more difficult and involves the closing lemma of Pugh [1967a, b]. We will not sketch this here, but see Palis and de Melo [1982], for example

EXERCISE 1.9.1. Sketch phase portraits for the family $\dot{x} = \mu + x^2 - xy$, $\dot{y} = y^2 - \frac{1}{2}x^2 - 1$ and show that a saddle connection exists for $\mu = 0$. What happens for $\mu > 0$; $\mu < 0$? (Cf. Guckenheimer [1973] and Section 6.1.)

Even though Peixoto's theorem guarantees that, in generic families of planar systems, the structurally stable ones occupy a set of full measure, the occurrence of infinitely many unstable (bifurcation) systems in some neighborhood cannot be excluded. The following example is due to Jacob Palis (who proposed it in connection with moduli of saddle connections, a topic

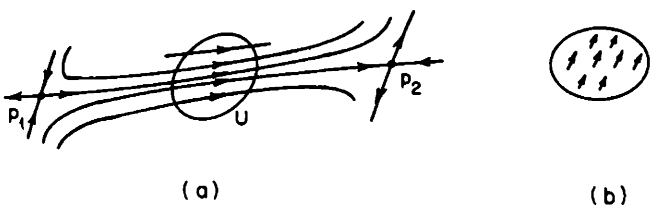


Figure 1.9.3. (a) A broken connection; (b) $(\varepsilon\phi_1, \varepsilon\phi_2)$.

which we do not discuss in this book). Consider the one-parameter family $\dot{x} = f_\mu(x)$, $x \in \mathbb{R}^2$, with phase portraits as indicated in Figure 1.9.4(a)–(c) for $\mu < 0$, $\mu = 0$, $\mu > 0$. For $\mu < 0$ there are two hyperbolic closed orbits, and at $\mu = 0$ these coalesce in a single "semistable" orbit, which is the ω limit set for nearby points inside it and the α limit set for nearby points outside. For $\mu > 0$ no closed orbits exist. In an annular strip containing the orbits, all we see is a local saddle-node bifurcation of closed orbits, but *globally* the stable and unstable manifolds of the saddle points r and q are involved in a crucial manner. For $\mu < 0$ the α limit set for points on the left-hand branch $W_l^s(r)$ of $W^s(r)$ is the repelling closed orbit γ_2, while the ω limit set for points on both branches of $W^u(q)$ is the attracting closed orbit γ_1. At $\mu = 0$ these two orbits merge in γ_0, which is the ω limit set for points in $W^u(q)$ and the α limit set for points in $W_l^s(r)$. To see what happens to $W_l^s(r)$ and $W^u(q)$ when $\mu > 0$, we take a local section, Σ, as indicated in Figure 1.9.4(b). For $\mu = 0$, γ_0 pierces Σ at p_0, and the sets $W_l^s(r) \cap \Sigma$ and $W_l^u(q) \cap \Sigma$, $W_r^u(q) \cap \Sigma$ are each countable sequences of points accumulating at p_0, the former from above, the latter two from below; cf. Figure 1.9.5(a), (b). Let $W_l^s(r) \cap \Sigma = \{r_i\}_{i=1}^\infty$, $W_l^u(q) \cap \Sigma = \{q_i\}_{i=1}^\infty$, $W_r^u(q) \cap \Sigma = \{q_i'\}_{i=1}^\infty$.

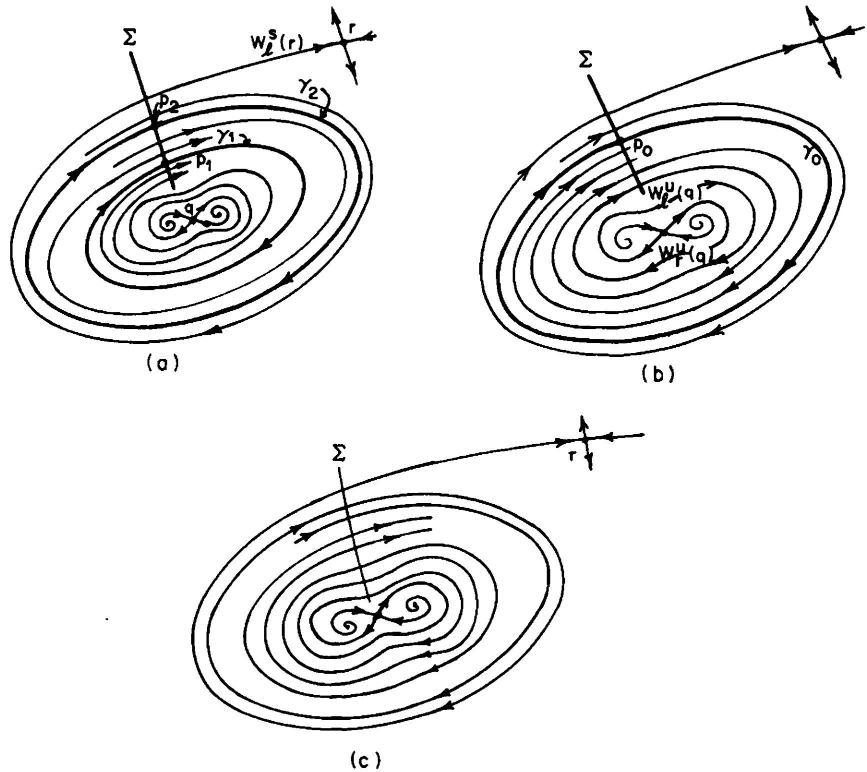


Figure 1.9.4. (a) $\mu < 0$; (b) $\mu = 0$; (c) $\mu > 0$.

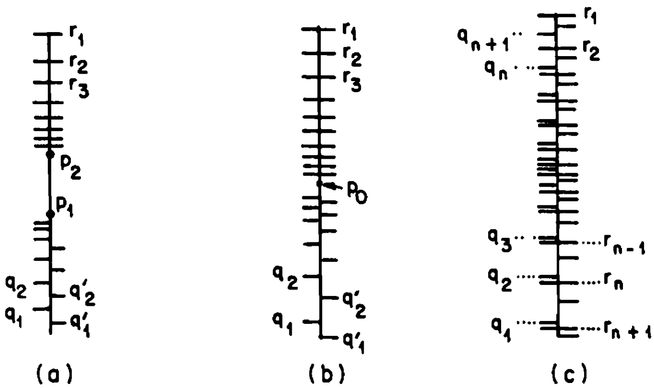


Figure 1.9.5. The cross section. (a) $\mu < 0$; (b) $\mu = 0$; (c) $\mu > 0$.

We have a Poincaré map defined on some neighborhood $U \subset \Sigma$ and, by our construction, $P^{-1}(r_i) = r_{i+1}$, $P(q_i) = q_{i+1}$, $P(q_i') = q_{i+1}'$. Clearly, for $\mu > 0$, all orbits pass through the annular region and thus leave the top of Σ going forward, and the bottom going backward, after a finite number of iterates. (This number goes to infinity on $\mu \to 0^+$.) Thus, between $\mu = 0$ and $\mu = \varepsilon$, for any $\varepsilon > 0$, a countably infinite set of points r_j, q_j, q_j', $j \geq N$, must pass each other on Σ. Hence, in the interval $\mu \in [0, \varepsilon]$, there are countably many heteroclinic saddle connection bifurcations, and the bifurcation set is a countable sequence of points μ_i accumulating on $\mu = 0$ from above. However, since the structurally unstable systems occur at isolated points μ_i, we still have an open dense set of structurally stable systems in the neighborhood of $\dot{x} = f_0(x)$.

With Peixoto's theorem in mind Smale proposed that one might study systems on compact n-manifolds satisfying conditions (1) and (3) of Theorem 1.9.1 but with (2) suitably modified in the light of Theorem 1.8.3. Such systems are now called *Morse–Smale* systems.

Definition. A Morse–Smale system is one for which:

(1) the number of fixed points and periodic orbits is finite and each is hyperbolic;
(2) all stable and unstable manifolds intersect transversally;
(3) the nonwandering set consists of fixed points and periodic orbits alone.

In the definition of transversal intersection, we include the empty set, for clearly if two manifolds do not intersect (i.e., are bounded away from each other), then a small perturbation cannot cause them to intersect.

The following conjectures were then proposed:

A system is structurally stable if and only if it is Morse–Smale:
Morse–Smale systems are dense in $\mathrm{Diff}^1(M)$ or $\mathscr{X}^1(M)$;
Structurally stable systems are dense in $\mathrm{Diff}^1(M)$ or $\mathscr{X}^1(M)$.

(Here $\mathrm{Diff}^r(M)$ (resp. $\mathscr{X}^r(M)$) denotes the set of all C^r diffeomorphisms (resp. vector fields) on finite dimensional manifolds M.) In the following pages we shall study examples of systems which show that all three conjectures are false. All that can be salvaged is part of Conjecture 1: Morse–Smale systems *are* structurally stable (the converse is false). One of the major contributions to the fall of Conjectures 1 and 2 was Smale's construction of the horseshoe map: a two-dimensional diffeomorphism with a complicated invariant set which was suggested by certain problems in forced oscillations. Before meeting this map in Chapter 5, we shall consider some examples of three-dimensional systems, including periodically forced single degree of freedom oscillators, which have very complicated solution structures. These systems provide additional counter-examples to the conjectures above, and they are therefore of historical as well as practical interest.