Problems on hyperbolic sets

1. Show that the map $(x, y) \mapsto (\bar{x}, \bar{y})$ where

$$\bar{x} = y, \qquad \bar{y} = 7 - x - 8\sin y$$

has a Smale horseshoe.

2. Show that the map $(x, y) \mapsto (\bar{x}, \bar{y})$ where

$$\bar{x} = y, \qquad \bar{y} = -x + 7y - y^3$$

has infinitely many periodic orbits. How many orbits of the least period 6 does the map have?

3. Let Λ_2 be the set of all points whose entire orbits by the map

$$f: \qquad \bar{x} = y, \qquad \bar{y} = 11 - x - y^2$$

are bounded, and Λ_3 be the set of all points whose entire orbits by the map

$$g: \qquad \bar{x} = y, \qquad \bar{y} = -x + 7y - y^3$$

are bounded. Compute the topological entropy of f on the set Λ_2 and of g on Λ_3. Show that f on Λ_2 is not topologically conjugate to g onΛ_3. Show that Λ_3 has a subset Λ' such that f on Λ_2 is topologically conjugate to g onΛ'. Show that Λ_2 has no subset on which f would be topologically conjugate to g onΛ_3.

4. Show that the sets Λ_2 and Λ_3 are homeomorphic. Hint: show that any zero-dimensional hyperbolic set generated by a Markov partition is homeomorphic to the standard Cantor set.

5. For the Markov partition shown in the figure, compute the topological entropy of the corresponding hyperbolic set Λ. Let P_k be the set of all points of period k in Λ. Compute

$$\limsup_{k\to+\infty} \frac{\ln P_k}{k} \quad \text{and} \quad \liminf_{k\to+\infty} \frac{\ln P_k}{k}.$$

Prove that no orbit is dense in Λ.