1. (i) Determine how many periodic orbits and equilibria can be born at the bifurcations of the zero equilibrium of the following system:

$$\begin{cases} \dot{x} = y - x^2, \\ \dot{y} = z + xy, \\ \dot{z} = -y - z + x^2 - xy + y^2 + z^2 - x^4. \end{cases}$$

(ii) Determine how many periodic orbits does the following system have at small $\varepsilon > 0$ in a small neighbourhood of zero:

$$\begin{cases} \dot{x} = y + y^2, \\ \dot{y} = \varepsilon y - x + x^3. \end{cases}$$

2. Consider the system

$$\begin{cases} \dot{x} = -2x + 16y^2, \\ \dot{y} = -z + xy - 4y^2 z - 2z^2 y, \\ \dot{z} = y. \end{cases}$$

(i) Write down the normal form up to the terms of the third order for the system on the center manifold near the equuilibrium $(0, 0, 0)$.

(ii) Is the equilibrium stable or unstable? How many stable periodic orbits can be born at the bifurcations of this equilibrium?

3. Consider the map

$$\bar{x} = a - bx - x^3.$$

(i) Find equations of the bifurcation curves for the fixed points of this map and draw these curves in the plane of parameters (a, b).

(ii) On the curve which corresponds to the existence of a fixed point with the multiplier equal to -1, find the points from which a bifurcation curve emanates which corresponds to a period-2 point with a multiplier $+1$.

4. It is known from numerical experiments that in the plane of parameters (a, b) there is a curve C such that the system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = ax - by - z + x^2 \end{cases}$$

has a homoclinic loop to the equilibrium state $O(0,0,0)$ at $(a, b) \in C$. This curve intersects the line $a = b + 2$ at a point $P^*(a^*, b^*)$ with $b^* > 0$ and $a^* > 2$.

Consider a sufficiently small neighbourhood U of P^* in the (a, b) plane. The line $a = b + 2$ divides U into two halves:
$U^+ = U \cap \{a = b + 2 + \varepsilon,\ \varepsilon > 0\}$ and $U^- = U \cap \{a = b + 2 + \varepsilon,\ \varepsilon < 0\}$.

(i) Let $a = b + 2 + \varepsilon$ and $b > 0$. For the eigenvalues $\lambda_{1,2,3}$ of the linearisation matrix at the equilibrium state O, find their expansion in ε up to the first order.

(ii) Show that infinitely many periodic orbits exist at $(a, b) \in C \cap U^+$.

(iii) Show that a single stable periodic orbit is born at the bifurcations of the homoclinic loop which exists at $(a, b) \in C \cap U^-$.

Solutions:

1. (i) (10 points) Determine how many periodic orbits and equilibria can be born at the bifurcations of the zero equilibrium of the following system:

$$\dot{x} = y - x^2, \quad \dot{y} = z + xy, \quad \dot{z} = -y - z + x^2 - xy + y^2 + z^2 - x^4.$$

Solution (seen similar): The linearisation matrix at the equilibrium is $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}$. The characteristic equation is $\lambda(\lambda^2 + \lambda + 1) = 0$. It has one zero root and two roots with non-zero real parts. The center manifold is therefore one-dimensional, hence no periodic orbits can be born at bifurcations.

In order to find equilibria of this system, one can find, from the first two equations, y and z as functions of x. Namely, $y = x^2$ and $z = -x^3$. Plugging this into the third equation, we obtain the equation $x^6 = 0$. Zero is a multiplicity 6 root of this equation, so up to 6 solutions (=up to 6 equilibria) can be obtained by small perturbations.

(ii) (10 points) Determine how many periodic orbits does the following system have at small $\varepsilon > 0$ in a small neighbourhood of zero:

$$\dot{x} = y + y^2, \qquad \dot{y} = \varepsilon y - x + x^3.$$

Solution (seen similar): The linearisation matrix is $\begin{pmatrix} 0 & 1 \\ -1 & \varepsilon \end{pmatrix}$. At $\varepsilon = 0$ there is a pair of pure imaginary eigenvalues, and at $\varepsilon \neq 0$ the real part of the eigenvalues becomes non-zero. At $\varepsilon = 0$ the system has an integral: $H(x, y) = \dfrac{y^2}{2} + \dfrac{y^3}{3} + \dfrac{x^2}{2} - \dfrac{x^4}{4}$, hence all orbits near zero are closed. It follows then from the Hopf theorem that no periodic orbits exist near zero at small $\varepsilon \neq 0$.

2. Consider the system

$$\dot{x} = -2x + 16y^2, \quad \dot{y} = -z + xy - 4y^2z - 2z^2y, \quad \dot{z} = y.$$

(i) (10 points) Write down the normal form up to the terms of the third order for the system on the center manifold near the equuilibrium $(0, 0, 0)$.

Solution (seen similar): The linearisation matrix is $\begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$. The eigenvalues are $\lambda_1 = -2$ and $\lambda_{2,3} = \pm i$. The center manifold is tangent to the (y, z) plane.

Let us make a coordinate transformation $X = x + ay^2 + byz + cz^2$ in order to kill the quadratic term in the first equation:

$$\dot{X} + 2X = -2x + 16y^2 + 2x + 2ay^2 + 2byz + 2cz^2 + 2ay\dot{y} + b\dot{y}z + by\dot{z} + 2cz\dot{z} =$$

$$= (16 + 2a)y^2 + 2byz + 2cz^2 - 2ayz - bz^2 + by^2 + 2cyz + \text{ third order terms.}$$

There will be now quadratic terms if $16 + 2a + b = 0$, $b - a + c = 0$ and $2c - b = 0$, i.e. if we choose $c = -2$, $b = -4$, $a = -6$. In this case the equation of the center manifold is $X = O(|y|^3 + |z|^3)$, hence $x = 6y^2 + 4yz + 2z^2 + O(|y|^3 + |z|^3)$. The system on the center manifold is

$$\dot{z} = y, \quad \dot{y} = -z + 6y^3 + O(y^4 + z^4).$$

Denote $u = y + iz$, hence $y = (u + u^*)/2$. The system takes the form

$$\dot{u} = iu + \frac{3}{4}(u + u^*)^3 + O(|u|^4).$$

By removing the non-resonant terms, we obtain the normal form

$$\dot{u} = iu + \frac{9}{4}u^2u^* + O(|u|^4).$$

(ii) (10 points) Is the equilibrium stable or unstable? How many stable periodic orbits can be born at the bifurcations of this equilibrium?

Solution (seen similar): The first Lyapunov coefficient $L_1 = 9/4$ is strictly positive. Therefore the equilibrium is unstable and no stable periodic orbits can be born.

3. Consider the map

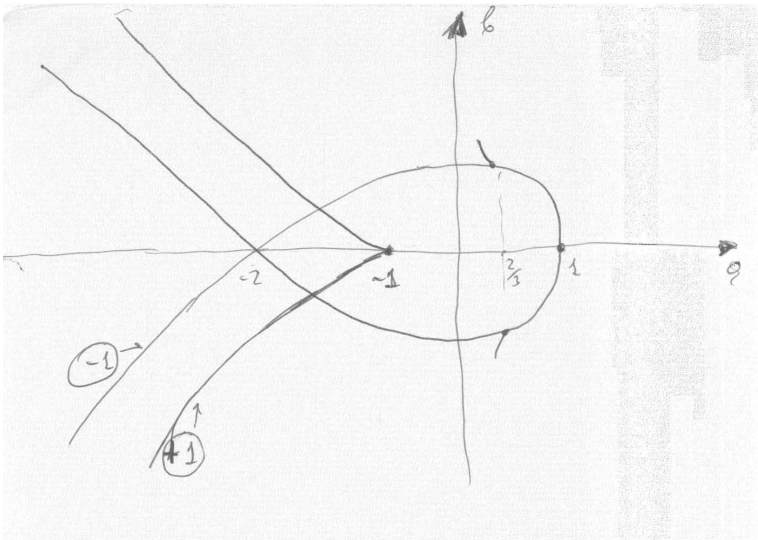
$$\bar{x} = a - bx - x^3.$$

(i) Find equations of the bifurcation curves for the fixed points of this map and draw these curves in the plane of parameters (a, b).

Solution (seen similar): The fixed point is given by the equation $a = (b+1)x + x^3$, and bifurcations happen at $-b - 3x^2 = \pm 1$. Express x from the second equation: $x^2 = -\dfrac{b \pm 1}{3}$. Then the first equation provides the equations for the bifurcation curves:

multiplier $+1$: $\quad a = \pm\dfrac{2}{3}(b+1)\sqrt{-\dfrac{b+1}{3}}, \quad b \leq -1,$

multiplier -1: $\quad a = \pm\dfrac{2}{3}(b+2)\sqrt{\dfrac{1-b}{3}}, \quad b \leq 1.$



(ii)(10 points) On the curve which corresponds to the existence of a fixed point with the multiplier equal to -1, find the points from which a bifurcation curve emanates which corresponds to a period-2 point with a multiplier $+1$.

Solution (unseen/ seen parts): Shift the origin to the fixed point $x_0 = \pm\sqrt{\dfrac{1-b}{3}}$, i.e. put $x = x_0 + y$. The map takes the form

$$\bar{y} = -y - 3y^2 x_0 - y^3.$$

The second iteration is

$$\bar{\bar{y}} = -\bar{y} - 3\bar{y}^2 x_0 - \bar{y}^3 = y + (2 - 18x_0^2)y^3 + O(y^4).$$

The period-2 points with multiplier $+1$ can be born only if the coefficient of y^3 vanishes, i.e. at $x_0^2 = 1/9$, which corresponds to $b = 2/3$, $a = \pm 16/27$.

4. It is known from numerical experiments that in the plane of parameters (a, b) there is a curve C such that the system

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = ax - by - z + x^2$$

has a homoclinic loop to the equilibrium state $(0, 0, 0)$ at $(a, b) \in C$. This curve intersects the line $a = b + 2$ at a point $P^*(a^*, b^*)$ with $b^* > 0$ and $a^* > 2$.

Consider a sufficiently small neighbourhood U of P^* in the (a, b) plane. The line $a = b + 2$ divides U into two halves:
$U^+ = U \cap \{a = b + 2 + \varepsilon,\ \varepsilon > 0\}$ and $U^- = U \cap \{a = b + 2 + \varepsilon,\ \varepsilon < 0\}$.

(i) (6 points) Let $a = b + 2 + \varepsilon$ and $b > 0$. For the eigenvalues $\lambda_{1,2,3}$ of the linearisation matrix at the equilibrium state O, find their expansion in ε up to the first order.

(ii) (7 points) Show that infinitely many periodic orbits exist at $(a, b) \in C \cap U^+$.

(iii) (7 points) Show that a single stable periodic orbit is born at the bifurcations of the homoclinic loop which exists at $(a, b) \in C \cap U^-$.

Solution (unseen): (i) The linearisation matrix is $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & -b & -1 \end{pmatrix}$. The characteristic equation is

$$\lambda^3 + \lambda^2 + b\lambda - a = 0,$$

or, at $a = b + 2 + \varepsilon$,

$$(\lambda - 1)(\lambda^2 + 2\lambda + b + 2) = \varepsilon. \qquad (*)$$

At $\varepsilon = 0$ and $b > 0$ the eigenvalues are $\lambda_1 = 1$ and $\lambda_{2,3} = -1 \pm i\sqrt{b+1}$. In order to see how the eigenvalues change as ε varies across zero, differentiate both sides of (*) with respect to ε at $\varepsilon = 0$. We obtain

$$\frac{d\lambda_1}{d\varepsilon}(\lambda_1^2 + 2\lambda_1 + b + 2) = \frac{d\lambda_1}{d\varepsilon}(b + 5) = 1 \implies \lambda_1 = 1 + \frac{\varepsilon}{b+5} + O(\varepsilon^2)$$

and

$$2\frac{d\lambda_2}{d\varepsilon}(\lambda_2^2 - 1) = -2(b + 1 + 2i\sqrt{b+1})\frac{d\lambda_2}{d\varepsilon} = 1 \implies \lambda_2 = -1 - \frac{\varepsilon(1 - 2i/\sqrt{b+1})}{2(b+5)} + O(\varepsilon^2).$$

(ii) We have

$$\lambda_1 + \mathrm{Re}\ \lambda_2 = \frac{\varepsilon}{2(b+5)} + O(\varepsilon^2).$$

This value is positive at small $\varepsilon > 0$, and $\lambda_{2,3}$ are not real, hence infinitely many periodic orbits coexist with the homoclinic loop.

(iii) At small $\varepsilon < 0$ we have $\lambda_1 + \mathrm{Re}\ \lambda_2 < 0$, hence a single stable periodic orbit is born from the loop.