1. (i) Let a system of differential equations have a homoclinic loop to an equilibrium state with the eigenvalues of the linearized matrix equal to $(1, -2, -3)$. What is the maximal number of periodic orbits which can be born from the loop?

(ii) The same question in the case where the eigenvalues are (2,-1,-1).

(iii) The same question in the case where the eigenvalues are (1,-2+i,-2-i).

(iv) Determine how many periodic orbits does the following system have at small $\varepsilon > 0$ in a small neighbourhood of zero:

$$\begin{cases} \dfrac{dx}{dt} = y + x^2 - y^3, \\ \\ \dfrac{dy}{dt} = \varepsilon y - x - y^2 + x^3. \end{cases}$$

2. Consider the following system:

$$\begin{cases} \dfrac{dx}{dt} = x + ay - y^3, \\ \\ \dfrac{dy}{dt} = b - 2y + x. \end{cases}$$

(i) How many periodic orbits does this system have?

(ii) Analyse the existence of equilibria, determine their stability and bifurcations, and draw the bifurcation diagram in the plane of parameters (a, b).

3. Consider the following map of the interval $[0, a]$:

$$\bar{x} = a - x^3,$$

with the parameter a varying from 0 to 1.
(i) Study bifurcations of the fixed points of the map.
(ii) Study bifurcations of points of period 2.
(iii) What other bifurcations may happen in this map?

4. Describe bifurcations of the zero equilibrium of the following system, as parameter ε changes:

$$\begin{cases} \dfrac{dx}{dt} = -2x + 4ay^2, \\[2ex] \dfrac{dy}{dt} = -z + xy, \\[2ex] \dfrac{dz}{dt} = \varepsilon z + y + 2yz + 2z^2, \end{cases}$$

for two fixed values of the parameter a: for $a = 1$ and for $a = 1/2$.

Solutions to M3A24/M4A24.

1.

(i) 1 periodic orbit, as the saddle value is strictly negative. (5 points)

(ii) infinitely many periodic orbits: as parameters change, the multiple eigenvalue -1 can became complex, and the corresponding saddle value will remain strictly positive, hence Shilnikov theorem will guarantee chaos. (5 points)

(iii) 1 periodic orbit, as the saddle value is strictly negative. (5 points)

(iv) no periodic orbits: at $\varepsilon = 0$ the system undergoes a Hopf bifurcation: the real part of the eigenvalues for the equilibrium at zero changes from negative to positive as ε changes sign. Since the system is time-reversible at $\varepsilon = 0$, namely it is symmetric with respect to $t \to -t$, $x \to y$, it follows that every orbit in a small neighbourhood of zero is closed at $\varepsilon = 0$. By Hopf theorem, there are no periodic orbits near zero at $\varepsilon \neq 0$. (5 points)

2. (i) (3 points) The vector field has negative divergence, so the system has no periodic orbits. Therefore, it remains to study bifurcations of the equilibria only.

(ii) (17 points) The equilibria are found as solutions of

$$\begin{cases} 0 = x + ay - y^3, \\ 0 = b - 2y + x \end{cases},$$

which gives

$$x = 2y - b$$

and

$$y^3 - (a+2)y + b = 0.$$

The last equation has a multiple root at

$$3y^2 = a + 2,$$

so the bifurcation curve L is

$$b = \pm 2\left(\frac{a+2}{3}\right)^{3/2}.$$

We have 3 equilibria for (a, b) inside the region $D_3 : b^2 < \frac{4}{27}(a+2)^3$, and 1 equilibrium in the region D_1 (the complement to $cl(D_3)$).

To determine the stability of the equilibria, write down the linearisation matrix

$$\begin{pmatrix} 1 & a - 3y^2 \\ 1 & -2 \end{pmatrix}.$$

The trace of the matrix is negative, so the sum of the eigenvalues is negative, hence the equilibria may be only stable or saddle, and if there is a zero eigenvalue, the other eigenvalue is negative. At the point $a = -3, b = 0$ from D_1

we have a single stable equilibrium at $y = 0$, so by continuity we have a single stable equilibrium everywhere in D_1, and 2 stable and 1 saddle equilibrium in the region D_3.

3. (i) (9 points) The fixed point x_a is found from the equation

$$a = x_a + x_a^3.$$

It is stable at $-3x_a^2 > -1$ and unstable at $-3x_a^2 < -1$. Thus, the fixed point is stable at $a < a^* := \frac{4}{3\sqrt{3}}$ and unstable at $a > a^*$. At $a = a^*$ the fixed point at $x = x_a^* := \frac{1}{\sqrt{3}}$ undergoes a period-doubling bifurcation. By denoting $y = x - x_a^*$, we recast the map at the bifurcation moment as follows:

$$\bar{y} = -y - \sqrt{3}y^2 - y^3.$$

The second iteration of the map is given by

$$y \mapsto y - (1 + 2\sqrt{3})y^3 + O(y^4).$$

As the coefficient of y^3 is strictly negative, the fixed point is stable at the bifurcation moment; moreover, only one period-2 orbit is born at the bifurcation, it is stable and exists in the region where the fixed point is unstable, i.e. at $a > a^*$.

(ii) (9 points) The period-2 orbit may bifurcate only when its multiplier equals to 1. This means that the bifurcating period-2 orbit (x_1, x_2) must satisfy the following system of equations:

$$a = x_1 + x_2^3, \quad a = x_2 + x_1^3, \quad (1 - 3x_1^2)(1 - 3x_2^2) = 1, \quad x_1 \neq x_2.$$

We obtain

$$x_1 - x_2 = x_1^3 - x_2^3, \quad 3(x_1x_2)^2 = x_1^2 + x_2^2, \quad x_1 \neq x_2.$$

This gives

$$1 = x_1^2 + x_2^2 + x_1x_2,$$

and

$$3(x_1x_2)^2 + x_1x_2 = 1 \Longrightarrow x_1x_2 = \frac{\sqrt{13} - 1}{6}.$$

We obtain then

$$(x_1 - x_2)^2 = 1 - 3x_1x_2 = \frac{3 - \sqrt{13}}{2} < 0,$$

i.e. there are no real solutions. This means that the points of period-2 do not bifurcate. Therefore, there is only one such orbit, it exists and remains stable at $a > a^*$.

(iii) (2 points) The map takes the interval into itself for all $a \in (0, 1)$. The derivative of the right-hand side is non-positive for all $x \in [0, a]$, so the map may have only a fixed point and points of period 2 - other orbits tend to them. So no bifurcations other than described in (i) and (ii).

4. (20 points) At $\varepsilon = 0$ the linearisation matrix at the origin is $\begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$, so we have a Hopf bifurcation, and the center manifold is tangent to the (y, z)-plane. After a transformation

$$x^{new} = x + Ay^2 + Byz + Cz^2,$$

we obtain $\frac{d}{dt}x^{new} + 2x^{new} = 4ay^2 - 2Ayz - Bz^2 + By^2 + 2Czy + 2Ay^2 + 2Byz + 2Cz^2 + ...$ where the dots stand for the 3d and higher order terms. Thus, if we take $B = C = -a$, $A = -2a$, all the quadratic terms in the x-equation will be killed. That means the center manifold is given by the equation

$$x^{new} = O(|z|^3 + |y|^3)$$

in this case, so the system on the center manifold is, at $\varepsilon = 0$, given by

$$\begin{cases} \dfrac{dy}{dt} = -z + 2ay^3 + ay^z + az^2y + \ldots, \\ \\ \dfrac{dz}{dt} = y + 2yz + 2z^2, \end{cases}$$

where the dots stand for the 4th and higher order terms.

Denote $u = y + iz$, so $y = (u + u^*)/2$, $z = (u - u^*)/2i$. The system takes the form

$$\frac{du}{dt} = iu + (1 - i)u^2/2 + iuu^* - (1 + i)(u^*)^2/2 + au^2u^*\frac{7 - i}{8} + \ldots,$$

where the dots stand for the irrelevant terms of order 3 and higher. Make the normalising transformation: $w = u + (1 + i)u^2/2 + uu^* + (i - 1)(u^*)^2/6$. We obtain

$$\frac{dw}{dt} - iw = u^2u^*(a\frac{7 - i}{8} - 1/2 - i7/6 + \ldots,$$

hence the first Lyapunov value is $7a/8 - 1/2$. It is positive at $a = 1$ - hence we have a saddle periodic orbit at $\varepsilon < 0$ and no periodic orbits at $\varepsilon \geq 0$ in this case, and it is negative at $a = 1/2$, i.e. we have a stable periodic orbit at $\varepsilon > 0$ and no periodic orbits at $\varepsilon \leq 0$ in this case.