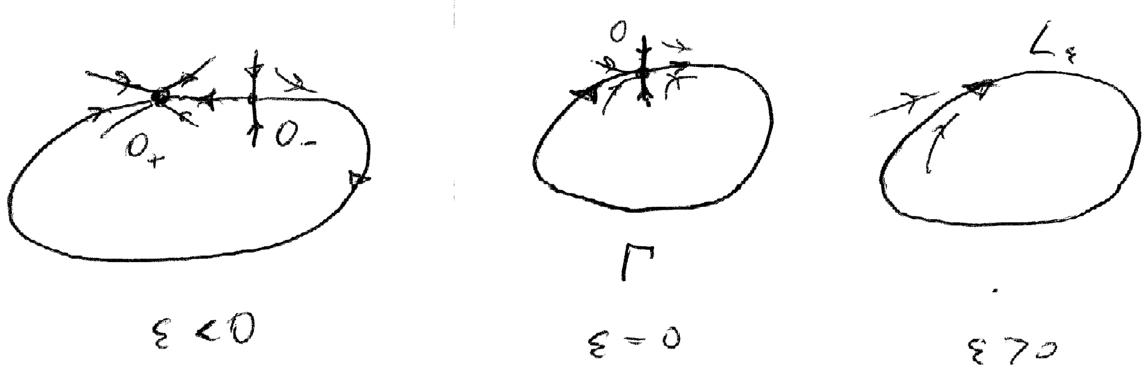
# 1 Homoclinic loops

Theory of global bifurcations studies bifurcations in a neighbourhood of closed connected sets which consist of several orbits. The basic example is given by a homoclinic loop - this is an equilibrium state and an orbit which tends to it both as $t \to +\infty$ and $t \to -\infty$. Not every equilibrium may have a homoclinic loop: there must exist at least one orbit which leaves the equilibrium at $t = -\infty$ and at least one orbit which enters the equilibrium at $t = +\infty$ (then the existence of a homoclinic loop means that such two orbits coincide). The first example of such type of equilibrium is given by a saddle-node: an equilibrium with one zero eigenvalue, the rest of eigenvalues to the left of the imaginary axis, and the first Lyaponov value non-zero. Locally, system near such equilibrium can be written as

$$\begin{cases} \dfrac{dx}{dt} = (A + f(x, z, \varepsilon))x \\ \\ \dfrac{dz}{dt} = \varepsilon + z^2 + h(x, z, \varepsilon) \end{cases} \qquad (*)$$

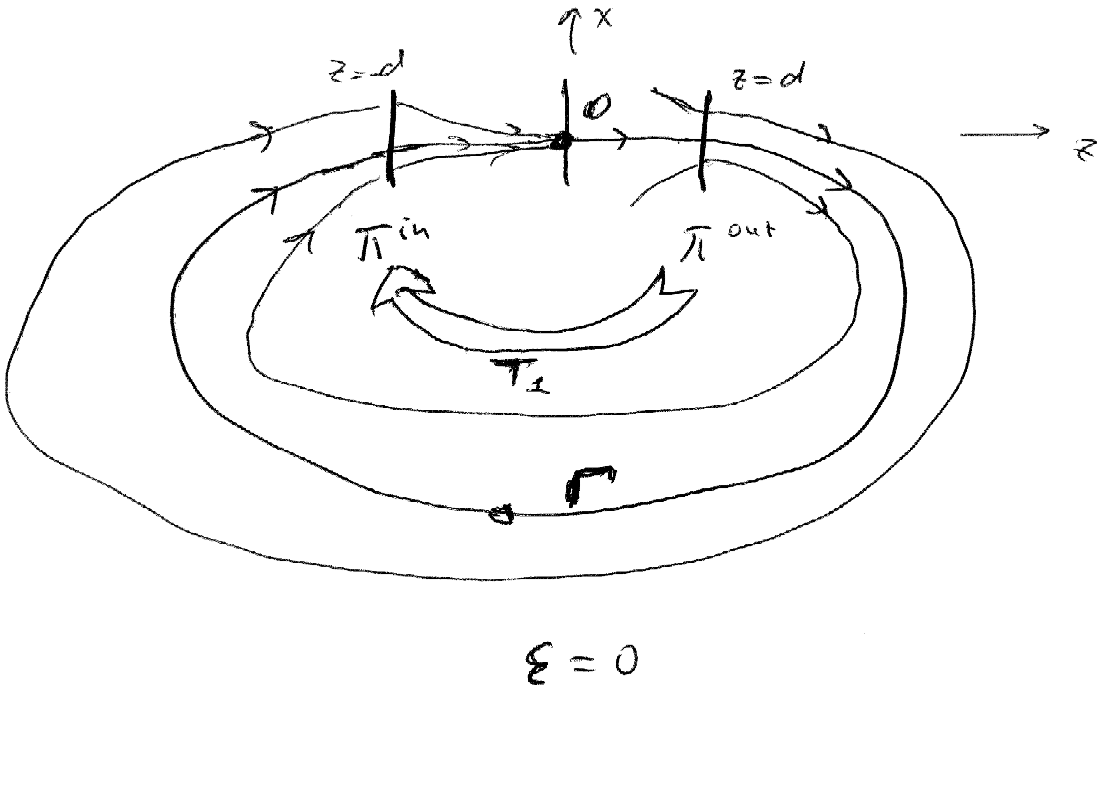
where $f(0,0,0) = 0$ and $h = O(|z|^3 + \|x\|^3)$. At $\varepsilon = 0$ the system has an equilibrium $O$ at zero; this equilibrium disappears at $\varepsilon > 0$. The only orbit which leaves $O$ at $t = -\infty$ is the positive part of the line $x = 0$. If, as time grows, this orbit returns to a small neighborhood of $O$ from the side of negative $z$, it will also tend to $O$ as $t \to +\infty$ (and it will be tangent to the $z$-axis at $O$). Suppose this is the case; denote the corresponding homoclinic loop as $\Gamma$. Let $U$ be a sufficiently small neighbourhood of $\Gamma$. At $\varepsilon = 0$ all the orbits in $U$ tend to $O$ as $t \to +\infty$. At small negative $\varepsilon$ the equilibrium $O$ splits into two equilibria: a stable $O_+$ and a saddle $O_-$; all the orbits in $U$, except for the orbits in the stable manifold of $O_-$, tend to $O_+$. The behaviour at small positive $\varepsilon$ is described by the following theorem.

**Theorem (Shilnikov)** At small $\varepsilon > 0$ all the orbits in $U$ tend to a stable periodic orbit $L_\varepsilon$. This orbit is homotopic to $\Gamma$ and tends to $\Gamma$ as $\varepsilon \to +0$.



Proof. Take some small $d > 0$ and consider two cross-sections to $\Gamma$: $\Pi^{out} : \{z = -d\}$ and $\Pi^{in} : \{z = d\}$. The homoclinic loop intersects $\Pi^{out}$ at the point $M^{out} = (x = 0, z = d)$. Since this orbit returns to $O$ from the side of negative $z$, it must also intersect $\Pi^{in}$ at some point $M^{in}$ (provided $d$ is chosen sufficiently small). Note also that $\frac{dz}{dt} \neq 0$ at both points $M^{in}$ and $M^{out}$. Therefore, every orbit which starts on $\Pi^{out}$ sufficiently close to $M^{out}$ will, after a finite time, reach $\Pi^{in}$ near $M^{in}$. Thus, a map $T_1 : \Pi^{out} \to \Pi^{in}$ is defined by the orbits of the system. Since the flight time from $\Pi^{out}$ to $\Pi^{in}$ is finite, the map $T_1$ is defined for all small $\varepsilon$ as well. Because on every finite interval of time the orbits of a smooth system of diferential equations depend smoothly on their initial conditions, the map $T_1$ is smooth. In particular, the derivative of $T_1$ is bounded - this is the only fact we need about the map $T_1$.

At $\varepsilon = 0$ the orbits which start on $\Pi^{in}$ tend to $O$. At positive $\varepsilon$ the point $O$ disappears, so the orbits which start on $\Pi^{in}$ at $\varepsilon > 0$ must reach $\Pi^{out}$. (In order to show this indeed, we have to prove that $\frac{dz}{dt} > 0$ all the time the orbit stays in $|z| \leq d$. This amounts to showing $|h| < \varepsilon + z^2$, i.e. $\varepsilon + z^2 \gg \|x\|^3$. It is enough to check that $\|x\|^2 \leq \varepsilon + z^2$, which is easy. Indeed, if $x^2 = \varepsilon + z^2$, then $\frac{dz}{dt} = (\varepsilon + z^2)(1 + o(1))$ hence $\frac{d(\varepsilon+z^2)}{dt} = 2z\frac{dz}{dt} = o(\varepsilon + z^2)$. On the other hand, as we showed in the Lemma in the previous Lecture, the coordinates in the $x$-space can be chosen in such a way that $\frac{dx^2}{dt} \leq -\alpha x^2$. Thus, if $x^2 = \varepsilon + z^2$, then $\frac{d}{dt}x^2 < \frac{d}{dt}(\varepsilon + z^2)$, which means that the orbits starting on the boundary of the region $x^2 \leq \varepsilon + z^2$ must enter the interior of this region, that is the orbits from inside this region cannot come to its boundary and have, therefore, remain inside. This proves the claim.) Thus, at $\varepsilon > 0$ we have a map $T_0 : \Pi^{in} \to \Pi^{out}$ by the orbits passing near zero.



Note that the flight time from $\Pi^{in}$ to $\Pi^{out}$ tends to infinity as $\varepsilon \to +0$ (would it remain finite, we would take a limit and obtain orbits which come from $\Pi^{in}$ to $\Pi^{out}$ at $\varepsilon = 0$, but such do not exist). This implies that the map $T_0$ is a strong contraction. Indeed, at $\varepsilon = 0$ at the point $O$ the linear part of the system is

$$\frac{dx}{dt} = Ax, \quad \frac{dz}{dt} = 0. \qquad (**)$$

The corresponding eigenvalues are zero and the eigenvalues of $A$. Thus the sum of any two different eigenvalues has *strictly negative* real part. This implies that system (**) *contracts two-dimensional areas exponentially.* This is a part of the following general statement, which we will leave without a proof:

**Lemma** Let $\gamma_1, \ldots, \gamma_m$ be eigenvalues of the matrix $B$, ordered so that $Re\ \gamma_1 \geq Re\ \gamma_2 \geq \ldots \geq Re\ \gamma_m$. Let $\beta$ be a number such that

$$\beta > Re\ \gamma_1 + Re\ \gamma_2 + \ldots + Re\ \gamma_k.$$

Denote as $R_t\underline{u}$ the time-$t$ shift of the vector $\underline{u}$ by the system

$$\frac{du}{dt} = Bu, \qquad (***)$$

and as $Vol(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k)$ we denote the volume spanned by the vectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k$. Then there exists a constant $C$ such that for every $k$ vectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k$

$$Vol(R_t\underline{v}_1, R_t\underline{v}_2, \ldots, R_t\underline{v}_k) \leq Ce^{\beta t} Vol(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k) \qquad (****)$$

for every $t \geq 0$.

Note that (****) holds true for every linear system close to (***), i.e. we may replace matrix $B$ by any (even time-dependent) matrix $\tilde{B}(t)$ uniformly close to $B$ and we will still have (****), with the same $\beta$ and with $C$ independent of $\tilde{B}$ and of the vectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k$. Indeed, (****) holds for every $\beta > Re\ \gamma_1 + Re\ \gamma_2 + \ldots + Re\ \gamma_k$. Thus, we may take $\beta'$ slightly smaller than $\beta$ and (****) will still be true with some other constant $C$:

$$Vol(R_t\underline{v}_1, R_t\underline{v}_2, \ldots, R_t\underline{v}_k) \leq C' e^{\beta' t} Vol(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k).$$

This inequality implies that there exists $\tau \geq 0$ such that

$$Vol(R_\tau\underline{v}_1, R_\tau\underline{v}_2, \ldots, R_\tau\underline{v}_k) \leq Ke^{\beta\tau} Vol(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k)$$

with $K < 1$. If we fix such $\tau$, then this inequality will obviously hold true for every linear system close to (***), i.e. by replacing matrix $B$ with any matrix $\tilde{B}(t)$ uniformly close to $B$ we will still, with the same $\tau$, have

$$Vol(R_\tau \underline{v}_1, R_\tau \underline{v}_2, \ldots, R_\tau \underline{v}_k) \leq e^{\beta\tau} Vol(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k)$$

for every $k$ vectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k$. By iterating this inequality $n$ times, we find

$$Vol(R_t \underline{v}_1, R_t \underline{v}_2, \ldots, R_t \underline{v}_k) \leq e^{\beta t} Vol(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k)$$

for $t = n\tau$. Thus, inequality (****) indeed holds true with $C = 1$ for any system close to (***) provided $t$ is divisible by $\tau$. As the interval between any two such consecutive moments of time is finite (it equals to $\tau$) the volume spanned by the vectors $R_t\underline{v}_1$, $R_t\underline{v}_2$, ..., $R_t\underline{v}_k$ can only acquire a bounded factor during this time interval, so we find that (****) indeed holds true for every linear system close to (***) and for all $t$ - with some common factor $C \geq 1$.

Thus there exist $\alpha > 0$ and $C > 0$ such that for any pair of vectors $\underline{u}$ and $\underline{v}$ in the $(x, z)$-space the area spanned by $R_t\underline{u}$ and $R_t\underline{v}$ decays exponentially:

$$Area(R_t\underline{u}, R_t\underline{v}) \leq Ce^{-\alpha t} Area(\underline{u}, \underline{v}).$$

Here $R_t$ is the time-$t$ shift by the system (**) or by any linear system close to (**), i.e., for all small $\varepsilon$, we have uniform exponential contraction of areas by the system linearised along any orbit which passes near $(x, z) = 0$ (for all times the orbit stays there).

We are now ready to prove that the map $T_0$ is strongly contracting. Namely, we will show that

$$\|\frac{d}{dx}T_0(x)\| \leq Qe^{-\alpha t(x)}$$

where $x$ is the coordinates on the cross-section $\Pi^{in} : \{z = -d\}$, $Q$ is a constant, $t(x)$ is the flight time from the point $x \in \Pi^{in}$ to the point $T_0(x) \in \Pi^{out} : \{z = d\}$, and $\alpha > 0$ is the same exponent as that entering the formula for the exponential contraction of areas by the linearised system. In order to prove this formula, we note that the derivative of $T_0$ is a linear map which acts on any vector $\underline{u}$ in $\Pi^{in}$ as follows: $T_0'(x)\underline{u}$ is the projection of the vector $R_{t(x)}\underline{u}$ (time-$t(x)$ shift of $\underline{u}$ by the system linearised along the orbit of $x$) to the cross-section $\Pi^{out}$ along the vector $\underline{v}_t$ which is tangent to the orbit at the point $T_0(x)$ (i.e. along the vector of the right-hand side of the system at the point $(T_0(x), z = d)$). Note that the vector $\underline{v}_t$ is bounded away from zero for all small $\varepsilon$ and the angle between $\underline{v}_t$ and $\Pi^{out}$ stays bounded away from zero as well (because the $z$-component of $\underline{v}_t$ is bounded away from zero - it equals to $\varepsilon + d^2 + \ldots$). Therefore, the length of the vector $T_0'(x)\underline{u}$ coincides with the area spanned by this vector and $\underline{v}_t$ - up to a factor bounded away from zero and infinity. Absolutely analogously, the length of the vector $\underline{u}$ coincides, up to a factor bounded away from zero and infinity, with the area spanned by this vector and the $\underline{v}_0$ which is tangent to the orbit at the point $x \in \Pi^{in} : z = -d$. Thus, to prove that $T_0'$ is contracting, it is enough to show that

$$Area(T_0'(x)\underline{u}, \underline{v}_t) \leq Ke^{-\alpha t(x)} Area(\underline{u}, \underline{v}_0)$$

for some constant $K > 0$ independent of $x$ and $\underline{u}$.

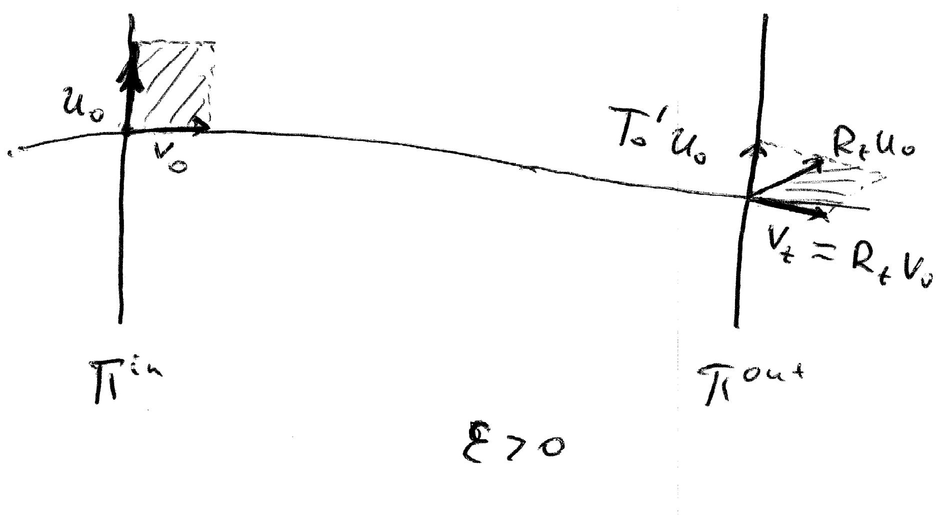
Recall that the vector $T_0'(x)\underline{u}$ is obtained from the vector $R_{t(x)}\underline{u}$ by projecting along the vector $\underline{v}_t$, therefore

$$Area(T_0'(x)\underline{u}, \underline{v}_t) = Area(R_{t(x)}\underline{u}, \underline{v}_t).$$

Thus, we are left to show that

$$Area(R_{t(x)}\underline{u}, \underline{v}_t) \leq Ke^{-\alpha t(x)} Area(\underline{u}, \underline{v}_0),$$

but this inequality follows from the earlier established fact that the time shift $R_t$ by the linearised system exponentially contracts areas - just note that $\underline{v}_t = R_t \underline{v}_0$ because both $\underline{v}_0$ and $\underline{v}_t$ are tangent to the same orbit.



Thus, we have shown that $T_0$ is a contracting map, with the contraction constant $O(e^{-\alpha t})$. As the derivative of $T_1$ is uniformly bounded for all small $\varepsilon$, we have that

$$\|\frac{d}{dx}T_1 \circ T_0(x)\| = O(e^{-\alpha t(x)})$$

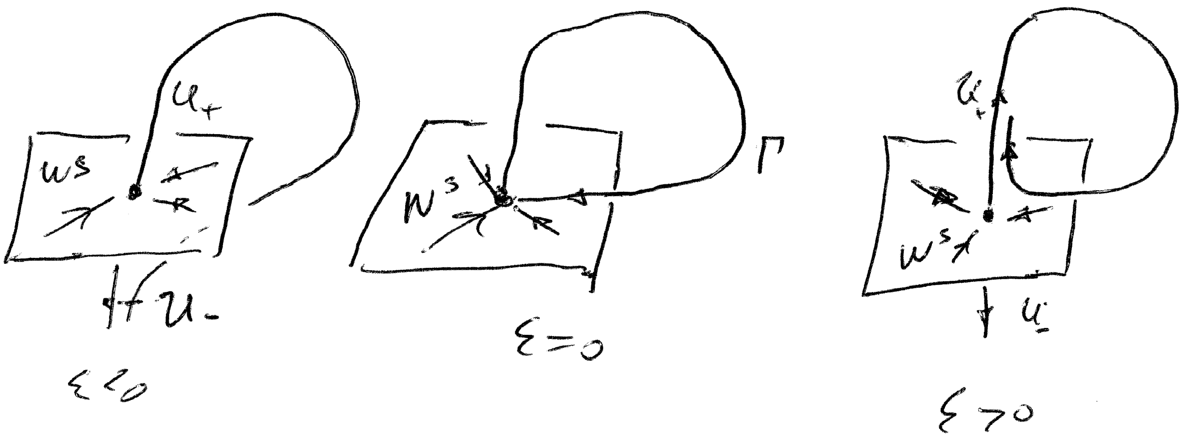
as well. Since the flight time $t$ tends to infinity as $\varepsilon \to +0$, we find that the map $T_1 \circ T_0 : \Pi^{in} \to \Pi^{out}$ is a contraction for all sufficiently small $\varepsilon > 0$. By Banach contracting map principle, this map has a unique fixed point which attracts iterations of all other points. Since the map $T_1 \circ T_0$ is defined by the orbits of the system under consideration, the fixed point corresponds to the sought stable periodic orbit. □

(See more about saddle-node homoclinics in Shilnikov,Shilnikov,Turaev,Chua, Ch.12).

Another example is given by a homoclinic loop to a saddle equilibrium state. Assume that a system of differential equations has a hyperbolic equilibrium state with one-dimensional unstable manifold and an $n$-dimensional stable manifold. We may write the system locally near the equilibrium in the following form

$$\begin{cases} \dfrac{dx}{dt} = (A + f(x,y))x, \\ \\ \dfrac{dy}{dt} = (\gamma + g(x,y))y, \end{cases} \qquad (*****)$$

where $y \in R^1$, $x \in R^n$, $\gamma > 0$, the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix $A$ lie strictly to the left of the imaginary axis, the functions $f$ and $g$ vanish at zero. The $y$-axis is the local unstable manifold $W^u$; it consists of three orbits: one is the equilibrium $O$ itself, and the other two are $y > 0$ and $y < 0$, these orbits tend to $O$ as $t \to -\infty$ and they leave a neighbourhood of $O$ as $t$ grows. We assume that one of this orbits - $\Gamma$ (let it be the orbit with $y > 0$) - forms a homoclinic loop, i.e. it returns to a small neighbourhood of $O$ and enters the local stable manifold $W^s : \{y = 0\}$, so it tends to $O$ as $t \to +\infty$. The stable manifold is an $n$-dimensional submanifold of $R^{n+1}$, so it divides a small neighbourhood of $O$ into two parts: $U_+ : \{y > 0\}$ and $U_- : \{y < 0\}$. If we perturb our system, the orbit $\Gamma$ may miss $W^s$ and arrive either in $U_+$ or in $U_-$. Let us consider a one-parameter family of systems which continuously depend on a parameter $\varepsilon$ in such a way that at $\varepsilon = 0$ we have our original system with the homoclinic loop $\Gamma$ while at $\varepsilon \neq 0$ the loop splits: at $\varepsilon > 0$ the orbit $\Gamma$ enters $U_+$ and at $\varepsilon < 0$ the orbit $\Gamma$ enters $U_-$.



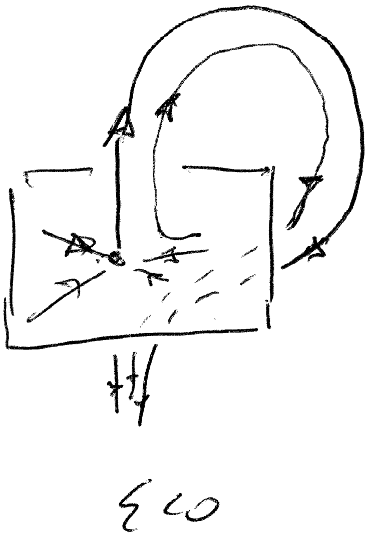

Denote as $\sigma$ the so-called *saddle value*:

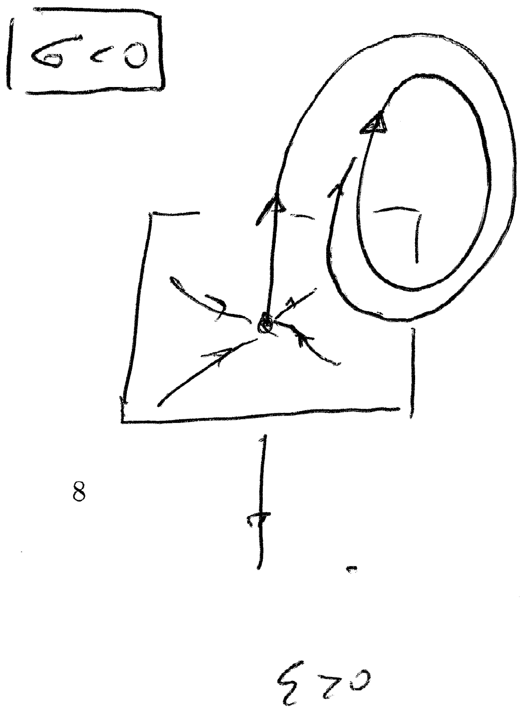
$$\sigma = \gamma + \max\{Re\ \lambda_1, \ldots, Re\ \lambda_n\}.$$

Assume that $\sigma < 0$. As all eigenvalues $\lambda$ have negative real parts, this condition may be rephrased as the real part of the sum of any two different eigenvalues of the linear part of the system at $O$ is strictly negative, i.e. the system linearised at $O$ contracts two-dimensional areas exponentially (as we explained above). If we consider a sufficiently small neighbourhood $V$ of $\Gamma$, any orbit in $V$ will spend most of the time in a small neighbourhood of $O$: each round along $\Gamma$ outside the small neighbourhood of $O$ is accompanied by a long time spent in the neighbourhood of $O$ before the orbit can leave the neighbourhood (indeed, when an orbit from $V$ enters the neighbourhood of $O$ it is close to $W^s$; as the orbits from $W^s$ stay in the neighbourhood of $O$ for all positive times, the orbits of points which are close to $W^s$ must spend a long time in the neighbourhood). It follows that the strong contraction of areas during the long time the orbit stays in a neighbourhood of $O$ overcomes any possible expansion of areas during a bounded time of the excursion out of the neighbourhood which the orbit makes following the homoclinic loop $\Gamma$. Thus, the system in $V$ exponentially contracts two-dimensional areas. This implies that the map $T$ defined on a small cross-section $\Pi$ to $\Gamma$ by the orbits of the system is *contracting* (see a proof of an analogous statement in the previous theorem). This property holds true for all small $\varepsilon$. Therefore, the map $T$ can have no more than one fixed point and this point must be stable. The fixed point of $T$ corresponds to a periodic orbit of the system, so there can be no more than one periodic orbit in $V$ and this orbit is stable. Note that the contracting map $T$ is defined only on the half of the cross-section $\Pi$, namely $T : \Pi_+ \to \Pi$ where $\Pi_+ = \Pi \cap \{y > 0\}$ (the orbits starting at $y < 0$ leave $V$ and do not return to $\Pi$). Moreover, it is not always true that $T(\Pi_+) \subset \Pi_+$ (for example, when $\varepsilon < 0$ the loop splits outwards, which means the point $\Gamma \cap \Pi$ lies in the region $y < 0$, so the orbits starting on $\Pi_+$ close to $W^s$ will, after one round in $V$ near $\Gamma$, arrive on $\Pi$ close to $\Gamma \cap \Pi$, i.e. outside of $\Pi_+$). Therefore, eventhough the map $T$ is contracting, we cannot immediately guarantee the existence of the fixed point. In fact, the fixed point always exist at small $\varepsilon > 0$ and disappears at $\varepsilon < 0$, which gives us the following theorem (see a proof and more discussion in Shilnikov,Shilnikov,Turaev,Chua, Ch.13):

**Theorem (Shilnikov)** At $\varepsilon > 0$ a stable periodic orbit is born from the homoclinic loop of a saddle equilibrium with a negative saddle value. This orbit attracts all the orbits from $V$ (except for the equilibrium $O$ itself and the orbits in its stable manifold) which do not leave $V$. At $\varepsilon < 0$ all the orbits except for those in $W^s(O)$ leave $V$ as time increases.

Thus, we have one more example of the birth of a stable periodic orbit from a homoclinic loop: it is born as the homoclinic loop splits inwards (as we saw, the condition $\sigma < 0$ which ensures the area-contraction property is of the most importance here).



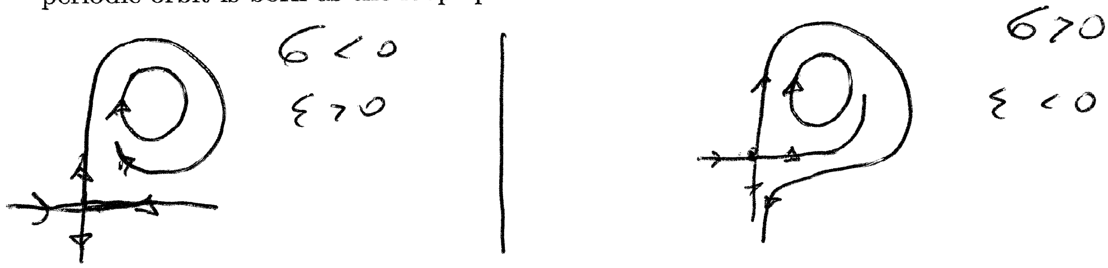



The case $\sigma > 0$ is pretty much different. We start with the systems on a plane. Let a system

$$\begin{cases} \dfrac{dx}{dt} = -\lambda x + o(x,y), \\ \\ \dfrac{dy}{dt} = \gamma y + o(x,y) \end{cases}$$

(with $\lambda > 0, \gamma > 0$) have a homoclinic loop to the saddle at zero. We assume that the saddle value $\sigma = \gamma - \lambda$ is non-zero. Behaviour at $\sigma < 0$ is described by the previous theorem. Since the case $\sigma > 0$ here is reduced to the case $\sigma < 0$ by the time reversion $t \to -t$, we have the following result:

**Theorem (Andronov-Leontovich)** At $\sigma < 0$ a single stable periodic orbit is born as the homoclinic loop splits inwards. At $\sigma > 0$ a single unstable periodic orbit is born as the loop splits outwards.




In higher dimensions we have, generically, two different cases of the behaviour near the loop in the case $\sigma > 0$. In the first case the nearest to the imaginary axis eigenvalue $\lambda_1$ of the matrix $A$ in (*****) is real:

$$0 > \lambda_1 > Re\lambda_j \quad (j \geq 2).$$

In the second case $\lambda_1$ is complex:

$$0 > Re\ \lambda_1 = Re\ \lambda_2 > Re\lambda_j \quad (j \geq 3).$$

**Theorem (Shilnikov)** In general, at $\sigma > 0$, if the nearest to the imaginary axis eigenvalue $\lambda_1$ is real, then a single saddle periodic orbit is born as the homoclinic loop splits, and if the nearest to the imaginary axis is a pair of complex-conjugate eigenvalues $\lambda_{1,2} = \rho \pm i\omega$, then dynamics near the homoclinic loop is chaotic and infinitely many saddle periodic orbits coexist with the loop.

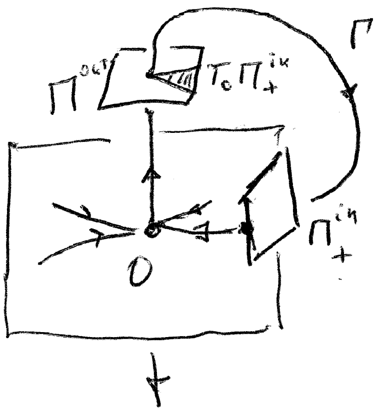
We will not give a proof here, just make some illustrations for a three-dimensional case. In a small neighbourhood of the equilibrium state $O$ we take two cross-sections to a homoclinic loop $\Gamma$: one cross-section, $\Pi^{out}$, to the piece of $\Gamma$ that comes out of $O$ and the second cross-section, $\Pi^{in}$, to the piece of $\Gamma$ that enters $O$. The stable manifold divides $\Pi^{in}$ into two halves; the orbits which start on one of the halves, $\Pi^{out}_{+}$ reach $\Pi^{out}$ after an unboundedly long time spent in the neighbourhood of $O$. The orbits which start on $\Pi^{out}$ follow $\Gamma$, hence they all arrive to $\Pi^{in}$ after a finite time. Thus, the orbits of the system define two maps: $T_0 : \Pi^{in}_{+} \to \Pi^{out}$ and $T_1 : \Pi^{out} \to \Pi^{in}$. The map $T_1$ corresponds to a finite flight time, so it is a smooth map; since the cross-section $\Pi^{out}$ is small, we may approximate $T_1$ by a linear map. To get an impression of map $T_0$ we assume that the system is linear in the neghbourhood of $O$. Thus, in the case of real eigenvalues, we write the system near $O$ in the form

$$\frac{dy}{dt} = \gamma y, \qquad \frac{dx_1}{dt} = \lambda_1 x_1, \qquad \frac{dx_1}{dt} = \lambda_2 x_2$$

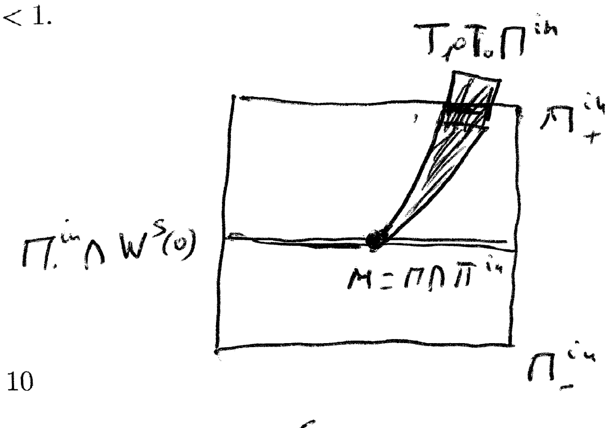
($\gamma > 0 > \lambda_1 > \lambda_2$). The solution is $y(t) = y(0)e^{\gamma t}$, $x_1(t) = x_1(0)e^{\lambda_1 t}$, $x_2(t) = x_2(0)e^{\lambda_2 t}$. We take the cross-section $\Pi^{in}$ be $x_1 = \delta$ and $\Pi^{out} : \{y = \delta\}$ for some small $\delta > 0$. Then the points in $\Pi^{in}$ are parametrised by $x_2$ and $y$, while the points in $\Pi^{in}$ are parametrised by $x_1$ and $x_2$. The flight time from $\Pi^{in}$ to $\Pi^{out}$ is found from the condition $y(0)e^{\gamma t} = \delta$, i.e. $t = -\frac{1}{\gamma}\ln\frac{y(0)}{\delta}$. Now, we find that the map $T_0 : \Pi^{in}_{+} \to \Pi^{out}$ is given by

$$x_1 = \delta\left(\frac{y}{\delta}\right)^{\nu}, \qquad x_2 = x_2\left(\frac{y}{\delta}\right)^{s}$$

where $\nu = |\lambda_1|/\gamma < s = |\lambda_2|/\gamma$. The positivity of the saddle value means




$\nu < 1.$






ε = 0

By assuming that the map $T_1 : \Pi^{out} \to \Pi^{in}$ is linear, we obtain the following formula for the composition map $T = T_1 \circ T_0 : \Pi^{in}_+ \to \Pi^{in}$:

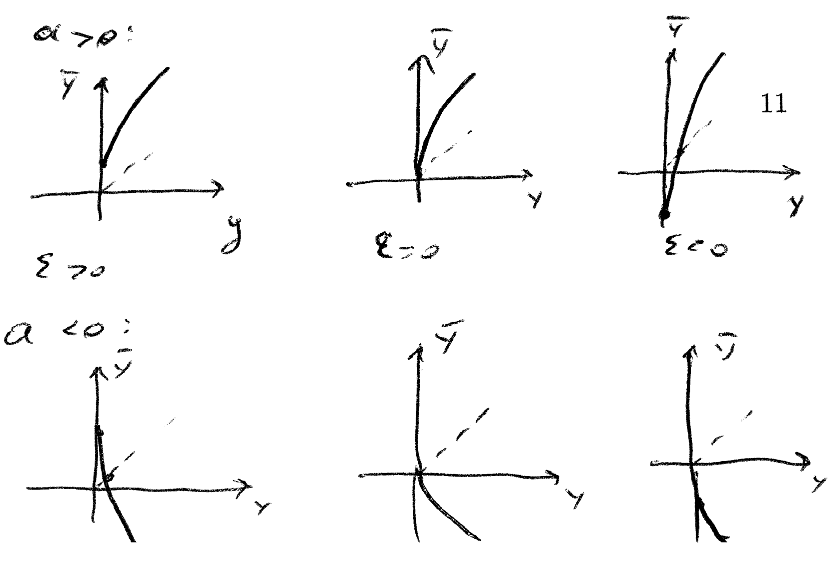
$$\begin{aligned} \bar{y} &= \varepsilon + a y^{\nu} + b x_2 y^{s}, \\ \bar{x}_2 &= x^{+} + c y^{\nu} + d x_2 y^{s}. \end{aligned} \qquad \text{(T)}$$

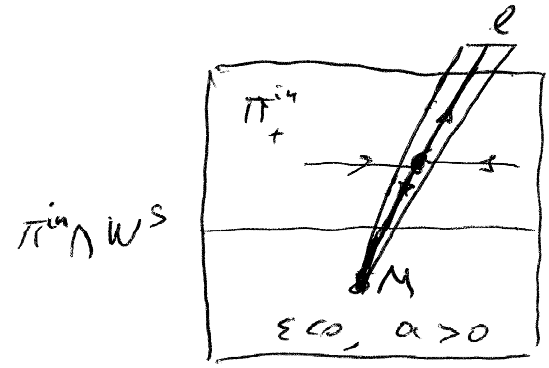
One can show that similar formulas (with insignificant corrections) are true indeed for the map $T$, without the assumptions that the system is locally linear (see a proof in Shilnikov,Shilnikov,Turaev,Chua, Ch.13). Therefore, our map is a good model for the true map $T$. Note that this map is defined only at $y > 0$. As $y \to +0$, the point in $\Pi^{in}_+$ tends to the stable manifold of $O$, therefore the corresponding orbit becomes closer and closer to $\Gamma$, hence its intersection point with $\Pi^{in}$ (this is the image of the original point by $T$) must tend to the point $M = \Gamma \cap \Pi^{in}$. As $(\bar{x}_2, \bar{y}) = (x^+, \varepsilon)$ at $y = 0$, we see that $M = (x^+, \varepsilon)$. The homoclinic loop corresponds to $M \in W^s$, i.e. to $\varepsilon = 0$. Positive $\varepsilon$ correspond to the loop split inwards, negative $\varepsilon$ correspond to the loop split outwards.

An important assumption is that the coefficient $a$ in (T) has to be non-zero (this is the "generality" assumption of Shilnikov theorem). As it is easy to see from (T), the image of any line $\{x_2 = const, y > 0\}$ by $T$ is a line tangent to the same line $\bar{x}_2 - x^+ = \frac{c}{a}(y - \varepsilon)$, hence the image of the whole of $\Pi^{in}_+$ is a thin wedge transverse to $W^s$. The map $T$ contracts in the $x$-direction, which implies, as one can show, that within the wedge there is a curve $\ell$ invariant with respet to $T$, such that the forward iterations of any point by the map $T$ tend to $\ell$ (unless they leave $\Pi^{in}_+$). The restricion of $T$ to $\ell$ has, obviously, the form

$$\bar{y} = \varepsilon + a y^{\nu} + o(y^{\nu}).$$

As $\nu < 1$ and $a \neq 0$, the derivative of this map at $y = 0$ is infinite, hence it is expanding at small $y$. It is easy to see that $T|_\ell$ has a single unstable fixed point at $a\varepsilon < 0$. The unstable fixed point of $T|_\ell$ corresponds to the saddle fixed point of $T$ (since the curve $\ell$ is attracting). Since the map $T$ is defined by the orbits of the system, its fixed point corresponds to a periodic orbit. Thus we have found that a saddle periodic orbit is indeed born as the homoclinic loop splits (outwards, if $a > 0$, and inwards if $a < 0$).



Similar computations show that in the case of complex $\lambda_1$ the image of $\Pi_+^{in}$ by $T$ is no longer a wedge, but rather a "snake". The intersection $T\Pi_+^{in} \cap \Pi_+^{in}$ consists of an infinite sequence of "half-coiles" $\pi_k$; the preimage $T^{-1}\pi_k$ is a horisontal strip which, as computations show, lies strictly *below* the top of $\pi_k$ if the saddle value $\sigma$ is positive. This hints that the map $T$ acts on each of the strips $T^{-1}\pi_k$ as the Smale horseshoe map - which implies chaos and, in particular, saddle fixed points exist on each of the infinite sequence of strips. The analysis of bifurcations of this structure as the loop splits is too complicated to be ever accomplished in full detail. We note only that any *finite* number of the Smale horseshoes (hence, chaos) survives any small perturbation of the system.

This describes the main ideas behind Shilnikov theorem. Note that if $\sigma < 0$, the snake in the case of complex $\lambda_1$ also exists, however the top of each of the half-coil $\pi_k$ lies below the corresponding preimage strip, so no Smale horseshoes exits in this case. We also note that unlike the case $\sigma < 0$, the above description of behaviour or/and bifurcations near the homoclinic loop is valid only under certain "generality" conditions. Thus, if condition $a = 0$ is violated in the case of real $\lambda_1$, then more than one periodic orbit can be born (e.g. a stable one is possible) and chaotic dynamics is also possible. Another obvious case where infinitely many periodic orbits can be born at the bifurcations of a system with a homoclinic loop to a saddle with real eigenvalues corresponds to a multiple eigenvalue: $\lambda_1 = \lambda_2 > \ldots > \lambda_n$. In this case, by a small perturbation of the system the eigenvalues $\lambda_{1,2}$ (the nearest to the imaginary axis) can be made complex and the perturbation can be made without destroying the loop. Then, by Shilnikov theorem, infinitely many saddle periodic orbits will immediately appear near the homoclinic loop.

(Further reading: Shilnikov,Shilnikov,Turaev,Chua, Ch.13)