BIFURCATION THEORY

3 Bifurcations on a plane. Andronov-Hopf bifurcation

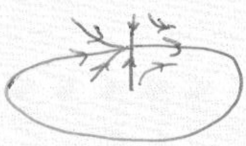
The subject of bifurcation theory is the study of structurally unstable systems. According to Andronov-Pontryagin theorem, bifurcation theory of systems on a plane has to deal with the following question: in a system which depends on parameters, as parameter change - what may happen to non-hyperbolic equilibria, periodic orbits, or to orbits that connect equilibria? The analysis starts with the so-called codimension-1 bifurcations. The idea is that if at some parameter value the system has, say, a non-hyperbolic equilibrium, it is unprobable that it has one more non-hyperbolic equilibrium, or a periodic orbit with the multiplier 1 exactly at the same moment. In other words, unless we deal with systems with symmetries or with some special structures (Hamiltonian, reversible), it seems to be reasonable to study first bifurcations of systems which have only one structurally-unstable orbit, and this orbit has to be of the least degenerate type. After the codimension-1 cases are studied, one may proceed to more complicated cases. The list of codimension-1 bifurcations on the plane is given below.

1. Andronov-Hopf bifurcation (an equilibrium with a pair of pure imaginary eigenvalues of the linearisation matrix: $\lambda = \pm i\omega$, $\omega \neq 0$).

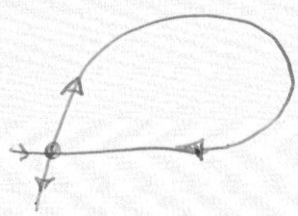
2. Semi-stable periodic orbit (a periodic orbit with the multiplier 1).

3. Saddle-node equilibrium (one eigenvalue of the linearisation matrix is zero, the other one is non-zero: $\lambda_1 = 0$, $\lambda_2 \neq 0$).

4. A homoclinic loop to a saddle-node (an orbit which tends to a saddle-node equiliobrium both as $t \to +\infty$ and $t \to -\infty$).



5. A homoclinic loop to a saddle equilibrium.



We start with the Andronov-Hopf bifurcation. Let O be an equilibrium at $(x, y) = 0$ of system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} + O(x^2 + y^2).$$

Assume that the eigenvalues of A are pure imaginary: $\lambda_{1,2} = \pm i\omega$, $\omega \neq 0$. Then by a linear coordinate transformation one can bring matrix A to the form $\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$, i.e. the system is rewritten as

$$\frac{dx}{dt} = \omega y + O(x^2 + y^2), \qquad \frac{dy}{dt} = -\omega y + O(x^2 + y^2).$$

Note that if we perturb the system slightly, the equilibrium cannot disappear, nor other equilibria can be born near it. Indeed, the following theorem holds true.

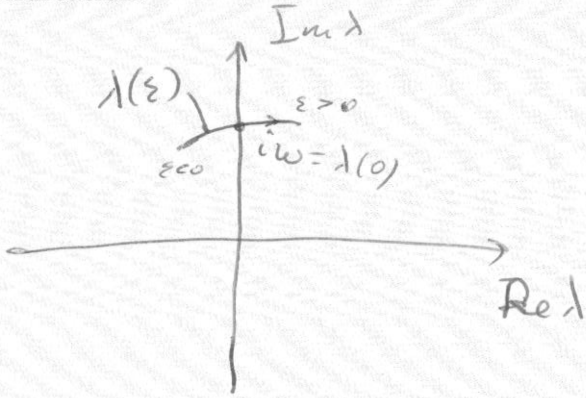
Theorem *Consider a family of systems*

$$\frac{d\underline{v}}{dt} = \underline{F}(\underline{v}, \varepsilon)$$

such that at $\varepsilon = 0$ there exists an equilibrium $\underline{v} = \underline{v}_0$, and all the eigenvlues of the linearisation matrix at this equilibrium are nonzero. Then the system has a unique equilibrium $\underline{v}_\varepsilon$ in a small neighbourhood of $\underline{v}_0$ for all small ε, and $\underline{v}_\varepsilon$ is a smooth function of ε.

Proof. Equilibria are found as zeros of $\underline{F}(\underline{v}, \varepsilon)$. As the matrix $\frac{\partial F}{\partial \underline{v}}(\underline{v}_0, 0)$ has no zero eigenvalues, it is non-degenerate, therefore equation $\underline{F}(\underline{v}, \varepsilon) = 0$ has a uniquely defined, smoothly depending on ε solution $\underline{v}_\varepsilon$ for all small ε, according to the Implicit Function Theorem. $\square$

Thus, the only thing which can happen to the equilibrium with pure imaginary eigenvalues as parameters change is the change of its stability type: when the eigenvalues move to the left of the imaginary axis the equilibrium will be stable, and when they move to the right the equilibrium will become unstable. However, as we will see below, this process is also accompanied by the birth of a small periodic orbit surrounding the equilibrium. This phenomenon is called Andronov-Hopf bifurcation.



In order to analyse what is going on here, it is convenient to introduce a complex variable $z = x + iy$ (which means $x = (z + \bar{z})/2$, $y = (z - \bar{z})/(2i)$), then the system will take the form

$$\frac{dz}{dt} = i\omega z + O(|z|^2).$$

More specifically, we will write

$$\frac{dz}{dt} = i\omega z + az^2 + bz\bar{z} + c\bar{z}^2 + O(|z|^3). \qquad (*)$$

Note that

$$\frac{d\bar{z}}{dt} = -i\omega\bar{z} + \bar{a}\bar{z}^2 + \bar{b}z\bar{z} + \bar{c}z^2 + O(|z|^3).$$

An important observation is that there exists a coordinate transformation $u = z + \alpha z^2 + \beta z\bar{z} + \gamma\bar{z}^2$ such that in the new coordinates there will be no quadratic terms in the equation. Namely, choose

$$\alpha = ia/\omega, \qquad \beta = -ib/\omega, \qquad \gamma = -ic/(3\omega).$$

Then

$$\frac{du}{dt} - i\omega u = \frac{dz}{dt} - i\omega z + 2\alpha z\frac{dz}{dt} + \beta\bar{z}\frac{dz}{dt} + \beta z\frac{d\bar{z}}{dt} + 2\gamma\bar{z}\frac{d\bar{z}}{dt} + O(|z|^3) - i\omega(\alpha z^2 + \beta z\bar{z} + \gamma\bar{z}^2) =$$

$$= (a + i\omega\alpha)z^2 + (b - i\omega\beta)z\bar{z} + (c - 3i\omega\gamma)\bar{z}^2 + O(|z|^3),$$

and we see that with our choice of the coefficients α, β, γ the equation for the new variable u takes the form

$$\frac{du}{dt} = i\omega u + O(|u|^3).$$

Let us write this as

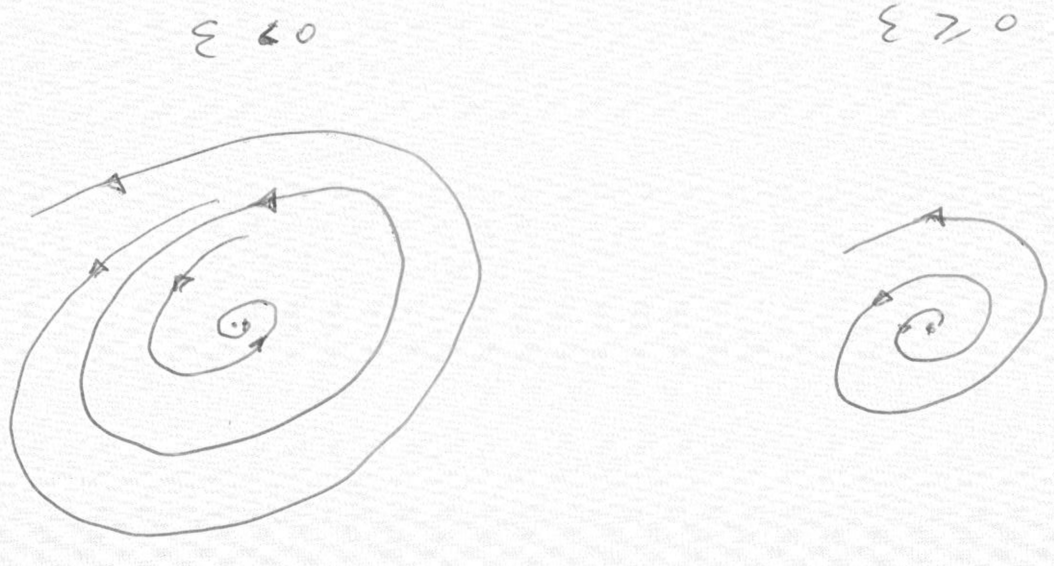
$$\frac{du}{dt} = i\omega u + a_{30}u^3 + a_{21}u^2\bar{u} + a_{12}u\bar{u}^2 + a_{03}\bar{u}^3 + O(|u|^4).$$

When the equation is brought to such form (i.e. with all quadratic terms killed), the most important coefficient is a_{21}. Its real part $L_1 = \operatorname{Re} a_{21}$ is called *the first Lyapunov value.*

Theorem *If $L_1 \neq 0$, then exactly one periodic orbit may be born at the bifurcations of an equilibrium with a pair of pure imaginary eigenvalues. If $L_1 < 0$, then the equilibrium is stable at the bifurcation moment, and if $L_1 > 0$, then the equilibrium is unstable at the bifurcation moment. When parameters vary in such a way that the equilibrium changes its stabiliy type, a periodic orbit is born from the equilibrium; the new-born orbit is stable if $L_1 < 0$ and unstable if $L_1 > 0$.*

$L_1 < 0$:

$L_1 > 0$:



Before starting proving the theorem, recall some facts from the *normal form* theory. Consider a system of differential equations near an equilibrium state at zero. We can always choose a basis such that the matrix of the linear part of the system will be in a diagonal or a Jordan form; i.e. the system is written as

$$\frac{dx_j}{dt} = \lambda_j x_j + \delta_j x_{j+1} + o(\|\underline{x}\|),$$

where x_j, $j = 1, \ldots, n$, are the components of vector $\underline{x}$; λ_j are the eigenvalues of the matrix of the linear part of the system at the equilibrium, and δ_j may take values 0 or 1 (all δ_j are zero when it is possible to bring the matrix to the diagonal form, e.g. when all λ's are different). Coordinate transformations of the form

$$y_j = x_j + \alpha x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \qquad (**)$$

will not, obviously, change the linear part of the system, but they can modify nonlinear terms. Thus, it is known that by performing a sequence of transformations of type (**) one can kill all *non-resonant* terms up to any given order M (the trasformations are made in the order of increasing value of $|m| = m_1 + \ldots + m_n$, i.e. we first make transformations with $|m| = 2$, then $|m| = 3$, etc., up to $|m| = M$). A term $a x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$ in the equation for $\frac{d}{dt} x_j$ is called non-resonant, if

$$m_1\lambda_1 + \ldots m_n\lambda_n \neq \lambda_j.$$

For example, in equation (*) we have $\lambda_1 = i\omega$ and $\lambda_2 = -i\omega$, so all quadratic terms are non-resonant:

$$2i\omega \neq i\omega, \qquad i\omega + (-i\omega) \neq i\omega, \qquad 2(-i\omega) \neq i\omega.$$

In general, the term $z^{m_1}\bar{z}^{m_2}$ in equation (*) is non-resonant if and only if

$$i\omega m_1 - i\omega m_2 \neq i\omega;$$

therefore all terms with $m_1 \neq m_2 + 1$ can be killed up to any given order. The recipee is as follows: when all the non-resonant terms up to the order M are killed, a non-resonant term $a_{m_1 m_2} z^{m_1}\bar{z}^{m_2}$ with $m_1 + m_2 = M + 1$ is killed by the transformation

$$u = z + \frac{i}{\omega(m_1 - m_2 - 1)} \, a_{m_1 m_2} z^{m_1} \bar{z}^{m_2};$$

such transformation does not affect terms of order M or less, nor it changes the terms of order $M+1$ other than $a_{m_1 m_2} z^{m_1} \bar{z}^{m_2}$, so we can indeed kill the non-resonant terms one by one in the order of the increase of $m_1 + m_2$.

Note that this is a part of a general formula: a non-resonant term $a x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$ in the equation for dx_j/dt is killed by the transformation (**) with

$$\alpha = \frac{a}{m_1\lambda_1 + \ldots + m_n\lambda_n - \lambda_j};$$

see Kuznetsov "Elements of applied bifurcation theory", Ch.3; Shilnikov, Shilnikov, Turaev, Chua "Methods of qualitative theory in nonlinear dynamics", Ch.2.

Coordinate transformations which kill non-resonant terms are called normalising transformations. As we just have seen, given any k, by a series of the normalising transformations we can bring equation (*) to the normal form

$$\frac{dz}{dt} = i\omega z + (L_1 + i\Omega_1) z^2 \bar{z} + (L_2 + i\Omega_2) z^3 \bar{z}^2 + \ldots + (L_k + i\Omega_k) z^{k+1} \bar{z}^k + o(|z|^{2k+1}). \qquad (***_0)$$

The numbers L_j (the real parts of the coefficients of $z^{k+1}\bar{z}^k$ in the normal form) are called *Lyapunov values.* Importantly, when we allow a small perturbation of the system, i.e. when we imbed the system into a family which depends smoothly on a parameter ε, the eigenvalues of the linearization matrix at the equilibrium will depend on ε continuously, so any term which was non-resonant will remain non-resonant for all sufficiently small parameter values. Therefore, a finite number of the non-resonant terms can be killed simultaneously for all systems close to the original one. It follows that given any k fixed, for all small ε we may bring the system that undergoes (at $\varepsilon = 0$) the Andronov-Hopf bifurcation to the form

$$\frac{dz}{dt} = (\mu + i\omega) z + (L_1 + i\Omega_1) z^2 \bar{z} + (L_2 + i\Omega_2) z^3 \bar{z}^2 + \ldots + (L_k + i\Omega_k) z^{k+1} \bar{z}^k + o(|z|^{2k+1}). \qquad (***_\varepsilon)$$

The difference with $(* * *_0)$ is that all the coefficients depend now on ε and the coefficient of z is no longer pure imaginary: its real part μ goes from negative to positive values as ε changes (at $\varepsilon = 0$ we have $\mu(\varepsilon) = 0$).

Let us write the system in the polar coordinates $z = re^{i\varphi}$:

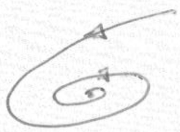
$$\frac{dz}{dt} = \frac{d}{dt}re^{i\varphi} = \frac{dr}{dt}e^{i\varphi}+ir\frac{d\varphi}{dt}e^{i\varphi} = (\mu+i\omega)re^{i\varphi}+(L_1+i\Omega_1)r^3e^{i\varphi}+(L_2+i\Omega_2)r^5e^{i\varphi}+\ldots+(L_k+i\Omega_k)r^{2k+1}e^{i\varphi}+o(r^{2k+1})$$

or

$$\begin{cases} \dfrac{dr}{dt} = \mu r + L_1 r^3 + L_2 r^5 + \ldots + L_k r^{2k+1} + o(r^{2k+1}), \\ \dfrac{d\varphi}{dt} = \omega + \Omega_1 r^2 + \Omega_2 r^4 + \ldots + \Omega_k r^{2k} + o(r^{2k}). \end{cases} \qquad (****)$$

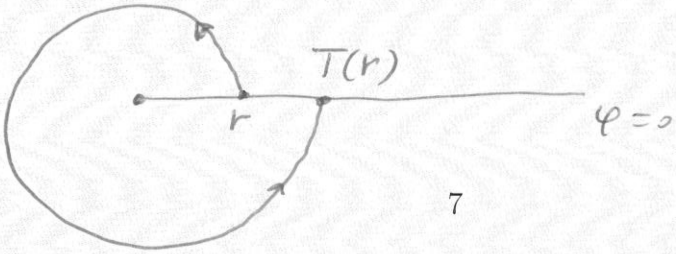
As we see, $d\varphi/dt$ is bounded away from zero in a small neigbourhood of the origin, therefore, every orbit (except for the equilibrium at $r = 0$) rotates around the origin, i.e. it is either a closed orbit or a spiral. When $\mu = 0$, $dr/dt > 0$ is positive for all small r if $L_1 > 0$ and it is negative for all small r if $L_1 < 0$. Therefore, if $L_1 < 0$ the value of r decreases to zero as time grows, which means the stability of the equilibrium at $\mu = 0$; analogously, the positivity of dr/dt at $\mu = 0$ and $L_1 > 0$ means that the equilibrium is unstable at the moment of bifurcation - all this is in agreement with the claim of the theorem.

$L_1 < 0$:



$L_1 > 0$:

In order to study the behaviour at $\mu \neq 0$, we note that as φ is a strictly monotone function of time, every orbit that starts at the ray $\varphi = 0$ $(r > 0)$ returns to it (at $\varphi = 2\pi$) after a finite time $t \sim 2\pi/\omega$. Thus, the Poincare map $T : \{\varphi = 0\} \to \{\varphi = 2\pi\}$ is defined by the orbits of the system; the fixed points of T with $r > 0$ correspond to periodic orbits, stablr fixed points correspond to stable periodic orbits, unstable correspond to unstable ones.



In order to compute the fixed points of T, we rewrite (****) as an integral equation

$$r(\varphi)=r(0)+\int_0^t \frac{\mu r(s)+L_1 r(s)^3+L_2 r(s)^5+\ldots+L_k r(s)^{2k+1}+o(r^{2k+1})}{\omega+\Omega_1 r(s)^2+\Omega_2 r(s)^4+\ldots+\Omega_k r(s)^{2k}+o(r^{2k})}ds$$

(we may consider $r(t)$ as a function of $\varphi(t)$ since $\varphi'(t)\neq 0$). When r and μ are small (this is exactly the case we consider) the derivative of the integral expression with respect to r is small, hence the integral operator in the right-hand side is *strongly contracting* on the space of continuous functions $r(s)$. Therefore the solution of the integral equation can be found by successive approximations: $r(\varphi)=\lim r_n(\varphi)$ where

$$r_{n+1}(\varphi)=r(0)+\int_0^t \frac{\mu r_n(s)+L_1 r_n(s)^3+L_2 r_n(s)^5+\ldots+L_k r_n(s)^{2k+1}+o(r_n^{2k+1})}{\omega+\Omega_1 r_n(s)^2+\Omega_2 r_n(s)^4+\ldots+\Omega_k r_n(s)^{2k}+o(r_n^{2k})}ds,$$

and each new approximation will approximate the solution with the increased order of accuracy in powers of $r(0)$ and μ. By taking the constant function $r_0(\varphi)=r_0$ as the first approximation, we will get

$$r_1(\varphi)=r(0)+\varphi\frac{\mu r(0)+L_1 r(0)^3+L_2 r(0)^5+\ldots+L_k r(0)^{2k+1}}{\omega+\Omega_1 r(0)^2+\Omega_2 r(0)^4+\ldots+\Omega_k r(0)^{2k}}+o(r(0)^{2k+1}).$$

One may check that the second approximation and, hence, all the rest give negligible contribution to the solution; namely, the solution has the form

$$r(\varphi)=r(0)+\frac{\varphi}{\omega}(\mu r(0)(1+\ldots)+L_1 r(0)^3(1+\ldots)+L_2 r(0)^5(1+\ldots)+\ldots+L_k r(0)^{2k+1}(1+\ldots))+o(r(0)^{2k+1})$$

where the dots stand for small terms of order $O(|\mu|+r(0)^2)$. The Poincare map $T: r(0)\mapsto r(2\pi)$ is thus

$$r(0)\mapsto r(0)+\frac{2\pi}{\omega}(\mu r(0)(1+\ldots)+L_1 r(0)^3(1+\ldots)+L_2 r(0)^5(1+\ldots)+\ldots+L_k r(0)^{2k+1}(1+\ldots))+o(r(0)^{2k+1})$$

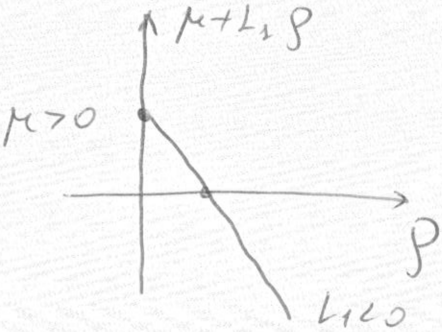
The fixed points of T correspond to positive zeros of $T(r_0)-r_0$, i.e. in order to find them we have to solve the equation

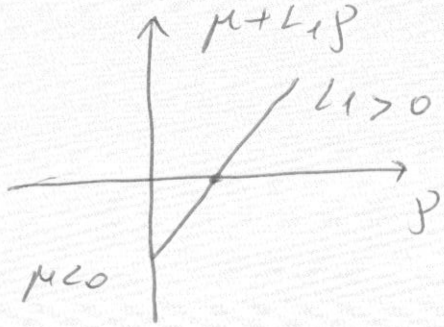
$$\mu(1+\ldots)+L_1\rho(1+\ldots)+L_2\rho^2(1+\ldots)+\ldots+L_k\rho^k(1+\ldots))+o(\rho^k)=0, \qquad (*****)$$

where $\rho\equiv r(0)^2$ has to be positive and small. When $L_1\neq 0$, the derivative of the left-hand side with respect to ρ is bounded away from zero, i.e. the left-hand side of the equation is a monotone function and cannot, hence, have more than one zero. This proves that no more than one periodic orbit can be born at the Andronov-Hopf bifurcation in the case $L_1 \neq 0$.

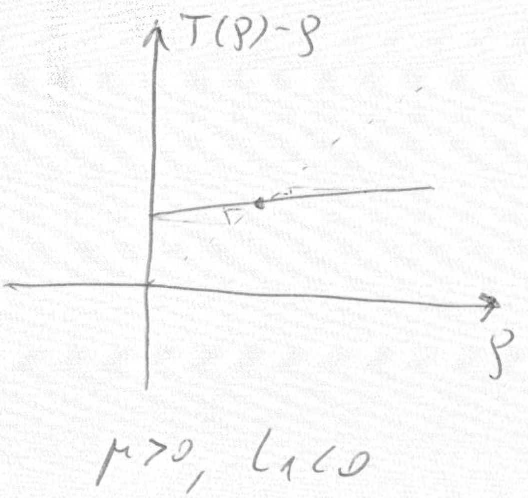
In fact, since equation (*****) has a zero at $\rho = 0$ at $\mu = 0$ and the derivative of the left-hand side with respect to ρ is non-zero, it follows by the Implicit Function Theorem that the equation has a root near zero at all small μ. We, however, are interested only in positive ρ - this corresponds to *sign* $\mu = -$*sign* L_1, and we have

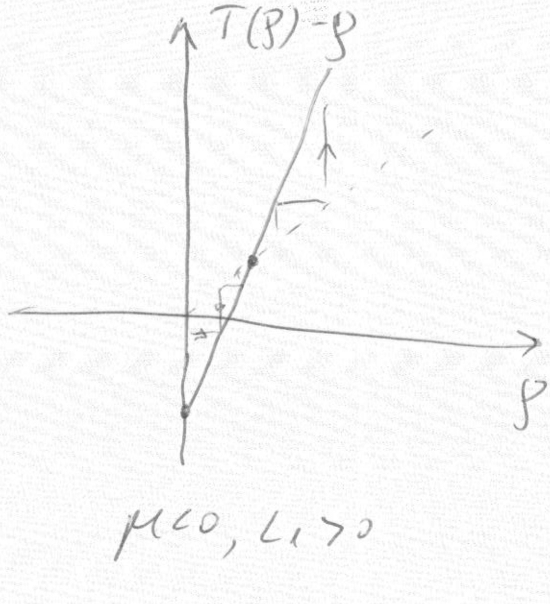
$$r^2 = \rho \sim -\mu/L_1.$$





For positive L_1 we therefore obtain the existence of the periodic orbit at $\mu < 0$, i.e. when the equilibrium state is stable, while for $L_1 < 0$ the periodic orbit is born at $\mu > 0$ which corresponds to the lost of stability by the equilibrium. Stability/instability or the newborn periodic orbits is checked by the evaluation of the derivative of the Poincare map T at the corresponding fixed point: the derivative less than 1 corresponds to a stable orbit (this is the case $L_1 < 0$) and the derivative greater than 1 corresponds to an unstable orbit (the case $L_1 > 0$). □





The above theorem describes the most general case of the Andronov-Hopf bifurcation. One can however ask what happens if $L_1 = 0$ at the bifurcation moment? Using the same arguments as in the previous case we may conclude that if all Lyapunov values from L_1 to L_{k-1} vanish at the moment of bifurcation, then
the stability of the equilibrium state at the bifurcation moment is determined by the sign of L_k: if $L_k < 0$, then the equilibrium is stable at the moment of bifurcation, and if $L_k > 0$, then the equilibrium at the moment of bifurcation is unstable
(to see this just note that the sign of dr/dt in (****) coincides with the sign of L_k when $\mu = L_1 = \ldots = L_{k-1} = 0$).

If we have an equilibrium state with a pair of pure-imaginary eigenvalues and the first $(k-1)$ Lyapunov exponents vanishing, when we perturb the system the values of L_1, ..., L_{k-1} may become non-zero, however they will be small. Therefore, if $L_k \neq 0$ at the moment of bifurcation, then the k-th derivative of the left-hand side of equation (*****) for the fixed points of the Poincare map T will stay bounded away from zero. This means that this equation cannot have more than k roots.

Exercise: show that it is always possible to chose the values of μ, L_1, ..., L_{k-1} such that this equation will have exactly j positive roots, where j is any number from 0 to k. See e.g. Shilnikov, Shilnikov, Turaev, Chua "Methods of qualitative theory in nonlinear dynamics", Ch.11

As we see, if $L_k \neq 0$ than *no more than k periodic orbits can be born at the Andronov-Hopf bifurcation.* These periodic orbits are always nested and surround zero. Typically, they have alternating stability types, with the most outer periodic orbit stable if $L_k < 0$ and unstable if $L_k > 0$ - it obviously has to inherit the stability of the equilibrium at the bifurcation moment.

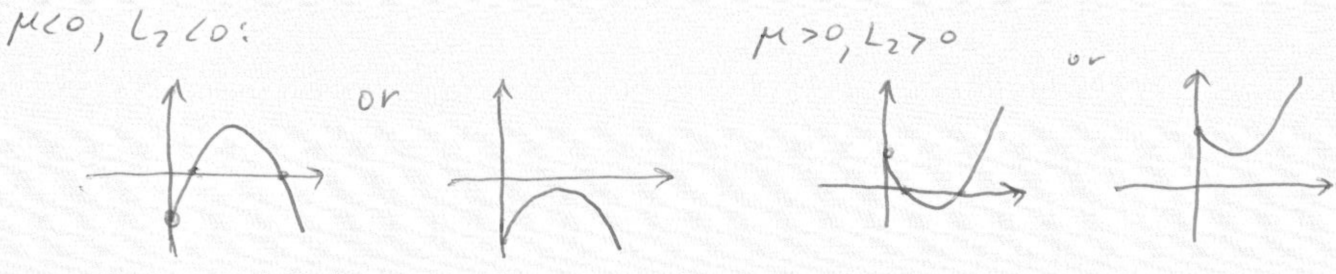
Another interesting fact. Suppose we have a family of systems which depend on a parameter ε and suppose that at $\varepsilon = 0$ the point $(x, y) = 0$ is an equilibrium state that undergoes Andronov-Hopf bifurcation. Denote as $\lambda(\varepsilon)$ an eigenvalue of the linearisation matrix at the equilibrium. By assumption $\mu(0) = Re\lambda(0) = 0$. Assume that $\mu'(0) \neq 0$, i.e. μ changes with a non-zero velocity as ε varies - namely, as ε changes sign, μ changes sign as well which means that the equilibrium changes its stability type. Under this assumption

the set of all periodic orbits that lie in a small neighbourhood of zero forms, in the space of variables x, y and ε, a smooth surface $\varepsilon = p(x, y)$ where $p(0,0) = 0$, $\nabla p(0,0) = 0$.
For a proof just note that equation (*****) for the fixed points of T is, under the assumption $\mu'(\varepsilon) \neq 0$, uniquely resolved with respect to ε.

In conclusion let us consider the case of Andronov-Hopf bifurcation with $L_1 = 0$ and $L_2 \neq 0$ in more detail. Equation (*****) for the fixed points of the Poincare map reads in this case as

$$\mu(1 + \dots) + L_1\rho(1 + \dots) + L_2\rho^2(1 + \dots) = 0, \qquad \rho > 0.$$

Since $L_2 \neq 0$, the second derivative of the left-hand side is non-zero, and it follows that if $\mu L_2 < 0$, then the equation has exactly one positive root.



If $\mu L_2 > 0$, then the equation has, in general, either 2 or 0 positive roots, and the boundary between these 2 cases correspond to a double root ρ^* where the left-hand side of the equation vanishes along with its first derivative. So, at these root we also have

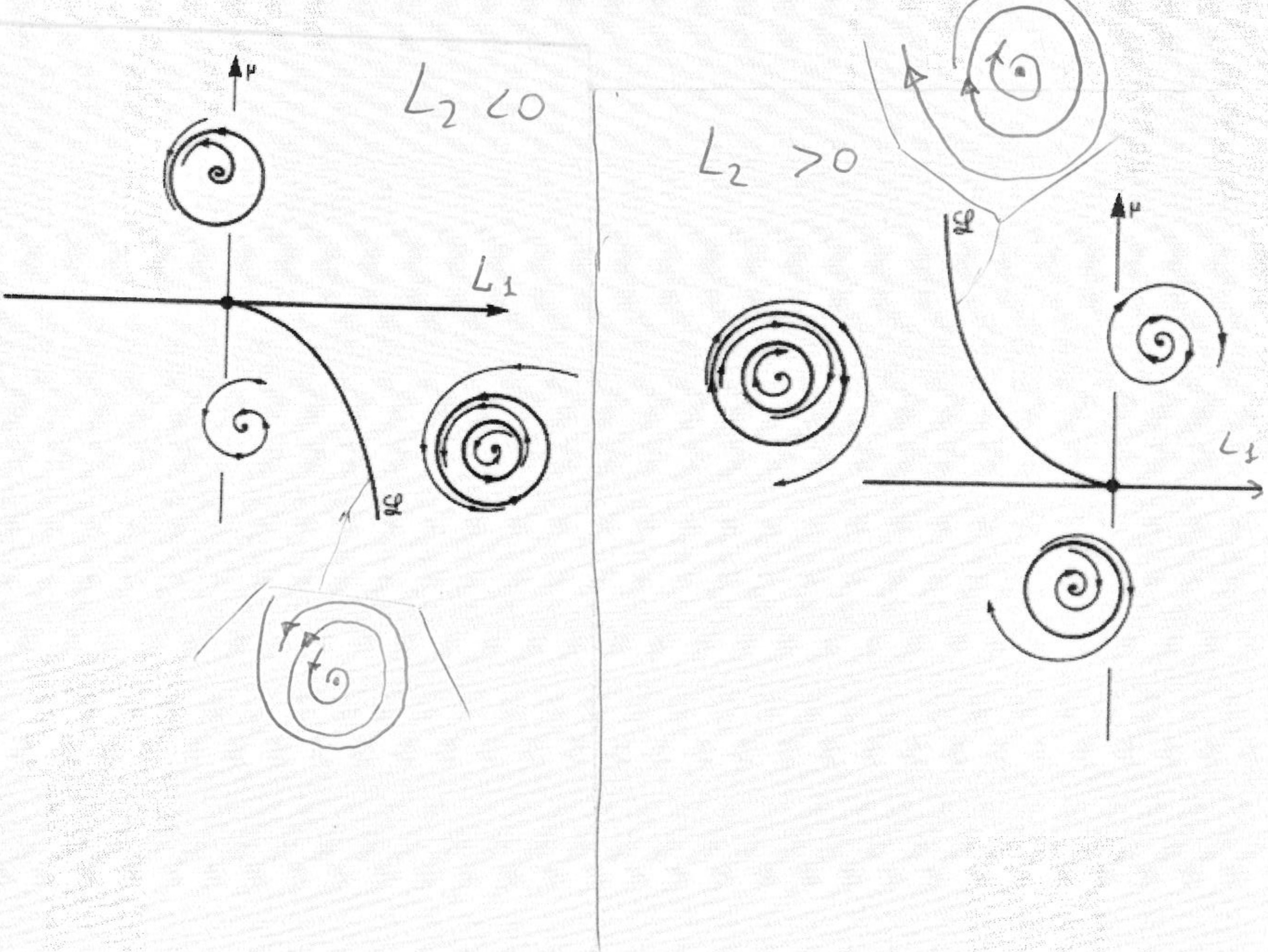
$$L_1 + 2L_2\rho^*(1 + \dots) + o(\mu) = 0.$$

This gives $\rho^* = -L_1/(2L_2) + o(|L_1| + |\mu|)$ (recall that μ and L_1 are small here). The positivity of ρ^* is achieved when $L_1L_2 < 0$. Plugging the expression for ρ^* into the original equation, we find the following equation for the curve $\mathcal{L}$

which separates on the (μ, L_1)-plane the region with 2 periodic orbits from the region with 0 periodic orbits:

$$\mu = \frac{L_1^2}{4L_2} + o(L_1^2), \qquad L_1 L_2 < 0.$$

This curve corresponds to a collision of 2 periodic orbits into 1, a semistable periodic orbit which will be considered in more detail in the next lecture. As we see, the plane of parameters (μ, L_1) is separated into 3 regions (corresponding to different numbers of periodic orbits) by two *bifucation curves*: the curve $\mu = 0$ and the curve $\mathcal{L}$ described above. When we cross $\mu = 0$ the equilibrium undergoes Andronov-Hopf bifurcation: at $L_1 < 0$ a stable periodic orbit is born, at $L_1 > 0$ an unstable periodic orbit is born. When we cross the curve $\mathcal{L}$ we have a bifurcation of a semi-stable periodic orbit: a stable and an unstable periodic orbits collide and disappear. The following pictures show what is called *bifurcation diagrams*: the plane of parameters is divided into regions of the same qualitative behaviour and the phase portraits are given for each of the regions and for each of the bifurcation curves which separate the regions. Further reading: Guckenheimer and Holmes, Ch.3; Kuznetsov "Elements of applied bifurcation theory", Chs.3,8; Shilnikov, Shilnikov, Turaev, Chua, Chs.2,9,11



Exercises.

1. The system $x' = y,\ y' = -x + \varepsilon y + x^2 + xy$ has an equilibrium at zero. The linearisation matrix is $A = \begin{pmatrix} 0 & 1 \\ -1 & \varepsilon \end{pmatrix}$. Its eigenvalues are $\lambda_{1,2} = -\frac{\varepsilon}{2} \pm i\sqrt{1-\frac{\varepsilon^2}{4}}$. At $\varepsilon = 0$ we have $Re\ \lambda = 0$, i.e. the equilibrium undergoes the Andronov-Hopf bifurcation. At $\varepsilon = 0$ the eigenvalues are $\pm i$ and the corresponding eigenvectors are $\begin{pmatrix} 1 \\ \pm i \end{pmatrix}$. To proceed to the basis of the eigenvectors means to introduce a new (complex) variable z such that $\begin{pmatrix} x \\ y \end{pmatrix} = z\begin{pmatrix} 1 \\ i \end{pmatrix} + \bar{z}\begin{pmatrix} 1 \\ -i \end{pmatrix}$, i.e. $x = z + \bar{z}, y = i(z - \bar{z}), z = (x - iy)/2$. The equation for z at $\varepsilon = 0$ is

$$z' = (x'-iy')/2 = iz-i(z+\bar{z})^2/2+(z^2-\bar{z}^2)/2 = iz+(1-i)z^2/2-iz\bar{z}-(1+i)\bar{z}^2/2.$$

Make the normalising transformation

$$u = z + i(1-i)z^2/2 - z\bar{z} + i(1+i)\bar{z}^2/6.$$

We obtain

$$u'-iu = z'-iz+(1-i)z^2/2+iz\bar{z}+(1+i)\bar{z}^2/6+i(1-i)zz'-z(\bar{z})'-z'\bar{z}+i(1+i)\bar{z}(\bar{z})'/3 =$$

$$= (1-i)z^2-(1+i)\bar{z}^2/3+i(1-i)z(iz-iz\bar{z}+\ldots)-z(-i\bar{z}+iz\bar{z}+\ldots)-(iz+(1-i)z^2/2+\ldots)\bar{z}+$$
$$+i(1+i)\bar{z}(-i\bar{z} - (1-i)z^2/2 + \ldots)/3 = z^2\bar{z}(1-5i)/2 + \ldots$$

where the dots stand for irrelevant terms (of order 4, or of order 3 - other than $z^2\bar{z}$). As we see, in the new coordinates the system at $\varepsilon = 0$ takes the form

$$u' = iu + u^2\bar{u}(1-5i)/2 + \ldots,$$

i.e. the first Lyapunov value $L_1 = 1/2 > 0$. How many periodic orbits is born at $\varepsilon > 0$? At $\varepsilon < 0$? Are they stable or unstable?

2. Show that all Lyapunov values are zero for the zero equilibrium of the system $x' = y,\ y' = -x + xy + x^3$. Hint: the system is *reversible*, i.e. it does not change when we change $t \to -t$, $x \to -x$; this means that every orbit which crosses the y-axis is symmetric with respect to it, so every orbit near zero is closed. Note that if we imbed our system into a family which depends on a parameter ε such that the equilibrium at zero will change its stability as ε varies across zero, the surface filled by the closed orbits in the

(x, y, ε)-space will be just the plane $\varepsilon = 0$, so no closed orbits will exist near zero at $\varepsilon \neq 0$.
3. Show that all Lyapunov values are zero for the zero equilibrium of the system $x' = y,\ y' = -x + x^2$. Hint: the system has a first integral $H = \frac{1}{2}(y^2 + x^2) - \frac{1}{3}x^3$; so every orbit is a level line of $H(x, y)$, and these level lines near zero are closed.
4. How many periodic orbits is born at $\varepsilon \neq 0$ near $(x, y) = 0$ in the system $x' = y,\ y' = -x + \varepsilon y + y^2$?

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5. Show that, given any n, system $x' = -x + O(x^2 + y^2),\ y' = \sqrt{2}y + O(x^2 + y^2)$ can be brought to a form $x' = -x + O(|x|^n + |y|^n),\ y' = \sqrt{2}y + O(|x|^n + |y|^n)$ by a polynomial coordinate transformation.
6. Show that if all the eigenvalues of a matrix A are strictly negative, any equation $\underline{x} = A\underline{x} + O(\|x\|^2)$ may have only a finite number of resonant terms.
7. Bring the system $x' = y + xz,\ y' = x^2 + y^2 + z^2,\ z' = -2z + xy$ to a normal form up to the second order (kill all nonresonant quadratic terms).