Mathematical Methods **Spring Term 2019**

Answers to Problem Sheet 8

1. i)
$$a)\ \ \delta_{ij}\delta_{ij} = 3 \quad b)\ \ \delta_{ij}\delta_{jk}\delta_{kl}\delta_{li} = 3 \quad c)\ \ \epsilon_{ijk}\epsilon_{jki} = 6.$$

   ii) $\epsilon_{ikl}\epsilon_{jkl} = 2\delta_{ij}$.

   Direct proof just insert values of k and l that could lead to a non-zero contribution:

$$\epsilon_{ikl}\epsilon_{jkl} = \epsilon_{i12}\epsilon_{j12} + \epsilon_{i21}\epsilon_{j21} + \epsilon_{i23}\epsilon_{j23} + \epsilon_{i32}\epsilon_{j32} + \epsilon_{i31}\epsilon_{j31} + \epsilon_{i13}\epsilon_{j13}$$

   Each of the six contributions is zero if $i \neq j$, eg. the first term is zero unless $i = j = 3$. For each $i = j$ two of the six terms are 1 with the four remaining terms vanishing. Hence $\epsilon_{ikl}\epsilon_{jkl} = 2\delta_{ij}$.

   Elegant proof Note that $\epsilon_{ikl}\epsilon_{jkl}$ is an isotropic tensor (not a pseudo-tensor). Any isotropic cartesian tensor of rank two is proportional to δ_{ij}. Taking $i = j = 1$ fixes the constant of proportionality to be 2.

   iii) Suppose U_i and V_i are axial. Under an orthogonal transformation $x'_i = R_{ij}x_j$ they transform according to

$$U'_i = \det R\ R_{ij}U_j, \qquad V'_i = \det R\ R_{ij}V_j.$$

   Therefore U_kV_j transform like a rank two tensor (as $(\det R)^2 = 1$). The cross product of $\mathbf{U}$ and $\mathbf{V}$ has components $(\mathbf{U} \times \mathbf{V})_i = \epsilon_{ijk}U_jV_k$. This is obtained by contracting the pseudo-tensor ϵ_{ijk} with the tensor U_jV_k giving a pseudo-tensor (or axial vector since the rank is one).

2. i) To show that
$$\det A = \epsilon_{ijk}A_{1i}A_{2j}A_{3j}.$$

   compute the right hand side (note that 21 of the 27 components of ϵ_{ijk} are zero)

$$\begin{aligned}\epsilon_{ijk}A_{1i}A_{2j}A_{3j} &= \epsilon_{123}A_{11}A_{22}A_{33} + \epsilon_{132}A_{11}A_{23}A_{32}\\ &\quad + \epsilon_{213}A_{12}A_{21}A_{33} + \epsilon_{231}A_{12}A_{23}A_{31}\\ &\quad + \epsilon_{312}A_{13}A_{21}A_{32} + \epsilon_{321}A_{13}A_{22}A_{31}\\ &= A_{11}A_{22}A_{33} - A_{11}A_{23}A_{32}\\ &\quad + A_{12}A_{21}A_{33} - A_{12}A_{23}A_{31}\end{aligned}$$

$$+A_{13}A_{21}A_{32} - A_{13}A_{22}A_{31}$$

which match the six terms in det A

ii) To show that under proper rotations ϵ_{ijk} is a rank 3 tensor we need to establish that

$$R_{ip}R_{jq}R_{kr}\epsilon_{pqr} = \epsilon_{ijk}$$

where R_{ij} is an orthogonal matrix with determinant one.

Setting $A_{st} = R_{st}$ in the result from part i) proves the result for $i = 1$, $j = 2$, $k = 3$ (using $\epsilon_{123} = 1$) To show that it holds for the other 26 components it is sufficient to show that $R_{ip}R_{jq}R_{kr}\epsilon_{pqr}$ is totally anti-symmetric (i.e., it changes sign under an interchange of two indices). Consider an interchange of i and j:

$$R_{jp}R_{iq}R_{kr}\epsilon_{pqr} = R_{jq}R_{ip}R_{kr}\epsilon_{qpr} = -R_{ip}R_{jq}R_{kr}\epsilon_{pqr}.$$

In the second step p was relabelled as q and vice versa. Similarly $R_{ip}R_{jq}R_{kr}\epsilon_{pqr}$ is anti-symmetric with respect to any other interchange of indices.

3. i) To prove $\epsilon_{ijp}\epsilon_{klp} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$ expand the left hand side

$$\text{LHS} = \epsilon_{ij1}\epsilon_{kl1} + \epsilon_{ij2}\epsilon_{kl2} + \epsilon_{ij3}\epsilon_{kl3}$$

This is zero for most choices of i, j, k, l (how many?). To get a non-zero result (which can only be ± 1) one must have $i \neq j$ and k and l must be the same two numbers as i and j

To get 1 requires $i \neq j$ and $k = i$ and $l = j$ (6 possibilities)

To get -1 requires $i \neq j$ and $k = j$ and $l = i$ (6 possibilities)

In all cases it is easy to check that the right hand side yields the same result.

ii) $[\mathbf{A}\times(\mathbf{B}\times\mathbf{C})]_i = \epsilon_{ijk}A_j(\mathbf{B}\times\mathbf{C})_k = \epsilon_{ijk}A_j\epsilon_{klm}B_lC_m = \epsilon_{ijk}\epsilon_{klm}A_jB_lC_m$ $= \epsilon_{ijk}\epsilon_{lmk}A_jB_lC_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})A_jB_lC_m = B_i(A_jC_j) - C_i(A_jB_j) =$ $[(\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}]_i$.

iii) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \partial_i(\mathbf{A}\times\mathbf{B})_i = \partial_i\epsilon_{ijk}A_jB_k = B_k\epsilon_{ijk}\partial_iA_j + A_j\partial_{ijk}B_k =$ $B_k\epsilon_{kij}\partial_iA_j - A_j\epsilon_{jik}\partial_iB_k = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$.

4.

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}, \quad \nabla\times\mathbf{B} = \mu_0\mathbf{j} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}, \quad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \quad \nabla\cdot\mathbf{B} = 0.$$

i) The trick is not to use the pseudo-tensors B_i and ϵ_{ijk}. This is possible as Maxwell's equations are parity invariant! The first three are straightforward (a quick calculation may be needed to work out the coefficients):

$$\partial_i E_i = \frac{\rho}{\epsilon_0}, \quad \partial_j F_{ij} = \mu_0 j_i + \mu_0 \epsilon_0 \frac{\partial E_i}{\partial t}, \quad \partial_i E_j - \partial_j E_i = -\frac{\partial F_{ij}}{\partial t}.$$

Finally, $\nabla \cdot \mathbf{B} = 0$ can be written as (this is not so obvious)

$$\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0.$$

Note that the four equation are written consecutively as scalar, vector, rank 2 tensor and rank 3 tensor equations.

ii)

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}, \qquad u = \frac{\epsilon_0}{2} \mathbf{E} \cdot \mathbf{E} + \frac{1}{2\mu_0} \mathbf{B} \cdot \mathbf{B}.$$

can be written in the form

$$S_i = \frac{1}{\mu_0} F_{ij} E_j, \quad u = \frac{\epsilon}{2} E_i E_i + \frac{1}{4\mu_0} F_{ij} F_{ij}.$$

S_i is polar since it is a contraction of two tensors (no pseudo-tensors involved). Alternatively, it is polar as it is the cross product of a polar and an axial vector.

iii)

$$\frac{\partial u}{\partial t} + \partial_i S_i = \epsilon_0 E_i \frac{\partial E_i}{\partial t} + \frac{1}{2\mu_0} F_{ij} \frac{\partial F_{ij}}{\partial t} + \frac{1}{\mu_0} \partial_i (F_{ij} E_j)$$

$$= -E_i j_i + \frac{1}{\mu_0} E_i \partial_j F_{ij} - \frac{1}{2\mu_0} F_{ij} (\partial_i E_j - \partial_j E_i) + \frac{1}{\mu_0} \partial_i (F_{ij} E_j)$$

using two of Maxwell's equation to eliminate the time derivatives. All term on the right hand side of the above equation cancel apart from the $-E_i j_i$ contribution. To show the cancellation use

$$F_{ij}(\partial_i E_j - \partial_j E_i) = 2 F_{ij} \partial_i E_j.$$

5. i) The Pauli matrices satisfy the commutation relations $[\sigma_1, \sigma_2] = 2i\sigma_3$, $[\sigma_2, \sigma_3] = 2i\sigma_1$, $[\sigma_3, \sigma_1] = 2i\sigma_2$. These are equivalent to $[\sigma_i, \sigma_j] =$

$2i\epsilon_{ijk}\sigma_k$ (9 equations, 3 of which give $0 = 0$, the other 6 give $\pm$ the commutation relations).

ii)

$$B'_i\sigma_i = e^{-\frac{1}{2}i\theta\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}}\ B_i\sigma_i\ e^{\frac{1}{2}i\theta\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}} = (I - \tfrac{1}{2}i\theta\hat{\mathbf{n}}\cdot\boldsymbol{\sigma} + ...)B_i\sigma_i(I + \tfrac{1}{2}i\theta\hat{\mathbf{n}}\cdot\boldsymbol{\sigma} + ...)$$

$$= B_i\sigma_i - \tfrac{1}{2}i\theta\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}B_i\sigma_i + B_i\sigma_i\tfrac{1}{2}i\theta\hat{\mathbf{n}}\cdot\boldsymbol{\sigma} + ...$$

Here I is the 2×2 identity matrix.

iii) Computing $\sigma_i B'_i$ for an infinitesimal rotation

$$\sigma_i B'_i = \sigma_i(\delta_{ij} - \theta\epsilon_{ijk}\ n_k)B_j$$

Now $\sigma_i\epsilon_{ijk} = \epsilon_{jki}\sigma_i = [\sigma_j, \sigma_k]/(2i)$ so that

$$\sigma_i B'_i = \sigma_i B_i + \frac{i\theta}{2}[\sigma_j, \sigma_k]B_j n_k = \sigma_i B_i - \frac{i\theta}{2}[\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}, B_i\sigma_i].$$