Mathematical Methods **Spring Term 2019**

Answers to Problem Sheet 6

1. If S is stationary then $y(x)$ satisfies the Euler-Lagrange equation

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = \frac{\partial L}{\partial y}.$$

Therefore

$$\frac{dH}{dx} = \frac{d}{dx}\left(y'\frac{\partial L}{\partial y'} - L\right) = y''\frac{\partial L}{\partial y'} + y'\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{dL}{dx} = y''\frac{\partial L}{\partial y'} + y'\frac{\partial L}{\partial y} - \frac{dL}{dx}.$$

We require the derivative of $L(y(x), y'(x))$ with respect to x; using the chain rule

$$\frac{dL}{dx} = \frac{\partial L}{\partial y}\frac{dy}{dx} + \frac{\partial L}{\partial y'}\frac{d}{dx}\left(\frac{dy}{dx}\right).$$

Hence H is constant.

2. Here

$$L = 2\pi y\sqrt{1 + y'^2},$$

giving the Euler Lagrange equation

$$\frac{d}{dx}\left(\frac{yy'}{\sqrt{1+y'^2}}\right) = \sqrt{1+y'^2}.$$

Now (dropping the factor of 2π in L)

$$H = y'\frac{\partial L}{\partial y'} - L = \frac{yy'^2}{\sqrt{1+y'^2}} - y\sqrt{1+y'^2} = \frac{y}{\sqrt{1+y'^2}}\left[y'^2 - (1 + y'^2)\right]$$

$$= -\frac{y}{\sqrt{1+y'^2}} = \text{constant}.$$

Squaring this gives

$$\frac{y^2}{1+y'^2} = C \quad \text{or} \quad y'^2 = \frac{y^2}{C} - 1.$$

where C is a positive constant. Therefore

$$\frac{\sqrt{C}dy}{\sqrt{y^2 - C}} = \pm dx.$$

which integrates to

$$\sqrt{C}\cosh^{-1}\frac{y}{\sqrt{C}} = \pm x + c,$$

so that

$$\frac{y}{\sqrt{C}} = \cosh\frac{\pm x + c}{\sqrt{C}}.$$

By symmetry $c = 0$ so that $y(x)$ has the stated form (on setting $C = 1/p^2$).

ii) $y(x) = p^{-1}\cosh px$. p is fixed using the boundary condition

$$y(-L/2) = y(L/2) = \frac{1}{p}\cosh\frac{pL}{2} = R.$$

It is convenient to define $q = pL/2$ so that q (and hence p) is fixed via

$$q^{-1}\cosh q = \frac{2R}{L}.$$

A quick sketch shows that for $q > 0$, $q^{-1}\cosh q$ has a positive minimum (Wolfram Alpha gives the minimum value as ≈ 1.50888 at $q \approx 1.19968$). So if $2R/L$ is less than 1.50888 the boundary conditions cannot be matched for any p. If $2R/L$ is greater than the critical value two values of q fit the boundary conditions (this is clear from a plot of $q^{-1}\cosh q$).

3. Take the line to be the x-axis and the endpoints to be $(a, 0)$ and $(b, 0)$. Maximize the integral

$$\int_a^b y(x)dx \qquad \text{with} \;\; y(a) = y(b) = 0,$$

subject to the constraint that the length

$$l = \int_a^b \sqrt{1 + y'^2}dx$$

is fixed. Implementing the constraint using a Lagrange multiplier λ. Consider the integral

$$S = \int_a^b L \, dx \quad \text{with} \quad L = y + \lambda \left(\sqrt{1 + y'^2} - \frac{l}{b-a} \right).$$

The Euler-Lagrange equation is

$$\frac{d}{dx} \frac{\lambda y'}{\sqrt{1 + y'^2}} = 1.$$

This can be solved using the result from question 1 *or* direct integration. Integrating the Euler-Lagrange equation gives

$$\frac{\lambda y'}{\sqrt{1 + y'^2}} = x + c.$$

Therefore

$$\frac{\lambda^2 y'^2}{1 + y'^2} = (x + c)^2 \quad \text{or} \quad y'^2 = \frac{(x + c)^2}{\lambda^2 - (x + c)^2},$$

so that

$$y' = \frac{\pm (x + c)}{\sqrt{\lambda^2 - (x + c)^2}}$$

which integrates to

$$y = \mp \sqrt{\lambda^2 - (x + c)^2} + d,$$

which is circular.