Mathematical Methods **Spring Term 2019**

Answers to Problem Sheet 3

1. i) The radius of convergence of the Taylor expansion for $f(z) = (1 - \cos z)/z^2$ about $z = 0$ is infinity as f is an entire function.

   ii) $f(z) = \ln z$. The derivatives of f are $f^{(m)}(z) = (-1)^{m+1}(m-1)!z^{-m}$. Taylor expansion

$$f(z) = f(e^{2\pi i/3}) + \sum_{m=1}^{\infty} \frac{f^{(m)}(e^{2\pi i/3})}{m!}(z - e^{2\pi i/3})^m$$

$$= 2\pi i/3 + \sum_{m=1}^{\infty} (-1)^{m+1} \frac{e^{-2\pi i m/3}}{m}(z - e^{2\pi i/3})^m$$

   $R = 1$ (distance between $e^{2\pi i/3}$ and singularity at origin).

2.

$$i) \quad f(z) = \frac{e^{iz}}{1+z^2} = \frac{e^{iz}}{(z+i)(z-i)}$$

   which has simple poles at $z = \pm i$. The residues are $\mathrm{Res}(f, i) = e^{-1}/(2i)$ and $\mathrm{Res}(f, -i) = e/(-2i)$.

$$ii) \quad f(z) = \frac{1}{(z+1)(z+2)(z+3)}$$

   has simple poles at -1, -2 and -3. The residues are

$$\mathrm{Res}(f, -1) = \frac{1}{(-1+2)(-1+3)} = \frac{1}{2}, \ \mathrm{Res}(f, -2) = \frac{1}{(-2+1)(-2+3)} = -1,$$

$$\mathrm{Res}(f, -3) = \frac{1}{(-3+1)(-3+2)} = \frac{1}{2}.$$

3. C is the unit circle with the orientation taken anti-clockwise.

$$a) \quad \oint_C \frac{e^z - 1}{z}\, dz = 0$$

   by Cauchy's theorem (the integrand is entire).

$$b) \quad \oint_C \frac{\cos 2z}{z^5}\, dz = 2\pi i \frac{2^4}{4!} = \frac{4\pi i}{3},$$

using $\cos 2z = 1 - (2z)^2/2! + (2z)^4/4! - ...$ to extract the residue of the integrand

$$c) \ \oint_C z^2 e^{1/z}\, dz = \frac{2\pi i}{3!} = \frac{\pi i}{3},$$

using $e^{1/z} = 1 + 1/z + 1/(2!z^2) + 1/(3!z^3) + ...$

4.

$$\oint_C \frac{e^{iz}}{(z-i)^2} dz$$

taken around a semi-circular path in the upper-half plane, with straight line section between $z = -R$ and $z = R$, $R > 1$. $f(z) = e^{iz}/(z-i)^2$ has a double pole at $z = i$. Expanding e^{iz} about i, $e^{iz} = e^{i[(z-i)+i]} = e^{-1}e^{i(z-i)} = e^{-1}(1 + i(z-i) + ...)$, so $\mathrm{Res}(f, i) = ie^{-1}$. By the residue theorem the contour integral is $-2\pi/e$. As $R \to \infty$ the contribution of the semi-circle to the contour integral vanishes. Accordingly,

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x-i)^2} dx = -\frac{2\pi}{e}.$$

Now

$$\mathrm{Re}\frac{e^{ix}}{(x-i)^2} = \frac{1}{2}\left[\frac{e^{ix}}{(x-i)^2} + \frac{e^{-ix}}{(x+i)^2}\right] = \frac{e^{ix}(x+i)^2 + e^{-ix}(x-i)^2}{2(x-i)^2(x+i)^2}$$

$$\frac{(x^2-1)(e^{ix}+e^{-ix}) + 2ix(e^{ix}-e^{-ix})}{2(x^2+1)^2} = \frac{2(x^2-1)\cos x - 4x\sin x}{2(x^2+1)^2}.$$

Therefore

$$\int_{-\infty}^{\infty} \frac{2x\sin x - (x^2-1)\cos x}{(1+x^2)^2} dx = \frac{2\pi}{e}.$$

As the integrand is even integrating from $-$ to ∞ yields π/e.

5. i) Consider the meromorphic function $f(z) = 1/(1+z^6)$ which has six simple poles at $e^{\pm i\pi/6}$, $\pm i$, $e^{\pm 5\pi i/6}$. Take the same contour as in question 4. The residues at the three poles inside the contour are

$$\mathrm{Res}(f, e^{i\pi/6}) = \frac{1}{6e^{5\pi i/6}} = \frac{e^{-5\pi i/6}}{6} = -\frac{e^{i\pi/6}}{6}, \quad \mathrm{Res}(f, i) = \frac{1}{6i^5} = -\frac{i}{6}$$

$$\mathrm{Res}(f, e^{5\pi i/6}) = \frac{1}{6e^{25\pi i/6}} = \frac{e^{-i\pi/6}}{6}$$

(write $f(z) = 1/g(z)$ where $g(z) = z^6 + 1$ with $g'(z) = 6z^5$, at a pole w, $\text{Res}(f, w) = 1/g'(w)$). For $R > 1$ the contour integral is

$$\oint_C f(z)dz = \frac{2\pi i}{6}\left[-e^{\pi i/6} - i + e^{-\pi i/6}\right] = \frac{\pi i}{3}\left[-2i\sin\frac{\pi}{6} - i\right] = \frac{2\pi}{3}.$$

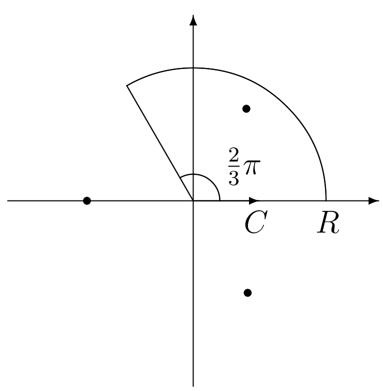
As $R \to \infty$ the semi-circle does not contribute so the contour integral reduces to the integral of $f(x)$ from $-\infty$ to ∞. This gives

$$\int_{-\infty}^{\infty} \frac{1}{1+x^6}dx = \frac{2\pi}{3}.$$

ii) The integral

$$I = \int_0^{\infty} \frac{dx}{1+x^3}$$

can be computed by integrating $f(z) = \ln(-z)/(1+z^3)$ over the keyhole contour discussed in the lectures. Alternatively, integrate $f(z) = 1/(1+z^3)$ over the contour C depicted below



$f(z)$ has simple poles at $e^{\pm i\pi/3}$ and -1 of which only $e^{i\pi/3}$ is enclosed by the contour. The residue at this pole is

$$\text{Res}(f, e^{i\pi/3}) = \frac{1}{3(e^{i\pi/3})^2} = \frac{e^{-2\pi i/3}}{3}.$$

By the Residue theorem

$$\oint_C f(z)dz = \frac{2\pi i e^{-2\pi i/3}}{3} = \frac{2\pi e^{-i\pi/6}}{3}.$$

The contour integral splits into three parts. The arc contribution vanishes as $R \to \infty$. Integrating f along the positive real axis gives the integral I we seek. The integral over the other line segment (denoted $\tilde{C}$

below) gives a contribution proportional to I. Parametrizing this line segment through $z = e^{2\pi i/3}t$ $(0 < t < R)$ gives

$$\int_{\tilde{C}} f(z)dz = -\int_0^R \frac{e^{2\pi i/3}dt}{1+(e^{2\pi i/3}t)^3} = -\int_0^R \frac{e^{2\pi i/3}dt}{1+t^3} \to -e^{2\pi i/3}I \quad \text{as} \quad R \to \infty.$$

(the minus sign follows from the anti-clockwise orientation of C). Accordingly

$$(1 - e^{2\pi i/3})I = \frac{2\pi e^{-i\pi/6}}{6}.$$

or

$$I = \frac{2\pi e^{-i\pi/2}}{e^{-\pi/3} - e^{\pi i/3}} = \frac{\pi}{3\sin(\frac{1}{6}\pi)} = \frac{2\pi}{3\sqrt{3}}.$$

Note that I is obviously real and positive - this provides a check on calculations of this type.

iii) *This problem is challenging!*

To compute

$$J = \int_0^\infty \frac{\ln x}{1+x^3}dx$$

integrate $g(z) = \ln z/(1+z^3)$ over the same contour as in part ii). As in part ii) there is a simple pole at $e^{i\pi/3}$- the residue is $\ln e^{\pi i/3} = i\pi/3$ multiplied by the residue of $f(z) = 1/(1+z^3)$, i.e. $\text{Res}(g, e^{\pi i/3}) = i\pi e^{-2\pi i/3}/9$. The contour integral is

$$\oint_C g(z)dz = -\frac{2\pi^2 e^{-2\pi i/3}}{9}.$$

As in part ii) integrating g along the positive real axis gives the integral J we seek. Much as in part ii) the integral of g along $\tilde{C}$ can be related to J

$$\int_{\tilde{C}} g(z)dz = -\int_0^R \frac{\ln(e^{2\pi i/3}t)e^{2\pi i/3}dt}{1+(e^{2\pi i/3}t)^3} = -e^{2\pi i/3}\int_0^R \frac{(\ln t \ + \frac{2}{3}\pi i)dt}{1+t^3}$$

$$\to -e^{2\pi i/3}J - \frac{2\pi i e^{2\pi i/3}}{3}I \quad \text{as} \quad R \to \infty$$

Accordingly,

$$(1 - e^{2\pi i/3})J = -\frac{2\pi^2 e^{-2\pi i/3}}{9} + \frac{2\pi i e^{2\pi i/3}}{3}I = -\frac{2\pi^2 e^{-2\pi i/3}}{9} + \frac{4\pi^2 i e^{2\pi i/3}}{9\sqrt{3}}$$

or

$$\left(\frac{3}{2}-\frac{\sqrt{3}i}{2}\right)J=\frac{2\pi^2}{9}\left(-e^{-2\pi i/3}+\frac{2ie^{2\pi i/3}}{\sqrt{3}}\right)=\frac{2\pi^2}{9}\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}-\frac{i}{\sqrt{3}}-1\right)$$

$$=-\frac{2\pi^2}{27}\left(\frac{3}{2}-\frac{\sqrt{3}i}{2}\right),$$

so that $J=-2\pi^2/27$.

iv) Let C be the unit circle (with anti-clockwise orientation)

$$\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta}=\oint_C\frac{1}{a+\frac{1}{2}b(z+z^{-1})}\frac{dz}{iz}=\frac{1}{i}\oint_C\frac{2}{2az+bz^2+b}dz.$$

$$=\frac{2}{ib}\oint_C\frac{1}{z^2+(2a/b)z+1}dz.$$

$z^2+(2a/b)z+1$ has roots at$z=-a/b\ \pm\sqrt{a^2/b^2\ -1}$ of which only $z=-a/b\ +\sqrt{a^2/b^2\ -1}$ is inside the unit circle. The residue of

$$\frac{1}{z^2+(2a/b)z+1}=\frac{1}{(z+a/b\ -\sqrt{a^2/b^2\ -1})(z+a/b\ +\sqrt{a^2/b^2\ -1}))}$$

at $z=-a/b\ +\sqrt{a^2/b^2\ -1}$ is

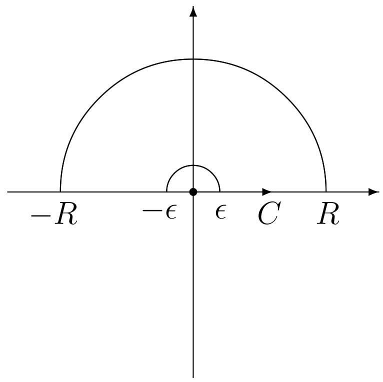
$$\frac{1}{2\sqrt{a^2/b^2\ -1)}}.$$

Using the residue theorem

$$\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta}=\frac{2}{ib}2\pi i\frac{1}{2\sqrt{a^2/b^2\ -1)}}=\frac{2\pi}{\sqrt{a^2-b^2}}.$$

v) $\sin^2 x=\frac{1}{2}(1-\cos 2x)=\mathrm{Re}\frac{1}{2}(1-e^{2ix})$. Consider the meromorphic function $f(z)=(1-e^{2iz})/(2z^2)$ which has a simple pole at the origin. $\mathrm{Res}(f,0)=-i$.

This problem can be tackled by using the contour from question 4 with a small semi-circular detour to avoid the pole at the origin:



There is no pole on or inside the contour C so by Cauchy's theorem

$$\oint_C f(z)dz = 0.$$

In the $R \to \infty$ limit the semi-circle does not contribute giving

$$0 = \left[\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right] \frac{1 - e^{2ix}}{2x^2} dx + \int_{C(\epsilon)} f(z)dz,$$

where $C(\epsilon)$ is the small semi-circle centred at the origin. Near $z = 0$ $f(z) \approx -i/z + ...$ so that for small ϵ

$$\int_{C(\epsilon)} f(z)dz = -\int_0^{\pi} \frac{-i}{\epsilon e^{i\theta}} i\epsilon e^{i\theta} d\theta = -\pi$$

(using $z = \epsilon e^{i\theta}$ with $0 \leq \theta \leq \pi$, the minus sign corrects the orientation). For small ϵ

$$\left[\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right] \frac{1 - e^{2ix}}{2x^2} dx \approx \pi.$$

Taking the real part and the $\epsilon \to 0$ limit gives

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx = \pi.$$

Alternatively, using the half residue rule Let C be the contour from question 4 which passes through the pole at the origin.

$$\mathrm{P} \oint f(z)dz = i\pi \mathrm{Res}(f, 0) = \pi$$

As $R \to \infty$ the semi-circle does not contribute giving

$$\mathrm{P} \int_{-\infty}^{\infty} \frac{1 - e^{2ix}}{2x^2} = \pi.$$

Taking the real part

$$\int_0^\infty \frac{\sin^2 x}{x^2} dx = \pi.$$

6. A particular solution of the ODE $\ddot{x}(t) + 3\dot{x}(t) + 2x(t) = \delta(t)$ is (see also Problem Sheet 4 Q3 ii))

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{-\omega^2 + 3i\omega + 2} d\omega.$$

Consider the meromorphic function

$$f(z) = \frac{e^{izt}}{-z^2 + 3iz + 2} = -\frac{e^{izt}}{(z-i)(z-2i)},$$

which has simple poles at i and $2i$. The residues are

$$\text{Res}(f, i) = -\frac{e^{-t}}{-i}, \quad \text{Res}(f, 2i) = -\frac{e^{-2t}}{i}.$$

If $t > 0$ take the same contour as in question 4. As $R \to \infty$ the contribution of the semicircle vanishes giving

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{-\omega^2 + 3i\omega + 2} d\omega = e^{-t} - e^{-2t}.$$

If $t < 0$ the semi-circle should be taken in the lower half-plane. As this contour neither encloses nor crosses any poles the integral is zero. The solution can be written as

$$x(t) = (e^{-t} - e^{-2t})\theta(t).$$

Check that this is a solution of the ODE!

7.

$$f(z) = \frac{e^{iz}}{\sinh z}$$

has simple poles at $i\pi n$ (n integer). Expanding $\sinh z$ about $in\pi$

$$\sinh z = \sinh[(z - i\pi n) + i\pi n] = -(1)^n \sinh(z - i\pi n) = (-1)^n \left[(z - in\pi) + ...\right].$$

Hence, the residues at the poles are $\text{Res}(f, in\pi) = (-1)^n e^{-n\pi}$

The contour integral over the given contour is zero as it does not enclose or pass through any poles.

The real integral can be computed by using the given contour or via the half-residue rule (note that these are essentially the same procedures)

Using the half-residue rule Take a rectangular contour (without the detours) that cuts through the simple poles at 0 and $i\pi$. Then

$$\mathrm{P}\oint_C f(z)dz = \pi i[\mathrm{Res}(f,0) + \mathrm{Res}(f,i\pi)] = i\pi(1 - e^{-\pi}).$$

As $R \to \infty$ the contour integral is

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{e^{ix}}{\sinh x}dx - \mathrm{P}\int_{-\infty}^{\infty} \frac{e^{i(x+i\pi)}}{\sinh(x+i\pi)} = (1 + e^{-\pi})\mathrm{P}\int_{-\infty}^{\infty} \frac{e^{ix}}{\sinh x}dx.$$

On taking the imaginary part

$$(1 + e^{-\pi})\int_{-\infty}^{\infty} \frac{\sin x}{\sinh x}dx = \pi(1 - e^{-\pi})$$

or

$$\int_0^{\infty} \frac{\sin x}{\sinh x}dx = \frac{\pi}{2}\tanh\frac{\pi}{2}.$$