Answers to Problem Sheet 2

1. $f(z) = \ln z$.

   i) Here $f(z) = u(x,y) + iv(x,y)$ with $u(x,y) = \frac{1}{2}\ln(x^2+y^2)$ and $v(x,y) = \tan^{-1}(y/x)$[1]. The first partial derivatives are

   $$u_x = \frac{x}{x^2+y^2}, \quad u_y = \frac{y}{x^2+y^2}, \quad v_x = \frac{-y/x^2}{1+(y/x)^2} = -\frac{y}{x^2+y^2},$$

   $$v_y = \frac{1/x}{1+(y/x)^2} = \frac{x}{x^2+y^2},$$

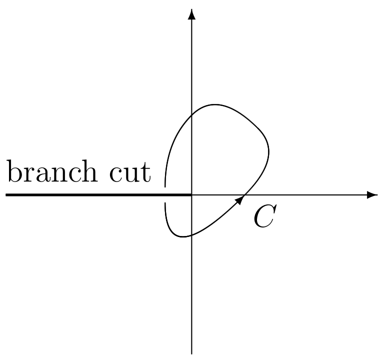
   which satisfy the Cauchy-Riemann equations.

   ii) Away from its branch cut

   $$\frac{df(z)}{dz} = \lim_{h\to 0}\frac{\ln(z+h)-\ln z}{h} = \lim_{h\to 0}\frac{1}{h}\ln\left(1+\frac{h}{z}\right) = \lim_{w\to 0}\frac{\ln(1+w)}{wz} = \frac{1}{z}$$

   setting $w = h/z$ in the last limit.

2. An anti-derivative for $f(z) = 1/z$ is $F(z) = \ln z$ which has a branch cut emanating from the origin (for example the branch cut can be taken to be along negative real axis as in the diagram below). A closed curve encircling the origin must pass through the branch cut of $F(z)$. Now remove a small part of the closed curve so that the end points are on opposite sides of the branch cut. Then $F(z)$ at the end points differs by $2\pi i$.



[1] In general $v(x,y) = \tan^{-1}(y/x) + n\pi$; the integer n changes when passing through the imaginary axis or a branch cut.. This does not affect the computation of the partial derivatives.

3. i) C is a circular contour of radius R centred at w which can be parametrized through $z = w + Re^{i\theta}$ with $0 \leq \theta \leq 2\pi$. This gives

$$\frac{1}{2\pi i}\oint_C \frac{\overline{f(z)}}{z-w}dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\overline{f(z+Re^{i\theta})}}{Re^{i\theta}} iRe^{i\theta}d\theta = \frac{1}{2\pi}\int_0^{2\pi} \overline{f(z+Re^{i\theta})}d\theta$$

$$= \overline{\frac{1}{2\pi}\int_0^{2\pi} f(z+Re^{i\theta})d\theta}.$$

This is the complex conjugate of the the average value of f over the circle. In the lectures it was shown that the average value of an analytic function over a circle is the value of the function at the centre. Accordingly

$$\frac{1}{2\pi i}\oint_C \frac{\overline{f(z)}}{z-w}dz = \overline{f(w)}.$$

ii) $\overline{f(z)}/(z-w)$ is not analytic (unless f is constant) so the deformation theorem does not apply. The analysis of part i) does not extend to non-circular contours.

4. Let $f(z)$ be a polynomial. Assume that f has no zero. Then the minimum value of $|f(z)|$ is non zero (as f is polynomial $|f(z)| \to \infty$ as $|z| \to \infty$). Therefore $|f(z)| > c$ for some positive constant c. Let $g(z) = 1/f(z)$ which is entire if f has no zero. But $|g(z)| < 1/c$ so that g is bounded. By Liouville's theorem g must be constant. Therefore f is constant. A contradiction. Therefore our assumption that f has no zero must be false.