Mathematical Methods **Spring Term 2017**

Answers to Easter Problem Sheet

1. i)

$$L = -mc^2\sqrt{1-\frac{\dot{x}^2}{c^2}},$$

$$p = \frac{\partial L}{\partial \dot{x}} = \frac{m\dot{x}}{\sqrt{1-\frac{\dot{x}^2}{c^2}}}.$$

p is constant since L does not depend on x (or x is cyclic). Hence $\dot{x}$ is constant.

ii)

$$L = -mc^2\sqrt{1-\frac{\dot{x}^2}{c^2}} + qEx,$$

The Euler Lagrange equation can be written

$$\frac{dp}{dt} = \frac{\partial L}{\partial x} = qE,$$

giving $p(t) = qEt$. The formula for p obtained in part i) is unchanged - rearranging this

$$\dot{x} = \frac{p}{\sqrt{m^2 + p^2c^{-2}}} = \frac{qEt}{\sqrt{m^2 + q^2c^{-2}E^2t^2}},$$

which can be integrated to give

$$x(t) = \frac{c^2}{qE}\sqrt{m^2 + q^2c^{-2}E^2t^2} + \text{constant}.$$

2.

$$i)\quad f(z) = \frac{e^{iz}}{1+z^2} = \frac{e^{iz}}{(z+i)(z-i)}$$

which has simple poles at $z = \pm i$. The residues are $\text{Res}(f, i) = e^{-1}/(2i)$ and $\text{Res}(f, -i) = e/(-2i)$.

$$ii)\quad f(z) = \frac{1}{(z+1)(z+2)(z+3)}$$

has simple poles at -1, -2 and -3. The residues are

$$\mathrm{Res}(f,-1) = \frac{1}{(-1+2)(-1+3)} = \frac{1}{2}, \quad \mathrm{Res}(f,-2) = \frac{1}{(-2+1)(-2+3)} = -1,$$

$$\mathrm{Res}(f,-3) = \frac{1}{(-3+1)(-3+2)} = \frac{1}{2}.$$

3. The (principal value of the) integral of $f(z) = 1/z$ over the given square contour, C, is $i\pi/2$. Here f has a simple pole at the origin with residue 1. There are no singularities inside C but the contour crosses the origin. Applying the half-residue rule gives $\mathrm{P}\oint_c f(z)dz = i\pi$. This is wrong because the half-residue rule only applies if the simple pole is on a smooth part of a contour. The half-residue rule avoids having to use a semi-circular indentation. As the pole is on a corner of the square the necessary indentation is a quarter-circle rather than a semi-circle.

4. i) Consider $g(x) = x^2 e^{-\alpha x^2}$. From Problem Sheet 4 $f(x) = e^{-ax^2}$ has Fourier transform $\hat{f}(k) = e^{-k^2/(4a)}/\sqrt{4\pi a}$. $g(x) = -\partial/\partial\alpha\ f(x)$. Accordingly

$$\hat{g}(k) = -\frac{\partial}{\partial\alpha}\hat{f}(k) = -\left(-\frac{1}{2a} + \frac{k^2}{4a^2}\right)\frac{1}{\sqrt{4\pi a}} e^{-k^2/(4a)}.$$

Setting $a = \frac{1}{2}$ yields the Fourier integral

$$x^2 e^{-\frac{1}{2}x^2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (1-k^2) e^{-\frac{1}{2}k^2} e^{ikx}\ dk.$$

ii) A particular solution to the ODE $\ddot{x}(t) + 3\dot{x}(t) + 2x(t) = t^2 e^{-\frac{1}{2}t^2}$ is

$$x(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{(1-\omega^2)e^{-\frac{1}{2}\omega^2}e^{i\omega t}}{-\omega^2 + 3i\omega + 2} dw.$$

5.

$$f'(x) = 2\delta(x) - \frac{2}{\pi}\frac{1}{1+x^2}.$$

Therefore

$$\hat{f'}(k) = ik\hat{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[2\delta(x) - \frac{2}{\pi}\frac{1}{1+x^2}\right] e^{-ikx}\ dx = \frac{1}{\pi} - \frac{e^{-|k|}}{\pi}.$$

In the last step the Fourier integral from Q1 of Problem Sheet 7 was used (alternatively use contour integration). Accordingly,

$$\hat{f}(k) = \frac{i(e^{-|k|} - 1)}{\pi k}.$$

6. i)

$$\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar}\psi(x)\ dx = \sqrt{\frac{2\pi}{\hbar}}\hat{\psi}(p/\hbar).$$

Therefore

$$\int_{-\infty}^{\infty} \tilde{\psi}^*(p)\tilde{\psi}(p)dp = \frac{2\pi}{\hbar}\int_{-\infty}^{\infty} \hat{\psi}^*(p/\hbar)\hat{\psi}(p/\hbar)dp = 2\pi\int_{\infty}^{\infty} |\hat{\psi}(k)|^2 dk$$

using the change of variable $k = p/\hbar$. By Parseval's formula this is equal to

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx$$

which is 1 if $\psi(x)$ is normalised.

ii) The momentum-space wave function is

$$\tilde{\psi}(\mathbf{p}) = \frac{1}{(2\pi\hbar)^{3/2}} \int e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar}\psi(\mathbf{r})\ d^3r$$

$$= \frac{1}{(2\pi\hbar)^{3/2}} \int_0^{2\pi} d\phi \int_0^{\infty} dr \int_0^{\pi} r^2 \sin\theta\ d\theta\ e^{-ipr\cos\theta/\hbar} e^{-r/a}.$$

$$= \frac{2\pi}{(2\pi\hbar)^{3/2}} \int_0^{\infty} dr\ \frac{r\hbar}{ip} e^{-ipr\cos\theta/\hbar} e^{-r/a}\Big|_{\theta=0}^{\theta=\pi}.$$

$$= \frac{1}{(2\pi\hbar)^{1/2}} \int_0^{\infty} dr \frac{r}{ip}\left(e^{+ipr/\hbar} - e^{-ipr/\hbar}\right) e^{-r/a}$$

$$= \frac{-i}{(2\pi\hbar)^{1/2}p}\left[\frac{1}{(a^{-1} - ip/\hbar)^2} - \frac{1}{(a^{-1} + ip/\hbar)^2}\right] = \frac{1}{(2\pi\hbar)^{1/2}} \frac{4a^{-1}/\hbar}{(a^{-2} + p^2/\hbar^2)^2}.$$

Here the following integral was used

$$\int_0^{\infty} re^{-br} dr = \frac{1}{b^2} \qquad (\text{Re}\ b > 0)$$

was used (derivation via parts).

$$\tilde{\psi}(\mathbf{p}) = \frac{1}{(2\pi\hbar)^{1/2}} \frac{4a^3\hbar^3}{(\hbar^2 + a^2|\mathbf{p}|^2)^2}$$

7. The divergence of a polar vector is a scalar. As a pseudo vector picks up an extra factor of $\det R$ the divergence of an axial vector is a pseudo scalar. Hence ρ_m is a pseudo scalar.

As both $\nabla \times \mathbf{E}$ and $\partial\mathbf{B}/\partial t$ are axial $\mathbf{j}_m$ must be axial as well.

From Problem Sheet 7 the two homogeneous Maxwell equations can be written as

$$\partial_i E_j - \partial_j E_i = -\frac{\partial F_{ij}}{\partial t},$$

and

$$\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0.$$

Including a magnetic density and current:

$$\partial_i E_j - \partial_j E_i = -\frac{\partial F_{ij}}{\partial t} + k_{ij},$$

and

$$\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = \lambda_{ijk}.$$

where k_{ij} is the rank 2 anti-symmetric tensor (not a pseudo tensor)

$$k_{ij} = \epsilon_{ijk}(\mathbf{j_m})_k,$$

and λ_{ijk} is the rank three totally anti-symmetric isotropic tensor (not a pseudo-tensor) $\lambda_{ijk} = \rho_m \epsilon_{ijk}$.