1. Consider the map

$$T : \bar{x} = y, \qquad \bar{y} = 7 - x - 8 \sin y.$$

Take the square $\Pi : \{0 \le x \le \pi,\ 0 \le x \le \pi\}$ and write the map in the cross-form on Π:

$$\bar{x} = f_j(x, \bar{y}), \qquad y = f_j(x, \bar{y})$$

where $j = 1$ or 2, and

$$f_1 = \arcsin(\frac{1}{8}(7 - x - \bar{y})), \qquad f_2 = \pi - \arcsin(\frac{1}{8}(7 - x - \bar{y})).$$

As $(x, \bar{y})$ run Π, the value of f_1 stays inside $(0, \frac{\pi}{2}$, and f_2 stays inside $(\frac{\pi}{2}, \pi)$, which implies that $T^{-1}(\Pi) \cap \Pi$ consists of 2 connected components, Π_1 and Π_2, where $\Pi_j : \{y \in \text{range } f_j\}$. These components form a Markov partition: to check this one must verify that

$$\|\frac{\partial f_j}{\partial x}\| + \|\frac{\partial f_j}{\partial y}\| < 1$$

on Π_j. This inequality reduces to

$$\frac{1}{4|\cos f|} < 1$$

or

$$|\sin f| < \frac{\sqrt{15}}{4} \iff \frac{1}{8}|7 - x - \bar{y}| < \frac{\sqrt{15}}{4},$$

which is, of course, true for $(x, \bar{y}) \in \Pi$: maximum of $7 - x - \bar{y}$ on Π is achieved at $(x, y) = (0, 0)$ and equals to $7 < 2\sqrt{15}$.

Thus, we have a Markov partition with 2 components, which gives the sought Smale horseshoe (we have checked condition 1-3 from the lecture notes).

2. Consider the map

$$T : \bar{x} = y, \qquad \bar{y} = -x + 7y - y^3.$$

Let us first find its points of period 2. Any such point (x, y) satisfies the relation $(x, y) = T(\bar{x}, \bar{y})$ where $(\bar{x}, \bar{y}) = T(x, y) \neq (x, y)$. Thus,

$$x = -x + 7y - y^3, \qquad y = -y + 7x - x^3.$$

By taking the sum and the difference of these equations, we get

$$5(x + y) = x^3 + y^3, \qquad 9(x - y) = x^3 - y^3.$$

Since $x \neq \bar{x} = y$, we find that

$$x + y = 0, \qquad 9 = x^2 + xy + y^2,$$

which gives $(x, y) = (3, -3)$ or $(x, y) = (-3, 3)$, or

$$5 = x^2 - xy + y^2, \qquad 9 = x^2 + xy + y^2,$$

which gives

$$xy = 2 \Longrightarrow (x + y)^2 = 11 \Longrightarrow (x + \frac{2}{x})^2 = 11 \Longrightarrow x^2 + \frac{4}{x^2} = 7,$$

so $(x, y) = \pm(\sqrt{\frac{7-\sqrt{33}}{2}}, \sqrt{\frac{7+\sqrt{33}}{2}})$ or $(x, y) = \pm(\sqrt{\frac{7+\sqrt{33}}{2}}, \sqrt{\frac{7-\sqrt{33}}{2}})$. Altogether, the map has 6 points of period 2, which means 3 orbits of period 2 (each of the orbits contains exactly 2 points). Note that all the points of period 2 satisfy the bounds $|x| \leq 3, |y| \leq 3$. The farthest of these points, $(x, y) = (-3, 3)$ and $(x, y) = (3, -3)$ are the 2 points of the same orbit: $T(-3, 3) = (3, -3)$.

Take the square $\Pi : \{|x| \leq 3, |y| \leq 3\}$ with the vertices at these two points. Let us show that T has no periodic points outside of Π. To this aim, it is enough to show that every orbit which starts outside of Π tends to infinity either at forward or backward iterations of T.

We will decompose the complement to Π into several regions, and check for each region that the orbits starting there tend to infinity indeed. We start with the regions $Q_1 : \{y \geq -x,\ y > 3$ and $Q_2 : \{y < -x,\ y \leq -3\}$. The region Q_1 is mapped inside Q_2. Indeed, for $(x, y) \in Q_1$ we have

$$\bar{y} = -x + 7y - y^3 \leq 8y - y^3 = -y + 9y - y^3 < -y + 9 \cdot 3 - 3^3 = -y < -3,$$

and

$$\bar{x} + \bar{y} = -x + 8y - y^3 \leq 9y - y^3 < 0,$$

i.e. $(\bar{x}, \bar{y}) \in Q_2$. Note also that

$$|\bar{y}| > |y|. \tag{1}$$

Similarly, if $(x, y) \in Q_2$, then $T(x, y) \in Q_1$, and (1) also holds (just note that the map T is symmetric with respect to the transformation $(x, y) \leftrightarrow (-x, -y)$ which interchanges Q_1 and Q_2). Thus, any forward orbit starting in $Q_1 \cup Q_2$ stays in this region forever, moreover the absolute value of the y coordinate grows monotonically with the iterations (while the sign of y alternates). It follows that the orbit tends to infinity: would the y coordinate stay bounded, $|y|$ would tend to a limit, and the orbit would tend to a point of period 2 inside $Q_1 \cup Q_2$, but all the points of period 2, as we have shown, do not lie there.

In the same way one shows that any backward orbit which starts in the region $Q_3 \cup Q_4$ where $Q_3 : \{x \geq -y,\ x > 3$ and $Q_4 : \{x < -y,\ x \leq -3\}$ never leaves this region and tend to infinity. Indeed, the backward orbits of T are the forward orbits of T^{-1}, and the map

$$T^{-1} : y = \bar{x}, \qquad x = -\bar{y} + 7\bar{x} - \bar{x}^3$$

coincides with T after the transformation $x \leftrightarrow y$, which maps $Q_1 \cup Q_2$ to $Q_3 \cup Q_4$, so everything we prove about the orbits of T in $Q_1 \cup Q_2$ holds also true for the orbits of T^{-1} in $Q_3 \cup Q_4$.

We have shown that for any point in $Q_1 \cup Q_2 \cup Q_3 \cup Q_4$ either its forward orbit or its backward orbit tends to infinity. Therefore, all bounded orbits stay in the complement to $Q_1 \cup Q_2 \cup Q_3 \cup Q_4$, i.e. in Π.

Thus, to answer the question about the structure of the set of periodic orbits of the map T it is enough to consider the behaviour of this map on Π. Let us show that $T^{-1}(\Pi) \cap \Pi$ consists of 3 connected components, and these components form a Markov partition. We write the map T in the cross-form:

$$\bar{x} = \varphi(x + \bar{y}), \qquad y = \varphi(x + \bar{y}), \tag{2}$$

where φ is inverse to the polynomial $P(y) = 7y - y^3$ (i.e. $P(\varphi(u)) = u$). The function φ is multivalued, namely it has 3 branches (as P is a polynomial of degree 3). In particular, for each $u \in [-6, 6]$ there exists exactly 3 values, $\varphi_{-1}(u)$, $\varphi_0(u)$ and $\varphi_1(u)$, such that $P(\varphi_j(u)) = u$ ($j = -1, 0, 1$). To see this, note that P has 3 intervals of monotonicity: from $-\infty$ to the minimum point at $y = -\sqrt{7/3} > -3$, from the minimum to the maximum at $y = +\sqrt{7/3} > -3$, and from the maximum to $+\infty$. The value of P at the minimum point is $-14\sqrt{7/27} < -6$, and the value of P at the maximum point is $+14\sqrt{7/27} > 6$; note also that $P(\pm 3) = \pm 6$. Thus, for $u \in [-6, 6]$ we indeed have 3 branches of the inverse to P: $\varphi_0(u) \in (-\sqrt{7/3}, \sqrt{7/3})$, $\varphi_{-1}(u) \in [-3, -\sqrt{7/3})$, $\varphi_1(u) \in (\sqrt{7/3}), 3]$.

It now follows from (2) that $T^{-1}(\Pi) \cap \Pi$ consists indeed of exactly 3 connected components, and the image by T of each of the components intersects all of them: when x and $\bar{y}$ run the interval $I = [-3, 3]$, the value of $u = x + \bar{y}$ runs from -6 to 6, so all 3 values of $\varphi(u)$ run within the same interval $I = [-3, 3]$ (and $\varphi_j(u) \neq \varphi_k(v)$ if $j \neq k$).

Thus, we will get that on the set of all orbits that never leave $\Pi = [-3, 3] \times [-3, 3]$ (i.e. on the set Λ of all bounded orbits) the map T is topologically conjugate to the topological Markov chain defined by the matrix

$$G = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

if we check the hyperbolicity property, which reads, in the case of map (2), as

$$|\varphi'(u)| < \frac{1}{2}$$

for all $u \in [-6, 6]$. Since φ is inverse to P, we can rewrite this condition as

$$|P'(y)| > 2 \quad \text{if} \quad |P(y)| \leq 3$$

(when u runs the interval $[-6, 6]$, the value of $y = \varphi(u)$ stays in $[-3, 3]$). In other words, we must show that

$$|P(y)| \leq 3 \implies |y| < \sqrt{5/3} \quad \text{or} \quad |y| > \sqrt{3}.$$

Now note that the points of extremum of P (the points $y = \pm\sqrt{7/3}$) lie in the intervals $(-3, -\sqrt{5/3})$ and $(\sqrt{5/3}, 3)$, i.e. P is monotone at $|y| < \sqrt{5/3}$ or at $|y| > \sqrt{3}$. Therefore, it is enough to check that $P(\sqrt{5/3}) > 3$, $P(-\sqrt{5/3}) < -3$, $P(\sqrt{3}) > 3$ and $P(-\sqrt{3}) < -3$, which is a trivial exercise. Therefore, the topological conjugacy of $T|_\Lambda$ to the topological Markov chain defined by G is established.

The eigenvalues of the matrix G are $(3, 0, 0)$, so the topological entropy of the Markov chain is $\log 3$, i.e. it is positive, which implies that the number of periodic orbits is infinite. The number of points of period k equals to

$$tr(G^k) = 3^k.$$

So T has 3 fixed points, 9 points of period 2, 27 points of period 3, and 729 points of period 6. Altogether we have $27 + 9 - 3 = 33$ different points of period 2 and 3 (since the fixed points are also points both of period 2 and period 3, we should not count them twice). These points are also points of period 6, i.e. we have 33 points of period 6 whose least period is smaller than 6. Thus, we have $729 - 33 = 696$ points whose least period is exactly 6. The number of corresponding periodic orbits is $696/6 = 116$ (each orbit consists of exactly 6 of these points).

3. As we just have proved, the map

$$g : \bar{x} = y, \qquad \bar{y} = -x + 7y - y^3$$

on the set Λ_3 of all points whose orbits are bounded is topologically conjugate to the walk along the edges of the graph defined by the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, i.e. to the shift on the set of all bi-infinite sequences of symbols $\{1, 2, 3\}$.

In the same way ones shows that the map

$$f : \bar{x} = y, \qquad \bar{y} = 11 - x - y^2$$

on the set Λ_2 of all points whose orbits are bounded is topologically conjugate to the shift on the set of all bi-infinite sequences of symbols $\{1, 2\}$. So $f|_{\Lambda_2}$ is topologically conjugate to g restricted to the subset of Λ_3 which corresponds to the sequences composed of 1's and 2's only.

On the other hand, the topological entropy of $f|_{\Lambda_2}$ is $\log 2$, and the topological entropy of f restricted to any invariant subset of Λ_2 cannot therefore exceed $\log 2$. Since the topological entropy of $f|_{\Lambda_3}$ is $\log 3 > \log 2$, we immediately obtain that $f|_{\Lambda_3}$ cannot be topologically conjugate to g restricted to Λ_2 or to any subset of Λ_2 (since topologically conjugate maps must have the same topological entropy).

4. Let us show that any two zero-dimensional uniformly-hyperbolic sets are homeomorphic to each other. Any such set is homeomorphic to the set of paths along the edges of a certain oriented graph G (defined by the Markov partition). The path is an infinite sequence $\alpha = \{\alpha_i\}_{i=-\infty}^{+\infty}$ of edges of G such that the end vertex of the edge α_i is the beginning of the edge α_{i+1}, for every i. To paths α and β are close if $\alpha_i = \beta_i$ for all $|i| \le N$ where N is sufficiently close.

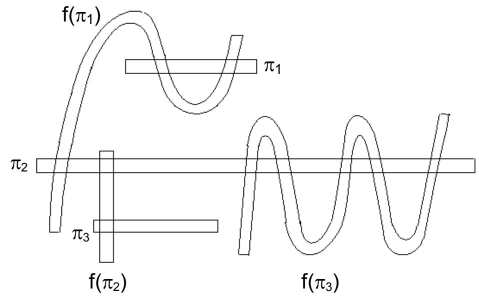
We will call a finite, of length $(2N+1)$, path $\{\alpha_i\}_{|i|\le N}$ along the edges of G a path of level N. We say that a path $\{\alpha_i\}_{|i|\le N+1}$ of level $N+1$ is subordinate to a path $\{\beta_i\}_{|i|\le N}$ of level N if $\alpha_i = \beta_i$ for all $|i| \le N$. Let us build a tree T_G whose vertices at a level N are all paths of level N, and from each vertex an edge is issued to each subordinate path. Thus, each infinite path $\{\alpha_i\}_{i=-\infty}^{+\infty}$ in the graph G corresponds to a path $\{\alpha_0\}, \{\alpha_{-1}\alpha_0\alpha_1\}, \dots, \{\alpha_{-N}\dots\alpha_0\dots\alpha_N\}, \dots$ along the edges of the tree T_G. This is a one-to-one correspondence, and if two paths in G are close, then their corresponding paths in T_G are also close. Thus, we have established a homeomorphism between the set of paths in G and the set of paths in T_G.

Let us show that the set of the paths in T_G is homeomorphic to the one-dimensional Cantor set C. The set C is an infinite closed subset of a segment I of a straight line such that it has no isolated points (we also assume that the end points of I are points of C). The complement to C in I is a countable sequence of open intervals $J_1, J_2, \dots$ which we order according to their lengths, i.e. $len(J_i) \le len(J_k)$ if $i \ge k$. When we remove any finite number of these open intervals from I, we obtain a finite set of closed segments such that C is contained within these closed segments and the end points of the segments are points of C. For each of these closed segments, the subset of C which lies in the segment is also a Cantor set; we will call these Cantor sets the Cantor subsets of C. Let us now establish a correspondence between some of such defined Cantor subsets and the vertices of the tree T_G. Let T_G have m vertices of zero level. Remove from I $(m-1)$ first of the intervals J_i (i.e. $(m-1)$ intervals of the maximal length). The Cantor set C will then be decomposed into m Cantor subsets. We will call these subsets the Cantor subsets of zero level and put them into a one-to-one correspondence with the zero level vertices of the tree T_G in an arbitrary way. Then we proceed inductively: if a vertex A of level N corresponds to a certain Cantor subset B, and this vertex has k subordinate vertices, then we take the segment bounded by the utmost right and utmost left points of B, consider those open intervals of J_i which lies in this segment, of them we pick up and remove $(k-1)$ of maximal length, after which the subset B becomes decomposed into k Cantor subsets, and we put these subsets, in an arbitrary way, into the correspondence to the k vertices of level $(N+1)$ which are subordinate to the vertex A.

In this way, each infinite path in the tree T_G corresponds to a sequence of Cantor subsets C_j such that $C_{j+1} \subset C_j$, moreover the diameter of C_j tends to zero as $j \to +\infty$. The latter fact means that there is only one point in C which belongs to all C_j, and it is obvious that every point in C can be obtained in this way. Thus, we built a one-to-one correspondence between the infinite paths in T_G and the points of C. If two paths are close, then the corresponding points in C belong to the same Cantor subset of a sufficiently high level, i.e. the corresponding points are close to each other, since the diameter of the Cantor subsets tends uniformly to zero as their level in the tree grows.

This proves that the set of infinite paths in the tree T_G (hence, the set of infinite path in the graph G, hence - the hyperbolic set under consideration) is homeomorphic to C. Since this is the same set C, independent of the graph G, it follows that the hyperbolic sets under consideration are all homeomorphic indeed.

5. For the Markov partition



shown in the figure, the map on the corresponding hyperbolic set Λ is topologically conjugate to the walk along the edges of the graph with 3 vertices, π_1, π_2, π_3, defined by the matrix

$$\mathcal{N} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 5 & 0 \end{pmatrix}.$$

This graph is not transitive – there is no path which leads from π_2 or π_3 to π_1, i.e. the orbits from Λ which leave π_1 can never return to π_1. This, in particular, implies that there is no dense orbit in Λ: a dense orbit must visit π_1, π_2 and π_3 infinitely often, which is impossible here, since once the orbit leaves π_1 for π_2 or π_3, there is no return.

Also, periodic orbits of Λ form 2 groups: those staying in π_1 and those staying in $\pi_2 \cup \pi_3$. The number of the period-k points of the first group is 2^k, and the number of the period-k points of the second group equals to $tr(\mathcal{N}')^k$ where we denote as $\mathcal{N}'$ the block of the matrix $\mathcal{N}$ on the inetsection of the second and third column with the second and third rows: $\mathcal{N}' = \begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix}$. Obviously,

$$(\mathcal{N}')^{2m} = \begin{pmatrix} 5^m & 0 \\ 0 & 5^m \end{pmatrix}$$

and

$$(\mathcal{N}')^{2m+1} = \begin{pmatrix} 0 & 5^m \\ 5^{m+1} & 0 \end{pmatrix}.$$

Thus, the number P_k of periodic points of the period k is $2^k + 5^{k/2}$ for even k, and 2^k for odd k. This gives us

$$\limsup_{k\to+\infty} \frac{\ln P_k}{k} = \ln\sqrt{5} \quad \text{and} \quad \liminf_{k\to+\infty} \frac{\ln P_k}{k} = \ln 2.$$